

Chapter 2

Optimal Control

In this chapter, optimal control problems for nonlinear and linear equations are considered. Necessary condition for optimality of control for nonlinear control problem is obtained and it is used for construction of the synthesis of optimal control for a linear-quadratic problem.

2.1 A Necessary Condition for Optimality of Control for Nonlinear System

2.1.1 Statement of the Problem

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a basic probability space, $\mathbf{E}$ be an expectation, $Z_0 = \{-h, -h + 1, \dots, -1, 0\}$, $h \in [0, \infty]$, and $Z = \{0, 1, \dots, N\}$ be a discrete time, $\mathfrak{F}_i \subset \mathfrak{F}$, $i \in Z$, be a family of σ -algebras, $\mathbf{E}_i = \mathbf{E}\{\cdot/\mathfrak{F}_i\}$ be a conditional expectation, H_0 be a space of $\mathfrak{F}_0$ -adapted random functions $\varphi(j) \in \mathbf{R}^n$, $j \in Z_0$, such that

$$\|\varphi\|^2 = \max_{j \in Z_0} \mathbf{E}|\varphi(j)|^2 < \infty, \quad (2.1)$$

H be a space of $\mathfrak{F}_i$ -adapted random functions $x(i) \in \mathbf{R}^n$, $i \in Z$, such that

$$\|x\|_i^2 = \max_{0 \leq j \leq i} \mathbf{E}|x(j)|^2 < \infty. \quad (2.2)$$

Consider the nonlinear controlled system

$$\begin{aligned} x(i+1) &= \eta(i+1) + \Phi(i+1, x_{i+1}) \\ &+ \sum_{j=0}^i a(i, j, x_j, u(j)) + \sum_{j=0}^i b(i, j, x_j, u(j))\xi(j+1), \\ i &= 0, 1, \dots, N-1, \quad x(j) = \varphi_0(j), \quad j \in Z_0, \quad \varphi_0 \in H_0, \end{aligned} \quad (2.3)$$

with the performance functional

$$J(u) = \mathbf{E} \left[F(x_N) + \sum_{j=0}^{N-1} G(j, x_j, u(j)) \right]. \quad (2.4)$$

Here, $x(i)$ is a value of the process x in the time moment i , x_i is a trajectory of $x(j)$ from the moment of time $j = -h$ until the moment of time $i \in Z$, $\eta \in H$, $\xi(i) \in \mathbf{R}^l$ is a sequence of $\mathfrak{F}_i$ -adapted Gaussian random variables, that are mutually independent and do not depend on $\eta(i)$ such that $\mathbf{E}\xi(i) = 0$, $\mathbf{E}\xi(i)\xi'(i) = I$, $i \in Z$, I is the identical matrix.

Definition 2.1 Arbitrary $\mathfrak{F}_i$ -adapted function $u(i) \in \mathbf{R}^m$, $i \in Z$, for which there exists the solution of the equation (2.3) and the performance functional (2.4) is bounded, is called an admissible control, $\mathbf{U}$ is the set of admissible controls.

Our nearest goal is to calculate the limit (1.9), (1.10) for the optimal control problem (2.3), (2.4). For this goal, we will assume that the functionals $\Phi(i, \varphi) \in \mathbf{R}^n$, $a(i, j, \varphi, u) \in \mathbf{R}^n$, $F(\varphi) \in \mathbf{R}^1$, $G(j, \varphi, u) \in \mathbf{R}^1$, and $n \times l$ -matrix $b(i, j, \varphi, u)$ have Gateaux derivatives with respect to φ denoted hereafter by ∇ and the functionals $a(i, j, \varphi, u)$, $b(i, j, \varphi, u)$, and $G(j, \varphi, u)$, $0 \leq j \leq i \leq N$ have derivatives with respect to u denoted hereafter by ∇_u , $u \in \mathbf{R}^m$, $\varphi \in \tilde{H}$, where $\tilde{H}$ is a space of the functions $\varphi(i) \in \mathbf{R}^n$, $i \in Z_0 \cup Z$.

Let X and Y be two normed spaces and $f(x)$ be a map of X to Y . Then the Gateaux derivative $\nabla f(\varphi_0)$ of this map for a fixed $\varphi_0 \in \tilde{H}$ is [110] a linear operator that maps X to Y . If $Y = \mathbf{R}^1$ then $\langle \nabla f(\varphi_0), \varphi \rangle$ is the value of the linear functional $\nabla f(\varphi_0)$ on the element $\varphi \in \tilde{H}$.

Also we assume that the functional $\Phi(i, \varphi)$ depends on the values of the function $\varphi(j)$ for $j = -h, -h+1, \dots, i$, the functionals $a(i, j, \varphi, u)$, $b(i, j, \varphi, u)$ depend on the values of the function $\varphi(l)$ for $l = -h, -h+1, \dots, j$, $\varphi \in \tilde{H}$ and satisfy the conditions

$$|\Phi(i, \varphi)| \leq \sum_{j=-h}^i (1 + |\varphi(j)|) K_0(j), \quad (2.5)$$

$$|a(i, j, \varphi, u)|^2 + |b(i, j, \varphi, u)|^2 \leq \sum_{l=-h}^j (1 + |u|^2 + |\varphi(l)|^2) K_1(l), \quad (2.6)$$

for arbitrary functions $\varphi_1, \varphi_2 \in \tilde{H}$

$$|\Phi(i, \varphi_1) - \Phi(i, \varphi_2)| \leq \sum_{j=-h}^i |\varphi_1(j) - \varphi_2(j)| K_0(j), \quad (2.7)$$

$$|a(i, j, \varphi_1, u) - a(i, j, \varphi_2, u)|^2 + |b(i, j, \varphi_1, u) - b(i, j, \varphi_2, u)|^2 \leq \sum_{l=-h}^j |\varphi_1(l) - \varphi_2(l)|^2 K_1(l), \quad (2.8)$$

$$|\nabla \Phi(i, \varphi_1) \varphi| \leq \sum_{j=0}^i |\varphi(j)| K_0(j), \quad (2.9)$$

$$|\nabla a(i, j, \varphi_1, u) \varphi|^2 + |\nabla b(i, j, \varphi_1, u) \varphi|^2 \leq \sum_{l=0}^j |\varphi(l)|^2 K_1(l), \quad (2.10)$$

$$|\nabla_u a(i, j, \varphi, u)| + |\nabla_u b(i, j, \varphi, u)| \leq C, \quad (2.11)$$

$$|(\nabla \Phi(i, \varphi_1) - \nabla \Phi(i, \varphi_2)) \varphi|^2 \leq \sum_{j=-h}^i |\varphi_1(j) - \varphi_2(j)|^2 |\varphi(j)|^2 K_0(j). \quad (2.12)$$

Here and everywhere below, it is supposed that

$$K_0 = \max_{-h \leq i \leq N} K_0(i) < 1, \quad K_1 = \max_{-h \leq i \leq N} K_1(i) < \infty, \\ \text{if } h = \infty \text{ then } \sum_{j=-h}^N K_l(j) < \infty, \quad l = 0, 1. \quad (2.13)$$

For arbitrary $u_1, u_2 \in \mathbf{U}$

$$(\nabla a(i, j, \varphi_1, u_1) - \nabla a(i, j, \varphi_2, u_2)) \varphi|^2 + |(\nabla b(i, j, \varphi_1, u_1) - \nabla b(i, j, \varphi_2, u_2)) \varphi|^2 \leq \sum_{l=0}^j \left(|\varphi_1(l) - \varphi_2(l)|^2 + |u_1 - u_2|^2 \right) |\varphi(l)|^2 K_1(l), \quad (2.14)$$

$$|\nabla_u a(i, j, \varphi, u_1) - \nabla_u a(i, j, \varphi, u_2)| + |\nabla_u b(i, j, \varphi, u_1) - \nabla_u b(i, j, \varphi, u_2)| \leq C|u_1 - u_2|. \quad (2.15)$$

Regarding the performance functional we assume that $F(\varphi)$ depends on the values of the function $\varphi(j)$ for $j = -h, -h + 1, \dots, N$, $G(i, \varphi, u)$ depends on

the values of the function $\varphi(j)$ for $j = -h, -h + 1, \dots, i$, $\varphi \in \tilde{H}$, and both these functionals satisfy the conditions

$$|F(\varphi)| \leq \sum_{j=-h}^N \left(1 + |\varphi(j)|^2\right) K_1(j), \quad (2.16)$$

$$|G(i, \varphi, u)| \leq \sum_{j=-h}^i \left(1 + |u|^2 + |\varphi(j)|^2\right) K_1(j), \quad (2.17)$$

for arbitrary functions $\varphi_1, \varphi_2 \in \tilde{H}$

$$|\langle \nabla F(\varphi_1), \varphi \rangle| \leq \sum_{j=0}^N (1 + |\varphi_1(j)|) |\varphi(j)| K_1(j), \quad (2.18)$$

$$|\langle \nabla G(i, \varphi_1, u), \varphi \rangle| \leq \sum_{j=0}^i (1 + |u| + |\varphi_1(j)|) |\varphi(j)| K_1(j), \quad (2.19)$$

$$|\langle \nabla F(\varphi_1) - \nabla F(\varphi_2), \varphi \rangle| \leq \sum_{j=0}^N |\varphi_1(j) - \varphi_2(j)| |\varphi(j)| K_1(j), \quad (2.20)$$

for arbitrary $\varphi_1, \varphi_2 \in \tilde{H}$, and $u_1, u_2 \in \mathbf{U}$

$$\begin{aligned} & |\langle \nabla G(i, \varphi_1, u_1) - \nabla G(i, \varphi_2, u_2), \varphi \rangle| \\ & \leq \sum_{j=0}^i (|\varphi_1(j) - \varphi_2(j)| + |u_1 - u_2|) |\varphi(j)| K_1(j), \end{aligned} \quad (2.21)$$

$$|\nabla_u G(i, \varphi, u)| \leq \sum_{j=0}^i (1 + |u| + |\varphi(j)|) K_1(j), \quad (2.22)$$

$$|\nabla_u G(i, \varphi, u_1) - \nabla_u G(i, \varphi, u_2)| \leq C|u_1 - u_2|. \quad (2.23)$$

2.1.2 Auxiliary Assertions

In order to calculate the limit (1.9), (1.10) for the optimal control problem (2.3), (2.4) we need the following auxiliary assertions.

Lemma 2.1 *Let the conditions (2.5), (2.6), (2.13), (2.16), (2.17) hold and $u \in H$, i.e., the control $u(i)$ satisfies the condition*

$$\|u\|_i^2 = \max_{0 \leq j \leq i} \mathbf{E}|u(j)|^2 < \infty, \quad i \in \mathbf{Z}. \quad (2.24)$$

Then the solution $x(i)$ of the Eq. (2.3) satisfies the condition (2.2) and the performance functional (2.4) is bounded.

Proof From the Eq. (2.3), we have

$$|x(i+1)| \leq |\eta(i+1)| + |\Phi(i+1, x_{i+1})| + \sum_{j=0}^i |a(i, j, x_j, u(j))| + \left| \sum_{j=0}^i b(i, j, x_j, u(j))\xi(j+1) \right|, \quad i = 0, 1, \dots, N-1. \quad (2.25)$$

Via (2.5), (2.13) we obtain

$$\begin{aligned} |\Phi(i+1, x_{i+1})| &\leq \sum_{j=-h}^{i+1} (1 + |x(j)|)K_0(j) \\ &\leq \sum_{j=-h}^{i+1} K_0(j) + \sum_{j=-h}^0 |\varphi_0(j)|K_0(j) \\ &\quad + \sum_{j=1}^i |x(j)|K_0(j) + |x(i+1)|K_0. \end{aligned} \quad (2.26)$$

From (2.25), (2.26) via (2.13), it follows that

$$\begin{aligned} 0 &\leq (1 - K_0)|x(i+1)| \\ &\leq |\eta(i+1)| + \sum_{j=-h}^{i+1} K_0(j) + \sum_{j=-h}^0 |\varphi_0(j)|K_0(j) + \sum_{j=1}^i |x(j)|K_0(j) \\ &\quad + \sum_{j=0}^i |a(i, j, x_j, u(j))| + \left| \sum_{j=0}^i b(i, j, x_j, u(j))\xi(j+1) \right|. \end{aligned}$$

Squaring the obtained inequality, calculating the mathematical expectation, and using the properties of the process $\xi(j)$, via (1.30), we obtain

$$\begin{aligned} (1 - K_0)^2 \mathbf{E}|x(i+1)|^2 &\leq 6 \left[\mathbf{E}|\eta(i+1)|^2 + \left(\sum_{j=-h}^{i+1} K_0(j) \right)^2 \right. \\ &\quad \left. + \sum_{l=-h}^0 K_0(l) \sum_{j=-h}^0 \mathbf{E}|\varphi_0(j)|^2 K_0(j) \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{l=1}^i K_0(l) \sum_{j=1}^i \mathbf{E}|x(j)|^2 K_0(j) \\
& + N \sum_{j=0}^i \left(\mathbf{E}|a(i, j, x_j, u(j))|^2 + \mathbf{E}|b(i, j, x_j, u(j))|^2 \right) \Bigg].
\end{aligned}$$

Note that $\varphi_0 \in H_0$, i.e., φ_0 satisfies the condition (2.1). So, via (2.6), (2.13) we have

$$\begin{aligned}
& \sum_{j=0}^i \left(\mathbf{E}|a(i, j, x_j, u(j))|^2 + \mathbf{E}|b(i, j, x_j, u(j))|^2 \right) \\
& \leq \sum_{j=0}^i \sum_{l=-h}^j (1 + \mathbf{E}|u(j)|^2 + \mathbf{E}|x(l)|^2) K_1(l) \\
& = \sum_{j=0}^i (1 + \mathbf{E}|u(j)|^2) \sum_{l=-h}^j K_1(l) \\
& \quad + \sum_{j=0}^i \left(\sum_{l=-h}^0 \mathbf{E}|\varphi_0(l)|^2 K_1(l) + \sum_{l=1}^j \mathbf{E}|x(l)|^2 K_1(l) \right) \\
& \leq \sum_{j=0}^i \left((1 + \mathbf{E}|u(j)|^2) \sum_{l=-h}^j K_1(l) + \|\varphi_0\|^2 \sum_{l=-h}^0 K_1(l) \right) \\
& \quad + N K_1 \sum_{l=1}^i \mathbf{E}|x(l)|^2.
\end{aligned}$$

As a result, using (2.13), we obtain

$$\begin{aligned}
& \mathbf{E}|x(i+1)|^2 \\
& \leq \frac{6}{(1-K_0)^2} \left[\mathbf{E}|\eta(i+1)|^2 + \left(\sum_{j=-h}^{i+1} K_0(j) \right)^2 + N^2 K_1 \sum_{l=1}^i \mathbf{E}|x(l)|^2 \right. \\
& \quad + \|\varphi_0\|^2 \left(\sum_{l=-h}^0 K_0(l) \right)^2 + K_0 \sum_{l=1}^N K_0(l) \sum_{j=1}^i \mathbf{E}|x(j)|^2 \\
& \quad \left. + N \sum_{j=0}^i \left((1 + \mathbf{E}|u(j)|^2) \sum_{l=-h}^j K_1(l) + \|\varphi_0\|^2 \sum_{l=-h}^0 K_1(l) \right) \right].
\end{aligned}$$

From this, it follows that

$$\mathbf{E}|x(i+1)|^2 \leq C \left(z(i) + \sum_{j=1}^i \mathbf{E}|x(j)|^2 \right),$$

where

$$\begin{aligned} C &= \frac{6}{(1-K_0)^2} \left(K_0 \sum_{l=1}^N K_0(l) + K_1 N^2 \right), \\ z(i) &= \left(K_0 \sum_{l=1}^N K_0(l) + K_1 N^2 \right)^{-1} \\ &\quad \times \left[\|\eta\|^2 + \|\varphi_0\|^2 \left(\sum_{l=-h}^0 K_1(l) \right)^2 + \left(\sum_{l=-h}^{i+1} K_0(l) \right)^2 \right. \\ &\quad \left. + N \sum_{j=0}^i \left(\|\varphi_0\|^2 \sum_{l=-h}^0 K_1(l) + (1 + \mathbf{E}|u(j)|^2) \sum_{l=-h}^j K_1(l) \right) \right]. \end{aligned}$$

Via $\eta, u \in H$, $\varphi_0 \in H_0$, and (2.13), we have that $z(i)$ is a nonnegative nondecreasing sequence and $\max_{0 \leq i \leq N} z(i) < \infty$. So, via Lemma 1.2 and (1.14) there exists $C_1 > 0$ such that

$$\mathbf{E}|x(i+1)|^2 \leq C_1 z(i) < \infty, \quad i = 0, 1, \dots, N-1, \quad (2.27)$$

and $x(i)$ satisfies (2.2).

For the performance functional (2.4) via (2.16), (2.17) we have

$$\begin{aligned} J(u) &\leq \sum_{j=-h}^N (1 + \mathbf{E}|x(j)|^2) K_1(j) \\ &\quad + \sum_{j=0}^{N-1} \sum_{l=-h}^j (1 + \mathbf{E}|u(j)|^2 + \mathbf{E}|x(l)|^2) K_1(l). \end{aligned}$$

From this, it follows that

$$\begin{aligned}
 J(u) \leq & \sum_{j=-h}^N K_1(j) + \sum_{j=-h}^0 \mathbf{E}|\varphi_0(j)|^2 K_1(j) + \sum_{j=1}^N \mathbf{E}|x(j)|^2 K_1(j) \\
 & + \sum_{j=0}^{N-1} (1 + |u(j)|^2) \sum_{l=-h}^{N-1} K_1(l) \\
 & + \sum_{j=0}^{N-1} \left(\sum_{l=-h}^0 \mathbf{E}|\varphi_0(l)|^2 K_1(l) + \sum_{l=1}^j \mathbf{E}|x(l)|^2 K_1(l) \right).
 \end{aligned}$$

As a result via (2.1), (2.2), (2.13), (2.24), (2.27) we obtain

$$\begin{aligned}
 J(u) & \leq (N+1)(1 + \|u\|_N^2 + \|\varphi_0\|^2 + \|x\|_N^2) \sum_{j=-h}^N K_1(j) \\
 & < \infty.
 \end{aligned}$$

The proof is completed.

Let us investigate now the behavior of the process

$$q_\varepsilon(i) = \frac{1}{\varepsilon} (x_\varepsilon(i) - x_0(i)), \quad (2.28)$$

where $x_\varepsilon(i)$ is the solution of the equation (2.3) under the control

$$u_\varepsilon(i) = u_0(i) + \varepsilon v(i), \quad \varepsilon \geq 0. \quad (2.29)$$

Definition 2.2 The process $q_\varepsilon(i)$ is called uniformly p -bounded, $p > 0$, with respect to $\varepsilon \geq 0$ if there exists $C > 0$ that does not depend on ε and such that

$$\max_{i \in Z} \mathbf{E}|q_\varepsilon(i)|^p \leq C.$$

If $p = 2$, then the process $q_\varepsilon(i)$ is called uniformly mean square bounded.

Lemma 2.2 Let the conditions (2.9)–(2.11), (2.13) hold and $\max_{i \in Z} \mathbf{E}|v(j)|^p < \infty$, $p \geq 2$. Then q_ε is uniformly p -bounded with respect to $\varepsilon \geq 0$.

Proof From the Eq. (2.3), it follows that

$$\begin{aligned}
 q_\varepsilon(i+1) &= \frac{1}{\varepsilon} [\Phi(i+1, x_{\varepsilon, i+1}) - \Phi(i+1, x_{0, i+1})] \\
 &+ \frac{1}{\varepsilon} \sum_{j=0}^i [a(i, j, x_{\varepsilon j}, u_\varepsilon(j)) - a(i, j, x_{0j}, u_0(j))] \\
 &+ \frac{1}{\varepsilon} \sum_{j=0}^i [b(i, j, x_{\varepsilon j}, u_\varepsilon(j)) - b(i, j, x_{0j}, u_0(j))] \xi(j+1). \quad (2.30)
 \end{aligned}$$

In order to transform the right-hand side of (2.30), put

$$\begin{aligned}
 \lambda_\varepsilon^\tau(i) &= x_0(i) + \tau \varepsilon q_\varepsilon(i), \quad u_\varepsilon^\tau(i) = u_0(i) + \tau \varepsilon v(i), \\
 \Phi_\varepsilon(i) &= \int_0^1 \nabla \Phi(i, \lambda_{\varepsilon i}^\tau) d\tau, \\
 A_\varepsilon(i, j) &= \int_0^1 \nabla a(i, j, \lambda_{\varepsilon j}^\tau, u_\varepsilon(j)) d\tau, \\
 a_\varepsilon(i, j) &= \int_0^1 \nabla_u a(i, j, x_{0j}, u_\varepsilon^\tau(j)) d\tau, \\
 B_\varepsilon(i, j) &= \int_0^1 \nabla b(i, j, \lambda_{\varepsilon j}^\tau, u_\varepsilon(j)) d\tau, \\
 b_\varepsilon(i, j) &= \int_0^1 \nabla_u b(i, j, x_{0j}, u_\varepsilon^\tau(j)) d\tau. \quad (2.31)
 \end{aligned}$$

Consider also the function

$$f(\tau) = \frac{1}{\varepsilon} \Phi(i, \lambda_{\varepsilon i}^\tau) = \frac{1}{\varepsilon} \Phi(i, x_{0i} + \tau \varepsilon q_{\varepsilon i}), \quad \tau \in [0, 1], \quad (2.32)$$

and note that via (2.32), (2.28)

$$\begin{aligned}
 f'(\tau) &= \nabla \Phi(i, \lambda_{\varepsilon i}^\tau) q_{\varepsilon i}, \\
 \int_0^1 f'(\tau) d\tau &= f(1) - f(0) = \frac{1}{\varepsilon} [\Phi(i, x_{\varepsilon i}) - \Phi(i, x_{0i})].
 \end{aligned}$$

From this and (2.31), it follows that the first summand in the right-hand side of (2.30) can be represented as follows:

$$\frac{1}{\varepsilon}[\Phi(i, x_{\varepsilon i}) - \Phi(i, x_{0i})] = \int_0^1 \nabla \Phi(i, \lambda_{\varepsilon i}^\tau) q_{\varepsilon i} d\tau = \Phi_\varepsilon(i) q_{\varepsilon i}. \quad (2.33)$$

Transform now the second summand in (2.30) by the following way:

$$\begin{aligned} & \frac{1}{\varepsilon}[a(i, j, x_{\varepsilon j}, u_\varepsilon(j)) - a(i, j, x_{0j}, u_0(j))] \\ &= \frac{1}{\varepsilon}[a(i, j, x_{\varepsilon j}, u_\varepsilon(j)) - a(i, j, x_{0j}, u_\varepsilon(j))] \\ & \quad + \frac{1}{\varepsilon}[a(i, j, x_{0j}, u_\varepsilon(j)) - a(i, j, x_{0j}, u_0(j))]. \end{aligned}$$

Similarly to (2.33) via (2.31), we obtain

$$\frac{1}{\varepsilon}[a(i, j, x_{\varepsilon j}, u_\varepsilon(j)) - a(i, j, x_{0j}, u_\varepsilon(j))] = A_\varepsilon(i, j) q_{\varepsilon i}.$$

Putting now

$$f(\tau) = \frac{1}{\varepsilon}a(i, j, x_{0j}, u_0(j) + \tau \varepsilon v(j)), \quad \tau \in [0, 1],$$

and note that

$$\begin{aligned} f'(\tau) &= \nabla_u a(i, j, x_{0j}, u_0(j) + \tau \varepsilon v(j)) v(j), \\ \int_0^1 f'(\tau) d\tau &= \frac{1}{\varepsilon}[a(i, j, x_{0j}, u_\varepsilon(j)) - a(i, j, x_{0j}, u_0(j))], \end{aligned}$$

we obtain

$$\frac{1}{\varepsilon}[a(i, j, x_{0j}, u_\varepsilon(j)) - a(i, j, x_{0j}, u_0(j))] = a_\varepsilon(i, j) v(j).$$

Continuing similar transformations, we represent (2.30) in the form

$$\begin{aligned} q_\varepsilon(i+1) &= \Phi_\varepsilon(i+1) q_{\varepsilon, i+1} + \sum_{j=1}^i A_\varepsilon(i, j) q_{\varepsilon j} + \sum_{j=1}^i B_\varepsilon(i, j) q_{\varepsilon j} \xi(j+1) \\ & \quad + \sum_{j=0}^i a_\varepsilon(i, j) v(j) + \sum_{j=0}^i b_\varepsilon(i, j) v(j) \xi(j+1), \quad q_{\varepsilon 0} = 0. \end{aligned} \quad (2.34)$$

From (2.34) via (2.31), (2.9) it follows that

$$\begin{aligned}
 & (1 - K_0(i+1))|q_\varepsilon(i+1)| \\
 & \leq \sum_{j=1}^i |q_\varepsilon(j)|K_0(j) + \sum_{j=1}^i |A_\varepsilon(i, j)q_{\varepsilon j}| + \sum_{j=0}^i |a_\varepsilon(i, j)v(j)| \\
 & \quad + \left| \sum_{j=1}^i B_\varepsilon(i, j)q_{\varepsilon j}\xi(j+1) \right| + \left| \sum_{j=0}^i b_\varepsilon(i, j)v(j)\xi(j+1) \right|.
 \end{aligned}$$

Calculating p th moments of the both parts of this inequality, via Lemma 1.7, we obtain

$$\begin{aligned}
 & (1 - K_0(i+1))^p \mathbf{E}|q_\varepsilon(i+1)|^p \\
 & \leq 5^{p-1} \left[\mathbf{E} \left(\sum_{j=1}^i |q_\varepsilon(j)|K_0(j) \right)^p + \mathbf{E} \left(\sum_{j=1}^i |A_\varepsilon(i, j)q_{\varepsilon j}| \right)^p \right. \\
 & \quad + \mathbf{E} \left(\sum_{j=0}^i |a_\varepsilon(i, j)v(j)| \right)^p + \mathbf{E} \left| \sum_{j=1}^i B_\varepsilon(i, j)q_{\varepsilon j}\xi(j+1) \right|^p \\
 & \quad \left. + \mathbf{E} \left| \sum_{j=0}^i b_\varepsilon(i, j)v(j)\xi(j+1) \right|^p \right].
 \end{aligned}$$

Note also that via Lemma 1.7, (2.13), and $M_p = \max_{j \in \mathbb{Z}} \mathbf{E}|\xi(j)|^p$ for $i \leq N-1$, we have

$$\begin{aligned}
 & \mathbf{E} \left(\sum_{j=1}^i |q_\varepsilon(j)|K_0(j) \right)^p \leq N^{p-1} K_0^p \sum_{j=1}^i \mathbf{E}|q_\varepsilon(j)|^p, \\
 & \mathbf{E} \left(\sum_{j=1}^i |A_\varepsilon(i, j)q_{\varepsilon j}| \right)^p \leq N^{p-1} \sum_{j=1}^i \mathbf{E}|A_\varepsilon(i, j)q_{\varepsilon j}|^p, \\
 & \mathbf{E} \left(\sum_{j=0}^i |a_\varepsilon(i, j)v(j)| \right)^p \leq N^{p-1} \sum_{j=0}^i \mathbf{E}|a_\varepsilon(i, j)v(j)|^p, \\
 & \mathbf{E} \left| \sum_{j=1}^i B_\varepsilon(i, j)q_{\varepsilon j}\xi(j+1) \right|^p \leq N^{p-1} \sum_{j=1}^i \mathbf{E}|B_\varepsilon(i, j)q_{\varepsilon j}|^p \mathbf{E}|\xi(j+1)|^p
 \end{aligned}$$

$$\begin{aligned}
&\leq N^{p-1} M_p \sum_{j=1}^i \mathbf{E} |B_\varepsilon(i, j) q_{\varepsilon j}|^p, \\
\mathbf{E} \left| \sum_{j=0}^i b_\varepsilon(i, j) v(j) \xi(j+1) \right|^p &\leq N^{p-1} \sum_{j=0}^i \mathbf{E} |b_\varepsilon(i, j) v(j)|^p \mathbf{E} |\xi(j+1)|^p \\
&\leq N^{p-1} M_p \sum_{j=0}^i \mathbf{E} |b_\varepsilon(i, j) v(j)|^p.
\end{aligned}$$

So, putting $\hat{M}_p = \max\{1, M_p\}$ and using (2.13), we get

$$\begin{aligned}
\mathbf{E} |q_\varepsilon(i+1)|^p &\leq \frac{(5N)^{p-1}}{(1-K_0)^p} \left[K_0^p \sum_{j=1}^i \mathbf{E} |q_\varepsilon(j)|^p \right. \\
&\quad + \hat{M}_p \sum_{j=1}^i \mathbf{E} (|A_\varepsilon(i, j) q_{\varepsilon j}|^p + |B_\varepsilon(i, j) q_{\varepsilon j}|^p) \\
&\quad \left. + \hat{M}_p \sum_{j=0}^i \mathbf{E} (|a_\varepsilon(i, j)|^p + |b_\varepsilon(i, j)|^p) |v(j)|^p \right].
\end{aligned}$$

Note that via (2.31)

$$\begin{aligned}
|A_\varepsilon(i, j) q_{\varepsilon j}|^p + |B_\varepsilon(i, j) q_{\varepsilon j}|^p &= \left| \int_0^1 \nabla a(i, j, \lambda_{\varepsilon j}^\tau, u_\varepsilon(j)) q_{\varepsilon j} d\tau \right|^p \\
&\quad + \left| \int_0^1 \nabla b(i, j, \lambda_{\varepsilon j}^\tau, u_\varepsilon(j)) q_{\varepsilon j} d\tau \right|^p \\
&\leq \int_0^1 (|\nabla a(i, j, \lambda_{\varepsilon j}^\tau, u_\varepsilon(j)) q_{\varepsilon j}|^p \\
&\quad + |\nabla b(i, j, \lambda_{\varepsilon j}^\tau, u_\varepsilon(j)) q_{\varepsilon j}|^p) d\tau. \tag{2.35}
\end{aligned}$$

Via (2.10), (2.13), $q_\varepsilon(0) = 0$, and Lemma 1.7, we obtain

$$\begin{aligned}
&|\nabla a(i, j, \lambda_{\varepsilon j}^\tau, u_\varepsilon(j)) q_{\varepsilon j}|^p + |\nabla b(i, j, \lambda_{\varepsilon j}^\tau, u_\varepsilon(j)) q_{\varepsilon j}|^p \\
&\leq \left(|\nabla a(i, j, \lambda_{\varepsilon j}^\tau, u_\varepsilon(j)) q_{\varepsilon j}|^2 + |\nabla b(i, j, \lambda_{\varepsilon j}^\tau, u_\varepsilon(j)) q_{\varepsilon j}|^2 \right)^{p/2}
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\sum_{l=1}^j |q_\varepsilon(l)|^2 K_1(l) \right)^{p/2} \\
&\leq K_1^{\frac{p}{2}} N^{\frac{p}{2}-1} \sum_{l=1}^j |q_\varepsilon(l)|^p.
\end{aligned} \tag{2.36}$$

Similarly to (2.35), (2.36) via (2.11) we have

$$\begin{aligned}
&|a_\varepsilon(i, j)|^p + |b_\varepsilon(i, j)|^p \\
&\leq \int_0^1 (|\nabla_u a(i, j, x_{0j}, u_\varepsilon^\tau(j))|^p + |\nabla_u a(i, j, x_{0j}, u_\varepsilon^\tau(j))|^p) d\tau \\
&\leq \int_0^1 (|\nabla_u a(i, j, x_{0j}, u_\varepsilon^\tau(j))| + |\nabla_u a(i, j, x_{0j}, u_\varepsilon^\tau(j))|)^p d\tau \\
&\leq C^p.
\end{aligned}$$

So, as a result we obtain

$$\begin{aligned}
\mathbf{E}|q_\varepsilon(i+1)|^p &\leq \frac{(5N)^{p-1}}{(1-K_0)^p} \left[K_0^p \sum_{j=1}^i \mathbf{E}|q_\varepsilon(j)|^p + \hat{M}_p K_1^{\frac{p}{2}} N^{\frac{p}{2}-1} \sum_{l=1}^i \sum_{j=l}^i \mathbf{E}|q_\varepsilon(l)|^p \right. \\
&\quad \left. + \hat{M}_p C^p \sum_{j=0}^i \mathbf{E}|v(j)|^p \right] \\
&\leq \frac{(5N)^{p-1}}{(1-K_0)^p} \left[\hat{M}_p C^p \sum_{j=0}^i \mathbf{E}|v(j)|^p + C_1 \sum_{j=1}^i \mathbf{E}|q_\varepsilon(j)|^p \right],
\end{aligned}$$

where $C_1 = K_0^p + \hat{M}_p (K_1 N)^{\frac{p}{2}}$. Putting also

$$C_2 = \frac{(5N)^{p-1} C_1}{(1-K_0)^p}, \quad y_\varepsilon(i) = \mathbf{E}|q_\varepsilon(i)|^p, \quad z(i) = \frac{\hat{M}_p C^p}{C_1} \sum_{j=0}^i \mathbf{E}|v(j)|^p,$$

rewrite the obtained inequality in the form

$$y_\varepsilon(i+1) \leq C_2 \left[z(i) + \sum_{j=1}^i y_\varepsilon(j) \right].$$

Via Lemma 1.2, (1.14), and $\max_{j \in \mathbb{Z}} \mathbf{E}|v(j)|^p < \infty$, we have $y_\varepsilon(i+1) \leq C_2(1 + C_2)^{N-1} z(i) < \infty$, i.e., q_ε is uniformly p -bounded with respect to $\varepsilon \geq 0$. The proof is completed.

Definition 2.3 The family of random variables $\{\xi_\varepsilon\}_{\varepsilon \geq 0}$ is called uniformly integrable, if $\sup_{\varepsilon \geq 0} \mathbf{E}[|\xi_\varepsilon| \chi\{|\xi_\varepsilon| > K\}] \rightarrow 0$ by $K \rightarrow \infty$ (here $\chi\{|\xi_\varepsilon| > K\}$ is the indicator of the set $\{w : |\xi_\varepsilon| > K\}$).

Lemma 2.3 Let the conditions (2.9), (2.10), (2.12)–(2.15) hold and the auxiliary process $v(j)$ in (2.29) satisfies the condition

$$\max_{i \in \mathbb{Z}} \mathbf{E}|v(i)|^4 < \infty. \quad (2.37)$$

Then the process $|q_\varepsilon(i)|^2$ is uniformly integrable with respect to $\varepsilon \geq 0$ and $\lim_{\varepsilon \rightarrow 0} \|q_\varepsilon - q_0\|_i = 0$, $i = 1, \dots, N$.

Proof Via (2.37) from Lemma 2.2 it follows that the process $\mathbf{E}|q_\varepsilon(i)|^4$ is uniformly bounded with respect to $\varepsilon \geq 0$. From this and [67, 184], it follows that the process $|q_\varepsilon(i)|^2$ is uniformly integrable with respect to $\varepsilon \geq 0$. Let us prove the second statement of the lemma. From (2.34), it follows that

$$\begin{aligned} q_\varepsilon(i+1) - q_0(i+1) &= \Phi_\varepsilon(i+1)q_{\varepsilon,i+1} - \Phi_0(i+1)q_{0,i+1} \\ &\quad + \sum_{j=1}^i [A_\varepsilon(i, j)q_{\varepsilon j} - A_0(i, j)q_{0j}] \\ &\quad + \sum_{j=1}^i [B_\varepsilon(i, j)q_{\varepsilon j} - B_0(i, j)q_{0j}] \xi(j+1) \\ &\quad + \sum_{j=0}^i [a_\varepsilon(i, j) - a_0(i, j)] v(j) \\ &\quad + \sum_{j=0}^i [b_\varepsilon(i, j) - b_0(i, j)] v(j) \xi(j+1). \end{aligned}$$

Rewrite this representation in the form

$$\begin{aligned} q_\varepsilon(i+1) - q_0(i+1) &= \Phi_0(i+1)(q_{\varepsilon,i+1} - q_{0,i+1}) + (\Phi_\varepsilon(i+1) - \Phi_0(i+1))q_{\varepsilon,i+1} \\ &\quad + \sum_{j=1}^i [A_0(i, j)(q_{\varepsilon j} - q_{0j}) + (A_\varepsilon(i, j) - A_0(i, j))q_{\varepsilon j}] \\ &\quad + \sum_{j=1}^i [B_0(i, j)(q_{\varepsilon j} - q_{0j}) + (B_\varepsilon(i, j) - B_0(i, j))q_{\varepsilon j}] \xi(j+1) \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=0}^i [a_\varepsilon(i, j) - a_0(i, j)]v(j) \\
& + \sum_{j=0}^i [b_\varepsilon(i, j) - b_0(i, j)]v(j)\xi(j+1).
\end{aligned}$$

From this via (2.9) it follows that

$$\begin{aligned}
& (1 - K_0(i+1))|q_\varepsilon(i+1) - q_0(i+1)| \\
& \leq \sum_{j=1}^i |q_\varepsilon(j) - q_0(j)|K_0(j) + |(\Phi_\varepsilon(i+1) - \Phi_0(i+1))q_{\varepsilon, i+1}| \\
& + \sum_{j=1}^i |A_0(i, j)(q_{\varepsilon j} - q_{0j})| + \sum_{j=0}^i |(A_\varepsilon(i, j) - A_0(i, j))q_{\varepsilon j}| \\
& + \left| \sum_{j=1}^i B_0(i, j)(q_{\varepsilon j} - q_{0j})\xi(j+1) \right| + \left| \sum_{j=1}^i (B_\varepsilon(i, j) - B_0(i, j))q_{\varepsilon j}\xi(j+1) \right| \\
& + \sum_{j=0}^i |(a_\varepsilon(i, j) - a_0(i, j))v(j)| + \left| \sum_{j=0}^i [b_\varepsilon(i, j) - b_0(i, j)]v(j)\xi(j+1) \right|.
\end{aligned}$$

Squaring and calculating the expectation, from this via Lemma 1.7 and (2.13) for $i \leq N-1$, we obtain

$$\begin{aligned}
& (1 - K_0)^2 \mathbf{E}|q_\varepsilon(i+1) - q_0(i+1)|^2 \\
& \leq 8 \left[NK_0^2 \sum_{j=1}^i \mathbf{E}|q_\varepsilon(j) - q_0(j)|^2 + \alpha_1(\varepsilon, i) \right. \\
& + N \sum_{j=1}^i \mathbf{E}|A_0(i, j)(q_{\varepsilon j} - q_{0j})|^2 + N\alpha_2(\varepsilon, i) \\
& \left. + \sum_{j=1}^i \mathbf{E}|B_0(i, j)(q_{\varepsilon j} - q_{0j})|^2 + N\alpha_3(\varepsilon, i) \right], \quad (2.38)
\end{aligned}$$

where

$$\begin{aligned}
\alpha_1(\varepsilon, i) &= \mathbf{E}|(\Phi_\varepsilon(i+1) - \Phi_0(i+1))q_{\varepsilon, i+1}|^2, \\
\alpha_2(\varepsilon, i) &= \sum_{j=1}^i \mathbf{E}|(A_\varepsilon(i, j) - A_0(i, j))q_{\varepsilon j}|^2 + \sum_{j=1}^i \mathbf{E}|(B_\varepsilon(i, j) - B_0(i, j))q_{\varepsilon j}|^2,
\end{aligned}$$

$$\alpha_3(\varepsilon, i) = \sum_{j=0}^i \mathbf{E}|(a_\varepsilon(i, j) - a_0(i, j))v(j)|^2 + \sum_{j=0}^i \mathbf{E}|(b_\varepsilon(i, j) - b_0(i, j))v(j)|^2. \quad (2.39)$$

So, similarly to (2.35), (2.36) with $p = 2$ from (2.38) it follows that

$$\begin{aligned} & (1 - K_0)^2 \mathbf{E}|q_\varepsilon(i + 1) - q_0(i + 1)|^2 \\ & \leq 8N \left[\sum_{k=1}^3 \alpha_k(\varepsilon, i) + K_0^2 \sum_{j=1}^i \mathbf{E}|q_\varepsilon(j) - q_0(j)|^2 \right. \\ & \quad \left. + \sum_{j=1}^i \mathbf{E} \left(|A_0(i, j)(q_{\varepsilon j} - q_{0j})|^2 + |B_0(i, j)(q_{\varepsilon j} - q_{0j})|^2 \right) \right] \\ & \leq 8N \left[\sum_{k=1}^3 \alpha_k(\varepsilon, i) + (K_0^2 + K_1 N) \sum_{j=1}^i \mathbf{E}|q_\varepsilon(j) - q_0(j)|^2 \right] \end{aligned}$$

or

$$y_\varepsilon(i + 1) \leq C_2 \left[z_\varepsilon(i) + \sum_{j=1}^i y_\varepsilon(j) \right], \quad (2.40)$$

where

$$\begin{aligned} C_2 &= \frac{8NC_1}{(1 - K_0)^2}, \quad z_\varepsilon(i) = \frac{1}{C_1} \sum_{k=1}^3 \alpha_k(\varepsilon, i), \\ C_1 &= K_0^2 + K_1 N, \quad y_\varepsilon(i) = \mathbf{E}|q_\varepsilon(i) - q_0(i)|^2. \end{aligned}$$

Via Lemma 1.2 we have $y_\varepsilon(i + 1) \leq C_2(1 + C_2)^{N-1} z_\varepsilon(i)$. So, it is enough to show that $\lim_{\varepsilon \rightarrow 0} z_\varepsilon(i) = 0, i \in Z$.

Let us show from the beginning that $\lim_{\varepsilon \rightarrow 0} \alpha_1(\varepsilon, i) = 0, i \in Z$. Represent $q_\varepsilon(i)$ in the form

$$q_\varepsilon(i) = q_\varepsilon(i) \chi_\varepsilon^K(i) + q_\varepsilon(i)(1 - \chi_\varepsilon^K(i)), \quad (2.41)$$

where $\chi_\varepsilon^K(i)$ is the indicator of the set $\{\omega : |q_\varepsilon(i)| > K\}$.

Then via Lemma 1.7

$$\begin{aligned} \alpha_1(\varepsilon, i - 1) &= \mathbf{E}|(\Phi_\varepsilon(i) - \Phi_0(i))q_{\varepsilon i}|^2 \\ &= \mathbf{E} \left| \Phi_\varepsilon(i) \left(q_\varepsilon \chi_\varepsilon^K \right)_i - \Phi_0(i) \left(q_\varepsilon \chi_\varepsilon^K \right)_i \right|^2 \end{aligned}$$

$$\begin{aligned}
& + \left| (\Phi_\varepsilon(i) - \Phi_0(i)) \left(q_\varepsilon \left(1 - \chi_\varepsilon^K \right) \right)_i \right|^2 \\
& \leq 3 \left[\mathbf{E} \left| \Phi_\varepsilon(i) \left(q_\varepsilon \chi_\varepsilon^K \right)_i \right|^2 + \mathbf{E} \left| \Phi_0(i) \left(q_\varepsilon \chi_\varepsilon^K \right)_i \right|^2 \right. \\
& \quad \left. + \mathbf{E} \left| (\Phi_\varepsilon(i) - \Phi_0(i)) \left(q_\varepsilon \left(1 - \chi_\varepsilon^K \right) \right)_i \right|^2 \right].
\end{aligned}$$

From this via (2.31), (2.9), (2.12) for $i \leq N$ we obtain that

$$\begin{aligned}
\alpha_1(\varepsilon, i-1) & \leq 3 \left[2(N+1) \sum_{j=1}^i \mathbf{E} |q_\varepsilon(j) \chi_\varepsilon^K(j)|^2 K_0^2(j) \right. \\
& \quad \left. + \sum_{j=1}^i \mathbf{E} |\varepsilon q_\varepsilon(j)|^2 |q_\varepsilon(j) (1 - \chi_\varepsilon^K(j))|^2 K_0(j) \right] \\
& \leq 3K_0 \left[2K_0(N+1) \sum_{j=1}^i \mathbf{E} |q_\varepsilon(j) \chi_\varepsilon^K(j)|^2 \right. \\
& \quad \left. + \varepsilon^2 \sum_{j=1}^i \mathbf{E} |q_\varepsilon(j)|^2 |q_\varepsilon(j) (1 - \chi_\varepsilon^K(j))|^2 \right] \\
& \leq 3K_0(N+1) \left[2K_0(N+1) \|q_\varepsilon \chi_\varepsilon^K\|_N^2 + \varepsilon^2 K^4 \right].
\end{aligned}$$

Since $|q_\varepsilon(i)|^2$ is uniformly integrable with respect to $\varepsilon \geq 0$ then $\lim_{K \rightarrow \infty} \|q_\varepsilon \chi_\varepsilon^K\|_N^2 = 0$. So, for arbitrary $\delta > 0$ there exists $K > 0$ such that $\|q_\varepsilon \chi_\varepsilon^K\|_N^2 < \delta(12K_0^2(N+1)^2)^{-1}$. Fixing this K , let us choose $\varepsilon > 0$ such that $\varepsilon^2 K^4 < \delta(6K_0(N+1))^{-1}$. Therefore, for arbitrary $\delta > 0$ there exists $\varepsilon > 0$ such that $\alpha_1(\varepsilon, i-1) < \delta$, i.e., $\lim_{\varepsilon \rightarrow 0} \alpha_1(\varepsilon, i-1) = 0$.

Note now that via (2.39), (2.41) we have

$$\begin{aligned}
\alpha_2(\varepsilon, i) & = \sum_{j=1}^i \mathbf{E} \left[|A_\varepsilon(i, j) \left(q_\varepsilon \chi_\varepsilon^K \right)_j - A_0(i, j) \left(q_\varepsilon \chi_\varepsilon^K \right)_j \right. \\
& \quad + (A_\varepsilon(i, j) - A_0(i, j)) (q_\varepsilon (1 - \chi_\varepsilon^K))_j|^2 \\
& \quad + |B_\varepsilon(i, j) \left(q_\varepsilon \chi_\varepsilon^K \right)_j - B_0(i, j) \left(q_\varepsilon \chi_\varepsilon^K \right)_j \\
& \quad \left. + (B_\varepsilon(i, j) - B_0(i, j)) \left(q_\varepsilon (1 - \chi_\varepsilon^K) \right)_j \right|^2 \right]
\end{aligned}$$

$$\begin{aligned}
&\leq 3 \sum_{j=1}^i \mathbf{E} \left[|A_\varepsilon(i, j) (q_\varepsilon \chi_\varepsilon^K)_j|^2 + |A_0(i, j) (q_\varepsilon \chi_\varepsilon^K)_j|^2 \right. \\
&\quad + |(A_\varepsilon(i, j) - A_0(i, j)) (q_\varepsilon (1 - \chi_\varepsilon^K))_j|^2 \\
&\quad + |B_\varepsilon(i, j) (q_\varepsilon \chi_\varepsilon^K)_j|^2 + |B_0(i, j) (q_\varepsilon \chi_\varepsilon^K)_j|^2 \\
&\quad \left. + |(B_\varepsilon(i, j) - B_0(i, j)) (q_\varepsilon (1 - \chi_\varepsilon^K))_j|^2 \right]
\end{aligned}$$

or in another form

$$\begin{aligned}
\alpha_2(\varepsilon, i) &\leq 3 \sum_{j=1}^i \mathbf{E} \left[|A_\varepsilon(i, j) (q_\varepsilon \chi_\varepsilon^K)_j|^2 + |B_\varepsilon(i, j) (q_\varepsilon \chi_\varepsilon^K)_j|^2 \right. \\
&\quad + |A_0(i, j) (q_\varepsilon \chi_\varepsilon^K)_j|^2 + |B_0(i, j) (q_\varepsilon \chi_\varepsilon^K)_j|^2 \\
&\quad + |(A_\varepsilon(i, j) - A_0(i, j)) (q_\varepsilon (1 - \chi_\varepsilon^K))_j|^2 \\
&\quad \left. + |(B_\varepsilon(i, j) - B_0(i, j)) (q_\varepsilon (1 - \chi_\varepsilon^K))_j|^2 \right].
\end{aligned}$$

From this and (2.31), (2.10), (2.14) we obtain

$$\begin{aligned}
\alpha_2(\varepsilon, i) &\leq 3 \sum_{j=1}^i \int_0^1 \mathbf{E} \left[|\nabla a(i, j, \lambda_{\varepsilon j}^\tau, u_\varepsilon(j)) (q_\varepsilon \chi_\varepsilon^K)_j|^2 \right. \\
&\quad \left. + |\nabla b(i, j, \lambda_{\varepsilon j}^\tau, u_\varepsilon(j)) (q_\varepsilon \chi_\varepsilon^K)_j|^2 \right] d\tau \\
&\quad + 3 \sum_{j=1}^i \mathbf{E} \left[|\nabla a(i, j, x_{0j}, u_0(j)) (q_\varepsilon \chi_\varepsilon^K)_j|^2 \right. \\
&\quad \left. + |\nabla b(i, j, x_{0j}, u_0(j)) (q_\varepsilon \chi_\varepsilon^K)_j|^2 \right] \\
&\quad + 3 \sum_{j=1}^i \int_0^1 \mathbf{E} \left[|(\nabla a(i, j, \lambda_{\varepsilon j}^\tau, u_\varepsilon(j)) - \nabla a(i, j, x_{0j}, u_0(j))) \right. \\
&\quad \times (q_\varepsilon (1 - \chi_\varepsilon^K))_j|^2 + |(\nabla b(i, j, \lambda_{\varepsilon j}^\tau, u_\varepsilon(j)) \\
&\quad \left. - \nabla b(i, j, x_{0j}, u_0(j))) (q_\varepsilon (1 - \chi_\varepsilon^K))_j|^2 \right] d\tau
\end{aligned}$$

$$\begin{aligned}
&\leq 3 \sum_{j=1}^i \sum_{l=1}^j \left[2\mathbf{E}|q_\varepsilon(l)\chi_\varepsilon^K(l)|^2 K_1(l) + \varepsilon^2 \mathbf{E} \left(|q_\varepsilon(l)|^2 + |v(j)|^2 \right) \right. \\
&\quad \left. \times |q_\varepsilon(l)(1 - \chi_\varepsilon^K(l))|^2 K_1(l) \right] \\
&\leq \frac{3}{2} K_1(N+1)(N+2) \left[2\|q_\varepsilon \chi_\varepsilon^K\|_N^2 + \varepsilon^2 K^2 (K^2 + \|v\|^2) \right].
\end{aligned}$$

Since $|q_\varepsilon(i)|^2$ is uniformly integrable with respect to $\varepsilon \geq 0$ then $\lim_{K \rightarrow \infty} \|q_\varepsilon \chi_\varepsilon^K\|_N = 0$. So, for arbitrary $\delta > 0$ there exists $K > 0$ such that $\|q_\varepsilon \chi_\varepsilon^K\|_N^2 < \delta(6K_1(N+1)(N+2))^{-1}$. Fixing this K , let us choose $\varepsilon > 0$ such that $\varepsilon^2 K^2 (K^2 + \|v\|^2) < \delta(3K_1(N+1)(N+2))^{-1}$. Therefore, for arbitrary $\delta > 0$ there exists $\varepsilon > 0$ such that $\alpha_2(\varepsilon, i) < \delta$, i.e., $\lim_{\varepsilon \rightarrow 0} \alpha_2(\varepsilon, i) = 0$.

Similarly, via (2.31) and (2.15) one can get that

$$\begin{aligned}
\alpha_3(\varepsilon, i) &\leq \sum_{j=0}^i \mathbf{E} \left(|a_\varepsilon(i, j) - a_0(i, j)|^2 + |b_\varepsilon(i, j) - b_0(i, j)|^2 \right) |v(j)|^2 \\
&\leq \sum_{j=0}^i \mathbf{E} \left[\int_0^1 |\nabla_u a(i, j, x_{0j}, u_\varepsilon^\tau(j)) - \nabla_u a(i, j, x_{0j}, u_0(j))|^2 d\tau \right. \\
&\quad \left. + \int_0^1 |\nabla_u b(i, j, x_{0j}, u_\varepsilon^\tau(j)) - \nabla_u b(i, j, x_{0j}, u_0(j))|^2 d\tau \right] |v(j)|^2 \\
&\leq \sum_{j=0}^i \mathbf{E} \int_0^1 [|\nabla_u a(i, j, x_{0j}, u_\varepsilon^\tau(j)) - \nabla_u a(i, j, x_{0j}, u_0(j))| \\
&\quad + |\nabla_u b(i, j, x_{0j}, u_\varepsilon^\tau(j)) - \nabla_u b(i, j, x_{0j}, u_0(j))|]^2 d\tau |v(j)|^2 \\
&\leq \varepsilon^2 C^2 \sum_{j=0}^i \mathbf{E} |v(j)|^4.
\end{aligned}$$

So, via (2.37) $\lim_{\varepsilon \rightarrow 0} \alpha_3(\varepsilon, i) = 0$.

As a result, we have $\lim_{\varepsilon \rightarrow 0} z_\varepsilon(i) = 0$, $i \in Z$, and, therefore, $\lim_{\varepsilon \rightarrow 0} \|q_\varepsilon - q_0\|_i^2 = 0$. The proof is completed.

2.1.3 Main Result

In this section, the bounded limit (1.9), (1.10) for the optimal control problem (2.3), (2.4) will be calculated. So, the necessary condition $J_0(u_0) \geq 0$ for optimality of the control u_0 will be obtained.

Theorem 2.1 *Let the conditions (2.5)–(2.15), (2.18)–(2.24), (2.37) hold. Then the limit (1.9), (1.10) for the optimal control problem (2.3), (2.4) there exists, equals*

$$J_0(u_0) = \mathbf{E} \left[\langle F_0(N), q_{0N} \rangle + \sum_{j=0}^{N-1} (\langle G_0(j), q_{0j} \rangle + v'(j)g_0(j)) \right] \quad (2.42)$$

and it is bounded.

Here, $x_0(j)$ is the solution of the equation (2.3) under the control $u_0(j)$, $q_0(j)$ is the solution of the linear stochastic difference equation

$$\begin{aligned} q_0(i+1) &= \eta_0(i+1) + \Phi_0(i+1)q_{0,i+1} \\ &\quad + \sum_{j=1}^i A_0(i, j)q_{0j} + \sum_{j=1}^i B_0(i, j)q_{0j}\xi(j+1), \\ q_{00} &= 0, \end{aligned} \quad (2.43)$$

where

$$\eta_0(i+1) = \sum_{j=0}^i a_0(i, j)v(j) + \sum_{j=0}^i b_0(i, j)v(j)\xi(j+1), \quad (2.44)$$

$$\begin{aligned} F_0(N) &= \nabla F(x_{0N}), & \Phi_0(i) &= \nabla \Phi(i, x_{0i}), \\ G_0(j) &= \nabla G(j, x_{0j}, u_0(j)), & g_0(j) &= \nabla_u G(j, x_{0j}, u_0(j)), \\ A_0(i, j) &= \nabla a(i, j, x_{0j}, u_0(j)), & a_0(i, j) &= \nabla_u a(i, j, x_{0j}, u_0(j)), \\ B_0(i, j) &= \nabla b(i, j, x_{0j}, u_0(j)), & b_0(i, j) &= \nabla_u b(i, j, x_{0j}, u_0(j)). \end{aligned} \quad (2.45)$$

Proof From (2.4), it follows that

$$\begin{aligned} J_\varepsilon(u_0) &= \frac{1}{\varepsilon} [J(u_\varepsilon) - J(u_0)] \\ &= \mathbf{E} \left[\frac{1}{\varepsilon} (F(x_{\varepsilon N}) - F(x_{0N})) + \frac{1}{\varepsilon} \sum_{j=0}^{N-1} (G(j, x_{\varepsilon j}, u_\varepsilon(j)) - G(j, x_{0j}, u_\varepsilon(j))) \right. \\ &\quad \left. + \frac{1}{\varepsilon} \sum_{j=0}^{N-1} (G(j, x_{0j}, u_\varepsilon(j)) - G(j, x_{0j}, u_0(j))) \right]. \end{aligned}$$

To transform the expression in square brackets, put

$$\begin{aligned} F_\varepsilon(N) &= \int_0^1 \nabla F(\lambda_{\varepsilon N}^\tau) d\tau, \\ G_\varepsilon(j) &= \int_0^1 \nabla G(j, \lambda_{\varepsilon j}^\tau, u_\varepsilon(j)) d\tau, \\ g_\varepsilon(j) &= \int_0^1 \nabla_u G(j, x_{0j}, u_\varepsilon^\tau(j)) d\tau, \end{aligned} \quad (2.46)$$

and consider the function

$$f(\tau) = \frac{1}{\varepsilon} F(x_{0N} + \varepsilon \tau q_{\varepsilon N}), \quad \tau \in [0, 1].$$

Since $f'(\tau) = \langle \nabla F(\lambda_{\varepsilon N}^\tau), q_{\varepsilon N} \rangle$, where λ_ε^τ is defined in (2.31), then via (2.46)

$$\frac{1}{\varepsilon} [F(x_{\varepsilon N}) - F(x_{0N})] = \langle F_\varepsilon(N), q_{\varepsilon N} \rangle.$$

Similarly, we have

$$\frac{1}{\varepsilon} [G(j, x_{\varepsilon j}, u_\varepsilon(j)) - G(j, x_{0j}, u_\varepsilon(j))] = \langle G_\varepsilon(j), q_{\varepsilon j} \rangle.$$

To transform the difference

$$\frac{1}{\varepsilon} [G(j, x_{0j}, u_\varepsilon(j)) - G(j, x_{0j}, u_0(j))],$$

consider the function

$$f(\tau) = \frac{1}{\varepsilon} G(j, x_{0j}, u_0(j) + \varepsilon \tau v(j)), \quad \tau \in [0, 1],$$

for which $f'(\tau) = v'(j) \nabla_u G(j, x_{0j}, u_\varepsilon^\tau(j))$. So, via (2.46) we obtain

$$\frac{1}{\varepsilon} [G(j, x_{0j}, u_\varepsilon(j)) - G(j, x_{0j}, u_0(j))] = v'(j) g_\varepsilon(j).$$

Thus, $J_\varepsilon(u_0)$ takes the representation

$$J_\varepsilon(u_0) = \mathbf{E} \left[\langle F_\varepsilon(N), q_{\varepsilon N} \rangle + \sum_{j=0}^{N-1} (\langle G_\varepsilon(j), q_{\varepsilon j} \rangle + v'(j) g_\varepsilon(j)) \right].$$

Rewrite $J_\varepsilon(u_0)$ in the form

$$J_\varepsilon(u_0) = \mathbf{E} \left[\langle F_0(N), q_{\varepsilon N} \rangle + \sum_{j=0}^{N-1} \langle G_0(j), q_{\varepsilon j} \rangle + v'(j)g_0(j) \right] + \sum_{i=1}^3 \beta_i(\varepsilon),$$

where $F_0(N)$, $G_0(j)$, $g_0(j)$ are defined in (2.45) and

$$\begin{aligned} \beta_1(\varepsilon) &= \mathbf{E} \langle F_\varepsilon(N) - F_0(N), q_{\varepsilon N} \rangle, \\ \beta_2(\varepsilon) &= \mathbf{E} \sum_{j=1}^{N-1} \langle G_\varepsilon(j) - G_0(j), q_{\varepsilon j} \rangle, \\ \beta_3(\varepsilon) &= \mathbf{E} \sum_{j=0}^{N-1} v'(j)(g_\varepsilon(j) - g_0(j)). \end{aligned}$$

Let us show that $\lim_{\varepsilon \rightarrow 0} \beta_i(\varepsilon) = 0$, $i = 1, 2, 3$. Via (2.46), (2.20), (2.31), (2.13) we have

$$\begin{aligned} |\beta_1(\varepsilon)| &\leq \mathbf{E} \int_0^1 |\langle \nabla F(\lambda_{\varepsilon N}^\tau) - \nabla F(x_{0N}), q_{\varepsilon N} \rangle| d\tau \\ &\leq \varepsilon \sum_{j=1}^N \mathbf{E} |q_\varepsilon(j)|^2 K_1(j) \\ &\leq \varepsilon K_1 N \|q_\varepsilon\|_N^2. \end{aligned}$$

From this via Lemma 2.2, it follows that $\lim_{\varepsilon \rightarrow 0} \beta_1(\varepsilon) = 0$.

Similarly, using (2.46), (2.21), (2.31), (2.29), (2.13), (2.23) we get

$$\begin{aligned} |\beta_2(\varepsilon)| &\leq \mathbf{E} \sum_{j=1}^{N-1} \int_0^1 |\langle \nabla G(j, \lambda_{\varepsilon j}^\tau, u_\varepsilon(j)) - \nabla G(j, x_{0j}, u_0(j)), q_{\varepsilon j} \rangle| d\tau \\ &\leq \varepsilon \mathbf{E} \sum_{j=1}^{N-1} \sum_{l=1}^j (|q_\varepsilon(l)| + |v(j)|) |q_\varepsilon(l)| K_1(l) \\ &\leq \frac{1}{2} \varepsilon K_1 (N-1)^2 (3 \|q_\varepsilon\|_N^2 + \|v\|_N^2), \end{aligned}$$

and

$$\begin{aligned} |\beta_3(\varepsilon)| &\leq \mathbf{E} \sum_{j=0}^{N-1} |v(j)| \int_0^1 |\nabla_u G(j, x_{0j}, u_\varepsilon^\tau(j)) - \nabla_u G(j, x_{0j}, u_0(j))| d\tau \\ &\leq \varepsilon C N \|v\|_N^2. \end{aligned}$$

From this and Lemmas 2.2 and 2.3, we obtain $\lim_{\varepsilon \rightarrow 0} |J_\varepsilon(u_0) - J_0(u_0)| = 0$. From the conditions (2.18), (2.19), (2.22), it follows also that $J_0(u_0) < \infty$. The proof is completed.

2.2 A Linear-Quadratic Problem

Here, the necessary condition for control optimality obtained above is used to construct, in an explicit form, the synthesis of optimal control for a linear-quadratic problem.

2.2.1 Synthesis of Optimal Control

Consider the optimal control problem for the linear equation

$$x(i+1) = \eta(i+1) + \sum_{j=0}^i a(i, j)x(j) + \sum_{j=0}^i b(i, j)u(j), \quad i = 0, \dots, N-1, \quad (2.47)$$

with the initial condition $x(0) = \varphi_0(0)$ and the performance functional

$$J(u) = \mathbf{E} \left[x'(N)Fx(N) + \sum_{j=0}^{N-1} u'(j)G(j)u(j) \right]. \quad (2.48)$$

Here $\eta \in H$, $x(i) \in \mathbf{R}^n$, $u(i) \in \mathbf{R}^m$, positive semidefinite matrix F , positive definite matrix $G(j)$, and matrices $a(i, j)$, $b(i, j)$, $i, j \in Z$, are nonrandom matrices of appropriate dimensions.

The optimal control problem (2.47), (2.48) is a particular case of the control problem (2.3), (2.4) with

$$\begin{aligned} \Phi(i+1, x_{i+1}) &= 0, & b(i, j, x_j, u(j)) &= 0, \\ a(i, j, x_j, u(j)) &= a(i, j)x(j) + b(i, j)u(j), \\ F(x_N) &= x'(N)Fx(N), & G(j, x_j, u(j)) &= u'(j)G(j)u(j). \end{aligned} \quad (2.49)$$

Let us calculate the coefficients of the Eqs. (2.43)–(2.45) for the optimal control problem (2.47), (2.48). Via (2.49), (2.45) we have

$$\begin{aligned} \Phi_0(i+1) &= 0, & B_0(i, j) &= 0, & b_0(i, j) &= 0, \\ A_0(i, j)q_{0j} &= a(i, j)q_{0j}, & a_0(i, j)v(j) &= b(i, j)v(j). \end{aligned}$$

From this and (2.43), (2.44) it follows that

$$\begin{aligned} q_0(i+1) &= \sum_{j=1}^i a(i, j)q_0(j) + \sum_{j=0}^i b(i, j)v(j), \\ q_0(0) &= 0. \end{aligned} \quad (2.50)$$

Besides, via (2.49), (2.45) we obtain

$$\begin{aligned} \langle F_0(N), q_{0N} \rangle &= 2x'_0(N)Fq_0(N), \\ \langle G_0(j), q_{0j} \rangle &= 0, \quad v'(j)g_0(j) = 2v'(j)G(j)u_0(j). \end{aligned}$$

Via Theorem 2.1 from this and (2.42), it follows that the limit (1.9), (1.10) for the optimal control problem (2.47), (2.48) there exists and equals

$$J_0(u_0) = 2\mathbf{E} \left[x'_0(N)Fq_0(N) + \sum_{j=0}^{N-1} u'_0(j)G(j)v(j) \right]. \quad (2.51)$$

Here, x_0 is a solution of the equation (2.47) under the control u_0 and $q_0(i)$ is a solution of the equation (2.50).

Note that for the linear-quadratic problem (2.47), (2.48) $q_\varepsilon \equiv q_0$. So, Lemma 2.3 became a trivial one and the condition of boundedness of the process $\mathbf{E}|q_0(i)|^2$ is a sufficient condition for existence of (2.51).

Definition 2.3 For arbitrary function $f(k)$, $0 \leq j \leq k \leq i$, the functional $\psi(i, j, f(\cdot))$ is defined by the following way

$$\psi(i, j, f(\cdot)) = f(i) + \sum_{k=j+1}^i R(i, k)f(k-1), \quad (2.52)$$

where $R(i, k)$ is the resolvent of the kernel $a(i, k)$ from the Eq. (2.47).

Lemma 2.4 *The necessary condition $J_0(u_0) \geq 0$ for optimality of the control u_0 of the problem (2.47), (2.48) has a unique solution*

$$u_0(j) = -G^{-1}(j)\psi'(N-1, j, b(\cdot, j))F\mathbf{E}_j x_0(N), \quad j = 0, \dots, N-1. \quad (2.53)$$

Proof Using the resolvent $R(i, k)$ of the kernel $a(i, k)$ and the representations (1.17), (2.52), rewrite the solution of the equation (2.50) in the form

$$\begin{aligned}
q_0(i+1) &= \sum_{j=0}^i b(i, j)v(j) + \sum_{k=1}^i R(i, k) \sum_{j=0}^{k-1} b(k-1, j)v(j) \\
&= \sum_{j=0}^i \left[b(i, j) + \sum_{k=j+1}^i R(i, k)b(k-1, j) \right] v(j) \\
&= \sum_{j=0}^i \psi(i, j, b(\cdot, j))v(j). \tag{2.54}
\end{aligned}$$

Substituting (2.54) into (2.51), we obtain

$$J_0(u_0) = 2\mathbf{E} \sum_{j=0}^{N-1} [\mathbf{E}_j x'_0(N)F\psi(N-1, j, b(\cdot, j)) + u'_0(j)G(j)]v(j). \tag{2.55}$$

The inequality $J_0(u_0) \geq 0$ holds for any $v \in \mathbf{U}$ if and only if the expression in the square brackets in (2.55) equals zero, that is equivalent to (2.53). The proof is completed.

Theorem 2.2 *The optimal control of the control problem (2.47), (2.48) is represented in the form*

$$\begin{aligned}
u_0(0) &= p(0) [\psi(N-1, 0, \mathbf{E}_0 \eta(\cdot+1)) + R(N-1, 0)\varphi_0(0)], \\
u_0(j+1) &= \alpha(j+1) + p(j+1)\psi(N-1, j, I)x_0(j+1) \\
&\quad + \sum_{k=0}^j \gamma(j, k)x_0(k) + Q(j, 0)p(0) [\psi(N-1, 0, \mathbf{E}_0 \eta(\cdot+1)) \\
&\quad + R(N-1, 0)\varphi_0(0)], \quad j = 0, 1, \dots, N-2. \tag{2.56}
\end{aligned}$$

Here, I is the identical matrix,

$$\begin{aligned}
p(j) &= -G^{-1}(j)\psi'(N-1, j, b(\cdot, j))F \\
&\quad \times \left[I + \sum_{k=j}^{N-1} \psi(N-1, k, b(\cdot, k))G^{-1}(k)\psi'(N-1, k, b(\cdot, k))F \right]^{-1}, \\
j &= 0, 1, \dots, N-1, \tag{2.57}
\end{aligned}$$

$$\begin{aligned}
\alpha(j+1) &= p(j+1)\psi(N-1, j, \beta(\cdot, j+1)) \\
&\quad + \sum_{k=1}^j Q(j, k)p(k)\psi(N-1, k-1, \beta(\cdot, k)), \quad j \geq 0, \tag{2.58}
\end{aligned}$$

$$\beta(i, j) = \mathbf{E}_j \eta(i+1) - \eta(j), \quad i \geq j > 0, \tag{2.59}$$

$$\gamma(j, k) = \begin{cases} p(j+1)\psi(N-1, j, a_j(\cdot, k)) + Q(j, k)p(k)\psi(N-1, k-1, I) \\ \quad + \sum_{l=k+1}^j Q(j, l)p(l)\psi(N-1, l-1, a_{l-1}(\cdot, k)), & j \geq k \geq 1, \\ p(j+1)\psi(N-1, j, a_j(\cdot, 0)) \\ \quad + \sum_{l=1}^j Q(j, l)p(l)\psi(N-1, l-1, a_{l-1}(\cdot, 0)), & j \geq k = 0, \end{cases} \quad (2.60)$$

$Q(j, k)$ is the resolvent of the kernel $p(j+1)\psi(N-1, j, b_j(\cdot, k))$,

$$\begin{aligned} a_j(i, k) &= a(i, k) - a(j, k), \\ b_j(i, k) &= b(i, k) - b(j, k), \\ 0 \leq k \leq j \leq i \leq N-1. \end{aligned} \quad (2.61)$$

Proof From (2.53), it follows that to get the representation of the optimal control $u_0(j)$ in the form (2.56), it is enough to calculate the conditional mathematical expectation $\mathbf{E}_j x_0(N)$.

To get $u_0(0)$ from (2.47), we have

$$\mathbf{E}_0 x(i+1) = \zeta(i+1) + \sum_{j=0}^i a(i, j)\mathbf{E}_0 x(j), \quad i = 0, \dots, N-1, \quad (2.62)$$

where

$$\zeta(i+1) = \mathbf{E}_0 \eta(i+1) + \sum_{j=0}^i b(i, j)\mathbf{E}_0 u(j), \quad i \geq 0, \quad \zeta(0) = x(0) = \varphi_0(0). \quad (2.63)$$

Using the resolvent $R(i, j)$ of the kernel $a(i, j)$, via (1.16), (1.17) from (2.62), we obtain

$$\mathbf{E}_0 x(i+1) = \zeta(i+1) + \sum_{k=1}^i R(i, k)\zeta(k) + R(i, 0)\varphi_0(0), \quad i = 0, \dots, N-1. \quad (2.64)$$

Substituting (2.63) into (2.64), we get

$$\begin{aligned}
 \mathbf{E}_0 x(i+1) &= \mathbf{E}_0 \eta(i+1) + \sum_{j=0}^i b(i, j) \mathbf{E}_0 u(j) + R(i, 0) \varphi_0(0) \\
 &\quad + \sum_{k=1}^i R(i, k) \left[\mathbf{E}_0 \eta(k) + \sum_{j=0}^{k-1} b(k-1, j) \mathbf{E}_0 u(j) \right] \\
 &= \mathbf{E}_0 \eta(i+1) + \sum_{k=1}^i R(i, k) \mathbf{E}_0 \eta(k) + R(i, 0) \varphi_0(0) \\
 &\quad + \sum_{j=0}^i \left[b(i, j) + \sum_{k=j+1}^i R(i, k) b(k-1, j) \right] \mathbf{E}_0 u(j).
 \end{aligned}$$

From this via (2.52) for $f(i) = \mathbf{E}_0 \eta(i+1)$ and $f(i) = b(i, j)$, it follows that

$$\mathbf{E}_0 x(i+1) = \psi(i, 0, \mathbf{E}_0 \eta(\cdot+1)) + \sum_{j=0}^i \psi(i, j, b(\cdot, j)) \mathbf{E}_0 u(j) + R(i, 0) \varphi_0(0).$$

Putting $i = N-1$, from this and (2.53) we obtain

$$\begin{aligned}
 \mathbf{E}_0 x(N) &= \psi(N-1, 0, \mathbf{E}_0 \eta(\cdot+1)) + R(N-1, 0) \varphi_0(0) \\
 &\quad + \sum_{j=0}^{N-1} \psi(N-1, j, b(\cdot, j)) \mathbf{E}_0 u(j) \\
 &= \psi(N-1, 0, \mathbf{E}_0 \eta(\cdot+1)) + R(N-1, 0) \varphi_0(0) \\
 &\quad - \sum_{j=0}^{N-1} \psi(N-1, j, b(\cdot, j)) G^{-1}(j) \psi'(N-1, j, b(\cdot, j)) F \mathbf{E}_0 x_0(N)
 \end{aligned}$$

or

$$\begin{aligned}
 \mathbf{E}_0 x(N) &= \left[I + \sum_{j=0}^{N-1} \psi(N-1, j, b(\cdot, j)) G^{-1}(j) \psi'(N-1, j, b(\cdot, j)) F \right]^{-1} \\
 &\quad \times [\psi(N-1, 0, \mathbf{E}_0 \eta(\cdot+1)) + R(N-1, 0) \varphi_0(0)].
 \end{aligned}$$

From this and (2.53), (2.57) it follows that $u_0(0)$ has the representation (2.56).

To get $u_0(j)$ for $j > 0$ from (2.47) for $i \geq j \geq 0$ via (2.61), we have

$$\begin{aligned} x_0(i+1) - x_0(j+1) &= \eta(i+1) - \eta(j+1) \\ &+ \sum_{k=0}^j a_j(i, k)x_0(k) + \sum_{k=j+1}^i a(i, k)x_0(k) \\ &+ \sum_{k=0}^j b_j(i, k)u_0(k) + \sum_{k=j+1}^i b(i, k)u_0(k). \end{aligned}$$

So, using (2.59) and putting

$$\zeta(i, j+1) = \begin{cases} x_0(j+1) + \beta(i, j+1) + \sum_{k=0}^j a_j(i, k)x_0(k) \\ + \sum_{k=0}^j b_j(i, k)u_0(k) + \sum_{k=j+1}^i b(i, k)\mathbf{E}_{j+1}u_0(k), & j < i, \\ x_0(i+1), & j = i, \end{cases} \quad (2.65)$$

we have

$$\mathbf{E}_{j+1}x_0(i+1) = \zeta(i, j+1) + \sum_{k=j+1}^i a(i, k)\mathbf{E}_{j+1}x_0(k), \quad i \geq j.$$

From this via the resolvent $R(i, k)$ of the kernel $a(i, k)$, we get

$$\mathbf{E}_{j+1}x_0(i+1) = \zeta(i, j+1) + \sum_{k=j+1}^i R(i, k)\zeta(k-1, j+1). \quad (2.66)$$

Substituting (2.65) into (2.66), we obtain

$$\begin{aligned} \mathbf{E}_{j+1}x_0(i+1) &= \left[I + \sum_{m=j+1}^i R(i, m) \right] x_0(j+1) \\ &+ \beta(i, j+1) + \sum_{m=j+1}^i R(i, m)\beta(m-1, j+1) \\ &+ \sum_{k=0}^j \left[a_j(i, k) + \sum_{m=j+1}^i R(i, m)a_j(m-1, k) \right] x_0(k) \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=0}^j \left[b_j(i, k) + \sum_{m=j+1}^i R(i, m) b_j(m-1, k) \right] u_0(k) \\
& + \sum_{k=j+1}^i \left[b(i, k) + \sum_{m=k+1}^i R(i, m) b(m-1, k) \right] \mathbf{E}_{j+1} u_0(k) \\
& = \psi(i, j, I) x_0(j+1) + \psi(i, j, \beta(\cdot, j+1)) \\
& + \sum_{k=0}^j \psi(i, j, a_j(\cdot, k)) x_0(k) + \sum_{k=0}^j \psi(i, j, b_j(\cdot, k)) u_0(k) \\
& + \sum_{k=j+1}^i \psi(i, k, b(\cdot, k)) \mathbf{E}_{j+1} u_0(k).
\end{aligned}$$

Putting now $i = N - 1$ and

$$\begin{aligned}
\zeta_0(j+1) & = \psi(N-1, j, I) x_0(j+1) + \psi(N-1, j, \beta(\cdot, j+1)) \\
& + \sum_{k=0}^j \psi(N-1, j, a_j(\cdot, k)) x_0(k), \quad j \geq 0,
\end{aligned} \tag{2.67}$$

from this we get

$$\begin{aligned}
\mathbf{E}_{j+1} x_0(N) & = \zeta_0(j+1) + \sum_{k=0}^j \psi(N-1, j, b_j(\cdot, k)) u_0(k) \\
& + \sum_{k=j+1}^{N-1} \psi(N-1, k, b(\cdot, k)) \mathbf{E}_{j+1} u_0(k), \quad j \geq 0.
\end{aligned} \tag{2.68}$$

Calculating the conditional mathematical expectation of (2.53), for $j < k$ we have

$$\mathbf{E}_{j+1} u_0(k) = -G^{-1}(k) \psi'(N-1, k, b(\cdot, k)) F \mathbf{E}_{j+1} x_0(N). \tag{2.69}$$

Substituting (2.69) into (2.68), we obtain

$$\begin{aligned}
\mathbf{E}_{j+1} x_0(N) & = \left[I + \sum_{k=j+1}^{N-1} \psi(N-1, k, b(\cdot, k)) G^{-1}(k) \psi'(N-1, k, b(\cdot, k)) F \right]^{-1} \\
& \times \left[\zeta_0(j+1) + \sum_{k=0}^j \psi(N-1, j, b_j(\cdot, k)) u_0(k) \right].
\end{aligned} \tag{2.70}$$

Substituting (2.70) into (2.53) and using (2.57), we get

$$\begin{aligned} u_0(j+1) &= w(j+1) + \sum_{k=0}^j p(j+1)\psi(N-1, j, b_j(\cdot, k))u_0(k), \\ w(j+1) &= p(j+1)\zeta_0(j+1), \quad j \geq 0, \quad w(0) = u_0(0). \end{aligned} \quad (2.71)$$

So, the optimal control $u_0(j)$ of the control problem (2.47), (2.48) satisfies the Eq. (2.71). Via (1.17) it admits the representation

$$u_0(j+1) = p(j+1)\zeta_0(j+1) + \sum_{k=1}^j Q(j, k)p(k)\zeta_0(k) + Q(j, 0)u_0(0), \quad j \geq 0, \quad (2.72)$$

where $Q(j, k)$ is the resolvent of the kernel $p(j+1)\psi(N-1, j, b_j(\cdot, k))$. Substituting $\zeta_0(j)$ given in (2.67) into (2.72), we obtain

$$\begin{aligned} u_0(j+1) &= p(j+1)\psi(N-1, j, \beta(\cdot, j+1)) \\ &\quad + p(j+1)\psi(N-1, j, I)x_0(j+1) \\ &\quad + p(j+1)\sum_{k=0}^j \psi(N-1, j, a_j(\cdot, k))x_0(k) + Q(j, 0)u_0(0) \\ &\quad + \sum_{k=1}^j Q(j, k)p(k)\psi(N-1, k-1, \beta(\cdot, k)) \\ &\quad + \sum_{k=1}^j Q(j, k)p(k)\psi(N-1, k-1, I)x_0(k) \\ &\quad + \sum_{k=1}^j Q(j, k)p(k)\sum_{l=0}^{k-1} \psi(N-1, k-1, a_{k-1}(\cdot, l))x_0(l). \end{aligned}$$

Changing the order of summation in the last expression and using (2.58), (2.60), we obtain the optimal control $u_0(j+1)$ in the form (2.56) for $j \geq 0$. The proof is completed.

Remark 2.1 Since (2.48) is a quadratic functional (i.e., a convex functional) and the control (2.56) has the bounded second moment, then the necessary condition of optimality is the sufficient condition too.

Remark 2.2 If the process $\eta(i)$ is a martingale with respect to σ -algebra $\mathfrak{F}_i$, i.e., $\mathbf{E}(\eta(i)/\mathfrak{F}_j) = \eta(j)$, $i > j$, then $\beta(i, j) = 0$, $i \geq j$, and therefore in (2.56) $\alpha(i) = 0$, $i \in \mathbb{Z}$.

Remark 2.3 Note that Theorem 2.2 can be easily generalized on the case of the more general performance functional

$$J(u) = \mathbf{E} \left[x'(N)Fx(N) + \sum_{j=0}^{N-1} (x'(j)F_1(j)x(j) + u'(j)G(j)u(j)) \right],$$

where F_1 is a positive semidefinite matrix and the matrices F and $G(j)$ satisfy the previous conditions. In particular, the representation (2.53) in this case takes the form

$$\begin{aligned} u_0(j) = & -G^{-1}(j) \left[\psi'(N-1, j, b(\cdot, j)) F \mathbf{E}_j x_0(N) \right. \\ & \left. + \sum_{i=j}^{N-2} \psi'(i, j, b(\cdot, j)) F_1(i+1) \mathbf{E}_j x_0(i+1) \right], \quad j = 0, \dots, N-1. \end{aligned}$$

Note that the performance functional of such type is considered below in Example 2.5, in Sect. 6.4 and also in Chap. 4 with $F = 0$, $N = \infty$.

2.2.2 Examples

Here, some examples of linear-quadratic optimal control problems of the type of (2.47) and (2.48) are considered.

Example 2.1 Consider the optimal control problem for the scalar linear stochastic difference equation

$$x(i+1) = \eta(i+1) + \sum_{j=0}^i a(i, j)x(j) + \sum_{j=0}^i b(i, j)u(j), \quad i \geq 0, \quad x(0) = \varphi_0(0), \quad (2.73)$$

with the quadratic performance functional

$$J(u) = \mathbf{E} \left[Fx^2(N) + \sum_{i=0}^{N-1} G(i)u^2(i) \right], \quad (2.74)$$

where $F \geq 0$, $G(i) > 0$, $i = 0, \dots, N-1$. The control problem (2.73), (2.74) is a particular case of the optimal control problem (2.47), (2.48). So, the optimal control of this problem is represented in the form (2.56)–(2.61). In this case, the function $\psi(N-1, j, b(\cdot, j))$ from (2.57) can be represented in some more simple form. Really, rewrite (2.50) for $q_0(i)$ in the matrix form

$$Q_0 = A Q_0 + B V. \quad (2.75)$$

Here, the matrices A and B have the dimension $(N + 1) \times (N + 1)$, the vectors Q_0 and V have the dimension $N + 1$ and, respectively, equal

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ a(0, 0) & 0 & \cdots & 0 & 0 \\ a(1, 0) & a(1, 1) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a(N-1, 0) & a(N-1, 1) & \cdots & a(N-1, N-1) & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ b(0, 0) & 0 & \cdots & 0 & 0 \\ b(1, 0) & b(1, 1) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b(N-1, 0) & b(N-1, 1) & \cdots & b(N-1, N-1) & 0 \end{pmatrix},$$

$$Q_0 = \begin{pmatrix} q_0(0) \\ q_0(1) \\ q_0(2) \\ \cdots \\ q_0(N) \end{pmatrix}, \quad V = \begin{pmatrix} v(0) \\ v(1) \\ v(2) \\ \cdots \\ v(N) \end{pmatrix}.$$

Since $\det(I - A) = 1$, then there exists the inverse matrix $D = (I - A)^{-1}$, and the solution of the equation (2.75) can be represented in the form

$$Q_0 = DBV. \quad (2.76)$$

Comparing (2.76) with (2.54), we obtain that $\psi(N - 1, j, b(\cdot, j)) = r(N, j)$, where $r(N, j)$, $j = 0, 1, \dots, N$, are elements of the last line of the matrix DB . Then the expression (2.57) can be written in the form

$$p(j) = -\frac{Fr(N, j)}{G(j)} \left[1 + F \sum_{k=j}^{N-1} \frac{r^2(N, k)}{G(k)} \right]^{-1}. \quad (2.77)$$

Remark 2.4 To calculate $r(N, j)$, it is enough to know the last line of the matrix D only.

Example 2.2 Consider the optimal control problem for the scalar linear stochastic difference equation

$$x(i + 1) = \eta(i + 1) + a \sum_{j=0}^i x(j) + b \sum_{j=0}^i u(j), \quad i \geq 0, \quad x(0) = \varphi_0(0), \quad (2.78)$$

with the quadratic performance functional

$$J(u) = \mathbf{E} \left[Fx^2(N) + \lambda \sum_{i=0}^{N-1} u^2(i) \right], \quad (2.79)$$

where

$$F > 0, \quad \lambda > 0, \quad \eta(i+1) = \varphi_0(0) + \sum_{j=0}^i \sigma(j)\xi(j+1), \quad i \in \mathbb{Z}, \quad (2.80)$$

$\sigma(j)$ are arbitrary constants, $\xi(j)$ are $\mathfrak{F}_j$ -adapted mutually independent random variables such that $\mathbf{E}\xi(j) = 0$, $\mathbf{E}\xi^2(j) = 1$, $j \in \mathbb{Z}$.

In this case, the matrices A , B , and D from Example 2.1 are respectively

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ a & 0 & 0 & \cdots & 0 & 0 \\ a & a & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a & a & a & \cdots & a & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ b & 0 & 0 & \cdots & 0 & 0 \\ b & b & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b & b & b & \cdots & b & 0 \end{pmatrix},$$

and

$$D = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ a & 1 & 0 & \cdots & 0 & 0 \\ a(a+1) & a & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a(a+1)^{N-1} & a(a+1)^{N-2} & a(a+1)^{N-3} & \cdots & a & 1 \end{pmatrix}.$$

To calculate the last line of the matrix DB note that

$$\begin{aligned} r(N, j) &= b \left[1 + a + a(a+1) + a(a+1)^2 + \cdots + a(a+1)^{N-2-j} \right] \\ &= b \left[1 + a \left(1 + (a+1) + (a+1)^2 + \cdots + (a+1)^{N-2-j} \right) \right] \\ &= b \left[1 + a \frac{(a+1)^{N-1-j} - 1}{a} \right] \\ &= b(a+1)^{N-1-j}. \end{aligned}$$

So, the last line of the matrix DB is

$$\left(b(a+1)^{N-1}, b(a+1)^{N-2}, \dots, b(a+1), b, 0 \right),$$

i.e.,

$$\begin{aligned} r(N, j) &= b(a+1)^{N-1-j} = b\psi(N-1, j, 1), \\ j &= 0, 1, \dots, N-1, \quad r(N, N) = 0. \end{aligned} \quad (2.81)$$

Let us calculate $p(j)$. Using (2.77), (2.81), we have

$$\begin{aligned} p(j) &= -\frac{Fb}{\lambda}(a+1)^{N-1-j} \left[1 + \frac{Fb^2}{\lambda} \sum_{k=j}^{N-1} (a+1)^{2(N-1-k)} \right]^{-1} \\ &= -b(a+1)^{N-1-j} \left[\frac{\lambda}{F} + \frac{b^2}{a(a+2)} \left((a+1)^{2(N-j)} - 1 \right) \right]^{-1}, \quad (2.82) \\ j &= 0, 1, \dots, N-1. \end{aligned}$$

Via (2.80) $\eta(i)$ is a martingale. So, $\mathbf{E}_0\eta(i) = \varphi_0(0)$, $i \geq 0$, and via Remark 2.2 $\beta(i, j) = 0$ and $\alpha(i) = 0$, $0 \leq j \leq i \leq N-1$. From (2.78), (2.61) it follows that $a_j(i, k) = b_j(i, k) = 0$. So, $Q(j, k) = 0$ and via (2.60) $\gamma(j, k) = 0$, $0 \leq k \leq j \leq i \leq N$. The resolvent $R(i, j)$ of the kernel $a(i, j) = a$ depends on one argument only, i.e., $R(i, j) = R(i-j)$, and can be obtained via the recurrence relation (1.20):

$$R(i) = a \left(1 + \sum_{j=0}^{i-1} R(j) \right).$$

So,

$$\begin{aligned} R(0) &= a, \\ R(1) &= a(1+a), \\ R(2) &= a[1+a+a(1+a)] \\ &= a(1+a)^2, \\ &\dots\dots\dots \\ R(i) &= a \left[1 + a + a(1+a) + \dots + a(1+a)^{i-1} \right] \\ &= a(1+a)^i, \end{aligned} \quad (2.83)$$

i.e., $R(i) = a(1+a)^i$, $i \geq 0$. Besides, via (2.56) the optimal control has the form

$$\begin{aligned} u_0(0) &= p(0)[\psi(N-1, 0, 1) + R(N-1)]\varphi_0(0), \\ u_0(j) &= p(j)\psi(N-1, j-1, 1)x_0(j), \\ j &= 1, \dots, N-1. \end{aligned} \quad (2.84)$$

Note that via (2.52), (2.83)

$$\begin{aligned} \psi(N-1, 0, 1) + R(N-1) &= 1 + \sum_{k=0}^{N-1} R(N-1-k) \\ &= 1 + a \left(1 + (1+a) + \dots + (1+a)^{N-1} \right) \\ &= 1 + a \frac{(1+a)^N - 1}{a} \\ &= (1+a)^N \end{aligned}$$

and

$$\begin{aligned} \psi(N-1, j-1, 1) &= 1 + \sum_{k=j}^{N-1} R(N-1-k) \\ &= 1 + a \left(1 + (1+a) + \dots + (1+a)^{N-1-j} \right) \\ &= 1 + a \frac{(1+a)^{N-j} - 1}{a} \\ &= (1+a)^{N-j}. \end{aligned}$$

As a result from this and (2.82), (2.84) we obtain

$$\begin{aligned} u_0(j) &= -q(j)x_0(j), \\ q(j) &= \frac{b(a+1)^{2(N-j)-1}}{\lambda F^{-1} + b^2[a(a+2)]^{-1}[(a+1)^{2(N-j)} - 1]}, \\ j &= 0, \dots, N-1. \end{aligned} \quad (2.85)$$

If, for example, $N = 1$, then via (2.85) we have

$$u_0(0) = -\frac{b(a+1)x(0)}{\lambda F^{-1} + b^2}. \quad (2.86)$$

Remark 2.5 Let us show that the control (2.86) can be obtained immediately from the optimal control problem (2.78)–(2.80). Really, from (2.78) for $i = 0$ we have

$$x(1) = \eta(1) + ax(0) + bu(0).$$

Substituting this into (2.79), by $N = 1$ we obtain

$$\begin{aligned} J(u) &= \mathbf{E}[Fx^2(1) + \lambda u^2(0)] \\ &= F\mathbf{E}[(\eta(1) + ax(0) + bu(0))^2 + \lambda F^{-1}u^2(0)] \\ &= F\mathbf{E}[\mathbf{E}_0(\eta(1) + ax(0))^2 + 2b(\mathbf{E}_0\eta(1) + ax(0))u(0) + (\lambda F^{-1} + b^2)u^2(0)]. \end{aligned}$$

Note now that via (2.80) $\mathbf{E}_0\eta(1) = \varphi_0(0)$ and a minimum in the square brackets is reached for $u(0) = u_0(0)$ given by (2.86).

Remark 2.6 Note that in the considered optimal control problem (2.78)–(2.80) the obtained optimal control $u_0(i)$ does not depend on the past, i.e., on $x_0(j)$ for $j < i$. Really, it is not a surprise since from (2.78), (2.80) it follows that

$$x(i+1) - x(i) = ax(i) + bu(i) + \sigma(i)\xi(i+1), \quad i \geq 0. \quad (2.87)$$

So, the initial equation (2.78) is an analogue of the Ito stochastic differential equation without delay

$$dx(t) = (ax(t) + bu(t))dt + \sigma(t)dw(t),$$

where $w(t)$ is the standard Wiener process, $\xi(i+1) = w(t_{i+1}) - w(t_i)$, $x(i) = x(t_i)$, $t_{i+1} - t_i = 1$.

Example 2.3 Let us show that a small modification of the optimal control $u_0(i)$ of the control problem (2.78)–(2.80) leads to an increase of the performance functional.

For this aim, note that via (2.85), (2.87) for $i = N - 1$ we obtain

$$x_0(N) = (1 + a - bq(N - 1))x_0(N - 1) + \sigma(N - 1)\xi(N). \quad (2.88)$$

Using (2.85), (2.88) and the properties of the process $\xi(i)$, represent the performance functional (2.79) in the form

$$\begin{aligned} J(u_0) &= \mathbf{E} \left[Fx_0^2(N) + \lambda \sum_{i=0}^{N-1} q^2(i)x_0^2(i) \right] \\ &= Q\mathbf{E}x_0^2(N - 1) + \sigma^2(N - 1) + \lambda \sum_{i=0}^{N-2} q^2(i)\mathbf{E}x_0^2(i), \end{aligned}$$

where

$$Q = F(1 + a - bq(N - 1))^2 + \lambda q^2(N - 1). \quad (2.89)$$

Note that via (2.85)

$$q(N-1) = \frac{b(1+a)}{\lambda F^{-1} + b^2}.$$

Substituting this into (2.89), we obtain

$$\begin{aligned} Q &= F \left(1 + a - \frac{b^2(1+a)}{\lambda F^{-1} + b^2} \right)^2 + \frac{\lambda b^2(1+a)^2}{(\lambda F^{-1} + b^2)^2} \\ &= (1+a)^2 \left[F \left(\frac{\lambda F^{-1}}{\lambda F^{-1} + b^2} \right)^2 + \frac{\lambda b^2}{(\lambda F^{-1} + b^2)^2} \right] \\ &= \frac{\lambda(1+a)^2}{\lambda F^{-1} + b^2}. \end{aligned}$$

Put now, for example,

$$q(N-1) = \frac{1+a}{b}, \quad (2.90)$$

and let all other $q(i)$ let be defined by (2.85). In this case for arbitrary $F > 0$, we obtain

$$Q = \frac{\lambda(1+a)^2}{b^2} > \frac{\lambda(1+a)^2}{\lambda F^{-1} + b^2},$$

i.e., the performance functional (2.79) is increasing.

For numerical simulation, put

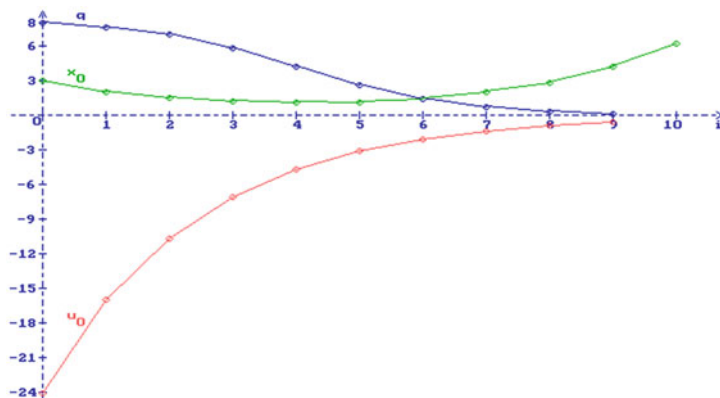
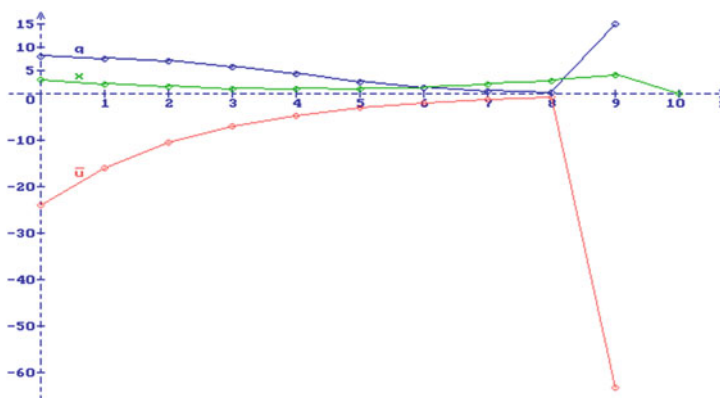
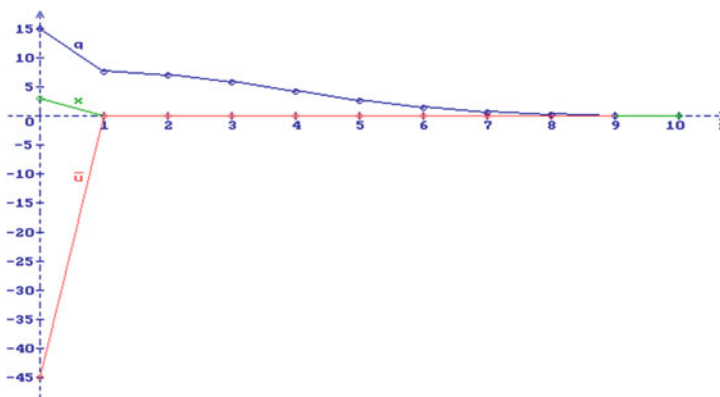
$$\begin{aligned} N &= 10, & a &= 0.5, & b &= 0.1, & F &= \lambda = 1, \\ x(0) &= 3, & \sigma(j) &= 0, & j &= 0, \dots, N-1. \end{aligned} \quad (2.91)$$

In Fig 2.1 one can see the trajectories of $q(j)$ defined in (2.85), of the optimal solution $x_0(j)$ and of the optimal control $u_0(j)$ given in (2.85), (2.85), respectively, by the values of the parameters (2.91). The minimal value of the performance functional is $J(u_0) = 1,085$.

Changing $q(j)$ in the one point $i = N-1$ only as it is given in (2.90), we obtain another picture (see Fig. 2.2). Via (2.88), (2.91) here $x(N) = 0$ but $\bar{u}(N-1) = -63.32$ and $J(\bar{u}) = 5,054 > J(u_0)$.

Consider also the case $q(0) = (1+a)b^{-1}$. By that via (2.87), (2.91), we obtain (see Fig. 2.3) $\bar{u}(j) = x(j) = 0$, $j > 0$, and $J(\bar{u}) = 2,025 > J(u_0)$.

Example 2.4 Consider the optimal control problem for the scalar linear stochastic difference equation

Fig. 2.1 $J(u_0) = 1,085$ Fig. 2.2 $J(\bar{u}) = 5,054$ Fig. 2.3 $J(\bar{u}) = 2,025$

$$\begin{aligned} x(i+1) &= \eta(i+1) + a_0x(i) + a_1x(i-1) + bu(i), \\ i \geq 0, \quad x(0) &= \varphi_0(0) = \eta(0), \quad x(-1) = \varphi_0(-1), \end{aligned} \quad (2.92)$$

where $\eta(i)$ is a martingale, and the quadratic performance functional

$$J(u) = \mathbf{E} \left[x^2(N) + \lambda \sum_{i=0}^{N-1} u^2(i) \right]. \quad (2.93)$$

The Eq. (2.92) can be represented in the form (2.47) as follows:

$$\begin{aligned} x(1) &= \hat{\eta}(1) + a_0x(0) + bu(0), \\ x(i+1) &= \hat{\eta}(i+1) + a_0x(i) + a_1x(i-1) + bu(i), \\ i \geq 1, \quad x(0) &= \varphi_0(0), \end{aligned}$$

where

$$\hat{\eta}(i) = \begin{cases} \eta(i), & i > 1, \\ \eta(1) + a_1\varphi_0(-1), & i = 1. \end{cases} \quad (2.94)$$

Note that via (1.21) and (2.92) the resolvent $R(i)$ is

$$\begin{aligned} R(0) &= a_0, \\ R(1) &= a_0R(0) + a_1 \\ &= a_1 + a_0^2, \\ R(2) &= a_0R(1) + a_1R(0) \\ &= 2a_0a_1 + a_0^3, \\ R(3) &= a_0R(2) + a_1R(1) \\ &= a_1^2 + 3a_0^2a_1 + a_0^4, \\ &\dots\dots\dots \\ R(i) &= a_0R(i-1) + a_1R(i-2), \quad i \geq 2. \end{aligned} \quad (2.95)$$

Note also that via (2.52), (2.57), (2.92)

$$\begin{aligned} \psi(N-1, j, b(\cdot, j)) &= \left(1 + \sum_{k=j+1}^{N-1} R(N-1-k) \right) b \\ &= \begin{cases} \psi(N-1, j, 1)b, & j = 0, 1, \dots, N-2, \\ b, & j = N-1, \end{cases} \end{aligned}$$

and

$$p(j) = \begin{cases} -\frac{\psi(N-1,j,1)b}{\lambda + b^2 \left(1 + \sum_{k=j}^{N-2} \psi^2(N-1,k,1)\right)}, & j = 0, 1, \dots, N-2, \\ -\frac{b}{\lambda + b^2}, & j = N-1, \end{cases} \quad (2.96)$$

Let be $N = 1$. Using that $\eta(i)$ is a martingale and (2.94), we have

$$\begin{aligned} \psi(0, 0, \mathbf{E}_0 \hat{\eta}(\cdot + 1)) &= \mathbf{E}_0 \hat{\eta}(1) \\ &= \mathbf{E}_0(\eta(1) + a_1 \varphi_0(-1)) \\ &= \varphi_0(0) + a_1 \varphi_0(-1). \end{aligned}$$

As a result from (2.56), (2.95), (2.96) we obtain

$$u_0(0) = -\frac{b}{\lambda + b^2} [(1 + a_0) \varphi_0(0) + a_1 \varphi_0(-1)].$$

Let be $N \geq 2$. Then

$$\begin{aligned} \psi(N-1, 0, \mathbf{E}_0 \hat{\eta}(\cdot + 1)) &= \mathbf{E}_0 \hat{\eta}(N) + \sum_{k=1}^{N-1} R(N-1-k) \mathbf{E}_0 \hat{\eta}(k) \\ &= \mathbf{E}_0 \eta(N) + \sum_{k=2}^{N-1} R(N-1-k) \mathbf{E}_0 \eta(k) \\ &\quad + R(N-2) \mathbf{E}_0 \hat{\eta}(1) \\ &= \left(1 + \sum_{k=2}^{N-1} R(N-1-k)\right) \varphi_0(0) \\ &\quad + R(N-2) (\varphi_0(0) + a_1 \varphi_0(-1)) \\ &= \left(1 + \sum_{k=1}^{N-1} R(N-1-k)\right) \varphi_0(0) \\ &\quad + R(N-2) a_1 \varphi_0(-1) \\ &= \psi(N-1, 0, 1) \varphi_0(0) + R(N-2) a_1 \varphi_0(-1). \end{aligned} \quad (2.97)$$

So, via (2.56), (2.97) for $N \geq 2$ we get

$$u_0(0) = p(0) [\psi(N-1, 0, 1) \varphi_0(0) + R(N-2) a_1 \varphi_0(-1)].$$

For example, for $N = 2$ we have

$$u_0(0) = -\frac{a_0 b}{\lambda + b^2(1 + a_0^2)} \left[(1 + a_0 + a_1 + a_0^2) \varphi_0(0) + a_0 a_1 \varphi_0(-1) \right].$$

To calculate $u_0(1)$ for $N = 2$ note that via (2.96) and (2.57)–(2.61)

$$\begin{aligned} p(1) &= -\frac{b}{\lambda + b^2}, & \psi(1, 0, 1) &= 1 + a_0, \\ \alpha(1) &= p(1)\psi(1, 0, \beta(\cdot, 1)) = p(1)\beta(1, 1) = -p(1)\varphi_0(-1), \\ \gamma(0, 0) &= p(1)\psi(1, 0, a_0(\cdot, 0)) = p(1)a_0(1, 0) = p(1)(a_1 - a_0), \\ \underline{Q}(0, 0) &= p(1)\psi(1, 0, b_0(\cdot, 0)) = p(1)b_0(1, 0) = -p(1)b. \end{aligned}$$

So, via (2.56) we obtain

$$u_0(1) = -\frac{b}{\lambda + b^2} [(1 + a_0)x_0(1) + (a_1 - a_0)\varphi_0(0) - \varphi_0(-1) - bu_0(0)].$$

Similarly, one can calculate the optimal control $u_0(0), u_0(1), \dots, u_0(N-1)$ of the problem (2.92), (2.93) for arbitrary N .

Example 2.5 Let us show that similarly to the control problem (2.47), (2.48) the optimal control can be obtained for linear-quadratic problems with linear equations that are different from the equation (2.47) and the quadratic performance functional that was considered in Remark 2.3. Consider, for example, the scalar controlled process with a noise in control

$$\begin{aligned} x(i+1) &= \eta(i+1) \sum_{j=0}^i [\beta(i, j) + \gamma(i, j)u(j)] \xi(j+1), \\ x(0) &= \varphi_0(0), \end{aligned} \tag{2.98}$$

and the quadratic performance functional

$$J(u) = \mathbf{E} \left[x^2(N) + \sum_{i=0}^{N-1} (\lambda_0 x^2(i) + \lambda_1 u^2(i)) \right]. \tag{2.99}$$

Here, $\lambda_0 \geq 0$, $\lambda_1 > 0$, $\eta(i) \in H$, $\xi(j)$, $j \in Z$, are $\mathfrak{F}_j$ -adapted Gaussian random variables that are mutually independent and are independent on $\eta(i)$, $\mathbf{E}\xi(j) = 0$, $\mathbf{E}\xi^2(j) = 1$.

Via (2.98), (2.99) in this case

$$q_0(i+1) = \sum_{j=0}^i \gamma(i, j)v(j)\xi(j+1),$$

$$J_0(u_0) = 2\mathbf{E} \left[x_0(N)q_0(N) + \sum_{i=0}^{N-1} (\lambda_0 x_0(i)q_0(i) + \lambda_1 u_0(i)v(i)) \right]. \quad (2.100)$$

Note that $\mathbf{E}\eta(i)\xi(j) = 0$, $i, j \in \mathbb{Z}$, and $\mathbf{E}\xi(i)\xi(j) = 0$, $i \neq j$. So, via (2.98), (2.100) we have

$$\begin{aligned} \mathbf{E}x_0(i)q_0(i) &= \mathbf{E} \left[\eta(i) + \sum_{j=0}^{i-1} [\beta(i-1, j) + \gamma(i-1, j)u_0(j)]\xi(j+1) \right] \\ &\quad \times \sum_{l=0}^{i-1} \gamma(i-1, l)v(l)\xi(l+1) \\ &= \sum_{j=0}^{i-1} \mathbf{E}[\beta(i-1, j) + \gamma(i-1, j)u_0(j)]\gamma(i-1, j)v(j) \end{aligned}$$

and

$$\begin{aligned} \sum_{i=0}^{N-1} \mathbf{E}x_0(i)q_0(i) &= \sum_{i=0}^{N-1} \sum_{j=0}^{i-1} \mathbf{E}[\beta(i-1, j) + \gamma(i-1, j)u_0(j)]\gamma(i-1, j)v(j) \\ &= \sum_{j=0}^{N-2} \sum_{i=j+1}^{N-1} \mathbf{E}[\beta(i-1, j) + \gamma(i-1, j)u_0(j)]\gamma(i-1, j)v(j) \end{aligned}$$

From this and (2.100), it follows that

$$\begin{aligned} J_0(u_0) &= 2\mathbf{E} \sum_{j=0}^{N-1} \left[[\beta(N-1, j) + \gamma(N-1, j)u_0(j)]\gamma(N-1, j) + \lambda_1 u_0(j) \right. \\ &\quad \left. + \lambda_0 \sum_{i=j+1}^{N-1} [\beta(i-1, j) + \gamma(i-1, j)u_0(j)]\gamma(i-1, j) \right] v(j). \end{aligned}$$

Thus, the necessary condition for optimality $J_0(u_0) \geq 0$ of the control u_0 holds if and only if the expression in the square brackets equals zero, i.e.,

$$\begin{aligned} &[\beta(N-1, j) + \gamma(N-1, j)u_0(j)]\gamma(N-1, j) + \lambda_1 u_0(j) \\ &+ \lambda_0 \sum_{i=j+1}^{N-1} [\beta(i-1, j) + \gamma(i-1, j)u_0(j)]\gamma(i-1, j) = 0. \end{aligned}$$

From this, it follows that the optimal control u_0 of the control problem (2.98), (2.99) has the form

$$u_0(j) = - \frac{\beta(N-1, j)\gamma(N-1, j) + \lambda_0 \sum_{i=j+1}^{N-1} \beta(i-1, j)\gamma(i-1, j)}{\lambda_1 + \gamma^2(N-1, j) + \lambda_0 \sum_{i=j+1}^{N-1} \gamma^2(i-1, j)},$$

$$j = 0, 1, \dots, N-1.$$

Optimal Control of Stochastic Difference Volterra
Equations

An Introduction

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