Chapter 2
Geometrically Nonlinear Behaviour

2.1 Fundamental Terms of Geometric Nonlinearities

The assumptions of a geometrically linear theory are:

1. equilibrium in the **un**deformed state
2. small rotations, i.e. linearised kinematics (s. Fig. 2.1)
3. small strain,
   i.e. it is useful and sufficient to define strain as changes in length relative to the **initial** length l_0.

   Step by step these assumptions will be given up in the following, i.e.

1. equilibrium in the **deformed** state
2. large rotations and
3. large strain

   will be considered.

2.2 Theory of Second Order, Equilibrium in the Deformed System

2.2.1 Motivation and FE-Formulation

Only assumption 1 is given up. This theory is sufficient for most of the civil engineering applications and is the base of Euler's theory of beam buckling and the usual solutions for plate buckling.

Consider the simply supported beam from Fig. 2.2. In the completely linear theory the transverse load q and the longitudinal force F are decoupled: the transverse load leads to shear force and bending moment M_0, the longitudinal

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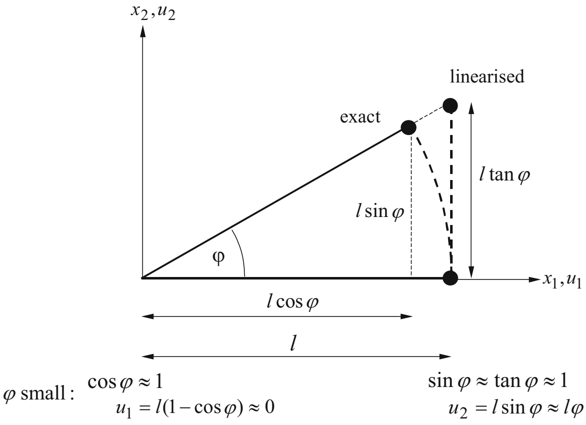

Fig. 2.1 Linearised kinematics

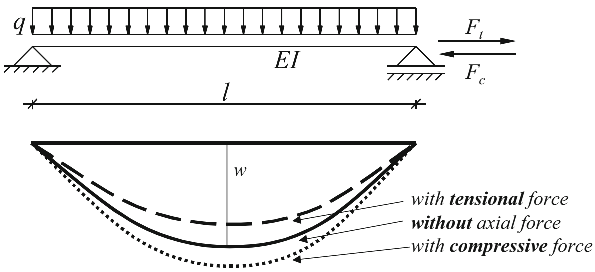

Fig. 2.2 Consequence of equilibrium in the deformed state

force to normal force. If the deformed system is considered for equilibrium the force F has a moment arm to any point on the deflection line of the beam. In the first approximation this arm is a result of the transverse load. This results in an additional moment of

$$\Delta M = -Fw \tag{2.1}$$

If a tensional force F_t is active this reduces the total moment. This leads to less deflection and thus to a reduced unloading in the final equilibrium state. The final moment will be in the range

$$M_0 - F_t w_0 < M < M_0 \tag{2.2}$$

This effect should be taken into account for economic reasons; maybe smaller cross sections can be used.

If a compressive force F_c is active the moment difference in the first approximation is

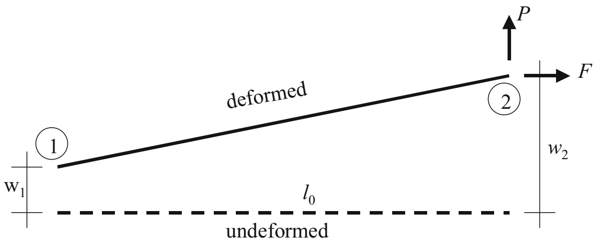

Fig. 2.3 Concerning the equilibrium in the deformed state for the link element

$$\Delta M = F_c w_0 \quad \Rightarrow \quad w_1 = w(M_0 + \Delta M) > w_0 \tag{2.3}$$

This causes an increase of the deflection and an additional moment and so forth. It depends on the size of the force whether the final deflection is finite or not. If Euler's critical load is exceeded the deflection becomes infinite and no equilibrium is possible. Thus, accounting for the effect of compressive forces is necessary for safety reasons.

For the simplest Finite element, the link element (Fig. 2.3), this effect can be formulated as follows:

Because of the missing bending stiffness no equilibrium with the load P is possible if the deformation is not taken into account. In the deformed state, however, the sum of moments around the left node yields:

$$F \underbrace{(w_2 - w_1)}_{\Delta w} = P l_0 \tag{2.4}$$

The calculation of Δw would be possible, but solving for P yields:

$$\frac{F}{l_0}(w_2 - w_1) = P \tag{2.5}$$

Still assumed that the rotations are small the longitudinal force F can be approximated by the normal force N which in turn can be expressed by stress σ times cross section area A:

$$\frac{\sigma A}{l_0}(w_2 - w_1) = P \tag{2.6}$$

In matrix notation this yields:

$$\frac{\sigma A}{l_0}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = P \tag{2.7}$$

Taking into account that a similar equation can be found for a load at the left node and that there are longitudinal displacements in addition to the transverse ones this relation can be extended to

$$\underbrace{\frac{\sigma A}{l_0}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \begin{bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ P_1 \\ 0 \\ P_2 \end{bmatrix} \tag{2.8}$$

A term which relates displacements and forces is called a stiffness. The matrix **S** also represents a stiffness, however it does not depend on material parameters but on stresses. That is why **S** is called "stress stiffening matrix". The stress, however, has an algebraic sign, i.e. a compressive stress leads to a weakening.

The matrix **S** is an addition to the stiffness matrix **K** according to the linear theory, in the case of the 2d link element:

$$\left(\frac{EA}{l}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\sigma A}{l}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} P_{1x} \\ P_{1z} \\ P_{2x} \\ P_{2z} \end{bmatrix} \tag{2.9}$$

2.2.2 Why Theory of Second Order?

In the previous chapter fully linearised kinematics has been used. Why is it named theory of second order? This is illustrated by solving the following stability problem in two ways, at first by formulating the equilibrium in the deformed state and using linearised kinematics (Fig. 2.4).

The equilibrium in the deformed state results in:

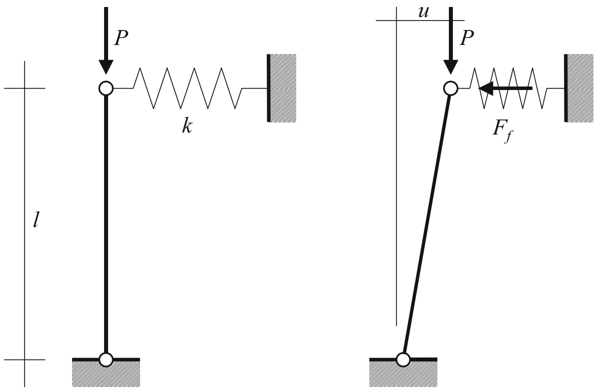

Fig. 2.4 Stability problem with linearised kinematics

$$Pu = F_f\, l \tag{2.10}$$

The spring force is expressed as

$$F_f = ku \tag{2.11}$$

thus

$$Pu = ku\, l \tag{2.12}$$

$$(P - kl)u = 0 \tag{2.13}$$

This equation has the trivial solution $u = 0$ and the non-trivial one

$$P = k\, l \tag{2.14}$$

This means the critical load of the system because a displacement without change in load becomes possible then.

Now the principle of the minimum of the potential energy is applied starting with the fully non-linear kinematics (Fig. 2.5).

The load P looses potential energy whereas the spring gains some. Together a minimum must be achieved:

$$-Pl(1 - \cos\varphi) + \frac{1}{2}k(l\sin\varphi)^2 \quad \rightarrow \quad Min. \tag{2.15}$$

Now the angular functions are replaced by their Taylor expansions truncated after the second order term:

$$\sin\varphi \approx \varphi \Big| - \frac{\varphi^3}{3!} + \cdots, \qquad \cos\varphi \approx 1 - \frac{\varphi^2}{2!} + \cdots \tag{2.16}$$

Then (2.15) becomes

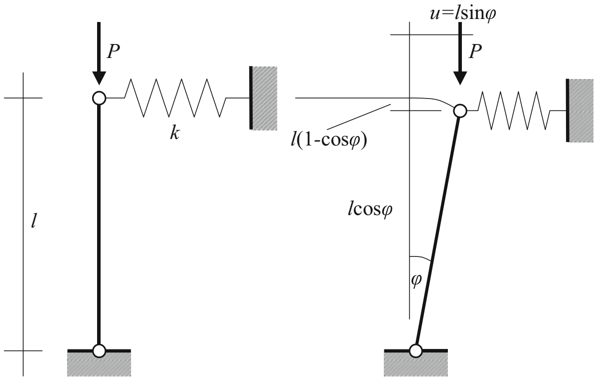

Fig. 2.5 Stability problem with exact kinematics

$$-Pl\left(\frac{\varphi^2}{2}\right) + \frac{1}{2}k(l\varphi)^2 \quad \rightarrow \quad Min. \tag{2.17}$$

As necessary condition the derivative must be zero:

$$-Pl\varphi + kl^2\varphi = 0 \tag{2.18}$$

$$(-P + kl)l\varphi = 0 \tag{2.19}$$

Again the critical load (2.14) is obtained as non-trivial solution. Thus this theory is called "of second order" because terms up to the second order of the Taylor expansions of the angular functions are necessary if energy methods are applied.

2.2.3 *Linear Buckling*

In symbolic matrix notation equation (2.9) reads:

$$(\mathbf{K} + \mathbf{S}(\boldsymbol{\sigma}))\hat{\mathbf{u}} = \mathbf{f}^{ext} \tag{2.20}$$

The matrix $\mathbf{S}$ is linearly depending on the stress, the stress linearly on the axial force. Therefore, the stress-stiffening due to a reference load $\mathbf{f}_0$ multiplied by a factor λ is

$$\mathbf{S}(\sigma(\lambda\mathbf{f}_0)) = \mathbf{S}(\lambda\sigma(\mathbf{f}_0)) = \lambda\mathbf{S}(\sigma(\mathbf{f}_0)) \tag{2.21}$$

A stability problem (buckling) occurs if a (further) deformation without change in loading is possible. Then (2.20) becomes

$$(\mathbf{K} + \lambda\mathbf{S}(\boldsymbol{\sigma}))\,\hat{\boldsymbol{\varphi}} = \mathbf{0} \tag{2.22}$$

This is a general matrix eigenvalue problem. $\boldsymbol{\varphi}$ is used instead of $\mathbf{u}$ to mark it as an eigenvector. The eigenvalue λ is the critical load multiplier for the applied load, i.e. the load which led to the stress σ. However,

$$\mathbf{f}_{cr,i} = \lambda\mathbf{f}_0 \tag{2.23}$$

is the critical load only under the assumption that there are no imperfections (pre-deflections, eccentricities not be taken into account) and the system behaviour is completely linear until buckling occurs. That's why this load level is called *i*deal *cr*itical load. In reality instability occurs at lower loads. How this is accounted for in the simulation is described in Chap. 3.

The eigenvector $\boldsymbol{\varphi}$ replacing the vector of the displacements describes the direction the system will follow when buckling begins. This state is called *buckling* *mode*. It can only be determined up to an arbitrary factor and is normalised, e.g. in such a way that the maximum displacement becomes 1.

The steps to follow in a FE buckling analysis are listed as Algorithm 2.1:

Algorithm 2.1 Linear Buckling Analysis

a) fully linear static analysis to determine the (pre-)stress state $\boldsymbol{\sigma}$
b) assembling of the stress-stiffening matrix $\mathbf{S}$
c) solving the eigenvalue problem,
   usually by vector iteration $\rightarrow \boldsymbol{\varphi} \rightarrow \lambda$

The mode $\boldsymbol{\varphi}$ can be plotted like a usual displacement state. Other results like strain and stress are of minor meaning; they are increments multiplied by an unknown factor, but they can be used for error estimation.

Example First Euler Case

As example the first Euler case is considered where one end is clamped and the other one is free (Fig. 2.6).

For the FE analysis an element is needed which includes a longitudinal and a bending stiffness. Before applying the boundary conditions the system of equations reads:

$$\left(\begin{bmatrix} \dfrac{EA}{l} & \mathbf{0} & -\dfrac{EA}{l} & \mathbf{0} \\ \mathbf{0} & \dfrac{EI}{l}\begin{bmatrix} \dfrac{12}{l^2} & \dfrac{6}{l} \\ \dfrac{6}{l} & 4 \end{bmatrix} & \mathbf{0} & \dfrac{EI}{l}\begin{bmatrix} -\dfrac{12}{l^2} & \dfrac{6}{l} \\ -\dfrac{6}{l} & 2 \end{bmatrix} \\ -\dfrac{EA}{l} & \mathbf{0} & \dfrac{EA}{l} & \mathbf{0} \\ \mathbf{0} & \dfrac{EI}{l}\begin{bmatrix} -\dfrac{12}{l^2} & -\dfrac{6}{l} \\ \dfrac{6}{l} & 2 \end{bmatrix} & \mathbf{0} & \dfrac{EI}{l}\begin{bmatrix} \dfrac{12}{l^2} & -\dfrac{6}{l} \\ -\dfrac{6}{l} & 4 \end{bmatrix} \end{bmatrix} + \mathbf{S}(\sigma)\right) \begin{bmatrix} u_1 \\ w_1 \\ \varphi_1 \\ u_2 \\ w_2 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -F \\ 0 \\ 0 \end{bmatrix} \tag{2.24}$$

The correct stress-stiffening matrix for the beam is derived not before Sect. 2.2.4. Here the matrix $\mathbf{S}$ for the link element is used because it makes the hand calculation easier. Taking the constraints of all degrees of freedom of the left node into account and introducing terms for $\mathbf{S}$ the equations read:

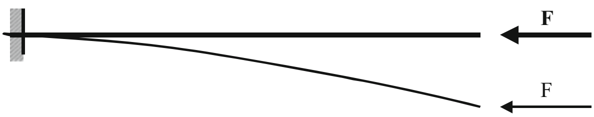

Fig. 2.6 Column buckling, first Euler case

$$\left(\begin{bmatrix} \frac{EA}{l} & \mathbf{0} \\ \mathbf{0} & \frac{EI}{l} \begin{bmatrix} \frac{12}{l^2} & -\frac{6}{l} \\ -\frac{6}{l} & 4 \end{bmatrix} \end{bmatrix} + \frac{\sigma A}{l} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} u_2 \\ w_2 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} -F \\ 0 \\ 0 \end{bmatrix} \tag{2.25}$$

In step a) of Algorithm 2.1 $\sigma = 0$ holds, furthermore it is obvious that the bending and the longitudinal part are decoupled and the right hand side for the bending part is $\mathbf{0}$. From the first row,

$$\frac{EA}{l} u_2 = -F \tag{2.26}$$

one obtains

$$u_2 = -\frac{Fl}{EA} \tag{2.27}$$

From there the axial stress follows as

$$\sigma = \frac{E}{l}(-u_1 + u_2) = \frac{E}{l}\left(-\frac{Fl}{EA}\right) = -\frac{F}{A} \quad \Rightarrow \quad \frac{\sigma A}{l} = -\frac{F}{l} \tag{2.28}$$

This result is used in step b) such that one obtains for step c) after multiplication of $\mathbf{S}$ by the load multiplier λ, addition of $\mathbf{K}$ and $\lambda\mathbf{S}$ and zeroing the right hand side (because a *change* in the displacements without a *further* load is requested):

$$\begin{bmatrix} \frac{EA}{l} & \mathbf{0} \\ \mathbf{0} & \frac{1}{l} \begin{bmatrix} \frac{12EI}{l^2} - \lambda F & -\frac{6EI}{l} \\ -\frac{6EI}{l} & 4EI \end{bmatrix} \end{bmatrix} \begin{bmatrix} u_2 \\ w_2 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.29}$$

For the hand calculation no vector iteration is performed as often in FE-calculations but the classic consideration is followed: Since the right hand side is zero this system of equations can only be solved in a non-trivial way ($\mathbf{u} = \mathbf{0}$ is the trivial solution) if the determinant of the system matrix equals zero. This is fulfilled if the lower left subdeterminant belonging to the bending part is zero:

$$\det \begin{bmatrix} \frac{12EI}{l^2} - \lambda F & -\frac{6EI}{l} \\ -\frac{6EI}{l} & 4EI \end{bmatrix} = 0 \tag{2.30}$$

$$\left(\frac{12EI}{l^2} - \lambda F \right) 4EI - \left(\frac{6EI}{l} \right)^2 = 0 \quad | \quad : EI \tag{2.31}$$

$$\frac{48EI}{l^2} - 4\lambda F - \frac{36EI}{l^2} = \frac{12EI}{l^2} - 4\lambda F = 0 \tag{2.32}$$

$$\lambda F = F_{cr,i} = \frac{12EI}{4l^2} \tag{2.33}$$

$$\boxed{F_{cr,i} = 3\frac{EI}{l^2}} \tag{2.34}$$

where the subscript *cr,i* means *ideal critical*.

Since the buckling length s_k is twice the length of the beam the analytical solution reads:

$$F_{cr,i}^{Euler} = \frac{\pi^2 EI}{(2l)^2} = 2.47\frac{EI}{l^2} \tag{2.35}$$

With two elements and again the simplified stress-stiffening matrix one obtains 2.60 EI/l^2.

In a hand calculation the result of (2.28) can be obtained from equilibrium considerations. Due to the decoupling in this and similar cases step a) can be omitted and the problem be solved based on

$$\left(\frac{EI}{l} \begin{bmatrix} \frac{12}{l^2} & \frac{6}{l} & -\frac{12}{l^2} & \frac{6}{l} \\ \frac{6}{l} & 4 & -\frac{6}{l} & 2 \\ -\frac{12}{l^2} & -\frac{6}{l} & \frac{12}{l^2} & -\frac{6}{l} \\ \frac{6}{l} & 2 & -\frac{6}{l} & 4 \end{bmatrix} + \frac{\sigma A}{l} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} w_1 \\ \varphi_1 \\ w_2 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.36}$$

i.e. the bending part without considering axial deformation.

For the calculation of the eigenvector, the buckling mode, solution (2.34) is introduced into (2.29) which makes the two lower rows of the system of equations linearly dependent. The solution is no longer unique. One unknown must be chosen:

$$w_2 = 1 \tag{2.37}$$

The third row of the system of equations (2.29) the reads:

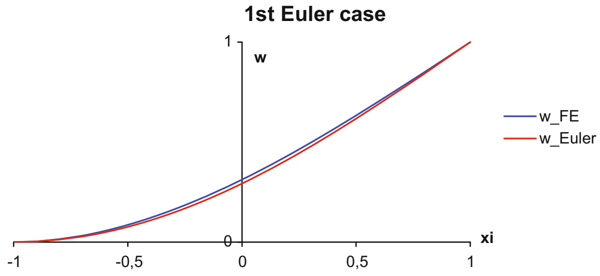

Fig. 2.7 Deflection line for the first Euler case with one beam element and analytical

$$-\frac{6EI}{l}\cdot 1 + 4EI\varphi_2 = 0 \tag{2.38}$$

$$\varphi_2 = \frac{3}{2l} \tag{2.39}$$

so that one obtains the eigenvector

$$\boldsymbol{\varphi} = \begin{bmatrix} 1 \\ \\ \frac{3}{2l} \end{bmatrix} \tag{2.40}$$

It is multiplied by the cubic shape functions of the beam element which are related to the right node (N_3 and N_4, complete set see Sect. 2.2.4):

$$w(\xi) = \frac{1}{4}\left(2 + 3\xi - \xi^3\right)\cdot 1 + \frac{1}{4}\left(-1 - \xi + \xi^2 + \xi^3\right)\frac{l}{2}\frac{3}{2l}\ , \qquad -1 \le \xi \le 1 \tag{2.41}$$

Thus one obtains the deflection line. It is compared with the analytical solution

$$w^{Euler}(\xi) = 1 - \sin\left(\pi\left(\frac{3}{4} + \frac{1}{4}\xi\right)\right) \tag{2.42}$$

See Fig. 2.7 for the similarity of the shapes.

2.2.4 *Correct Stress-Stiffness Matrix for the Bernoulli-Beam*

2.2.4.1 Derivation

Here a formal way is followed resulting from the differential equation of the beam in the second-order theory:

$$EIw^{iv} + Nw'' = EIw^{iv} + \sigma A w'' = 0 \tag{2.43}$$

The corresponding minimisation problem (weak form) reads:

$$\frac{1}{2}\int\limits_{(l)} w'' EI w'' \, dx + \frac{1}{2}\int\limits_{(l)} w' \sigma A w' \, dx \quad \rightarrow \quad Min. \tag{2.44}$$

The first part leads to the well known stiffness matrix, the second one to the stress-stiffness matrix **S**. For the Bernoulli-beam element the displacement function reads:

$$w(\xi) = \mathbf{N}(\xi)\hat{\mathbf{u}} \,, \qquad -1 \le \xi \le 1 \tag{2.45}$$

$$\begin{array}{llll|l} w & = \frac{1}{4}\left(2 - 3\xi + \xi^3\right) & & \cdot w_1 & = N_1(\xi) w_1 \\ & + \frac{1}{4}\left(1 - \xi - \xi^2 + \xi^3\right)\frac{l}{2} & & \cdot \varphi_1 & + N_2(\xi)\varphi_1 \\ & + \frac{1}{4}\left(2 + 3\xi - \xi^3\right) & & \cdot w_2 & + N_3(\xi) w_2 \\ & + \frac{1}{4}\left(-1 - \xi + \xi^2 + \xi^3\right)\frac{l}{2} & & \cdot \varphi_2 & + N_4(\xi)\varphi_2 \end{array} \tag{2.46}$$

The derivatives with respect to the real coordinate x read:

$$w'(\xi) = \frac{dw}{dx} = \frac{dw}{d\xi}\frac{d\xi}{dx} = \frac{2}{l}\frac{d\mathbf{N}}{d\xi}\hat{\mathbf{u}} = \frac{2}{l}\mathbf{N}'\hat{\mathbf{u}} \tag{2.47}$$

with

$$\begin{aligned} \frac{d\mathbf{N}}{d\xi} &= \mathbf{N}' \\ &= \left[\frac{1}{4}(-3 + 3\xi^2) \quad \frac{1}{4}(-1 - 2\xi + 3\xi^2)\frac{l}{2} \quad \frac{1}{4}(3 - 3\xi^2) \quad \frac{1}{4}(-1 + 2\xi + 3\xi^2)\frac{l}{2}\right] \end{aligned} \tag{2.48}$$

Thus, the stress-stiffening matrix is calculated with $dx = \frac{l}{2} d\xi$ as

$$\mathbf{S} = \int\limits_{-1}^{1} \sigma \frac{4}{l^2} \mathbf{N}'^T \mathbf{N}' A \frac{l}{2} d\xi = \sigma A \frac{2}{l} \int\limits_{-1}^{1} \mathbf{N}'^T \mathbf{N}' d\xi \tag{2.49}$$

The product of the derivatives of the shape functions creates the matrix

$$
\begin{array}{c|c}
\mathbf{N}'^{T}\ \mathbf{N}' & \begin{bmatrix} N'_1 & N'_2 & N'_3 & N'_4 \end{bmatrix} \\
\hline
\begin{bmatrix} N'_1 \\ N'_2 \\ N'_3 \\ N'_4 \end{bmatrix} & \begin{bmatrix} N'^2_1 & \cdots & \cdots & N'_1\,N'_4 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ N'_4\,N'_1 & \cdots & \cdots & N'^2_4 \end{bmatrix}
\end{array}
\tag{2.50}
$$

As example the matrix element S_{12} is calculated as

$$
\begin{aligned}
16\int_{-1}^{1} N_1' N_2' d\xi &= \int_{-1}^{1} \left(-3+3\xi^2\right)\left(-1-2\xi+3\xi^2\right)\frac{l}{2}d\xi \\
&= \int_{-1}^{1} \left(3+6\xi-9\xi^2-3\xi^2-6\xi^3+9\xi^4\right)\frac{l}{2}d\xi \\
&= \int_{-1}^{1} \left(3+6\xi-12\xi^2-6\xi^3+9\xi^4\right)\frac{l}{2}d\xi \\
&= 3\xi-4\xi^3+\frac{9}{5}\xi^5\Big|_{-1}^{1}\frac{l}{2} = \left(6-8+\frac{18}{5}\right)\frac{l}{2} = \frac{8}{5}\,\frac{l}{2}
\end{aligned}
\tag{2.51}
$$

and

$$
S_{12} = \sigma A \frac{2}{l}\frac{8}{5}\frac{l}{2}\frac{1}{16} = \frac{1}{10}\sigma A = 3l\frac{\sigma A}{30l}
\tag{2.52}
$$

The complete stress-stiffening matrix reads:

$$
\mathbf{S} = \frac{\sigma A}{30l}\begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l. & -l^2 & -3l & 4l^2 \end{bmatrix}
\tag{2.53}
$$

2.2.4.2 Application to the First Euler Case

After the introduction of the boundary conditions and the determination of the axial stress the eigenvalue problem reads:

$$(\mathbf{K}+\mathbf{S})\,\boldsymbol{\varphi} = \left(\frac{EI}{l}\begin{bmatrix} \frac{12}{l^2} & -\frac{6}{l} \\ -\frac{6}{l} & 4 \end{bmatrix} - \frac{F}{30l}\begin{bmatrix} 36 & -3l \\ -3l & 4l^2 \end{bmatrix}\right)\boldsymbol{\varphi} = \mathbf{0} \tag{2.54}$$

$$\frac{1}{l}\left(\begin{bmatrix} \frac{12EI}{l^2} & -\frac{6EI}{l} \\ -\frac{6EI}{l} & 4EI \end{bmatrix} - \begin{bmatrix} \frac{6}{5}F & -\frac{l}{10}F \\ -\frac{l}{10}F & \frac{2l^2}{15}F \end{bmatrix}\right)\boldsymbol{\varphi} = \mathbf{0} \tag{2.55}$$

$$\begin{bmatrix} \frac{12EI}{l^2} - \frac{6}{5}F & -\frac{6EI}{l} + \frac{l}{10}F \\ -\frac{6EI}{l} + \frac{l}{10}F & 4EI - \frac{2l^2}{15}F \end{bmatrix}\boldsymbol{\varphi} = \mathbf{0} \tag{2.56}$$

The determinant of the matrix is

$$\left(\frac{12EI}{l^2} - \frac{6}{5}F\right)\left(4EI - \frac{2l^2}{15}F\right) - \left(-\frac{6EI}{l} + \frac{l}{10}F\right)^2 = 0 \tag{2.57}$$

$$\frac{48(EI)^2}{l^2} - \frac{32EI}{5}F + \frac{4l^2}{25}F^2 - \left(\frac{36(EI)^2}{l^2} - \frac{6EI}{5}F + \frac{l^2}{100}F^2\right)^2 = 0 \tag{2.58}$$

$$\frac{12(EI)^2}{l^2} - \frac{26EI}{5}F + \frac{3l^2}{20}F^2 = 0 \tag{2.59}$$

Thus, one obtains as governing equation for the critical load:

$$F^2 - \frac{104EI}{3l^2}F + \frac{80(EI)^2}{l^4} = 0 \tag{2.60}$$

with the solutions

$$F_{cr,i\,1/2} = \frac{52EI}{3l^2} \pm \sqrt{\left(\frac{52EI}{3l^2}\right)^2 - \frac{80(EI)^2}{l^4}} \tag{2.61a}$$

$$F_{cr,i\,1/2} = \frac{EI}{3l^2}\left(52 \pm \sqrt{52^2 - 720}\right) \tag{2.61b}$$

$$F_{cr,i\,1/2} = \frac{EI}{3l^2}(52 \pm 44.54) \tag{2.61c}$$

The smaller and thus relevant value is

$$F_{cr,i\,1} = 2.486\frac{EI}{l^2} \tag{2.62}$$

which is close to the analytical solution of 2.47 from Eq. (2.35).

2.3 Large Rotations I: Strain Measure

2.3.1 Kinematic Effects

Large rotation must be accounted for if the linearised kinematics (see assumption 2 in Sect. 2.1) cannot be applied any longer. This is surely the case if rotation angles $>4\ldots5°$ occur. Figure 2.8 shows the kinematic differences. In the linear theory (red model) the middle of the free end of the cantilever moves perpendicularly to the original beam axis, i.e. vertically downwards. It seems as if the system becomes thicker, however, it is nothing but the fact that the vertical dimension, here noted as h, remains constant. The fully geometrically non-linear theory leads to a reasonable deformation behaviour (blue model); here the original height of the cross section is kept in the deformed system and the free end also moves to the left in the horizontal direction.

Taking large rotations into account can become necessary for smaller rotations than mentioned above, especially if the bending deformation has a significant influence on the axial stress as it can be the case for a rope under a perpendicular load. Figure 2.9 shows such a rope which has not been pre-stressed and which gets its axial stress by the vertical deformation. It is to be seen that the forces increase over-proportionally when doubling the vertical displacement and that the ratio between the horizontal and vertical force components changes significantly.

This effect cannot be accounted for in the theory of second order because only the equilibrium in the deformed state is considered but the linear kinematics is kept. However, if a pre-stressing force is present and is only changed by small portions due to the deformation, second order theory and fully non-linear theory lead to nearly the same force reactions (Fig. 2.10) while second order leads to proportional increments whereas in fully non-linear theory differences occur for larger rotations. Then the second order theory is no longer valid.

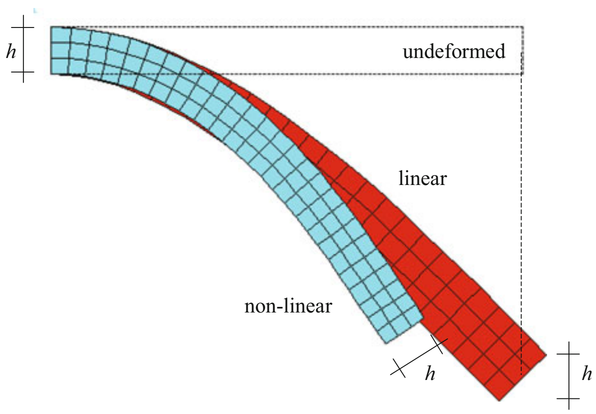

Fig. 2.8 Linear and non-linear kinematics

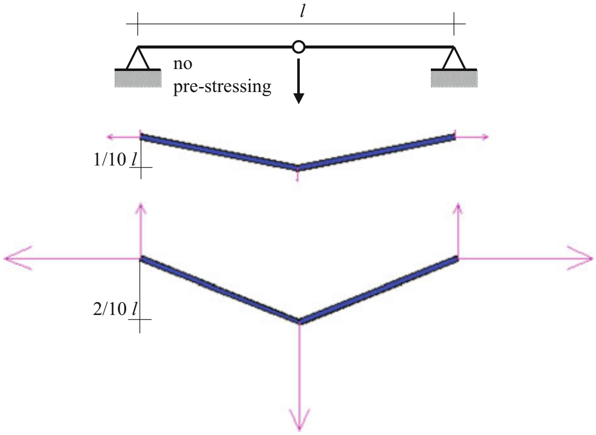

Fig. 2.9 Rope fixed at both ends under perpendicular load, geometrically non-linear

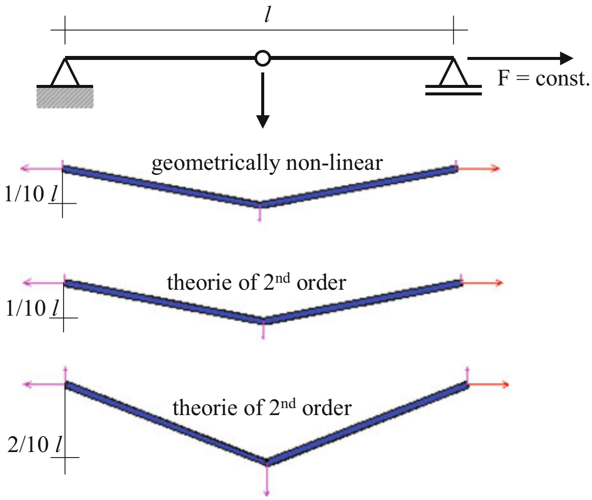

Fig. 2.10 Rope being prestressed by a constant force under perpendicular load, different theories

2.3.2 Appropriate Strain Measure: Green-Lagrange Strain

2.3.2.1 Exemplary Derivation

Green-Lagrange strains are appropriate to describe the strains, i.e. the relative deformations of a body undergoing large rotations, in the coordinates determined by its initial configuration. In the derivation the change of the square of the distance of two neighboured points is considered. For one direction in the plane this can be illustrated as follows:

The relative change of the squares of the deformed length l and the undeformed one l_0 is

$$\Delta = \frac{l^2 - l_0^2}{l_0^2} = \frac{(l_0 + u)^2 + v^2 - l_0^2}{l_0^2} = \frac{l_0^2 + 2l_0 u + u^2 + v^2 - l_0^2}{l_0^2} \\ = 2\frac{u}{l_0} + \left(\frac{u}{l_0}\right)^2 + \left(\frac{v}{l_0}\right)^2 \tag{2.63}$$

The transition to the infinitesimal small element dx yields

$$\frac{u}{l_0} \to \frac{\partial u}{\partial x} = u' \quad and \quad \frac{v}{l_0} \to \frac{\partial v}{\partial x} = v' \tag{2.64}$$

and thus

$$\Delta = 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \tag{2.65}$$

For small deformations the quadratic terms become negligible so that only the first term remains which is twice the linear or engineering strain. Therefore, one half of Δ is defined as the Green-Lagrange strain (often only called Green's strain):

$$\varepsilon_{xx}^{GL} = \frac{\Delta}{2} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 = u' + \frac{1}{2}u'^2 + \frac{1}{2}v'^2 \tag{2.66}$$

Generalised:

$$\varepsilon_{ij}^{GL} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{k=1}^{n_{dim}} \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \tag{2.67}$$

with n_{dim}—considered dimension; i, j—directions

e.g. $\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$ in three dimensions

Since the derivatives of the displacements are zero for rigid body *translations* the strain is independent of that motion, but that was already the case for engineering strain.

GL strain is not a series truncated after the quadratic term, but a measure making sure that a rigid body *rotation* with arbitrary angles does not cause any strain, as it is shown in the following example:

$$\begin{aligned} u &= x(\cos\varphi - 1) - y\sin\varphi \\ v &= x\sin\varphi + y(\cos\varphi - 1) \end{aligned} \tag{2.68}$$

describes the displacements of arbitrary points in the plane with the coordinates x and y due to a rotation around the origin by an angle of φ while keeping the distance to the centre of rotation. With

$$\begin{aligned} \frac{\partial u}{\partial x} &= \cos\varphi - 1 \quad & \frac{\partial u}{\partial y} &= -\sin\varphi \\ \frac{\partial v}{\partial x} &= \sin\varphi & \frac{\partial v}{\partial y} &= \cos\varphi - 1 \end{aligned} \tag{2.69}$$

one normal strain becomes

$$\begin{aligned} \varepsilon_{xx} &= \cos\varphi - 1 + \frac{1}{2}(\cos\varphi - 1)^2 + \frac{1}{2}(\sin\varphi)^2 \\ &= \cos\varphi - 1 + \frac{1}{2}\left(\cos^2\varphi - 2\cos\varphi + 1 + \sin^2\varphi\right) \end{aligned} \tag{2.70}$$

Due to $\cos^2\varphi + \sin^2\varphi = 1$ one obtains:

$$\varepsilon_{xx}^{GL} = \cos\varphi - 1 - \cos\varphi + 1 = 0 \tag{2.71}$$

whereas the engineering strain is

$$\varepsilon_{xx}^{eng} = \frac{\partial u}{\partial x} = \cos\varphi - 1 \tag{2.72}$$

which approaches zero for $\varphi \to 0$ only, i.e. for small rotations. The same holds for the other strain components.

The strain components keep their direction even in the case of a large rotation; in the example of Fig. 2.11 x is always the direction of the spar in its initial position in which the length l_0 is named.

In one dimension the strain is

$$\varepsilon^{GL} = u' + \frac{1}{2}u'^2 = \varepsilon^{eng} + \frac{1}{2}\varepsilon^{eng\,2} \tag{2.73}$$

Neither the direction nor the one-dimensional measure are very obvious, but the Green-Lagrange strain is suitable for arbitrary rotations.

There is no "natural" definition of strain. Strain cannot be measured directly, not even by strain gauges; they measure differences in length.

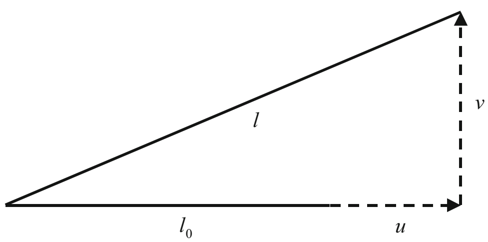

Fig. 2.11 Illustrating the Green-Lagrange strain

2.3.2.2 Example Truss Element

As example

- the link element
- initially located parallel to the x-axis
- with displacement degrees of freedom in the x-y-plane
- with linear shape functions

is formulated for large rotations using Green-Lagrange strain.
The displacement function reads:

$$u(x) = \left[1-\frac{x}{l} \quad 0 \quad \frac{x}{l} \quad 0\right]\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} \quad \text{and} \quad v(x) = \left[0 \quad 1-\frac{x}{l} \quad 0 \quad \frac{x}{l}\right]\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} \tag{2.74}$$

with $\xi = x/l$:

$$\mathbf{u}(\xi) = \mathbf{N}(\xi)\,\hat{\mathbf{u}} \tag{2.75}$$

The derivatives with respect to x needed for the strain in the direction of the spar then are

$$\begin{aligned} \frac{\partial u}{\partial x} = u' = \frac{1}{l}[-1 \quad 0 \quad 1 \quad 0]\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} \quad &\text{and} \quad \frac{\partial v}{\partial x} = v' = \frac{1}{l}[0 \quad -1 \quad 0 \quad 1]\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} \\ u' = \mathbf{C}\hat{\mathbf{u}} \qquad\qquad &\qquad\qquad v' = \mathbf{D}\hat{\mathbf{u}} \end{aligned} \tag{2.76}$$

Remark Unlike in the linear theory this relation is not named **B** because the B-Matrix will get a different or better generalised meaning.

The Green-Lagrange strain now reads:

$$\varepsilon = \mathbf{C}\hat{\mathbf{u}} + \frac{1}{2}\hat{\mathbf{u}}^T\mathbf{C}^T\mathbf{C}\hat{\mathbf{u}} + \frac{1}{2}\hat{\mathbf{u}}^T\mathbf{D}^T\mathbf{D}\hat{\mathbf{u}} \tag{2.77}$$

2.3.3 The Principle of Virtual Work for Geometrically Nonlinear Problems

2.3.3.1 General

The internal virtual work of an element reads:

$$\delta W_{int} = \int_{(V)} \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV \tag{2.78}$$

In dV area and length are united; stress times area yields force, strain times length yields displacement, force times displacement yields work. In case of the virtual work the force is already fully developed and is dislocated by a small virtual displacement without influence on its magnitude. Thus there is no factor of ½ as it is known from the total internal energy.

$\delta\boldsymbol{\varepsilon}$ is the virtual strain, the strain resulting from the virtual displacement

$$\delta \mathbf{u}(\boldsymbol{\xi}) = \mathbf{N}(\boldsymbol{\xi}) \delta \hat{\mathbf{u}} \tag{2.79}$$

Since this must be kinematically possible and small the virtual strain can be derived by linearisation:

$$\delta \boldsymbol{\varepsilon} = \underbrace{\frac{\partial \boldsymbol{\varepsilon}}{\partial \hat{\mathbf{u}}}}_{\mathbf{B}(\hat{\mathbf{u}})} \delta \hat{\mathbf{u}} \tag{2.80}$$

The derivative of the strain with respect to the nodal displacements is newly called B-matrix. This is no contradiction to but a generalisation of the B-matrix in the linear theory.

After forming the derivative of the Finite Element formulation of linear strain

$$\boldsymbol{\varepsilon}^{lin} = \mathbf{B} \hat{\mathbf{u}} \tag{2.81}$$

with respect to the nodal displacements it is obvious that (2.80) is also valid in the linear case, i.e. the B-Matrix has only be generalised.

By introducing (2.80) into (2.78) the internal virtual work becomes

$$\delta W_{int} = \delta \hat{\mathbf{u}}^T \int_{(V)} \mathbf{B}^T(\hat{\mathbf{u}}) \boldsymbol{\sigma} dV \tag{2.82}$$

The virtual nodal displacements are constant with respect to the integration variables and can thus be put outside the integral. Since the total term means work the integral means forces, namely the internal nodal forces:

$$\mathbf{f}_{int} = \int_{(V)} \mathbf{B}^T(\hat{\mathbf{u}})\boldsymbol{\sigma} dV \tag{2.83}$$

This relation is very general and will be used several times in the following.

2.3.3.2 Application to Link Element

For the link element with Green-Lagrange strain the B-matrix reads:

$$\begin{aligned}\mathbf{B} &= \frac{\partial \varepsilon}{\partial \hat{\mathbf{u}}} = \frac{\partial \varepsilon}{\partial u'}\frac{\partial u'}{\partial \hat{\mathbf{u}}} + \frac{\partial \varepsilon}{\partial v'}\frac{\partial v'}{\partial \hat{\mathbf{u}}} = \left(1 + u'\right)\mathbf{C} + v'\mathbf{D} \\ &= \mathbf{C} + \hat{\mathbf{u}}^T\mathbf{C}^T\mathbf{C} + \hat{\mathbf{u}}^T\mathbf{D}^T\mathbf{D}\end{aligned} \tag{2.84}$$

while the stress becomes

$$\sigma = E\varepsilon = E\left(u' + \frac{1}{2}u'^2 + \frac{1}{2}v'^2\right) = E\left(\mathbf{C}\hat{\mathbf{u}} + \frac{1}{2}\hat{\mathbf{u}}^T\mathbf{C}^T\mathbf{C}\hat{\mathbf{u}} + \frac{1}{2}\hat{\mathbf{u}}^T\mathbf{D}^T\mathbf{D}\hat{\mathbf{u}}\right) \tag{2.85}$$

In case of the link element the infinitesimal small volume element dV is replaced by Adx. For the integration one can make use of the fact that all integrands are constant in case of linear shape functions. Thus

$$\mathbf{f}_{int} = \left(\left(1 + u'\right)\mathbf{C}^T + v'\mathbf{D}^T\right)\sigma Al \tag{2.86}$$

Now the first part of the element formulation is done. The remaining question is how to fulfil the equilibrium of the internal with the external forces $\mathbf{f}_{ext}$. Since the stress depends on the nodal displacements a non-linear system of equations must be solved.

2.3.4 Solution of the Nonlinear Equation by the Newton-Raphson Method

In the preceding chapter the internal forces became dependent on the nodal displacements in a non-linear way, i.e. they cannot be formulated as a product of a stiffness matrix and the displacement vector. The equilibrium condition, however, remains that the difference between internal and external forces must be zero. These equations are iteratively solved by the Newton-Raphson scheme from Sect. 1.3.

2.3.4.1 Application to the Nonlinear FEM

For a discretised mechanical problem the equilibrium equation reads in general terms:

$$\mathbf{f}_{int}(\hat{\mathbf{u}}) = \mathbf{f}_{ext}(\hat{\mathbf{u}}) \quad \Leftrightarrow \quad \mathbf{d}(\hat{\mathbf{u}}) := \mathbf{f}_{int}(\hat{\mathbf{u}}) - \mathbf{f}_{ext}(\hat{\mathbf{u}}) = \mathbf{0} \tag{2.87}$$

An example for displacement-dependent loads is pressure being perpendicular to the deformed surface.

The tangential matrix, the derivatives of the internal and external forces, is obtained in the general case as

$$\mathbf{K}_T = \frac{\partial}{\partial \hat{\mathbf{u}}}\mathbf{f}_{int} - \frac{\partial}{\partial \hat{\mathbf{u}}}\mathbf{f}_{ext} = \frac{\partial}{\partial \hat{\mathbf{u}}} \int\limits_{(V)} \mathbf{B}^T(\hat{\mathbf{u}})\ \boldsymbol{\sigma}\, dV - \frac{\partial}{\partial \hat{\mathbf{u}}}\mathbf{f}_{ext} \tag{2.88}$$

$$\mathbf{K}_T = \int\limits_{(V)} \mathbf{B}^T(\hat{\mathbf{u}})\ \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}}\frac{\partial \boldsymbol{\varepsilon}}{\partial \hat{\mathbf{u}}} dV + \int\limits_{(V)} \frac{\partial \mathbf{B}^T(\hat{\mathbf{u}})}{\partial \hat{\mathbf{u}}} \boldsymbol{\sigma}\, dV - \frac{\partial}{\partial \hat{\mathbf{u}}}\mathbf{f}_{ext} \tag{2.89}$$

After (2.80) the derivative of the strain is the B-matrix. The derivative of the stresses, the material tangent, strongly depends on the type of the material law (see Chaps. 5–8). It can be a non-symmetric matrix. In case of Hooke's law it is the elasticity matrix $\mathbf{E}$ so that one obtains as tangential matrix

$$\mathbf{K}_T = \underbrace{\int\limits_{(V)} \mathbf{B}^T(\hat{\mathbf{u}})\ \mathbf{E}\mathbf{B}(\hat{\mathbf{u}})dV}_{\mathbf{K}_u} + \underbrace{\int\limits_{(V)} \frac{\partial \mathbf{B}^T(\hat{\mathbf{u}})}{\partial \hat{\mathbf{u}}} \boldsymbol{\sigma}\, dV}_{\mathbf{K}_\sigma} \underbrace{- \frac{\partial}{\partial \hat{\mathbf{u}}}\mathbf{f}_{ext}}_{\mathbf{K}_p} \tag{2.90}$$

$\mathbf{K}_u$ can be called *initial displacement matrix* (this expression is not really fix) and formally equals the linear stiffness matrix. The material tangent is multiplied from the left and the right by the B-matrix and its transposed. For linear elasticity it is positive definite if sufficient constraints exist to suppress the rigid body motions, for non-linear material it is positive definite as long as the material tangent is positive definite.

In mathematical terms positive definite means that the matrix has only positive eigenvalues with the effect that all pivot elements remain positive during Gaussian elimination process. In mechanical terms that means that increasing displacements resp. strains cause increasing forces resp. stresses.

$\mathbf{K}_\sigma$ is sometimes called *geometric matrix* because only in case of geometric non-linearity the B-matrix depends on the displacements with the consequence that its derivative and thus $\mathbf{K}_\sigma$ exist. More often the expression *initial stress matrix* is used because it directly depends on the known stresses. "Initial" means the beginning of the iteration step. Since the stresses can be negative $\mathbf{K}_\sigma$ can get negative eigenvalues. This can cause that the total tangential matrix $\mathbf{K}_T$ looses its positive definiteness. This is an indicator for a physical instability. There is a certain analogy to the stress-stiffening matrix **S** in the FE-formulation of the theory of second order from Sect. 2.2 and the stability problem from Sect. 2.2.3.

$\mathbf{K}_\sigma$ is always symmetric because it is the second derivative of the strain (each component times a stress component, resulting in a scalar) with respect to the same vector (the nodal displacements).

The load matrix $\mathbf{K}_p$ is symmetric because it is the second derivative of the external work (a scalar) with respect to the displacement vector.

2.3.4.2 Application to the Truss Example

For the link element with Green-Lagrange strain one obtains the components of the tangential matrix as

$$\begin{aligned}\mathbf{K}_u &= \left[(1+u')\mathbf{C}^T + v'\mathbf{D}^T\right]\left[\left(1+u'\right)\mathbf{C} + v'\mathbf{D}\right]EAl \\ &= \left[\left(1+u'\right)^2\mathbf{C}^T\mathbf{C} + v'\left(1+u'\right)\mathbf{D}^T\mathbf{C} + \left(1+u'\right)v'\mathbf{C}^T\mathbf{D} + v'^2\mathbf{D}^T\mathbf{D}\right]EAl\end{aligned} \tag{2.91}$$

$$\mathbf{K}_u = \left[\left(1+u'\right)^2\mathbf{C}^T\mathbf{C} + \left(v' + u'v'\right)\left(\mathbf{D}^T\mathbf{C} + \mathbf{C}^T\mathbf{D}\right) + v'^2\mathbf{D}^T\mathbf{D}\right]EAl \tag{2.92}$$

$$\mathbf{K}_\sigma = \left[\frac{\partial}{u'}\left(1+u'\right)\mathbf{C}^T\frac{\partial u'}{\hat{\mathbf{u}}} + \frac{\partial}{v'}v'\mathbf{D}^T\frac{\partial v'}{\partial\hat{\mathbf{u}}}\right]\sigma Al \tag{2.93}$$

$$\mathbf{K}_\sigma = \left(\mathbf{C}^T\mathbf{C} + \mathbf{D}^T\mathbf{D}\right)\sigma Al \tag{2.94}$$

Herein is

$$\begin{array}{c|c} \mathbf{C}^T\mathbf{C} & \begin{bmatrix}-1 & 0 & +1 & 0\end{bmatrix}\frac{1}{l} \\ \hline \frac{1}{l}\begin{bmatrix}-1\\0\\+1\\0\end{bmatrix} & \begin{bmatrix}+1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & +1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\frac{1}{l^2}\end{array} \tag{2.95}$$

and analogously

$$\mathbf{D}^T\mathbf{D} = \frac{1}{l^2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{2.96}$$

$$\mathbf{C}^T\mathbf{C} + \mathbf{D}^T\mathbf{D} = \frac{1}{l^2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad \text{and}$$

$$\mathbf{C}^T\mathbf{D} + \mathbf{D}^T\mathbf{C} = \frac{1}{l^2}\begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \tag{2.97}$$

After addition one obtains the total tangential stiffness matrix as

$$\mathbf{K}_T = \frac{EA}{l}\begin{bmatrix} (1+u')^2 & (v'+u'v') & -(1+u')^2 & -(v'+u'v') \\ (v'+u'v') & v'^2 & -(v'+u'v') & -v'^2 \\ -(1+u')^2 & -(v'+u'v') & (1+u')^2 & (v'+u'v') \\ -(v'+u'v') & -v'^2 & (v'+u'v') & v'^2 \end{bmatrix} + \frac{\sigma A}{l}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{2.98}$$

This derivation is made in the coordinate system of the element in the initial configuration (superscript ^{e} or ^{elem}). The nodal displacement vector is $\hat{\mathbf{u}} = \hat{\mathbf{u}}^e$. Before composing the total system (superscript ^{g} for *global*) the transformation

$$\hat{\mathbf{u}}^e = \mathbf{T}\hat{\mathbf{u}}^g \quad \textit{resp.} \;\; \delta\hat{\mathbf{u}}^{eT} = \delta\hat{\mathbf{u}}^{gT}\mathbf{T}^T \tag{2.99}$$

must be carried out. With $c := \cos\alpha$ and $s := \sin\alpha$ the transformation matrix reads:

$$\mathbf{T} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \tag{2.100}$$

The internal forces become

$$\mathbf{f}_{int} = \mathbf{T}^T \mathbf{f}^e_{int} \tag{2.101}$$

and the tangential stiffness matrix

$$\mathbf{K}_T = \frac{\partial \mathbf{f}_{int}}{\partial \mathbf{u}_g} = \mathbf{T}^T \frac{\partial \mathbf{f}^e_{int}}{\partial \mathbf{u}_e} \frac{\partial \mathbf{u}_e}{\partial \mathbf{u}_g} = \mathbf{T}^T \mathbf{K}^{elem}_T \mathbf{T} \tag{2.102}$$

2.3.5 Test Problem Two-Legged Truss

The link element is used for a simple example. The two-legged truss from Fig. 2.12 is discretised by one element for symmetry reasons.

As the only global degree of freedom ${v_2}^g$ is remaining. In the element coordinate system (superscript ^{e}) this is split up into

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}^e = \begin{bmatrix} s \\ c \end{bmatrix} v^g_2 \tag{2.103}$$

Thus

$$\frac{\partial u}{\partial x} = u^{'} = \frac{1}{l} u_2 = \frac{s}{l} v^g_2 \quad \text{and} \quad \frac{\partial v}{\partial x} = v^{'} = \frac{1}{l} v_2 = \frac{c}{l} v^g_2 \tag{2.104}$$

$$\sigma = E\left(u^{'} + \frac{1}{2} u^{'2} + \frac{1}{2} v^{'2}\right) \tag{2.105}$$

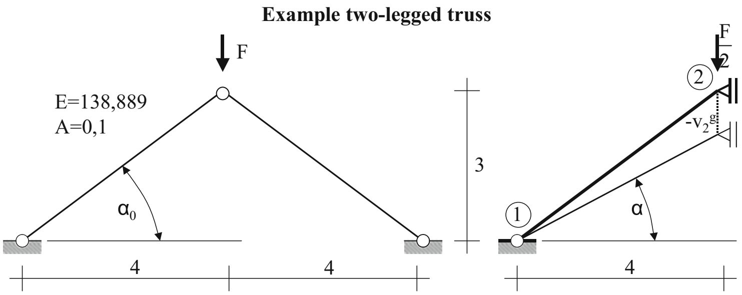

Fig. 2.12 Two-legged truss

Out of the vectors and matrices on element level only the terms in position 3 and 4 are of interest because they are related to the active degrees of freedom:

$$\mathbf{f}_{int} = \mathbf{T}^T\left(\mathbf{C}^T + u'\mathbf{C}^T + v'\mathbf{D}^T\right)\sigma A l$$
$$= [\,s \quad c\,]\left(\begin{bmatrix}1\\0\end{bmatrix}\left(1+u'\right) + \begin{bmatrix}0\\1\end{bmatrix}v'\right)\sigma A \tag{2.106}$$

$$\mathbf{f}_{int} = \left[s\left(1+u'\right) + cv'\right]\sigma A \tag{2.107}$$

$$\mathbf{K}_T = \frac{EA}{l}[\,s \quad c\,]\begin{bmatrix}\left(1+u'\right)^2 & \left(v'+u'v'\right)\\ \left(v'+u'v'\right) & v'^2\end{bmatrix}\begin{bmatrix}s\\c\end{bmatrix}$$
$$+\frac{\sigma A}{l}[\,s \quad c\,]\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}s\\c\end{bmatrix} \tag{2.108}$$

$$\mathbf{K}_T = \frac{EA}{l}\left[\, s(1+u')^2 + c(v'+u'v') \quad s(v'+u'v') + cv'^2\,\right]\begin{bmatrix}s\\c\end{bmatrix} + \frac{\sigma A}{l}[\,s \quad c\,]\begin{bmatrix}s\\c\end{bmatrix}$$
$$= \frac{EA}{l}\left[s^2\left(1+u'\right)^2 + 2cs\left(v'+u'v'\right) + c^2v'^2\right] + \frac{\sigma A}{l}\left[s^2+c^2\right] \tag{2.109}$$

$$\mathbf{K}_T = \frac{EA}{l}\left[s^2\left(1+u'\right)^2 + 2cs\left(1+u'\right)v' + c^2v'^2\right] + \frac{\sigma A}{l} \tag{2.110}$$

Because of symmetry the vector of external loads in global system reduces to

$$\mathbf{f}^{ext} = -\frac{1}{2}F \tag{2.111}$$

Furthermore necessary are

$$l = l_0 = \sqrt{4^2+3^2} = 5 \;, \quad c = \cos\alpha = \frac{4}{5} \;, \quad s = \sin\alpha = \frac{3}{5} \tag{2.112}$$

With this formulae one obtains the iteration course of Table 2.1. Therein the convergence exponent κ is shown. It means the following: If the norm of the right hand side **d** (disequilibrium forces) decreases by a factor of a from iteration step $i-2$ to $i-1$ it is reduced by a^κ from $i-1$ to i. The formula for κ is (1.48).

Close to the final solution κ tends to 2. This is called "quadratic convergence".

A Newton-Raphson scheme shows quadratic convergence in the vicinity of the solution. In practice the iteration is often considered as converged and thus the iteration is aborted before this effect can fully be seen.

Figure 2.13 shows how the forces and displacements develop during the solution.

Table 2.1 Newton-Raphson iteration for the two-legged truss with Green-Lagrange strain

$\mathbf{f}_{ext}$	v_2^g	Right hand side	$\mathbf{K}_T$	Δv	κ
−0.5	0	−0.5	1	−0.5	
−0.5	−0.5	−0.11805556	0.54166667	−0.21794872	
−0.5	−0.71794872	−0.01921719	0.36795968	−0.05222635	1.25764431
−0.5	−0.77017506	−0.0010295	0.32868654	−0.00313217	1.61221173
−0.5	−0.77330724	−3.6442E-06	0.32636011	−1.1166E-05	1.92832425
−0.5	−0.7733184	−4.6273E-11	0.32635182	−1.4179E-10	1.99764891
−0.55	−0.7733184	−0.05	0.32635182	−0.15320889	
−0.55	−0.92652729	−0.00851134	0.21654818	−0.03930462	
−0.55	−0.96583191	−0.0005305	0.18963997	−0.00279738	1.56743928
−0.55	−0.96862929	−2.6518E-06	0.18774449	−1.4124E-05	1.90915951
−0.55	−0.96864341	−6.7543E-11	0.18773493	−3.5978E-10	1.9963818
−0.57	−0.96864341	−0.02	0.18773493	−0.10653319	
−0.57	−1.0751766	−0.00377525	0.11749085	−0.03213227	
−0.57	−1.10730887	−0.00032938	0.09704662	−0.00339405	1.46287675
−0.57	−1.11070293	−3.6317E-06	0.09490724	−3.8265E-05	1.84809966
−0.57	−1.11074119	−4.6106E-10	0.09488314	−4.8593E-09	1.99037628
−0.58	−1.1107412	−0.01	0.09488314	−0.1053928	
−0.58	−1.216134	−0.0034325	0.03036298	−0.11304878	
−0.58	−1.32918278	−0.00371938	**−0.03472831**	0.10709932	−0.0750662
−0.58	−1.22208346	−0.00326237	0.0268312	−0.12158855	−1.63332064
−0.58	−1.34367201	−0.00428086	−0.04276293	0.10010671	−2.0724101
−0.58	−1.2435653	−0.00282218	0.01417714	−0.19906513	−1.53347465

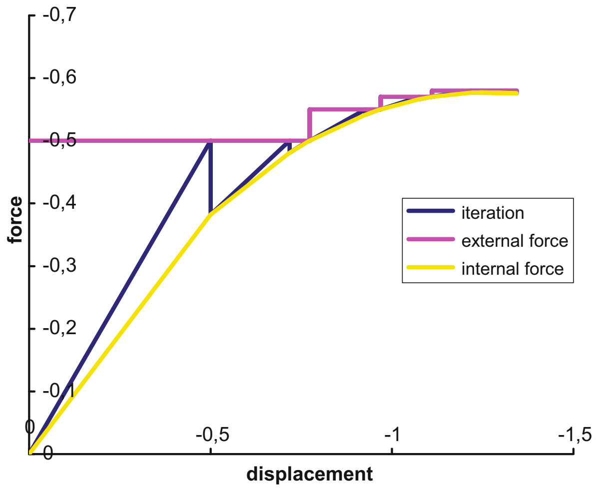

Fig. 2.13 Course of iteration in the Newton-Raphson scheme

2.3.6 Notation in Continuum Mechanical Symbols

The symbolic notation of continuum mechanics is not in the focus of this book. Details can be found e.g. in Wriggers [27] or Belytschko et al. [2]. It is only sketched here, but we use it to extend the theory to two or three dimensions.

The deformation gradient

$$\mathbf{F} = \left[\frac{\partial x_i}{\partial x_{0j}}\right] \tag{2.113}$$

contains the derivatives of the coordinates x_i of the deformed system with respect to the initial coordinates x_{0j}. The *polar decomposition* means a multiplicative split of the deformation gradient into a

rotation $\mathbf{R}$ and a

stretching $\mathbf{U}$:

$$\mathbf{F} = \mathbf{R}\mathbf{U} \tag{2.114}$$

The rotation is an orthogonal tensor, i.e.

$$\mathbf{R}^T = \mathbf{R}^{-1} \quad \Rightarrow \quad \mathbf{R}^T\mathbf{R} = \mathbf{I} \tag{2.115}$$

In case of a uniform deformation over the length in a one-dimensional element like in Sect. 2.3.2.1 there is only one component of the (so-called *right*) stretch tensor $\mathbf{U}$:

$$U_{11} = \frac{l}{l_0} \tag{2.116}$$

Taking (2.63) to (2.65) into account results in

$$\varepsilon^{GL} = \frac{1}{2}\frac{l^2 - l_0^2}{l_0^2} = \frac{1}{2}\left(\frac{l^2}{l_0^2} - \frac{l_0^2}{l_0^2}\right) = \frac{1}{2}\left(\frac{l}{l_0}\,\frac{l}{l_0} - 1\right) = \frac{1}{2}(U_{11}U_{11} - 1) \tag{2.117}$$

In two and three dimensions this strain measure is obtained as

$$\boldsymbol{\varepsilon}^{GL} = \frac{1}{2}\left(\mathbf{U}^T\mathbf{U} - \mathbf{I}\right) \tag{2.118}$$

Making use of (2.115):

$$\boldsymbol{\varepsilon}^{GL} = \frac{1}{2}\left(\mathbf{U}^T\mathbf{I}\mathbf{U} - \mathbf{I}\right) = \frac{1}{2}\left(\underbrace{\mathbf{U}^T\mathbf{R}^T}_{\mathbf{F}^T}\,\underbrace{\mathbf{R}\mathbf{U}}_{\mathbf{F}} - \mathbf{I}\right) \tag{2.119}$$

$$\boldsymbol{\varepsilon}^{GL} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) \tag{2.120}$$

Thus the expression $\mathbf{F}^T\mathbf{F}$ is equal to the square of the stretch tensor, $\mathbf{U}^2$, and directly independent of the rigid body rotation as it is shown exemplarily in the chapters above. In this representation the components of strain tensor form a matrix instead of a vector. The different choices for the notation and their advantages are discussed in Sect. 5.3.

In detail the deformation gradient comprises terms with the same indices on the main diagonal:

$$F_{ii} = \frac{\partial x_i}{\partial x_{0i}} = \frac{\partial(x_{0i} + u_i)}{\partial x_{0i}} = 1 + \frac{\partial u_i}{\partial x_{0i}} \tag{2.121}$$

and with different indices on the secondary diagonals:

$$F_{ij} = \frac{\partial x_i}{\partial x_j} = \frac{\partial(x_{0i} + u_i)}{\partial x_{0j}} = \frac{\partial u_i}{\partial x_{0j}} \quad , \quad j \neq i \tag{2.122}$$

Together, while leaving out the index 0 because only displacements and *initial* coordinates are present:

$$\mathbf{F} = \begin{bmatrix} 1 + \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & 1 + \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & 1 + \frac{\partial w}{\partial z} \end{bmatrix} = \mathbf{I} + \underbrace{\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}}_{\mathbf{H}} \tag{2.123}$$

$\mathbf{H}$ is called *displacement gradient*. With this quantity the strain can be written as

$$\boldsymbol{\varepsilon}^{GL} = \frac{1}{2}\left[\left(\mathbf{I} + \mathbf{H}^T\right)(\mathbf{I} + \mathbf{H}) - \mathbf{I}\right] = \frac{1}{2}\left[\mathbf{I} + \mathbf{H}^T + \mathbf{H} + \mathbf{H}^T\mathbf{H} - \mathbf{I}\right] \tag{2.124}$$

$$\boldsymbol{\varepsilon}^{GL} = \frac{1}{2}\left[\mathbf{H}^T + \mathbf{H} + \mathbf{H}^T\mathbf{H}\right] \tag{2.125}$$

In detail:

$$\boldsymbol{\varepsilon}^{GL} = \frac{1}{2}\left(\underbrace{\begin{bmatrix} 2\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} & 2\dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y} \\ \dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x} & \dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y} & 2\dfrac{\partial w}{\partial z} \end{bmatrix}}_{\boldsymbol{\varepsilon}^{eng}} + \mathbf{H}^T\mathbf{H}\right) \tag{2.126}$$

The second part requires multiplication:

$$\begin{array}{ccc|ccc} & \mathbf{H}^T\mathbf{H} & & \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ & & & \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ & & & \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \\ \hline \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} & \dfrac{\partial w}{\partial x} & H_{11} & H_{12} & etc. \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} & \dfrac{\partial w}{\partial y} & & & \\ \dfrac{\partial u}{\partial z} & \dfrac{\partial v}{\partial z} & \dfrac{\partial w}{\partial z} & & & \end{array} \tag{2.127}$$

As examples:

$$\begin{aligned} H_{11} &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 = \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2 = \sum_{i=1}^{3}\left(\frac{\partial u_i}{\partial x_1}\right)^2 \\ H_{12} &= \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = \frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2} = \sum_{i=1}^{3}\frac{\partial u_i}{\partial x_1}\frac{\partial u_i}{\partial x_2} \end{aligned} \tag{2.128}$$

The general formula is given as Eq. (2.67)

For the B-Matrix, necessary to obtain the internal forces and $\mathbf{K}_u$, as well as for the initial stress matrix $\mathbf{K}_\sigma$ we switch to the index notation (rules in Sect. 1.1):

$$\varepsilon_{ij}^{GL} = \frac{1}{2}\left(F_{ki}F_{kj} - \delta_{ij}\right) \ , \quad i,j,k = 1\ldots 3 \ (number\ of\ spatial\ dimensions) \tag{2.129}$$

For the B-matrix we note:

$$B_{ijn} = \frac{\partial \varepsilon_{ij}^{GL}}{\partial \hat{u}_n} = \frac{1}{2}\left(\frac{\partial F_{ki}}{\partial \hat{u}_n} F_{kj} + F_{ki}\frac{\partial F_{kj}}{\partial \hat{u}_n}\right) \tag{2.130}$$

where the index n runs over all degrees of freedom of the element. The deformation gradient is expressed as

$$F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial x_{0j}} \tag{2.131}$$

Here we have to distinguish between u_i as one direction of the displacement as a function over the element and $\hat{u}_n$ as one degree of freedom of the element nodes. The relation is (for 2d)

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & N_1 & N_2 & N_3 & \cdots \end{bmatrix} \begin{bmatrix} \hat{u}_{x1} \\ \hat{u}_{x2} \\ \hat{u}_{x3} \\ \vdots \\ \hat{u}_{y1} \\ \hat{u}_{y2} \\ \hat{u}_{y3} \\ \vdots \end{bmatrix} \tag{2.132}$$

as long as the order in the nodal displacement vector is

1. all d.o.f.s of the same direction
2. all directions.

This order is unusual for programming (common is firstly all d.o.f.s of a node, then all nodes) but good for matrix representation. In index notation (2.132) reads:

$$u_k = N_{km} \hat{u}_m \tag{2.133}$$

where m runs over all d.o.f.s of the element including the zeros in N_{km}. The derivative of the deformation gradient with respect to the degrees of freedom is calculated as

$$\frac{\partial F_{ki}}{\partial \hat{u}_n} = \frac{\partial}{\partial \hat{u}_n}\frac{\partial u_k}{\partial x_{0i}} = \frac{\partial}{\partial \hat{u}_n}\frac{\partial N_{km}}{\partial x_{0i}}\hat{u}_m = \frac{\partial N_{km}}{\partial x_{0i}}\delta_{mn} = \frac{\partial N_{kn}}{\partial x_{0i}} \tag{2.134}$$

n also runs over all d.o.f.s. The derivative of $\hat{u}_m$ with respect to $\hat{u}_n$ is one if $m = n$ and zero otherwise which is expressed by Kronecker's delta. The sum over m has only contributions if $m = n$. By exchanging index i by j one obtains

$$\frac{\partial F_{kj}}{\partial \hat{u}_n} = \frac{\partial N_{kn}}{\partial x_{0j}} \tag{2.135}$$

thus

$$B_{ijn} = \frac{1}{2}\left(\frac{\partial N_{kn}}{\partial x_{0i}} F_{kj} + F_{ki}\frac{\partial N_{kn}}{\partial x_{0j}}\right) \tag{2.136}$$

For the internal forces we need

$$B_{ijn}\sigma_{ij} = \frac{1}{2}\left(\frac{\partial N_{kn}}{\partial x_{0i}} F_{kj}\sigma_{ij} + F_{ki}\frac{\partial N_{kn}}{\partial x_{0j}}\sigma_{ij}\right) \tag{2.137}$$

and for $\mathbf{K}_\sigma$

$$K_{nm}^{\sigma} = \int\limits_{(V)} k_{nm}^{\sigma} dV \tag{2.138}$$

with m running over all degrees of freedom and

$$k_{nm}^{\sigma} = \frac{\partial B_{ijn}}{\partial \hat{u}_m}\sigma_{ij} = \frac{1}{2}\left(\frac{\partial N_{kn}}{\partial x_{0i}}\frac{\partial N_{km}}{\partial x_{0j}}\sigma_{ij} + \frac{\partial N_{km}}{\partial x_{0i}}\frac{\partial N_{kn}}{\partial x_{0j}}\sigma_{ij}\right) \tag{2.139}$$

For the symmetric stress tensor this results in

$$k_{nm}^{\sigma} = \frac{\partial N_{kn}}{\partial x_{0i}}\sigma_{ij}\frac{\partial N_{km}}{\partial x_{0j}} \tag{2.140}$$

It should be kept in mind that in index notation only scalars are handled such that the order can be changed. The representation above makes it more evident that a symmetric matrix $\mathbf{K}_\sigma$ is obtained and how the terms can be prepared for matrix multiplication routines. Equation (2.140) includes three sums, over k, i and j, each over the coordinate resp. displacement directions, whereas n and m span the matrix. The sum over k is expressed for 2d:

$$k_{nm}^{\sigma} = \frac{\partial N_{1n}}{\partial x_{0i}}\sigma_{ij}\frac{\partial N_{1m}}{\partial x_{0j}} + \frac{\partial N_{2n}}{\partial x_{0i}}\sigma_{ij}\frac{\partial N_{2m}}{\partial x_{0j}} \tag{2.141}$$

Now we go a step back to matrix notation. With

$$\mathbf{N} = [N_1 \quad N_2 \quad N_3 \quad \cdots] \quad \text{and} \quad \hat{\mathbf{u}}_i^T = [\hat{u}_{i1}\, \hat{u}_{i2}\, \hat{u}_{i3} \cdots] \tag{2.142}$$

i running over the directions, (2.132) can be written as

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \mathbf{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}}_1 \\ \hat{\mathbf{u}}_2 \end{bmatrix} \tag{2.143}$$

and

$$
\begin{aligned}
\mathbf{k}^{\sigma} &:= \left[k^{\sigma}_{nm}\right] \\
&= \begin{bmatrix} \dfrac{\partial \mathbf{N}^T}{\partial x_{01}} & \dfrac{\partial \mathbf{N}^T}{\partial x_{02}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \mathbf{N}}{\partial x_{01}} & \mathbf{0} \\ \dfrac{\partial \mathbf{N}}{\partial x_{02}} & \mathbf{0} \end{bmatrix} \\
&\quad + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \dfrac{\partial \mathbf{N}^T}{\partial x_{01}} & \dfrac{\partial \mathbf{N}^T}{\partial x_{02}} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \dfrac{\partial \mathbf{N}}{\partial x_{01}} \\ \mathbf{0} & \dfrac{\partial \mathbf{N}}{\partial x_{02}} \end{bmatrix}
\end{aligned} \tag{2.144}
$$

$$
\mathbf{K}_{\sigma} = \begin{bmatrix} \displaystyle\int\limits_{(V)} \begin{bmatrix} \dfrac{\partial \mathbf{N}^T}{\partial x_{01}} & \dfrac{\partial \mathbf{N}^T}{\partial x_{02}} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \mathbf{N}}{\partial x_{01}} \\ \dfrac{\partial \mathbf{N}}{\partial x_{02}} \end{bmatrix} dV & \mathbf{0} \\ \mathbf{0} & \displaystyle\int\limits_{(V)} \begin{bmatrix} \dfrac{\partial \mathbf{N}^T}{\partial x_{01}} & \dfrac{\partial \mathbf{N}^T}{\partial x_{02}} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \mathbf{N}}{\partial x_{01}} \\ \dfrac{\partial \mathbf{N}}{\partial x_{02}} \end{bmatrix} dV \end{bmatrix} \tag{2.145}
$$

In fact the same terms occur as many times as directions (dimensions) are considered. They can be calculated once and then scattered to the final matrix according to the order in the vector of degrees of freedom.

For the link element only one coordinate direction but two displacement components per node are accounted for. Thus only

$$
\frac{\partial \mathbf{N}}{\partial x} = \frac{1}{l_0}\begin{bmatrix} -1 & 1 \end{bmatrix} \tag{2.146}
$$

exists. Then

$$
\int\limits_{(l)} \frac{\partial \mathbf{N}^T}{\partial x_0} \sigma_{11} \frac{\partial \mathbf{N}}{\partial x_{01}} A_0 dx = \frac{1}{l_0^2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} \sigma_{11} A_0 l_0 = \frac{\sigma_{11} A_0}{l_0} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{2.147}
$$

must be evaluated and scattered twice to form $\mathbf{K}_{\sigma}$ as in (2.98).

2.4 Large Rotations II: Co-rotational Formulation

2.4.1 Basic Idea

If an element undergoes large rotations but only small relative rotations inside the element, which can be assumed for sufficiently fine meshes, the following way of an element formulation is an appropriate alternative. Again the link element serves as an example.

A nodal displacement state $\mathbf{u}$ results in

- a rigid body translation
- a rigid body rotation
- a displacement $\mathbf{u}_{def}$ leading to a deformation.

Only the latter one causes strain.

The nodal coordinates of the deformed system, $\mathbf{x}$, can be calculated from those of the undeformed state, $\mathbf{x}_0$, by adding the displacements:

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{u} \tag{2.148}$$

Inversely the displacement is the coordinate difference

$$\mathbf{u} = \mathbf{x} - \mathbf{x}_0 \tag{2.149}$$

As shown in Fig. 2.14 one obtains the *deformatoric* displacement by determining the coordinate difference in a coordinate system moving and rotating with the element, .i.e. following its rigid body motions:

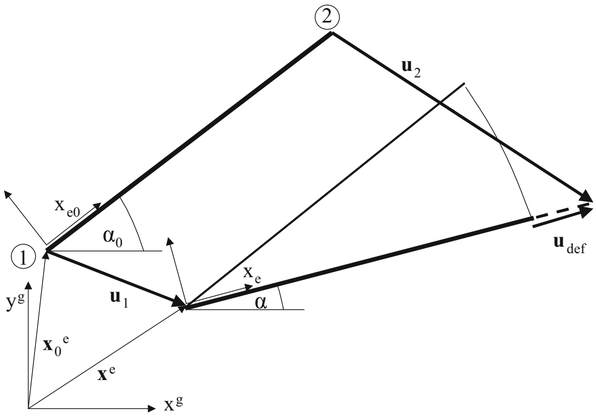

Fig. 2.14 Concerning the principle of the co-rotational formulation

$$\mathbf{u}_{def} = \mathbf{T}(\mathbf{x} - \mathbf{x}^e) - \mathbf{T}_0\left(\mathbf{x}_0 - \mathbf{x}_0^e\right) \tag{2.150}$$

Herein $\mathbf{x}_0{}^{\mathrm{e}}$ and $\mathbf{x}^{\mathrm{e}}$ denote the origin of the coordinate system in the undeformed resp. deformed state. This holds for each node:

$$\mathbf{u}_{1def} = \mathbf{T}(\mathbf{x}_1 - \mathbf{x}^e) - \mathbf{T}_0\left(\mathbf{x}_{10} - \mathbf{x}_0^e\right) \tag{2.151}$$

$$\mathbf{u}_{2def} = \mathbf{T}(\mathbf{x}_2 - \mathbf{x}^e) - \mathbf{T}_0\left(\mathbf{x}_{20} - \mathbf{x}_0^e\right) \tag{2.152}$$

Since only a difference of the nodal displacements yields a strain, in case of the spar

$$\varepsilon = \frac{1}{l}\left(\mathbf{u}_{2def} - \mathbf{u}_{1def}\right) \tag{2.153}$$

the rigid body *translation* causes no problem because it is the same for all nodes. One obtains:

$$\begin{aligned}\mathbf{u}_{2def} - \mathbf{u}_{1def} &= \mathbf{T}(\mathbf{x}_2 - \mathbf{x}^e) - \mathbf{T}_0\left(\mathbf{x}_{20} - \mathbf{x}_0^e\right) - \left[\mathbf{T}(\mathbf{x}_1 - \mathbf{x}^e) - \mathbf{T}_0\left(\mathbf{x}_{10} - \mathbf{x}_0^e\right)\right] \\ &= \mathbf{T}\mathbf{x}_2 - \mathbf{T}\mathbf{x}^e - \mathbf{T}_0\mathbf{x}_{20} + \mathbf{T}_0\mathbf{x}_0^e - \mathbf{T}\mathbf{x}_1 + \mathbf{T}\mathbf{x}^e + \mathbf{T}_0\mathbf{x}_{10} - \mathbf{T}_0\mathbf{x}_0^e \end{aligned} \tag{2.154}$$

$$\mathbf{u}_{2def} - \mathbf{u}_{1def} = \mathbf{T}\mathbf{x}_2 - \mathbf{T}_0\mathbf{x}_{20} - [\mathbf{T}\mathbf{x}_1 - \mathbf{T}_0\mathbf{x}_{10}] \tag{2.155}$$

The location of the moving coordinate system is eliminated, only the orientation is still of importance. Thus, for the calculation of the deformatoric displacement it is sufficient to transform the coordinates into a rotated system parallel to the element coordinate system with origin in global origin, so that instead of (2.150) it is newly defined:

$$\mathbf{u}_{def} := \mathbf{T}\mathbf{x} - \mathbf{T}_0\mathbf{x}_0 \tag{2.156}$$

Figure 2.15 tries to illustrate that the same difference of the deformatoric displacements is obtained. For the link element the transformation equation reads:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_e = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix}_g \tag{2.157}$$

$$\mathbf{x}_e = \mathbf{T}\mathbf{x}$$

The location of the link element in the plane can be described in the deformed state by

$$\Delta x = x_{20} + u_2 - x_{10} - u_1 \quad , \quad \Delta y = y_{20} + v_2 - y_{10} - v_1 \tag{2.158}$$

(see Fig. 2.16). The length can be calculated as

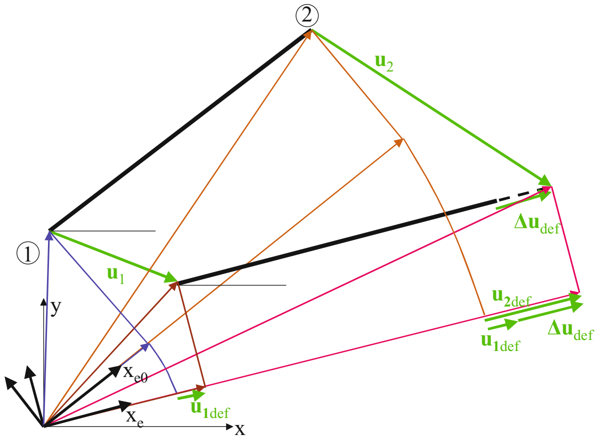

Fig. 2.15 Effect of the rotating only coordinate system

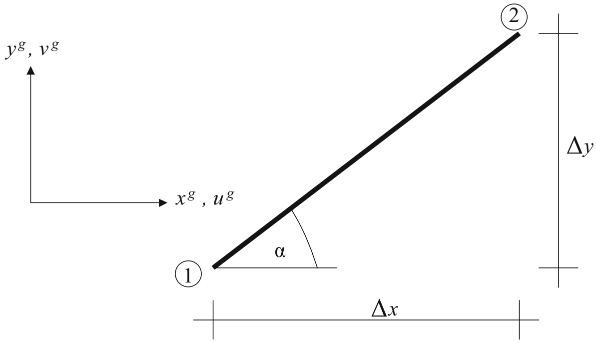

Fig. 2.16 Geometric relations of the link element

$$l = \sqrt{\Delta x^2 + \Delta y^2} \tag{2.159}$$

The trigonometric functions can be expressed as

$$\cos\alpha = \frac{\Delta x(\mathbf{u})}{l(\mathbf{u})} \quad , \quad \sin\alpha = \frac{\Delta y(\mathbf{u})}{l(\mathbf{u})} \tag{2.160}$$

In the undeformed state the trigonometric functions only depend on the initial coordinates, here:

$$\cos\alpha_0 = \frac{(x_{20} - x_{10})}{l_0} \quad , \qquad \sin\alpha_0 = \frac{(y_{20} - y_{10})}{l_0} \tag{2.161}$$

The transformation matrix of the deformed state depends on the global displacements, thus:

$$\mathbf{u}_{def} = \mathbf{T}(\mathbf{u}) \;\; \mathbf{x} - \mathbf{T}_0\mathbf{x}_0 \tag{2.162}$$

2.4.2 Strain, Internal Forces, Tangential Stiffness Matrix

For the co-rotational formulation the linear strain measure remains appropriate (for accounting for large strain in this formulation see Sect. 2.5.4). With the symbols of the FE-formulation the strain now reads:

$$\boldsymbol{\varepsilon} = \mathbf{B}_{lin}\mathbf{u}_{def} = \mathbf{B}_{lin}[\mathbf{T}(\mathbf{u})(\mathbf{x}_0 + \mathbf{u}) - \mathbf{T}_0\mathbf{x}_0] \tag{2.163}$$

Thus the relation between strain and nodal displacements has become non-linear.

This equation can be extended to two- and three-dimensional elements with the only change that the strain ε becomes the vector $\boldsymbol{\varepsilon}$.

Again the general B-matrix is obtained as the derivative of the strain with respect to the (global) nodal displacements:

$$\begin{aligned} \mathbf{B} &= \frac{\partial\boldsymbol{\varepsilon}}{\partial\mathbf{u}} = \frac{\partial}{\partial\mathbf{u}}\left(\mathbf{B}_{lin}[\mathbf{T}(\mathbf{u})(\mathbf{x}_0 + \mathbf{u}) - \mathbf{T}_0\mathbf{x}_0]\right) \\ &= \mathbf{B}_{lin}\underbrace{\left[\frac{\partial\mathbf{T}}{\partial\mathbf{u}}(\mathbf{x}_0 + \mathbf{u}) + \mathbf{T}(\mathbf{u})\right]}_{=:\mathbf{T}^*} \end{aligned} \tag{2.164}$$

It should be emphasized that the general B-matrix $\mathbf{B}$ differs from the linear B-matrix $\mathbf{B}_{lin}$ by a term depending on the transformation only. Then the internal nodal forces in the global system read:

$$\begin{aligned} \mathbf{f}_{int} &= \int\limits_{(V)} \mathbf{B}^T\boldsymbol{\sigma}dV = \int\limits_{(V)} \mathbf{T}^{*T}{\mathbf{B}_{lin}}^T\boldsymbol{\sigma}dV \\ &= \int\limits_{(V)} \left[(\mathbf{x}_0 + \mathbf{u})^T\frac{\partial\mathbf{T}^T}{\partial\mathbf{u}^T} + \mathbf{T}^T\right]{\mathbf{B}_{lin}}^T\mathbf{E}\mathbf{B}_{lin}[\mathbf{T}(\mathbf{x}_0 + \mathbf{u}) - \mathbf{T}_0\mathbf{x}_0]\,dV \end{aligned} \tag{2.165}$$

The transformation matrix is constant for the whole element, $\mathbf{x}$ and $\mathbf{u}$ are independent of the integrand so that they can be written in front of the integral:

$$\mathbf{f}_{int} = \underbrace{\left[(\mathbf{x}_0 + \mathbf{u})^T \frac{\partial \mathbf{T}^T}{\partial \mathbf{u}^T} + \mathbf{T}^T \right]}_{\mathbf{T}^{*T}} \underbrace{\int\limits_{(V)} \mathbf{B}_{lin}{}^T \boldsymbol{\sigma} dV}_{\mathbf{f}_{int}^{elem}} \tag{2.166}$$

The remaining integral looks like the internal forces from the linear element formulation but one must keep in mind that the stress is calculated using (2.163) and the material model in a non-linear way. If the deformatoric displacements are passed to the element routine instead of the total ones nothing else must be changed for this part.

The symbol $\hat{\mathbf{u}}_g$ in the following terms emphasises that the nodal displacements in the global coordinate systems are meant whereas the superscript ^{elem} expresses that the terms are related to the element coordinate system and are calculated like in the linear theory.

Under the assumption of constant loads the tangential stiffness matrix then is

$$\begin{aligned} \mathbf{K}_T &= \frac{\partial}{\partial \hat{\mathbf{u}}_g} \mathbf{f}_{int} = \frac{\partial}{\partial \hat{\mathbf{u}}_g} \int\limits_{(V)} \mathbf{B}^T \boldsymbol{\sigma} dV \\ &= \mathbf{T}^{*T} \int\limits_{(V)} \mathbf{B}_{lin}^T \mathbf{E} \mathbf{B}_{lin} \, dV \, \mathbf{T}^* + \frac{\partial \mathbf{T}^{*T}}{\partial \hat{\mathbf{u}}_g} \int\limits_{(V)} \mathbf{B}_{lin}^T \boldsymbol{\sigma} dV \end{aligned} \tag{2.167}$$

$$\mathbf{K}_T = \underbrace{\mathbf{T}^{*T} \mathbf{K}_{lin}^{elem} \mathbf{T}^*}_{\mathbf{K}_u} + \underbrace{\frac{\partial \mathbf{T}^{*T}}{\partial \hat{\mathbf{u}}_g} \mathbf{f}_{int}^{elem}}_{\mathbf{K}_\sigma} \tag{2.168}$$

or

$$\mathbf{K}_T = \underbrace{\int\limits_{(V)} \mathbf{B}^T \mathbf{E} \mathbf{B} \, dV}_{\mathbf{K}_u} + \underbrace{\frac{\partial \mathbf{T}^{*T}}{\partial \hat{\mathbf{u}}_g} \mathbf{f}_{int}^{elem}}_{\mathbf{K}_\sigma} \tag{2.169}$$

In (2.168) it can again be seen that this method makes use of the element forces and stiffness matrix of a geometrically linear formulation. That means once the terms from the rotation are known this method can be applied to any existing element.

In order to understand the structure of the following terms it must be kept in mind that

- a first derivative with respect to the vector of global degrees of freedom $\hat{\mathbf{u}}_g$ has been formed where the derivatives with respect to each degree form a row,

- a transposition has been carried out such that the derivatives with respect to $\hat{\mathbf{u}}_g$ now form a column,
- a further derivative with respect to $\hat{\mathbf{u}}_g$ has been formed where these derivatives with respect to each degree form a row.

The meaning of the notation is explained in Sect. 1.2.

$$\frac{\partial \mathbf{T}^{*T}}{\partial \hat{\mathbf{u}}_g}\mathbf{f}_{int}^{elem} = \left(\mathbf{x}_0 + \hat{\mathbf{u}}_g\right)^T \frac{\partial^2 \mathbf{T}^T}{\partial \hat{\mathbf{u}}_g^T \partial \hat{\mathbf{u}}_g}\mathbf{f}_{int}^{elem} + \frac{\partial \mathbf{T}^T}{\partial \hat{\mathbf{u}}_g^T}\mathbf{f}_{int}^{elem} + \frac{\partial \mathbf{T}^T}{\partial \hat{\mathbf{u}}_g}\mathbf{f}_{int}^{elem} \tag{2.170}$$

The last two summands are not equal but it will be shown below by means of the index notation that the second one is the transposed of the third one:

$$\frac{\partial \mathbf{T}^{*T}}{\partial \hat{\mathbf{u}}_g}\mathbf{f}_{int}^{elem} = \left(\mathbf{x}_0 + \hat{\mathbf{u}}_g\right)^T \frac{\partial^2 \mathbf{T}^T}{\partial \hat{\mathbf{u}}_g^T \partial \hat{\mathbf{u}}_g}\mathbf{f}_{int}^{elem} + \left[\frac{\partial \mathbf{T}^T}{\partial \hat{\mathbf{u}}_g}\mathbf{f}_{int}^{elem}\right]^T + \frac{\partial \mathbf{T}^T}{\partial \hat{\mathbf{u}}_g}\mathbf{f}_{int}^{elem} \tag{2.171}$$

The expressions leading to the B-matrix, the internal forces and the tangential matrix comprise derivatives of the transformation matrix with respect to a vector. This leads to a three-dimensional matrix (hypermatrix), the second derivative to a four-dimensional one. Then usual matrix notation is not sufficient to describe how the summations must be performed. One way to overcome this difficulty is at first to execute the matrix multiplication in the given context, then form the derivative of the components of $\mathbf{T}$ within these products. The first term of (2.170) form a scalar after multiplying $\mathbf{T}^T$ by the pre- and succeeding vectors, thus a matrix after forming the two derivatives.

This is shown in the following by the example of the link element. An alternative way is to use the index notation including the sum convention (rules explained in Sect. 1.1). The strain then reads:

$$\varepsilon_i = B_{ij}^{lin}\left[T_{jk}\left(x_{0k}^g + u_k^g\right) - T_{jk}x_{0k}^g\right] \tag{2.172}$$

where j runs—over the degrees of freedom in element coordinates, k—over the degrees of freedom in global coordinates.

Here the strain components form a column matrix (Voigt notation). The total B-matrix can be expressed as

$$B_{il} = \frac{\partial \varepsilon_i}{\partial u_l^g} = B_{ij}^{lin}\left[\frac{\partial T_{jk}}{\partial u_l^g}\left(x_{0k}^g + u_k^g\right) + T_{jk}\frac{\partial\left(x_{0k}^g + u_k^g\right)}{\partial u_l^g}\right] \tag{2.173}$$

where l runs over the degrees of freedom in global coordinates.

The derivative of a displacement component with respect to a displacement component is 1 if they carry the same index, otherwise 0.

$$B_{il} = B_{ij}^{lin}\left[\frac{\partial T_{jk}}{\partial u_l^g}\left(x_{0k}^g + u_k^g\right) + T_{jk}\delta_{kl}\right] \tag{2.174}$$

The last product has non-zero contributions only if $k = l$:

$$B_{il} = B_{ij}^{lin}\left[\frac{\partial T_{jk}}{\partial u_l^g}\left(x_{0k}^g + u_k^g\right) + T_{jl}\right] \tag{2.175}$$

For the multiplication with $\mathbf{B}^T$ the first index must be used for summation, for $\mathbf{K}_u$:

$$K_{lm}^u = \int_{(V)} B_{il}E_{ij}B_{jm}dV \tag{2.176}$$

for the internal forces:

$$f_l^{int} = \int_{(V)} B_{il}\sigma_i dV \tag{2.177}$$

For the initial stress matrix the following term is needed (keep in mind that only scalars are handled such that their order in products, but not the indices, can be changed):

$$\frac{\partial B_{il}}{\partial u_m^g}\sigma_i = \left[\frac{\partial^2 T_{jk}}{\partial u_l^g \partial u_m^g}\left(x_{0k}^g + u_k^g\right) + \frac{\partial T_{jk}}{\partial u_l^g}\frac{\partial\left(x_{0k}^g + u_k^g\right)}{\partial u_m^g} + \frac{T_{jl}}{\partial u_m^g}\right]B_{ij}^{lin}\sigma_i \tag{2.178}$$

$$\frac{\partial B_{il}}{\partial u_m^g}\sigma_i = \left[\frac{\partial^2 T_{jk}}{\partial u_l^g \partial u_m^g}\left(x_{0k}^g + u_k^g\right) + \frac{\partial T_{jk}}{\partial u_l^g}\delta_{km} + \frac{T_{jl}}{\partial u_m^g}\right]B_{ij}^{lin}\sigma_i \tag{2.179}$$

$$K_{lm}^\sigma = \left[\frac{\partial^2 T_{jk}}{\partial u_l^g \partial u_m^g}\left(x_{0k}^g + u_k^g\right) + \frac{\partial T_{jm}}{\partial u_l^g} + \frac{T_{jl}}{\partial u_m^g}\right]\underbrace{\int_{(V)} B_{ij}^{lin}\sigma_i dV}_{f_j^{int,elem}} \tag{2.180}$$

There are sums over i, j and k whereas l and m span the resulting matrix.

$$K_{lm}^\sigma = \left[f_j^{int,elem}\frac{\partial^2 T_{jk}}{\partial u_l^g \partial u_m^g}\left(x_{0k}^g + u_k^g\right) + f_j^{int,elem}\frac{\partial T_{jm}}{\partial u_l^g} + f_j^{int,elem}\frac{T_{jl}}{\partial u_m^g}\right] \tag{2.181}$$

The first summand produces a scalar before forming the derivatives, the second and third one a vector, all three terms matrices after executing the derivative(s) where the second and third term have exchanged indices l and m, i.e. the second one forms the transposed of the third one when being ordered in matrices.

Instead of one coordinate system per element one system per integration point can be chosen. This increases the accuracy and can even account for curved elements.

2.4.3 Direction of Strain and Stress

In the co-rotational formulation stresses and strains are calculated in the rotated element coordinate system. This is very helpful for beams and shells using the beam axis resp. the mid-surface as reference for kinematic assumptions and is useful for the interpretation of results for other types, too. The latter can be seen in Fig. 2.18 compared with Fig. 2.17. Especially in beam- or shell-*like* structures stress and

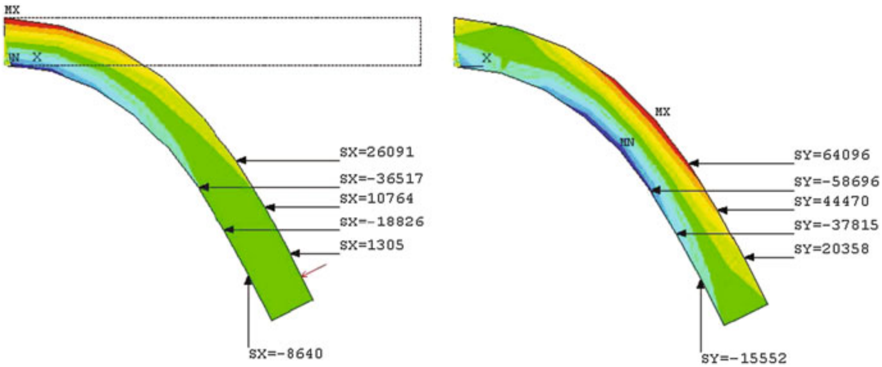

Fig. 2.17 Stress components in the initial coordinate system

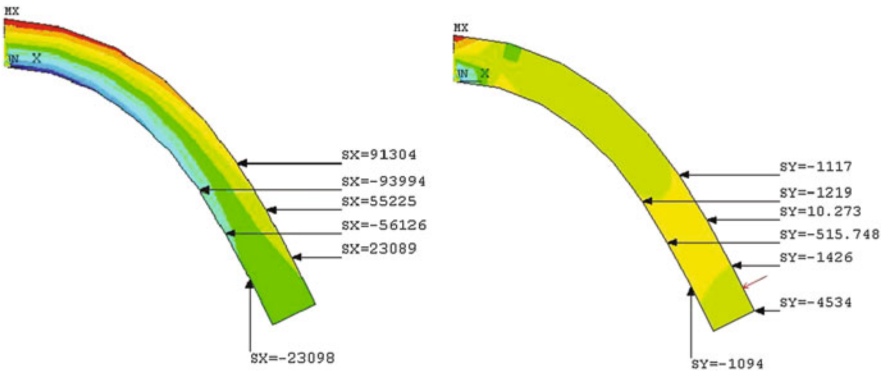

Fig. 2.18 Stress components in the rotated coordinate system

strain components being parallel or perpendicular to the edges after a large rotation are most meaningful. Another example are anisotropy axes.

2.4.4 Example Link Element

For the link element dV becomes $A_0 dx$. In case of the linear shape functions all terms in the integrand are independent of x. Thus:

$$\mathbf{f}_{int}^{elem} = \mathbf{B}_{lin}^T \boldsymbol{\sigma} A_0 l_0 \tag{2.182}$$

$$\mathbf{f}_{int} = \mathbf{T}^{*T} \mathbf{f}_{int}^{elem} \tag{2.183}$$

$$\mathbf{K}_T = \mathbf{B}^T \mathbf{B} E A_0 l_0 + \frac{\partial \mathbf{T}^{*T}}{\partial \mathbf{u}} \mathbf{f}_{int}^{elem} = \mathbf{T}^{*T} \mathbf{B}_{lin}^T \mathbf{B}_{lin} \mathbf{T}^* E A_0 l_0 + \frac{\partial \mathbf{T}^{*T}}{\partial \mathbf{u}} \mathbf{f}_{int}^{elem} \tag{2.184}$$

Furthermore, the stress has only one component which is related to the nodal displacements by

$$\sigma = E \mathbf{B}_{lin} \left[\mathbf{T} \left(\mathbf{x}_0 + \hat{\mathbf{u}}_g \right) - \mathbf{T}_0 \mathbf{x}_0 \right] \tag{2.185}$$

The B-Matrix on element level is

$$\mathbf{B}_{lin} = \frac{1}{l_0} \begin{bmatrix} -1 & 1 \end{bmatrix} \tag{2.186}$$

where l_0 is the original length which the derivative is not formed from.

With the abbreviations

$$\begin{aligned} c &:= \cos \alpha \\ s &:= \sin \alpha \end{aligned} \tag{2.187}$$

the transformation matrix reads:

$$\mathbf{T} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \tag{2.188}$$

Together this means:

$$\sigma = \frac{E}{l_0}\begin{bmatrix} -1 & 1 \end{bmatrix}\left(\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}\begin{bmatrix} x_{10}+u_1 \\ y_{10}+v_1 \\ x_{20}+u_2 \\ y_{20}+v_2 \end{bmatrix} - \begin{bmatrix} c_0 & s_0 & 0 & 0 \\ 0 & 0 & c_0 & s_0 \end{bmatrix}\begin{bmatrix} x_{10} \\ y_{10} \\ x_{20} \\ y_{20} \end{bmatrix}\right) \tag{2.189}$$

$$\sigma = \frac{E}{l_0}\left(\begin{bmatrix} -c & -s & c & s \end{bmatrix}\begin{bmatrix} x_{10}+u_1 \\ y_{10}+v_1 \\ x_{20}+u_2 \\ y_{20}+v_2 \end{bmatrix} - \begin{bmatrix} -c_0 & -s_0 & c_0 & s_0 \end{bmatrix}\begin{bmatrix} x_{10} \\ y_{10} \\ x_{20} \\ y_{20} \end{bmatrix}\right) \tag{2.190}$$

Furthermore from (2.186) follows:

$$\mathbf{B}_{lin}^T \sigma = \frac{1}{l_0}\begin{bmatrix} -\sigma \\ \sigma \end{bmatrix} \tag{2.191}$$

and thus:

$$\mathbf{f}_{lin}^{elem} = \frac{1}{l_0}\begin{bmatrix} -\sigma \\ \sigma \end{bmatrix} A_0 l_0 = \begin{bmatrix} -\sigma A_0 \\ \sigma A_0 \end{bmatrix} \tag{2.192}$$

Some terms contain the derivative of the transformation matrix with respect to the nodal displacements. This would be a hypermatrix, difficult to show on paper. Therefore, it is recommended either to use the index notation or to multiply it by the preceding or succeeding vector:

$$\begin{aligned} \frac{\partial \mathbf{T}}{\partial \mathbf{u}}(\mathbf{x}_0+\mathbf{u}) &= \frac{\partial}{\partial \mathbf{u}}\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}\begin{bmatrix} x_{10}+u_1 \\ y_{10}+v_1 \\ x_{20}+u_2 \\ y_{20}+v_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial c}{\partial \mathbf{u}}(x_{10}+u_1) + \frac{\partial s}{\partial \mathbf{u}}(y_{10}+v_1) \\ \frac{\partial c}{\partial \mathbf{u}}(x_{20}+u_2) + \frac{\partial s}{\partial \mathbf{u}}(y_{20}+v_2) \end{bmatrix} \end{aligned} \tag{2.193}$$

$$\frac{\partial \mathbf{T}}{\partial \mathbf{u}}(\mathbf{x}_0+\mathbf{u}) = \begin{bmatrix} \frac{\partial c}{\partial u_1}(x_{10}+u_1)+\frac{\partial s}{\partial u_1}(y_{10}+v_1) & \cdots & \frac{\partial c}{\partial v_2}(x_{10}+u_1)+\frac{\partial s}{\partial v_2}(y_{10}+v_1) \\ \frac{\partial c}{\partial u_1}(x_{20}+u_2)+\frac{\partial s}{\partial u_1}(y_{20}+v_2) & \cdots & \frac{\partial c}{\partial v_2}(x_{20}+u_2)+\frac{\partial s}{\partial v_2}(y_{20}+v_2) \end{bmatrix} \tag{2.194}$$

Within (2.164) one term becomes

$$\mathbf{T}^* = \begin{bmatrix} \cdots + c & \cdots + s & \cdots & \frac{\partial c}{\partial v_2}(x_{10}+u_1)+\frac{\partial s}{\partial v_2}(y_{10}+v_1) \\ \cdots & \cdots & \cdots + c & \frac{\partial c}{\partial v_2}(x_{20}+u_2)+\frac{\partial s}{\partial v_2}(y_{20}+v_2)+s \end{bmatrix} \tag{2.195}$$

Herein the derivatives are:

$$\frac{\partial \Delta x}{\partial \mathbf{u}} = [-1 \quad 0 \quad 1 \quad 0]\ , \quad \frac{\partial \Delta y}{\partial \mathbf{u}} = [0 \quad -1 \quad 0 \quad 1] \tag{2.196}$$

$$\frac{\partial l}{\partial \mathbf{u}} = \frac{1}{2\sqrt{\Delta x^2+\Delta y^2}}\left(2\Delta x\frac{\partial \Delta x}{\partial \mathbf{u}} + 2\Delta y\frac{\partial \Delta y}{\partial \mathbf{u}}\right) = \frac{1}{l}\left(\Delta x\frac{\partial \Delta x}{\partial \mathbf{u}} + \Delta y\frac{\partial \Delta y}{\partial \mathbf{u}}\right) \tag{2.197}$$

$$\frac{\partial l}{\partial \mathbf{u}} = \frac{1}{l}[-\Delta x \quad -\Delta y \quad \Delta x \quad \Delta y] = [-c \quad -s \quad c \quad s] \tag{2.198}$$

$$\begin{aligned} \frac{\partial \cos\alpha}{\partial \mathbf{u}} &= \frac{1}{l^2}\left(\frac{\partial \Delta x}{\partial \mathbf{u}}l - \Delta x\frac{\partial l}{\partial \mathbf{u}}\right) \\ &= \frac{1}{l^2}\left([-1 \quad 0 \quad 1 \quad 0]l - \Delta x\frac{1}{l}[-\Delta x \quad -\Delta y \quad \Delta x \quad \Delta y]\right) \end{aligned} \tag{2.199}$$

$$\frac{\partial \cos\alpha}{\partial \mathbf{u}} = \frac{1}{l}[-1 \quad 0 \quad 1 \quad 0] - \frac{1}{l^3}\Delta x[-\Delta x \quad -\Delta y \quad \Delta x \quad \Delta y] \tag{2.200}$$

$$\begin{aligned} \frac{\partial \cos\alpha}{\partial \mathbf{u}} &= \frac{1}{l}([-1 \quad 0 \quad 1 \quad 0] - c[-c \quad -s \quad c \quad s]) \\ &= \frac{1}{l}[-1+c^2 \quad cs \quad 1-c^2 \quad -cs] \end{aligned} \tag{2.201}$$

$$\begin{aligned}\frac{\partial \sin\alpha}{\partial \mathbf{u}} &= \frac{1}{l}[\,0 \quad -1 \quad 0 \quad 1\,] - \frac{1}{l^3}\Delta y[\,-\Delta x \quad -\Delta y \quad \Delta x \quad \Delta y\,] \\ &= \frac{1}{l}\left[\,cs \quad -1+s^2 \quad -cs \quad 1-s^2\,\right]\end{aligned} \tag{2.202}$$

Furthermore,

$$\mathbf{B} = \mathbf{B}_{lin}\mathbf{T}^* = \frac{1}{l_0}[\,-1 \quad 1\,]\mathbf{T}^* \tag{2.203}$$

If one defines

$$\mathbf{T}^* =: \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \end{bmatrix} \tag{2.204}$$

the B-matrix becomes

$$\mathbf{B} = \frac{1}{l_0}[\,-t_{11}+t_{21} \quad \cdots \quad \cdots \quad -t_{14}+t_{24}\,] \tag{2.205}$$

and the internal forces in the global coordinate system follow as

$$\mathbf{f}_{int} = \mathbf{B}^T\sigma A l_0 = \begin{bmatrix} (-t_{11}+t_{21})\sigma A \\ (-t_{12}+t_{22})\sigma A \\ (-t_{13}+t_{23})\sigma A \\ (-t_{14}+t_{24})\sigma A \end{bmatrix} \tag{2.206}$$

The derivative of $\mathbf{T}^{*T}$ (2.195) is multiplied by ${\mathbf{f}_{int}}^{elem}$:

$$\begin{aligned}\mathbf{K}_\sigma &= \frac{\partial \mathbf{T}^{*T}}{\partial \hat{\mathbf{u}}_g}\mathbf{f}_{int}^{elem} \\ &= \left(\mathbf{x}_0 + \hat{\mathbf{u}}_g\right)^T \frac{\partial^2 {\mathbf{T}^T}^2}{\partial \hat{\mathbf{u}}_g}\mathbf{f}_{int}^{elem} + \left[\frac{\partial \mathbf{T}^T}{\partial \hat{\mathbf{u}}_g}\mathbf{f}_{int}^{elem}\right]^T + \frac{\partial \mathbf{T}^T}{\partial \hat{\mathbf{u}}_g}\mathbf{f}_{int}^{elem}\end{aligned} \tag{2.207}$$

The first term yields:

$$
\begin{array}{c|c|c}
\left(\mathbf{x}_0+\hat{\mathbf{u}}_g\right)^T \dfrac{\partial^2 \mathbf{T}^T}{\partial \hat{\mathbf{u}}_g^2}\mathbf{f}_{\text{int}}^{elem} &
\begin{array}{c:c}
\dfrac{\partial^2 c}{\partial \hat{\mathbf{u}}_g^2} & 0 \\ \hdashline
\dfrac{\partial^2 s}{\partial \hat{\mathbf{u}}_g^2} & 0 \\ \hdashline
0 & \dfrac{\partial^2 c}{\partial \hat{\mathbf{u}}_g^2} \\ \hdashline
0 & \dfrac{\partial^2 s}{\partial \hat{\mathbf{u}}_g^2}
\end{array} &
\begin{array}{c}
-\sigma A \\ \hdashline \sigma A
\end{array} \\ \hline
\begin{array}{cccc}
x_{10}+u_1 & & x_{20}+u_2 & \\
 & y_{10}+v_1 & & y_{20}+v_2
\end{array} &
\begin{array}{c:c}
(x_{10}+u_1)\dfrac{\partial^2 c}{\partial \hat{\mathbf{u}}_g^2} & (x_{20}+u_2)\dfrac{\partial^2 c}{\partial \hat{\mathbf{u}}_g^2} \\
+(y_{10}+v_1)\dfrac{\partial^2 s}{\partial \hat{\mathbf{u}}_g^2} & +(y_{20}+v_2)\dfrac{\partial^2 s}{\partial \hat{\mathbf{u}}_g^2}
\end{array} &
\begin{array}{c}
-(x_{10}+u_1)\dfrac{\partial^2 c}{\partial \hat{\mathbf{u}}_g^2}\sigma A \\
-(y_{10}+v_1)\dfrac{\partial^2 s}{\partial \hat{\mathbf{u}}_g^2}\sigma A \\
+(x_{20}+u_2)\dfrac{\partial^2 c}{\partial \hat{\mathbf{u}}_g^2}\sigma A \\
+(y_{20}+v_2)\dfrac{\partial^2 s}{\partial \hat{\mathbf{u}}_g^2}\sigma A
\end{array}
\end{array}
$$

Herein is for example:

$$
\frac{\partial^2 c}{\partial \hat{\mathbf{u}}_g^2} = \begin{bmatrix}
\dfrac{\partial^2 c}{\partial u_1^2} & \dfrac{\partial^2 c}{\partial u_1 v_1} & \dfrac{\partial^2 c}{\partial u_1 u_2} & \dfrac{\partial^2 c}{\partial u_1 v_2} \\
 & \dfrac{\partial^2 c}{\partial v_1^2} & & \vdots \\
 & & \dfrac{\partial^2 c}{\partial u_2^2} & \vdots \\
symm. & & & \dfrac{\partial^2 c}{\partial v_2^2}
\end{bmatrix} \tag{2.208}
$$

Therein for example:

$$
\frac{\partial^2 \cos\alpha}{\partial v_2^2} = \frac{3}{l^4}\underbrace{\frac{\partial l}{\partial v_2}}_{\Delta y/l}\Delta x\Delta y - \frac{1}{l^3}\underbrace{\frac{\partial \Delta x}{\partial v_2}}_{0}\Delta y - \frac{1}{l^3}\Delta x\underbrace{\frac{\partial \Delta y}{\partial v_2}}_{1} \tag{2.209}
$$

$$
\frac{\partial^2 \cos\alpha}{\partial v_2^2} = \frac{3}{l^5}\Delta x\Delta y^2 - \frac{1}{l^3}\Delta x = \frac{1}{l^2}\left(3cs^2 - c\right) \tag{2.210}
$$

and in the same way:

$$\frac{\partial^2 \sin\alpha}{\partial v_2^2} = -\frac{1}{l^2}\frac{\partial l}{\partial v_2} + \frac{3}{l^4}\frac{\partial l}{\partial v_2}\Delta y^2 - \frac{1}{l^3}\frac{\partial \Delta y}{\partial v_2}\Delta y - \frac{1}{l^3}\Delta y\frac{\partial \Delta y}{\partial v_2} \tag{2.211}$$

$$\frac{\partial^2 \sin\alpha}{\partial v_2^2} = -\frac{3}{l^3}\Delta y + \frac{3}{l^5}\Delta y^3 = \frac{1}{l^2}\left(-3s + 3s^3\right) \tag{2.212}$$

The third term of the initial stress matrix (2.207) yields:

$$\frac{\partial \mathbf{T}^T}{\partial \hat{\mathbf{u}}_g}\mathbf{f}_{int}^{elem} = \frac{\partial}{\partial \hat{\mathbf{u}}_g}\begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix}\begin{bmatrix} -\sigma A \\ \sigma A \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial c}{\partial \hat{\mathbf{u}}_g} \\ -\dfrac{\partial s}{\partial \hat{\mathbf{u}}_g} \\ \dfrac{\partial c}{\partial \hat{\mathbf{u}}_g} \\ \dfrac{\partial s}{\partial \hat{\mathbf{u}}_g} \end{bmatrix}\sigma A \tag{2.213}$$

its transposed:

$$\left[\frac{\partial \mathbf{T}^T}{\partial \hat{\mathbf{u}}_g}\mathbf{f}_{int}^{elem}\right]^T = \begin{bmatrix} -\dfrac{\partial c}{\partial \hat{\mathbf{u}}_g^T} & -\dfrac{\partial s}{\partial \hat{\mathbf{u}}_g^T} & \dfrac{\partial c}{\partial \hat{\mathbf{u}}_g^T} & \dfrac{\partial s}{\partial \hat{\mathbf{u}}_g^T} \end{bmatrix}\sigma A \tag{2.214}$$

The total tangential stiffness matrix then reads:

$$\begin{aligned} \mathbf{K}_T &= \mathbf{B}^T\mathbf{B}EA_0 l_0 + \Delta x\frac{\partial^2 c}{\partial \hat{\mathbf{u}}_g^2}\sigma A_0 + \Delta y\frac{\partial^2 s}{\partial \hat{\mathbf{u}}_g^2}\sigma A_0 \\ &+ \begin{bmatrix} -\dfrac{\partial c}{\partial \mathbf{u}_g^T} & -\dfrac{\partial s}{\partial \mathbf{u}_g^T} & \dfrac{\partial c}{\partial \mathbf{u}_g^T} & \dfrac{\partial s}{\partial \mathbf{u}_g^T} \end{bmatrix}\sigma A_0 + \begin{bmatrix} -\dfrac{\partial c}{\partial \hat{\mathbf{u}}_g} \\ -\dfrac{\partial s}{\partial \hat{\mathbf{u}}_g} \\ \dfrac{\partial c}{\partial \hat{\mathbf{u}}_g} \\ \dfrac{\partial s}{\partial \hat{\mathbf{u}}_g} \end{bmatrix}\sigma A_0 \end{aligned} \tag{2.215}$$

2.4.5 Numerical Example Two-Legged Truss

For the two-legged truss from Fig. 2.12 the coordinates and constraints are

$$\begin{matrix} x_{10} = 0 & y_{10} = 0 & x_{20} = 4 & y_{20} = 3 \\ u_1 = 0 & v_1 = 0 & u_2 = 0 & \end{matrix} \tag{2.216}$$

Thus, v_2 is the only degree of freedom and only the derivatives with respect to v_2 must be taken into account. The equilibrium at node 2 in y-direction only is of interest. That means:

$$\mathbf{f}_{ext} = -\frac{F}{2} \quad \hat{\mathbf{u}} = v_2$$

$$\Delta x = 4 \ , \quad \Delta y = 3 + v_2$$

$$\Delta y_0 = 3 + v_2 \ , \quad l_0 = 5, \quad \cos\alpha_0 = \frac{4}{5} \ , \quad \sin\alpha_0 = \frac{3}{5} \ ,$$

$$l = \sqrt{4^2 + (3 + v_2)^2}$$

$$\cos\alpha = \frac{4}{l} \ , \quad \sin\alpha = \frac{3 + v_2}{l}$$

$$\sigma = \frac{E}{5}(4c + (3 + v_2)s - 4c_0 - 3s_0) = \frac{E}{5}(4c + (3 + v_2)s - 5)$$

$$\frac{\partial l}{\partial v_2} = \frac{1}{l}(3 + v_2)$$

$$\frac{\partial}{\partial v_2}\cos\alpha = -\frac{4}{l^3}(3 + v_2) \ , \quad \frac{\partial}{\partial v_2}\sin\alpha = \frac{1}{l} - \frac{1}{l^3}(3 + v_2)^2$$

$$\mathbf{T}^* = \begin{bmatrix} t_{11} & \cdots & t_{13} & 0 \\ \cdots & \cdots & t_{23} & \frac{\partial c}{\partial v_2}\cdot 4 + \frac{\partial s}{\partial v_2}(3 + v_2) + s \end{bmatrix}$$

$$\mathbf{B} = \frac{1}{5}t_{24}$$

$$\mathbf{f}_{int} = t_{24}\sigma A_0$$

$$\mathbf{K}_u = \left(\frac{1}{5}t_{24}\right)^2 EA_0 \cdot 5 = \frac{1}{5}t_{24}^2 EA_0$$

$$\frac{\partial^2}{\partial v_2^2}\cos\alpha = \left(\frac{3}{l^5}(3 + v_2)^2 - \frac{1}{l^3}\right)\cdot 4$$

$$\frac{\partial^2}{\partial v_2^2}\sin\alpha = \left(-\frac{3}{l^3} + \frac{3}{l^5}(3 + v_2)^2\right)(3 + v_2)$$

$$\mathbf{K}_\sigma = \left(4\frac{\partial^2 c}{\partial v_2^2} + (3+v_2)\frac{\partial^2 s}{\partial v_2^2} + \frac{\partial s}{\partial v_2} + \frac{\partial s}{\partial v_2}\right)\sigma A_0 = \left(4\frac{\partial^2 c}{\partial v_2^2} + (3+v_2)\frac{\partial^2 s}{\partial v_2^2} + 2\frac{\partial s}{\partial v_2}\right)\sigma A_0$$

$$\mathbf{K}_T = \mathbf{K}_u + \mathbf{K}_\sigma$$

Then the following algorithm leads to the solution:

given: $\mathbf{f}_{ext}$

set $\quad i=1,\ \Delta\hat{\mathbf{u}}_0 = \mathbf{0},\ \hat{\mathbf{u}}_1 = \hat{\mathbf{u}}_{converged}$ *from last load increment* resp. $\hat{\mathbf{u}}_1 = \mathbf{0}$

Table 2.2 Newton-Raphson iteration for the two-legged truss in the co-rotational formulation

External load	Δv_2	v_2	K_T	r.h.s.	Convergence
−0.5	0	0	1.0000008	−0.5	
−0.5	−0.4999996	−0.4999996	0.66042952	−0.08334799	
−0.5	−0.6262023	−0.6262023	0.56949548	−0.00572057	1.4953006
−0.5	−0.63624727	−0.63624727	0.56218932	−3.6687E-05	1.88483506
−0.5	−0.63631253	−0.63631253	0.56214182	−1.5497E-09	1.99471873
−0.5	−0.63631254	−0.63631254	0.56214182	−1.0547E-15	1.40986474
−0.6	0	−0.63631254	0.56214182	−0.1	
−0.6	−0.17789105	−0.81420359	0.43143732	−0.01159205	
−0.6	−0.20475949	−0.84107203	0.41152375	−0.00026745	1.74914856
−0.6	−0.20540939	−0.84172192	0.41104167	−1.5665E-07	1.97462355
−0.6	−0.20540977	−0.8417223	0.41104139	−5.2958E-14	2.00197669
−0.6	−0.20540977	−0.8417223	0.41104139	2.2204E-16	0.36740652
−0.7	0	−0.8417223	0.41104139	−0.1	
−0.7	−0.2432845	−1.08500681	0.22992876	−0.02201854	
−0.7	−0.33904694	−1.18076924	0.15882583	−0.00340757	1.23300293
−0.7	−0.36050169	−1.202224	0.14295845	−0.00017027	1.60588135
−0.7	−0.36169271	−1.20341502	0.14207845	−5.2405E-07	1.93016356
−0.71	0	−1.20341502	0.14207845	−0.01000052	
−0.71	−0.07038734	−1.27380235	0.09025931	−0.00182599	
−0.71	−0.09061782	−1.29403284	0.07544379	−0.00014993	1.46997863
−0.71	−0.09260511	−1.29602012	0.07399056	−1.4441E-06	1.85728841
−0.71	−0.09262462	−1.29603964	0.07397629	−1.3925E-10	1.99166208
−0.71	−0.09262462	−1.29603964	0.07397629	−4.4409E-16	1.3686783
−0.71	−0.09262462	−1.29603964	0.07397629	−8.8818E-16	−0.05476934
−0.72	0	−1.29603964	0.07397629	−0.01	
−0.72	−0.13517844	−1.43121809	−0.02379526	−0.0066341	
−0.72	0.14362066	−1.15241898	0.17983348	−0.02820801	−3.52708527
−0.72	−0.01323561	−1.30927525	0.06430777	−0.00908488	−0.7827879
−0.72	−0.15450753	−1.45054718	−0.03757155	−0.00722727	0.20189861
−0.72	0.03785274	−1.25818691	0.10172079	−0.01332489	−2.67441291

1) calculate $\mathbf{f}_{int}(\hat{\mathbf{u}}_i)$

2) solve $\Delta\hat{\mathbf{u}}_i = \Delta\hat{\mathbf{u}}_{i-1} + \mathbf{K}_T^{-1}(\mathbf{f}_{ext} - \mathbf{f}_{int})$

3) calculate $\hat{\mathbf{u}}_{i+1} = \hat{\mathbf{u}}_1 + \Delta\hat{\mathbf{u}}_i$

when converged:

new $\mathbf{f}_{ext}$, $i = 1$

continue with 1)

The iteration progress is shown in Table 2.2 and illustrated in Fig. 2.19. The convergence exponent approaches two in the vicinity of the solution which is the typical value in the Newton-Raphson scheme (quadratic convergence). When reaching the processor accuracy it can become worth. Usually the iteration is terminated before.

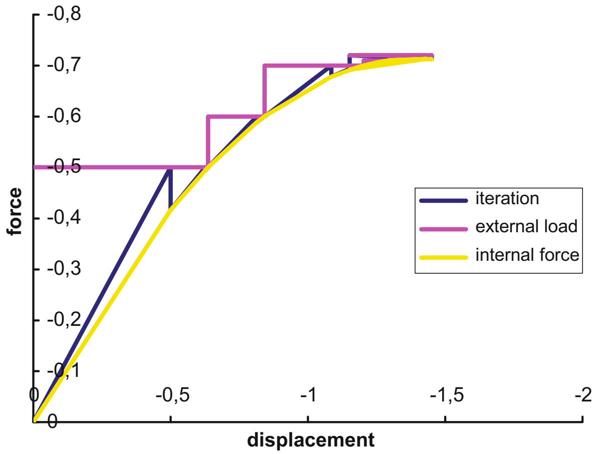

Fig. 2.19 Course of the iteration in the Newton-Raphson scheme for the co-rotational formulation

2.4.6 Comparison with Green Strain

The results being different from Sect. 2.3.5 can amongst others be explained by the fact that different strain measures are used which already differ in one dimension, i.e. deliver different values for the same changes in length.

In the last converged state in Table 2.2 the external and for equilibrium reasons the internal force is

$$\mathbf{f}^{int} = -0.71 \tag{2.217}$$

and the global displacement

$$v_2^g = -1.2960 \tag{2.218}$$

For the same system and the same $v_2{}^g$ but with *Green's strain* the displacement components after (2.103) in the element coordinate system would follow as

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}^e = \begin{bmatrix} \sin\alpha_0 \\ \cos\alpha_0 \end{bmatrix} v_2^g = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix} (-1.2960) = \begin{bmatrix} -0.7776 \\ -1.0368 \end{bmatrix} \tag{2.219}$$

According to (2.104) the derivatives with respect to the x-coordinate are

$$u' = \frac{1}{5}(-0.7776) = -0.1555 \quad , \quad v' = \frac{1}{5}(-1.0368) = -0.2074 \tag{2.220}$$

After (2.66) the Green-Lagrange strain is

$$\varepsilon^{GL} = u' + \frac{1}{2}u'^2 + \frac{1}{2}v'^2 = -0.1219 \tag{2.221}$$

and the stress reversely calculated from the internal force after (2.107), the value taken from the corotational solution:

$$\sigma = \frac{\mathbf{f}_{int}}{A \cdot (s(1+u') + cv')} = \frac{-0.71}{0.1 \cdot (0.6 \cdot (1 - 0.1555) + 0.8 \cdot (-0.2074))}$$
$$= -20.83 \tag{2.222}$$

In order to fulfil this stress–strain relation Young's modulus would have to be

$$E_{mod} = \frac{\sigma}{\varepsilon} = \frac{-20.83}{-0.1219} = 170.87 \tag{2.223}$$

If one extrapolates the last converged solution of the system with Green-Lagrange strain, $\mathbf{f}^{ext} = 0.58$, for the modified modulus one obtains

$$\mathbf{f}^{ext}_{mod} = 0.58 \cdot \frac{170.87}{138.89} = 0.714 \tag{2.224}$$

which means the same load-carrying capacity as in case of the co-rotational formulation. The force-displacement curve for the system with Green's strain and modified Young's modulus is shown in Fig. 2.22 in Sect. 2.6.3.

However, Young's modulus is a material parameter and one may be in doubt whether it may be changed for a different theory for large rotations. On the other hand it is obtained from a force-displacement measurement from which a certain strain and a certain stress measure is calculated (so-called *engineering* measures). In the *corotational formulation* they are used directly. After (2.190) one obtains in the considered state

$$\varepsilon^{eng} = -0.1304 \tag{2.225}$$

Green-Lagrange strains are defined in a different way. The one-dimensional relation (2.73) delivers:

$$\varepsilon^{GL} = -0.1304 + \frac{1}{2}(-0.1304)^2 = -0.1219 \tag{2.226}$$

like in (2.221). The *strain*-displacement relation is non-linear such that a linear *force*-displacement characteristic can only be modelled by a non-linear stress–strain relation. Indeed Hooke's law is not valid for such large strain.

For the stress there are differences, too. For the *co-rotational formulation* one obtains after the formulae following (2.216):

$$l = \sqrt{4^2 + (3 - 1.2960)^2} = 4.3478 \tag{2.227}$$

$$\cos\alpha = \frac{4}{4.3478} = 0.9200 \quad , \quad \sin\alpha = \frac{3 - 1.2960}{4.3478} = 0.3919 \tag{2.228}$$

$$\sigma = 138.889/5 \cdot (4 \cdot 0.9200 + (3 - 1.2960) \cdot 0.3919 - 5) = -18.12 \tag{2.229}$$

This is the *engineering* stress being calculated from the deformatoric displacements whereas the stress (2.222) (−20.83) is the second Piola-Kirchhoff stress which is explained in Sect. 2.6.3.

2.4.7 Determination of the Element Coordinate Systems

In two dimensions the orientation of an element can be determined by the distance vector of two nodes in their actual configuration, i.e. taking the displacement into account, as could be seen in the example above. In three dimensions three orientation vectors are necessary, e.g.

- the distance vector $\mathbf{\Delta}_1$ of two nodes
- the distance vector $\mathbf{\Delta}_2{}^*$ from the first to a third node
- the cross product $\mathbf{\Delta}_3 = \mathbf{\Delta}_1 \times \mathbf{\Delta}_2^*$
- the cross product $\mathbf{\Delta}_2 = \mathbf{\Delta}_3 \times \mathbf{\Delta}_1$

completing the orthogonal base. After normalising the vectors to unit length getting $\overline{\mathbf{\Delta}}_i$ one obtains the transformation matrix

$$\mathbf{T} = \begin{bmatrix} \overline{\mathbf{\Delta}}_1 & \overline{\mathbf{\Delta}}_2 & \overline{\mathbf{\Delta}}_3 \end{bmatrix} \tag{2.230}$$

Following the isoparametric concept where real coordinates {x,y,z} can be calculated for any point in unit coordinates {ξ,η,ζ} by using the shape functions

$$\mathbf{\Delta}_1 = \left\{ \frac{\partial x}{\partial \xi}, \frac{\partial y}{\partial \xi}, \frac{\partial z}{\partial \xi} \right\} \quad \text{and} \quad \mathbf{\Delta}_2^* = \left\{ \frac{\partial x}{\partial \eta}, \frac{\partial y}{\partial \eta}, \frac{\partial z}{\partial \eta} \right\} \tag{2.231}$$

can also be chosen. These are the tangents to the unit coordinate lines. They can be determined at different points in the element, e.g. at integration points.

Another technique is based on the polar decomposition of the deformation gradient $\mathbf{F}$ (see Sect. 2.3.6):

$$\mathbf{F}^T\mathbf{F} = \mathbf{U}^T\mathbf{R}^T\mathbf{R}\mathbf{U} = \mathbf{U}^2 \tag{2.232}$$

Then

$$\mathbf{U}^{-1} = \left(\mathbf{F}^T\mathbf{F}\right)^{-\frac{1}{2}} \tag{2.233}$$

which can be calculated e.g. after the theorem of Cayley-Hamilton or the spectral decomposition like in (2.247). Now the Rotation matrix can be determined as

$$\mathbf{R} = \mathbf{R}\mathbf{U}\mathbf{U}^{-1} = \mathbf{F}\mathbf{U}^{-1} = \mathbf{F}\left(\mathbf{F}^T\mathbf{F}\right)^{-\frac{1}{2}} \tag{2.234}$$

R is used as the transformation matrix **T**. The advantage of this method is that the rotation matrix can be determined at each integration point, thus taking into account bending of the element or curved elements. Therefore, this method has a higher accuracy and is appropriate for higher order elements. The same holds for the orientations of (2.231).

2.5 Large Strain

2.5.1 One-Dimensional Considerations

Engineering strains are calculated from changes in length relative to the **initial** length l_0. However, this is not suitable for every order of magnitude of the strain appearing in technical applications as the following example shows.

In Fig. 2.20 three cases are shown for which the strain—at first engineering strain—must be determined. In case a) this is

$$\varepsilon^{eng} = \frac{\Delta l}{l_0} = \frac{\Delta L}{L} \tag{2.235}$$

in case b)

$$\varepsilon^{eng} = \frac{\Delta L}{2L} \tag{2.236}$$

because the initial length is twice as large.

In case c) the spar is deformed to the double length in the first step and then by another ΔL. After the definition of engineering strain the increment is

$$\Delta\varepsilon^{eng} = \frac{\Delta L}{L} \tag{2.237}$$

i.e. comparable to case a), because the initial length is L. More appropriate, however, would be the same result as in case b), because the lengths before the

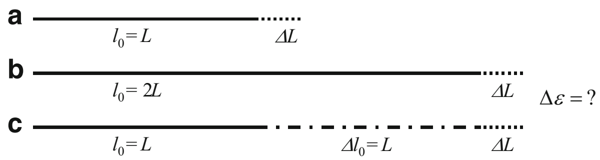

Fig. 2.20 Concerning the introduction of logarithmic strain

deformation by ΔL are equal. The actual, i.e. the deformed length l would have to be taken to achieve this result:

$$\Delta\varepsilon = \frac{\Delta l}{l} = \frac{\Delta L}{2L} \tag{2.238}$$

This leads to the principle:

$$\varepsilon = \sum \Delta\varepsilon = \sum \frac{\Delta l}{l} \tag{2.239}$$

In terms of infinitesimal small increments one obtains:

$$\varepsilon = \int_{l_0}^{l} d\varepsilon = \int_{l_0}^{l} \frac{1}{l} dl = [\ln l]_{l_0}^{l} = \ln l - \ln l_0 = \ln\left(\frac{l}{l_0}\right) \tag{2.240}$$

These strains are called logarithmic strains. They can be transformed in the following way to get a better comparison with engineering strains:

$$\varepsilon^{log} = \ln\left(\frac{l}{l_0}\right) = \ln\left(\frac{l_0 + \Delta l}{l_0}\right) = \ln\left(1 + \frac{\Delta l}{l_0}\right) \tag{2.241}$$

$$\varepsilon^{log} = \ln\left(1 + \varepsilon^{Ing}\right) \tag{2.242}$$

Especially in three dimensions the logarithmic strains are also called *Hencky* strains (cf. 2.5.2). In Table 2.3 and Fig. 2.21 the different strain measures are compared with each other. The values of the Green-Lagrange strain seem not to be very useful; it is made for large rotations. The logarithmic strain shows a "non-symmetry" between tension and compression. Remarkable is that for engineering strain -1 the logarithmic strain tends to $-\infty$. Engineering strain -1 means that a part is compressed to the length of 0. This is the largest imaginable deformation. Thus, a strain measure of $-\infty$ is appropriate.

Table 2.3 Comparison of different strain measures

Engineering strain	Green-Lagrange strain	Logarithmic strain
−1	−0.5000	$-\infty$
−0.99	−0.5000	−4.6052
−0.5	−0.3750	−0.6931
−0.3	−0.2550	−0.3567
−0.1	−0.0950	−0.1054
−0.05	−0.0488	−0.0513
−0.03	−0.0296	−0.0305
−0.01	−0.0100	−0.0101
−0.001	−0.0010	−0.0010
0	0.0000	0.0000
0.001	0.0010	0.0010
0.01	0.0101	0.0100
0.03	0.0305	0.0296
0.05	0.0513	0.0488
0.1	0.1050	0.0953
0.3	0.3450	0.2624
0.5	0.6250	0.4055
1	1.5000	0.6931

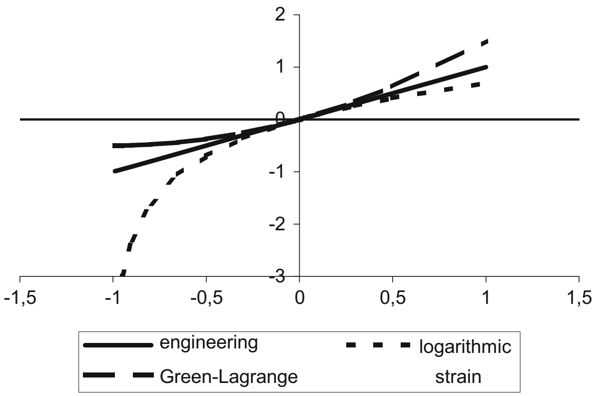

Fig. 2.21 Graphic comparison of different strain measures

2.5.2 Transition to Two- and Three-Dimensional Systems

Since it is probable that large strain occurs in combination with large rotations the two phenomena must be represented within the same theory.

There a several ways. One is—based on the relation between logarithmic and engineering strain in 1d (2.242)—to replace the engineering strain by a measure accounting for large rotations, here the Green-Lagrange strain:

$$\varepsilon^{log} = \ln\left(\frac{l}{l_0}\right) = \ln\left[\left(\frac{l}{l_0}\right)^2\right]^{\frac{1}{2}} = \frac{1}{2}\ln\left(\frac{l^2}{l_0^2}\right) = \frac{1}{2}\ln\left(\frac{l^2 - l_0^2}{l_0^2} + \frac{l_0^2}{l_0^2}\right) = \frac{1}{2}\ln\left(\frac{l^2 - l_0^2}{l_0^2} + 1\right)$$
$$\varepsilon^{log} = \frac{1}{2}\ln\left(2\varepsilon^{GL} + 1\right) \tag{2.243}$$

In 3d the measure defined in that way is called *Hencky* strain:

$$\boldsymbol{\varepsilon}^{Hencky} = \frac{1}{2}\ln\left(2\boldsymbol{\varepsilon}^{GL} + \mathbf{I}\right) \tag{2.244}$$

where $\mathbf{I}$ denotes the unit matrix.

One can immediately see that this measure is suitable for large rotations because for arbitrary rigid body rotations the Green-Lagrange strain becomes $\mathbf{0}$ so that one obtains

$$\boldsymbol{\varepsilon}^{Hencky} = \frac{1}{2}\ln(\mathbf{I}) = \mathbf{0} \tag{2.245}$$

The remaining question is how to determine the logarithm of a matrix. Mathematically this is defined in the following way:

A symmetric matrix $\mathbf{A}$ can be represented by the matrix $\mathbf{Q}$ of its normalised eigenvectors and the diagonal matrix

$$\boldsymbol{\Lambda} = diag[\lambda_i] \tag{2.246}$$

of its eigenvalues:

$$\mathbf{A} = \mathbf{Q}\ \boldsymbol{\Lambda}\ \mathbf{Q}^T \tag{2.247}$$

Since this covers the total spectrum of the eigenvalues this is called *spectral decomposition*.

A function of the matrix $\mathbf{A}$ is calculated by applying the function to the eigenvalues, again forming a diagonal matrix from the results and multiplying it by the eigenvectors from both sides:

$$f(\mathbf{A}) = \mathbf{Q}\ \ diag[f(\lambda_i)]\ \ \mathbf{Q}^T \tag{2.248}$$

This method is rather complicated (keep in mind that for the iteration in a FE code derivatives with respect to the displacements are necessary) but it is for example the base of the formulation of the (legacy) elements VISCO106 to 108 in ANSYS.

For the second method one has to remember the incremental form (2.239) of the logarithmic strain. The strain increment is determined by calculating engineering strain but with respect to a deformed reference configuration. The rigid body rotation can be accounted for in the same way as for small strain. If this is done by the co-rotational formulation the strain increment is called *Green-Naghdi* rate.

The definition of the logarithmic strain and true stress with their relation to the *actual* configuration requires to use the *actual* volume (or Adx for the link element) when integrating over dV. In the numerical integration this means that the Jacobian $\mathbf{J}$ and its determinant accounting for the relation between real and unit coordinates must be formed from the coordinates $\mathbf{x}_0 + \mathbf{u}$ of the (deformed) reference configuration. The limits of the unit coordinates and thus the numerical integration procedure itself are not directly affected but element distortions must be measured in deformed, not initial coordinates to judge the accuracy.

2.5.3 Hencky Strain in Terms of Continuum Mechanical Symbols

Following Eq. (2.241) the one-dimensional logarithmic strain can be written infinitesimally as

$$\varepsilon^{log} = \ln\left(\frac{dx}{dx_0}\right) = \ln(F_{11}) = \frac{1}{2}\ln\left(F_{11}^2\right) \tag{2.249}$$

With the considerations from Sect. 2.3.6 one obtains for the three-dimensional case

$$\boldsymbol{\varepsilon}^{Hencky} = \frac{1}{2}\ln\left(\mathbf{F}^T\mathbf{F}\right) = \frac{1}{2}\ln\left(\mathbf{U}^2\right) \tag{2.250}$$

2.5.4 Logarithmic Strain and Corotational Formulation

Since the co-rotational formulation accounts for large rotations only the incrementally changing reference configuration must be added according to (2.239). Strains are composed from derivatives in the rotated system, now strain increments. Changes in length must be calculated from the deformatoric displacements $\mathbf{u}_{def}$.

The basic formula for the calculation of strain in the co-rotational formulation has been

$$\boldsymbol{\varepsilon} = \mathbf{B}_{lin}\mathbf{u}_{def} = \mathbf{B}_{lin}[\mathbf{T}(\mathbf{u})(\mathbf{x}_0 + \mathbf{u}) - \mathbf{T}_0\mathbf{x}_0] \tag{2.163}$$

Herein $\mathbf{B}_{lin}$ was determined from the derivatives of the shape functions $\mathbf{N}$ with respect to the initial coordinates in the element coordinate system

$$\mathbf{x}_0^e = \mathbf{T}_0\mathbf{x}_0 \tag{2.251}$$

Instead of that the element coordinates in a deformed reference state

$$\mathbf{x}_{ref}^{e} = \mathbf{T}\left(\mathbf{u}_{ref}\right)\left(\mathbf{x}_0 + \mathbf{u}_{ref}\right) \tag{2.252}$$

are now used to calculate a strain *increment*:

$$\begin{aligned}\Delta\varepsilon &= \mathbf{B}_{lin}\left(\mathbf{x}_{ref}^{e}\right)\Delta\mathbf{u}_{def} \\ &= \mathbf{B}_{lin}\left(\mathbf{x}_{ref}^{e}\right)\left[\mathbf{T}(\mathbf{u}_{i+1})(\mathbf{x}_0 + \mathbf{u}_{i+1}) - \mathbf{T}(\mathbf{u}_i)(\mathbf{x}_0 + \mathbf{u}_i)\right]\end{aligned} \tag{2.253}$$

Herein i is the last converged solution and $i + 1$ the new one in the actual iteration. $\mathbf{u}_{ref}$ can be chosen between $\mathbf{u}_i$ and $\mathbf{u}_{i+1}$, see below.

When forming derivatives with respect to the real coordinates the element geometry is usually accounted for by the inverse of the Jacobian matrix $\mathbf{J}$. In case of the corotational formulation with small strain $\mathbf{J}$ is determined from the initial coordinates:

$$\mathbf{J}_0 = \left[\frac{\partial x_{0i}^{e}}{\partial \xi_j}\right] \tag{2.254}$$

For large strain the coordinates of a reference configuration actualised by deformatoric displacements must be used, see (2.252):

$$\mathbf{J} = \left[\frac{\partial x_{ref,i}^{e}}{\partial \xi_j}\right] \tag{2.255}$$

Analogously the related Jacobian determinant det$\mathbf{J}$ is used for the integration over the element volume. This will be done numerically with n_{GP} Gaussian points, e.g. the internal forces in 3d:

$$\int_{(V)} \mathbf{B}^T \sigma dV \approx \sum_{i=1}^{n_{GP}} w_i \mathbf{B}^T\left(\frac{\partial \mathbf{N}(\xi_i, \eta_i, \zeta_i)}{\partial \mathbf{x}}\right)\sigma(\xi_i, \eta_i, \zeta_i)\det\mathbf{J}\left(\mathbf{x}_{ref}^{e}; \xi_i, \eta_i, \zeta_i\right) \tag{2.256}$$

where w_i—means the weighting factor and ξ_i,η_i,ζ_i—the coordinates of the Gaussian point i

For higher accuracy and stability an implicit method is preferred, i.e. the reference configuration depends on the deformation at the end of the load increment which is the goal of the iteration (that makes it implicit). One choice is the midpoint rule, i.e. the reference configuration is located in the middle between the beginning (last converged solution $\mathbf{u}_i$) and the actual end of the load increment (iterative solution $\mathbf{u}_{i+1}$):

In one and two dimensions it must be made sure that the correct (deformed) volume is calculated. In general this depends on the material model used. In case of Von-Mises plasticity (s. Chap. 8) and dominant plastic strain the volume is constant, in 1d:

$$V = Al = A_0 l_0 \tag{2.257}$$

The strain increment for the link element then reads:

$$\begin{aligned}\Delta\varepsilon &= \mathbf{B}_{lin}\left(l_{Ref}\right)\Delta\mathbf{u}_{def} \\ &= \frac{1}{l_{Ref}}\begin{bmatrix}-1 & 1\end{bmatrix}\left[\mathbf{T}(\mathbf{u}_{i+1})(\mathbf{x}_0 + \mathbf{u}_{i+1}) - \mathbf{T}(\mathbf{u}_i)(\mathbf{x}_0 + \mathbf{u}_i)\right]\end{aligned} \tag{2.258}$$

$$\varepsilon_{i+1} = \varepsilon_i + \Delta\varepsilon \tag{2.259}$$

When using the midpoint rule the reference length is taken as

$$l_{Ref} = \frac{l_{i+1} + l_i}{2} \tag{2.260}$$

For comparison with the small-strain formulation it is written:

$$\mathbf{B}_{lin}\left(l_{Ref}\right) = \frac{l_0}{l_{Ref}}\mathbf{B}_{lin} = \frac{2l_0}{l_{i+1} + l_i}\mathbf{B}_{lin} = \frac{2l_0}{l(\mathbf{u}_{i+1}) + l_i}\mathbf{B}_{lin} \tag{2.261}$$

$$\varepsilon_{i+1} = \varepsilon_i + \frac{2l_0}{l(\mathbf{u}_{i+1}) + l_i}\underbrace{\mathbf{B}_{lin}\left[\mathbf{T}(\mathbf{u}_{i+1})(\mathbf{x}_0 + \mathbf{u}_{i+1}) - \mathbf{T}(\mathbf{u}_i)(\mathbf{x}_0 + \mathbf{u}_i)\right]}_{\Delta\varepsilon^{small}} \tag{2.262}$$

All terms with index i are constant during the actual load increment and not subject to differentiation. Therefore, the derivative of $\Delta\varepsilon^{small}$ with respect to the displacement vector $\mathbf{u}$ is the B-matrix for small strain (2.164). For large strain $\mathbf{B}$ then reads following the product rule:

$$\begin{aligned}\mathbf{B}^{large} &= \frac{2l_0}{l_{i+1} + l_i}\mathbf{B}^{small} - \frac{2l_0}{(l_{i+1} + l_i)^2}\frac{\partial l_{i+1}}{\partial \mathbf{u}}\Delta\varepsilon^{small} \\ &= \frac{l_0}{l_{ref}}\mathbf{B}^{small} - \frac{l_0}{2l_{ref}^2}\frac{\partial l_{i+1}}{\partial \mathbf{u}}\Delta\varepsilon^{small}\end{aligned} \tag{2.263}$$

Analogously the second derivative of $\boldsymbol{\varepsilon}$, including the derivative of $\mathbf{B}^{large}$, needed for $\mathbf{K}_\sigma$ can be formed.

In general derivatives of the Jacobian matrix are needed, all other terms are known from the small strain formulation.

2.6 Related Stress

2.6.1 General 1d-Relation to Strain

In the two spar examples with different formulations (Sects. 2.3.5 and 2.4.5) different force-displacement relations, especially different limit loads, were obtained although the behaviour was qualitatively similar. The reason was the use of the same stress–strain and stress–force relation. For the maximum load this was explained in Sect. 2.4.6.

It is unlikely that Hooke's law applies to the whole range of a deformation of such size but more general is the fact that stresses must be calculated from forces and cross sections in different ways according to the strain measure used, so-called *conjugate* stresses. Base is the relation

$$\mathbf{f}_{int} = \int_{(V)} \mathbf{B}^T \boldsymbol{\sigma} dV \tag{2.83}$$

$\mathbf{f}_{int}$ means the internal nodal forces. Since $\mathbf{B}$ is the derivative of $\boldsymbol{\varepsilon}$ with respect to the nodal displacements there is a different relation between force and stress for each distinct strain measure. In 1d from (2.83) one obtains

$$\mathbf{f}_{int} = \int_{(l)} \mathbf{B}^T \sigma A dx \tag{2.264}$$

for constant strain and stress over the element length

$$\mathbf{f}_{int} = \mathbf{B}^T \sigma A l \tag{2.265}$$

and for one degree of freedom u only (the other one be fixed)

$$f_{int} = F = \frac{d\varepsilon}{du} \sigma A l \tag{2.266}$$

This formula is solved for σ:

$$\sigma = \frac{F}{A l \frac{d\varepsilon}{du}} \tag{2.267}$$

where for small strain the undeformed area A_0 and the undeformed length l_0 must be used.

The one dimensional consideration is important in particular because it is often the base for the determination of material parameters from experiments.

2.6.2 Engineering Quantities

Companion to *engineering strain* is *engineering stress*, in one dimension:

$$\sigma^{eng} = \frac{F}{A_0} \tag{2.268}$$

This definition is common. Nevertheless Eq. (2.267) is applied to test its validity. The change in length here is u, the strain and its derivative consequently

$$\varepsilon = \frac{u}{l_0} \quad \Rightarrow \quad \frac{d\varepsilon}{du} = \frac{1}{l_0}, \tag{2.269}$$

thus

$$\sigma = \frac{F}{A_0 l_0 \frac{1}{l_0}} = \frac{F}{A_0} \tag{2.270}$$

The engineering stress is also valid in the basic form of the co-rotational formulation, naturally in the rotated coordinate system, where the deformatoric displacement is determined and hence the engineering strain. As a test the results of the two-legged truss in Sect. 2.4.5 are taken. For the last converged state one obtained $\mathbf{f}_{int} = -0.71$, $\sin\alpha = 0.3919$ and $\sigma = -18.12$. The force in the truss F can be calculated from the internal force in the global system as

$$F = \frac{\mathbf{f}_{int}}{\sin\alpha} \quad \Rightarrow \quad \sigma = \frac{F}{A_0} = \frac{\mathbf{f}_{int}}{\sin\alpha A_0} \frac{1}{} = \frac{-0.71}{0.3919 \cdot 0.1} = -18.12 \tag{2.271}$$

2.6.3 Green-Lagrange Strain

Companion to the *Green-Lagrange strains* are the *second Piola-Kirchhoff stresses*. In one dimension the strain after (2.63) in conjunction with (2.65) and its derivative read:

$$\varepsilon^{GL} = \frac{1}{2}\frac{l^2 - l_0^2}{l_0^2} = \frac{1}{2}\frac{(l_0 + u)^2 - l_0^2}{l_0^2} \quad \Rightarrow \quad \frac{d\varepsilon}{du} = \frac{1}{2}\frac{2(l_0 + u)}{l_0^2} = \frac{l}{l_0^2} \tag{2.272}$$

Then the related stress after (2.267) is

$$\sigma^{PK} = \frac{F}{A_0 l_0 \frac{l}{l_0^2}} = \frac{F\, l_0}{A_0\, l} = \sigma^{eng} \frac{l_0}{l} \tag{2.273}$$

or, in order to include the engineering *strain*, too:

$$\sigma^{PK} = \sigma^{Ing} \frac{1}{\frac{l_0+u}{l_0}} = \sigma^{Ing} \frac{1}{1 + \varepsilon^{Ing}} \tag{2.274}$$

Its physical interpretation is limited in the same degree as the meaning of this strain which is suitable for large rotations—nothing else.

Test: For the two-legged truss the deformed length in the last converged state has been calculated as $l = 4.3478$ at the end of Sect. 2.3.6. The initial length has been $l_0 = 5$. By solving (2.107) for the stress $\sigma = -20.83$ has been calculated for this deformation and the context of Green-Lagrange strain whereas engineering stress had been $\sigma^{eng} = -18.12$. By applying (2.273) one now obtains:

$$\sigma^{PK} = -18.12 \frac{5}{4.3478} = -20.83 \tag{2.275}$$

If the 1d-relation between Green-Lagrange and engineering strain (2.73) is solved for ε^{GL} via the mixed quadratic equation

$$-2\varepsilon^{GL} + 2\varepsilon^{eng} + \varepsilon^{eng^2} = 0 \tag{2.276}$$

one obtains:

$$\varepsilon^{eng}_{1/2} = -1 \pm \sqrt{1 + 2\varepsilon^{GL}} \tag{2.277}$$

Since the engineering strain is limited to $\varepsilon^{eng} > -1$ the Green-Lagrange strain has as lower bound

$$\varepsilon^{GL} > -\frac{1}{2} \tag{2.278}$$

Thus the square root is always real. From the same condition follows that only the positive sign is meaningful:

$$\varepsilon^{eng} = -1 + \sqrt{1 + 2\varepsilon^{GL}} \tag{2.279}$$

Replacing the engineering stress in (2.274) by Young's modulus times engineering strain according to Hooke's law:

$$\sigma^{PK} = E \frac{\varepsilon^{eng}}{1 + \varepsilon^{eng}} \tag{2.280}$$

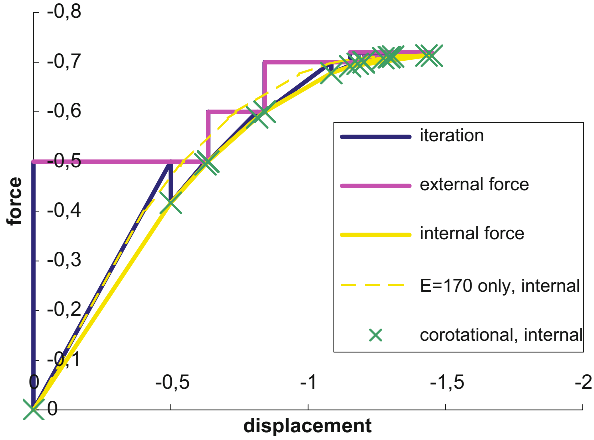

Fig. 2.22 Course of the iteration when using the modified Young's modulus (2.223) (*dashed line*) resp. the correct relation between GL-strain and displacement as well as second Piola-Kirchhoff stress and force from Sect. 2.6.3

Inserting (2.279) for the engineering strain:

$$\sigma^{PK} = E\frac{-1+\sqrt{1+2\varepsilon^{GL}}}{\sqrt{1+2\varepsilon^{GL}}} \tag{2.281}$$

This is a non-linear material law but leading to a linear force-displacement relation. After Sect. 2.3.4.1 the derivative of the stress with respect to the strain is needed for the tangential matrix:

$$\frac{d\sigma^{PK}}{d\varepsilon^{GL}} = \frac{E\frac{\sqrt{1+2\varepsilon^{GL}}}{\sqrt{1+2\varepsilon^{GL}}} - \frac{-1+\sqrt{1+2\varepsilon^{GL}}}{\sqrt{1+2\varepsilon^{GL}}}}{1+2\varepsilon^{GL}} = \frac{E}{(1+2\varepsilon^{GL})^{\frac{3}{2}}} \tag{2.282}$$

In this way Young's modulus in the example of the two-legged truss can remain at $E = 138.889$ leading to the solution in Fig. 2.22. Like in the co-rotational formulation the limit load is calculated as 0.71 at a tip displacement of 1.296. Now even the curves of the internal force as well as the converged solutions match those from the co-rotational example.

2.6.4 *Logarithmic Strain*

The *logarithmic* strain is used as a measure for large deformations. Thus the volume of the *deformed* body, in (2.267) the deformed cross section area A and the deformed length l must be considered. The derivative of the strain is

$$\frac{d\varepsilon^{log}}{du} = \frac{d}{du}\ln\left(\frac{l}{l_0}\right) = \frac{d}{du}\ln\left(\frac{1}{l_0}(l_0+u)\right) = \frac{l_0}{l_0+u}\,\frac{1}{l_0} = \frac{1}{l_0+u} = \frac{1}{l} \tag{2.283}$$

By introducing in (2.267) one obtains

$$\sigma = \frac{F}{\frac{1}{l}Al} = \frac{F}{A} \tag{2.284}$$

Thus the appropriate stress measure for *logarithmic* strain where the change in length is related to the *deformed* length is the so-called *"true" stress*, the force divided by the *deformed* area, in 1d:

$$\sigma^{true} = \frac{F}{A} \tag{2.285}$$

These stresses are also called—especially in two or three dimensions—*Cauchy* stress.

A uniaxial stress state usually produces a triaxial strain state. From this fact the deformed cross section area can be calculated. Poisson's ratio ν in Hooke's law yields for a uniaxial stress state:

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x \tag{2.286}$$

(2.241) gives for the loading direction

$$\varepsilon_x = \ln\left(1 + \frac{\partial u_x}{\partial x}\right) \tag{2.287}$$

Analogously this relation delivers for the transverse directions:

$$\varepsilon_y = \ln\left(1 + \frac{\partial u_y}{\partial y}\right) \tag{2.288}$$

Inserted into (2.286) this means:

$$\ln\left(1 + \frac{\partial u_y}{\partial y}\right) = -\nu\ln\left(1 + \frac{\partial u_x}{\partial x}\right) = \ln\left[\left(1 + \frac{\partial u_x}{\partial x}\right)^{-\nu}\right] \tag{2.289}$$

Applying the exponential function to both sides:

$$\left(1 + \frac{\partial u_y}{\partial y}\right) = \left(1 + \frac{\partial u_x}{\partial x}\right)^{-\nu} \tag{2.290}$$

This intermediate result leads to the following effect:

If a cube of edge length l is stretched by l for $\nu = 0.3$ one obtains as change in length in transverse direction:

Table 2.4 Comparison of strain and stress measures

Point	ε^{eng}	σ^{eng}	ε^{log}	σ^{Cauchy}
1	0.00168	348	0.00167859	348.58464
2	0.0386	348	0.03787365	361.4328
3	0.04	371	0.03922071	385.84
4	0.072	428	0.06952606	458.816
5	0.101	455	0.09621886	500.955
6	0.143	467	0.13365638	533.781
7	0.192	471	0.17563257	561.432
8	0.272	463	0.24059046	588.936

$$1+\frac{\Delta l_y}{l} = \left(1+\frac{l}{l}\right)^{-0.3} \frac{\Delta l_y}{l} = \left(1+\frac{l}{l}\right)^{-0.3} - 1 \Delta l_y = \left[\left(1+\frac{l}{l}\right)^{-0.3} - 1\right] l$$
$$= -0.1877 l \tag{2.291}$$

whereas for engineering strain the result would be $-0.3\,l$.

More important, however, is that the cross section area of the deformed system is

$$A = A_0\left(1+\frac{\partial u_y}{\partial y}\right)\left(1+\frac{\partial u_z}{\partial z}\right) = A_0\left(1+\frac{\partial u_x}{\partial x}\right)^{-2\nu} \tag{2.292}$$

that means

$$\sigma^{Cauchy} = \frac{F}{A_0} \frac{1}{\left(1+\frac{\partial u_x}{\partial x}\right)^{-2\nu}} = \frac{F}{A_0}\left(1+\frac{\partial u_x}{\partial x}\right)^{2\nu} \tag{2.293}$$

Hooke's law does not hold for strains in a range where a significant difference between the strain measures can be noticed. More important is e.g. plasticity of metals where it is assumed that the plastic strain

- dominates the elastic one and
- is incompressible, i.e. no volume change occurs.

This is equivalent to a Poisson's ratio of 0.5, thus

$$\sigma^{Cauchy} = \frac{F}{A_0}\left(1+\frac{\partial u_x}{\partial x}\right)^{2\cdot 0,5} = \frac{F}{A_0}\left(1+\frac{\partial u_x}{\partial x}\right) \tag{2.294}$$

$$\sigma^{Cauchy} = \sigma^{eng}\left(1+\varepsilon_x^{eng}\right) \tag{2.295}$$

If a FE-program uses large strain the measured yield curves (usually engineering measures) must be transformed into true stress vs. logarithmic strain via (2.242) and (2.295).

Table 2.4 and Fig. 2.23 show stress–strain data for a certain type of steel. One can see that the stresses differ more than the strain. Furthermore, the Cauchy stress

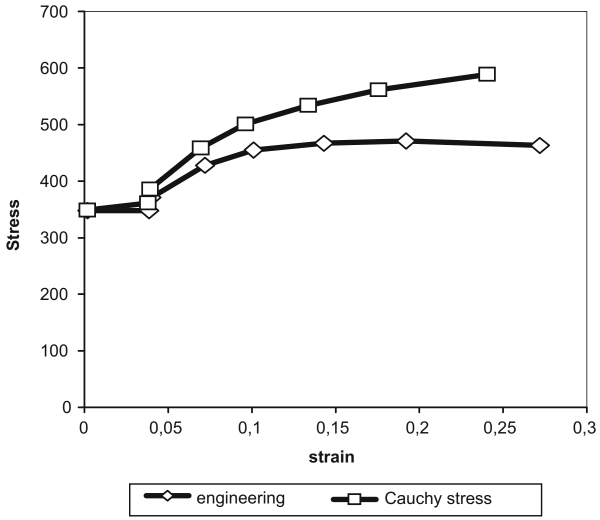

Fig. 2.23 Comparison of strain and stress measures

shows hardening where the engineering stress indicates softening which physically is not the case: The decrease in stress is caused by a reduction of the cross section area.

2.6.5 Continuum Mechanics Aspect

Equation (2.295) can also be written as

$$\sigma^{true} = \sigma^{eng}\left(1 + \frac{l - l_0}{l_0}\right) = \sigma^{eng}\frac{l}{l_0} \tag{2.296}$$

whereas (2.273) can be solved for

$$\sigma^{eng} = \frac{l}{l_0}\sigma^{PK} \tag{2.297}$$

thus

$$\sigma^{true} = \frac{l}{l_0}\sigma^{PK}\frac{l}{l_0} \tag{2.298}$$

In Sect. 2.3.6 l/l_0 was identified as the 1d representation of the stretch tensor $\mathbf{U}$. Thus the 3d extension is

$$\boldsymbol{\sigma}^{true} = \mathbf{U}\boldsymbol{\sigma}^{PK}\mathbf{U}^{(T)} \tag{2.299}$$

but this hold for the measure in the initial coordinate system due to the nature of the Piola-Kirchhoff stress and of **U**. For the actual configuration a rotation is necessary:

$$\boldsymbol{\sigma} = \underbrace{\mathbf{R}\mathbf{U}}_{\mathbf{F}}\,\boldsymbol{\sigma}^{PK}\,\underbrace{\mathbf{U}^{(T)}\mathbf{R}^{T}}_{\mathbf{F}^{T}} \tag{2.300}$$

$$\boldsymbol{\sigma} = \mathbf{F}\boldsymbol{\sigma}^{PK}\mathbf{F}^{T} \tag{2.301}$$

This is called push-forward operation. The result, however, is called *Kirchhoff* stress tensor.

2.7 Updated-Lagrange Formulation

2.7.1 Classic Approach

Lagrange formulation—in contrast to Euler's approach dominating fluid dynamics—means that the motion of a material point is observed. If the kinematics of a system is totally described in terms of the initial configuration this method is called Total-Lagrange formulation.

A simple but less accurate way to account for large rotations and—more or less as a side effect—for large strain is the following:

- perform a geometrically linear analysis for a load increment evolving small rotations only
- add the displacements to the initial coordinates to get new coordinates
- add a new load increment
- sum up the strain and stress increments.

In terms of "time"-integration this is an explicit method which can show a larger error and even numerical instability when the increment is chosen too large.

Example
The stiffness matrix of a linear link element rotated by an angle of α reads (with the abbreviations below)

c: cos α and
s: sin α:

$$\mathbf{K} = \mathbf{T}^{T}\mathbf{K}^{elem}\mathbf{T} = \frac{EA}{l}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \tag{2.302}$$

Be $\mathbf{u} = \mathbf{0}$ and $\varepsilon = 0$ initial values of displacement and strain.

In the first load increment the displacement in the global system can be calculated by solving

$$\mathbf{K}\Delta\mathbf{u} = \mathbf{f}_{ext} \tag{2.303}$$

The displacement is *updated*:

$$\mathbf{u} \leftarrow \mathbf{u} + \Delta\mathbf{u} \tag{2.304}$$

Now a new transformation matrix can be determined:

$$\mathbf{T}_1 = \mathbf{T}(\mathbf{x}_0 + \mathbf{u}) \tag{2.305}$$

The displacement increment in the element coordinate system reads:

$$\Delta\mathbf{u}_e = \mathbf{T}_1\Delta\mathbf{u} \tag{2.306}$$

The strain can then be *updated* to

$$\varepsilon \leftarrow \varepsilon + \mathbf{B}_{lin}\Delta\mathbf{u}_e \tag{2.307}$$

This strain leads to the stress

$$\sigma = E\varepsilon \tag{2.308}$$

Thus the internal forces read:

$$\mathbf{f}_{int} = \mathbf{T}_1^T\mathbf{B}_{lin}^T\sigma V = \begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix}\frac{1}{l}[\,-1 \quad 1\,]\,\sigma V \tag{2.309}$$

Under the assumption of constant volume in large strain one obtains:

$$\mathbf{f}_{int} = \begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix}\frac{1}{l}[\,-1 \quad 1\,]\,\sigma A_0 l_0 = \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix}\sigma A_0\frac{l_0}{l} \tag{2.310}$$

In this position a new load increment is applied leading to a new external force $\mathbf{f}_{ext}$. Now the displacement increment is determined by solving

$$\mathbf{K}\Delta\mathbf{u} = \mathbf{f}_{ext} - \mathbf{f}_{int} \tag{2.311}$$

and the procedure starts again with Eq. (2.304).

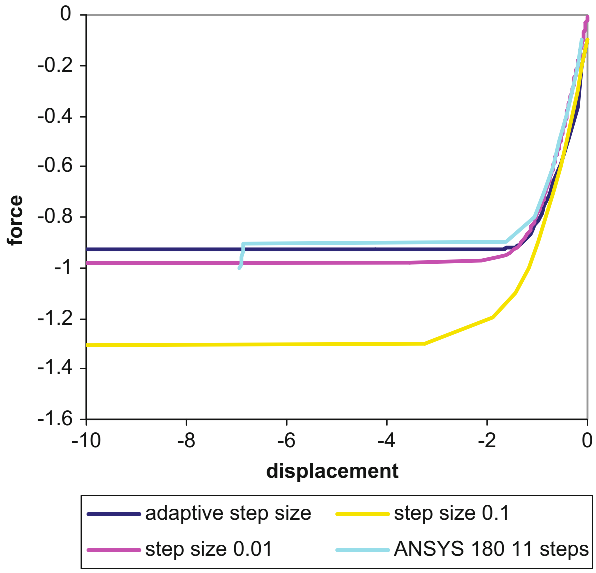

Fig. 2.24 Behaviour of the classic updated-Lagrange approach

What is needed for the two-legged truss is listed in the chapter above. Some values are repeated here:

$$l = \sqrt{4^2 + (3 + v_2)^2} \tag{2.312}$$

$$c = \cos\alpha = \frac{4}{l}\,, \quad s = \sin\alpha = \frac{3 + v_2}{l} \tag{2.313}$$

With these values **K** can be formed. Only

$$k_{44} = \frac{EA}{l}s^2 \tag{2.314}$$

is needed for this example. The displacement on element level has one component only:

$$\Delta u_{e2} = s\Delta v_2 \tag{2.315}$$

$$\varepsilon + \Delta\varepsilon = \varepsilon + \frac{\Delta u_{e2}}{l} \tag{2.316}$$

$$\sigma = E\varepsilon \tag{2.317}$$

$$\mathbf{f}_{int} = s\sigma A_0 \frac{l_0}{l} \tag{2.318}$$

The results, especially the maximum load, strongly depend on the step size as shown in Fig. 2.24. The behaviour is compared with the ANSYS LINK180 element with co-rotational formulation for large strain.

It can be seen that too large a step size leads to large errors in the result when the behaviour of the system becomes strongly non-linear. The non-linearity can be measured in terms of internal and external forces because the internal forces in the updated configuration i + 1 do not match exactly the external forces from the configuration before (i). A certain difference is remaining enlarging the right hand side of (2.241). Therefore, for the curve marked as "adaptive step size" the increments of the external forces are chosen so that the error is restricted to a certain fraction of the external load:

$$\mathbf{f}_i^{ext} - \mathbf{f}_{i+1}^{int} = c\,\mathbf{f}_i^{ext} \tag{2.319}$$

If this is not the case the last load increment is scaled to get the next result nearly in the desired range:

$$\Delta\mathbf{f}_{i+1}^{ext} = \Delta\mathbf{f}_i^{ext} \frac{c\,\mathbf{f}_i^{ext}}{\mathbf{f}_i^{ext} - \mathbf{f}_{i+1}^{int}} \tag{2.320}$$

When choosing $c = 0.01$ the result shown in Fig. 2.24 is obtained with significantly less increments then with step size 0.01 but with higher accuracy.

2.7.2 Generalisation

Nowadays the term "updated Lagrange" is used for nearly every incremental method, nearly everything which is not formulated based on the initial configuration. Such methods can be of high accuracy. Co-rotational with large strain is of this type because the strain is updated.

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