Chapter 2
Similitude Theory and Applications

Wikipedia definition of similitude is a concept that is used in the testing of engineering models. A model is said to have similitude with the real application if the two share geometric similarity, kinematic similarity and dynamic similarity. Similarity and similitude are interchangeable in this context. The term dynamic similitude is often used as a catchall because it implies that geometric and kinematic similitude has already been met. Similitude's main application is in hydraulic and aerospace engineering to test fluid flow conditions with scaled models. It is also the primary theory behind many textbook formulas in fluid mechanics [1, 2, 3].

2.1 Introduction

To describe similitude we continue using what Wikipedia is considering as an overview of it. Engineering models are used to study complex fluid dynamics problems where calculations and computer simulations are not reliable. Models are usually smaller than the final design, but not always. Scale models allow testing of a design prior to building, and in many cases are a critical step in the development process. Over all the conditions that a model of a phenomenon reproduces all aspects of behavior of the prototypes represented by it are known as conditions of *similitude*.

A primary goal of any experiment is to provide the result as part of prototype and final build of any application result. To achieve that end, the concept of *similitude* is often used so that measurements made one system in the laboratory environment can be used to describe the behavior of other similar system in real world and outside of the laboratory. The laboratory built systems are often thought as model while the first build of the similar systems based on behavior its model, beyond laboratory frame called prototype.

Construction of a scale model, however, must be accompanied by an analysis to determine what conditions it is tested under. While the geometry may be simply scaled, other parameters, such as pressure, temperature or the velocity and type of fluid may need to be altered. Similitude is achieved when testing conditions are created such that the test results are applicable to the real design.

B. Zohuri, *Dimensional Analysis and Self-Similarity Methods for Engineers and Scientists,* DOI 10.1007/978-3-319-13476-5_2

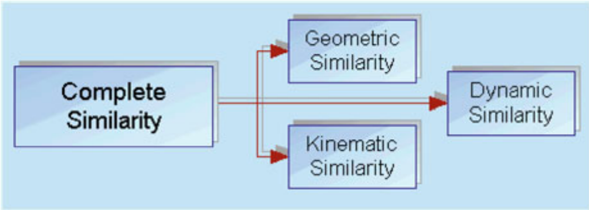


Fig. 2.1 Illustration of complete similarity

Two systems, described by the same physics, operating under different sets of conditions are said to be physically similar in respect of certain specified physical quantities; when the ratio of corresponding magnitudes of these quantities between the two systems is the same everywhere.

In the field of mechanics and hydraulic models as in sand box model, there are three concepts of types of similarities, which constitute the complete similarity between problems of same kind. We recognize these three concepts as *geometric similarity*, *kinematics similarity* and third one is known as *dynamic similarity*. These similarities are depicted in Fig. 2.1, while described in more details as well. There are other types of similarity as well, all which must be satisfied in order to have a *complete similarity* totally to exist between the flow phenomena in the two systems of fluids for example. However, for purpose of our arguments Fig. 2.1 seems to be sufficient.

The following criteria are required to achieve similitude and represents types of physical similarity;

- **Geometric similarity**—The model is the same shape as the application, usually scaled. In other words, if the specified physical quantities are geometrical dimensions, the similarity is called *Geometric Similarity*.
- **Kinematic similarity**—Fluid flow of both the model and real application must undergo similar time rates of change motions (i.e. fluid streamlines are similar). In other words, If the quantities are related to motions, the similarity is called *Kinematic Similarity*.
- **Dynamic similarity**—Ratios of all forces acting on corresponding fluid particles and boundary surfaces in the two systems are constant. In other words, if the quantities refer to forces, then the similarity is termed as *Dynamic Similarity*.

2.1.1 Geometric Similarity

Basically we can claim the *Geometric Similitude* will exit between *Model* (*m*) and *Prototype* (*p*) if the ratios of all corresponding dimensions in both model and prototype are equal and, mathematically can be presented as follows;

$$\frac{L_{\text{mod}el}}{L_{prototype}} = L_{ratio} \quad \text{or} \quad \frac{L_{\text{m}}}{L_p} = L_r \tag{2.1}$$

and

$$\frac{A_{\text{mod}el}}{A_{prototype}} = \frac{L^2{}_{\text{mod}el}}{L^2{}_{prototype}} = L^2{}_{ratio} = L_r^2 \tag{2.2}$$

All these can be summarized into following two statements

- Geometric Similarity implies the similarity of shape such that, the ratio of any length in one system to the corresponding length in other system is the same everywhere.
- This ratio is usually known as scale factor.

Therefore, geometrically similar objects are similar in their shapes, i.e., proportionate in their physical dimensions, but differ in size.

In investigations of physical similarity,

- The full size or actual scale systems are known as prototypes
- The laboratory scale systems are referred to as models
- Use of the same fluid with both the prototype and the model is not necessary
- Model need not be necessarily smaller than the prototype. The flow of fluid through an injection nozzle or a carburetor, for example, would be more easily studied by using a model much larger than the prototype.
- The model and prototype may be of identical size, although the two may then differ concerning other factors such as velocity, and properties of the fluid.

If l_1 and l_2 are the two characteristic physical dimensions of any object, then the requirement of geometrical similarity is

$$\text{Model Ratio} = \frac{l_{1_m}}{l_{1_p}} = \frac{l_{2_m}}{l_{2_p}} = l_r \tag{2.3}$$

(The second suffices m and p refer to model and prototype respectively) where l_r is the scale factor or sometimes known as the model ratio. Figure 2.2 shows three pairs of geometrically similar objects, namely, a right circular cylinder, a parallelopiped, and a triangular prism.

Geometric similarity is perhaps the most obvious requirement in a model system designed to correspond to a given prototype system.

A perfect geometric similarity is not always easy to attain. Problems in achieving perfect geometric similarity are:

- For a small model, the surface roughness might not be reduced according to the scale factor (unless the model surfaces can be made very much smoother than those of the prototype). If for any reason, the scale factor is not the same throughout, a distorted model results.
- Sometimes it may so happen that to have a perfect geometric similarity within the available laboratory space, physics of the problem changes. For example, in case of large prototypes, such as rivers, the size of the model is limited by the available floor space of the laboratory; but if a very low scale factor is used in

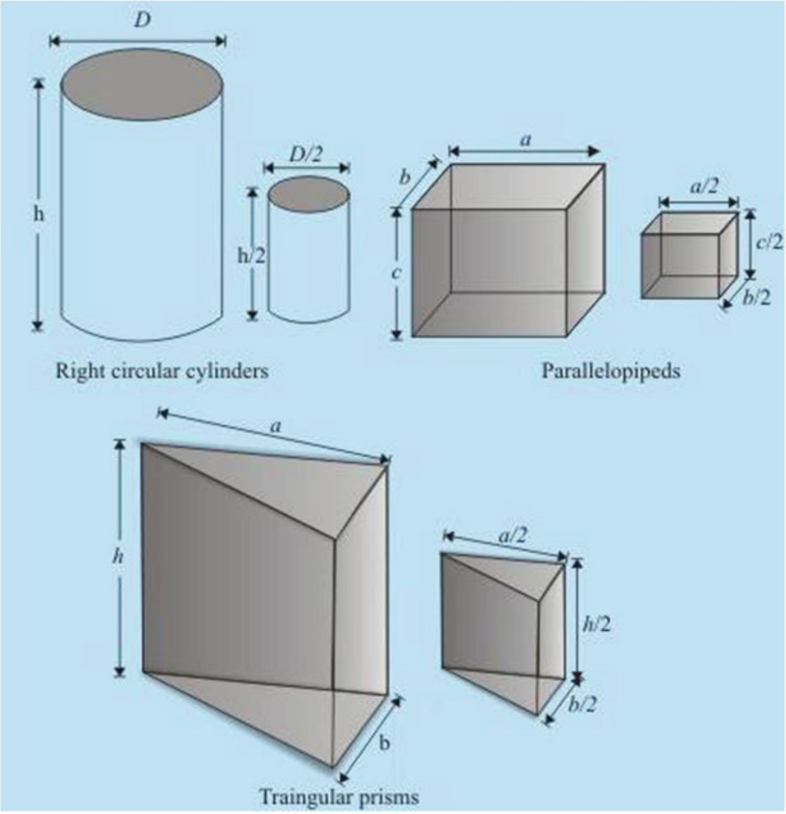


Fig. 2.2 Geometrically similar objects. (In the entire above cases, model ratio is ½)

reducing both the horizontal and vertical lengths. This may result in a stream so shallow that surface tension has a considerable effect and further more; the flow may be laminar instead of turbulent. In this situation, a distorted model may be unavoidable (*a lower scale factor for horizontal lengths while a relatively higher scale factor for vertical lengths*). The extent to which perfect geometric similarity should be sought therefore depends on the problem being investigated, and the accuracy required from the solution.

2.1.2 Kinematic Similarity

Kinematic similarity refers to similarity of motion.

Since motions are described by distance and time, it implies similarity of lengths (i.e., geometrical similarity) and, in addition, similarity of time intervals.

If the corresponding lengths in the two systems are in a fixed ratio, the velocities of corresponding particles must be in a fixed ratio of magnitude of corresponding time intervals.

If the ratio of corresponding lengths, known as the scale factor, is l_r given Eq. 2.1, and the ratio of corresponding time intervals is t_r, then the magnitudes of corresponding velocities are in the ratio l_r/t_r and the magnitudes of corresponding accelerations are in the ratio l_r/t_r^2.

A well-known example of kinematic similarity is found in a planetarium. Here the galaxies of stars and planets in space are reproduced in accordance with a certain length scale and in simulating the motions of the planets, a fixed ratio of time intervals (and hence velocities and accelerations) is used.

When fluid motions are kinematically similar, the patterns formed by streamlines are geometrically similar at corresponding times.

Since the impermeable boundaries also represent streamlines, kinematically similar flows are possible only past geometrically similar boundaries.

Therefore, geometric similarity is a necessary condition for the kinematic similarity to be achieved, but not the sufficient one.

For example, geometrically similar boundaries may ensure geometrically similar streamlines in the near vicinity of the boundary but not at a distance from the boundary. In summary such similarity will exist if

1. The paths of homologous moving particles are geometrically similar.
2. The ratios of the velocities of homologous particles are equal.

We can define few useful ratios between model and prototype, using the relations in Eq. 2.1 for Kinematic Similarity as well;

$$\text{Velocity:} \quad \frac{V_m}{V_p} = \frac{L_m/T_m}{L_p/T_p} = \frac{L_m}{L_p} \div \frac{T_m}{T_p} = \frac{L_r}{T_r} \tag{2.4a}$$

$$\text{Acceleration:} \quad \frac{a_m}{a_p} = \frac{L_m/T_m^2}{L_p/T_p^2} = \frac{L_m}{L_p} \div \frac{T_m^2}{T_p^2} = \frac{L_r}{T_r^2} \tag{2.4b}$$

$$\text{Discharge:} \quad \frac{Q_m}{Q_p} = \frac{L_m^3/T_m}{L_p^3/T_p} = \frac{L_m^3}{L_p^3} \div \frac{T_m}{T_p} = \frac{L_r^3}{T_r} \tag{2.4c}$$

Note that: it is impossible to maintain kinematic similarity in a distorted model.

2.1.3 Dynamic Similarity

Dynamic similarity is the similarity of forces.

In dynamically similar systems, the *magnitudes of forces* at correspondingly similar points in each system are in a fixed ratio. Later on in this chapter, we will provide more details about magnitude of forces. In this regime forces at homologous points and times acting on homologous elements of fluid mass must be in the same ratio through the two systems. In addition, we therefore require geometric and kinematic similarity. Here forces are those of pressure, gravity, friction or viscosity, elasticity and surface tension. In addition, it should be known that, the physical properties involved are density, viscosity, elasticity, etc. As an example, expressing the force due to inertia by $f_i = \rho V^2 l^2$ and that due to viscosity by $f_\upsilon = \mu V l$, and requiring that their ratio remains constant at all homologous points of model and the prototype, leads to:

$$\frac{(f_i)_{\text{model}}}{(f_i)_{\text{prototype}}} = \frac{(f_\upsilon)_{\text{model}}}{(f_\upsilon)_{\text{prototype}}} \tag{2.5a}$$

or

$$\begin{cases} \left(\dfrac{f_i}{f_\upsilon}\right)_{\text{model}} = \left(\dfrac{f_i}{f_\upsilon}\right)_{\text{prototype}} = \left(\dfrac{Vl\rho}{\mu}\right)_{\text{model}} = \left(\dfrac{Vl\rho}{\mu}\right)_{\text{prototype}} \\ (\text{Re})_{\text{model}} = (\text{Re})_{\text{prototype}} \end{cases} \tag{2.5b}$$

where $\text{Re} = Vl/v$ is defined as the *Reynolds number,* V being a characteristic velocity, l a characteristic length and $v = \mu/\rho$ is the *kinematic viscosity* (m^2/s), μ is the *dynamic viscosity* of the fluid (Pas or N s/m^2 or kg/(m s)) and ρ is the *density* of the fluid (kg/m$\geq$).

To satisfy all three above conditions the application of each case is analyzed;

1. All parameters required to describe the system are identified using principles from continuum mechanics.
2. Dimensional analysis is used to express the system with as few independent variables and as many dimensionless parameters as possible.
3. The values of the dimensionless parameters are held to be the same for both the scale model and application. This can be done because they are dimensionless and will ensure dynamic similitude between the model and the application. The resulting equations are used to derive scaling laws, which dictate model-testing conditions.

It is often impossible to achieve strict similitude during a model test. The greater the departure from the application's operating conditions, the more difficult achieving similitude is. In these cases, some aspects of similitude may be neglected, focusing on only the most important parameters (Fig. 2.3).

Similitude is a term used widely in fracture mechanics relating to the strain life approach. Under given loading conditions the fatigue damage in an un-notched specimen is comparable to that of a notched specimen. Similitude suggests that the component fatigue life of the two objects will also be similar. In summary we have;

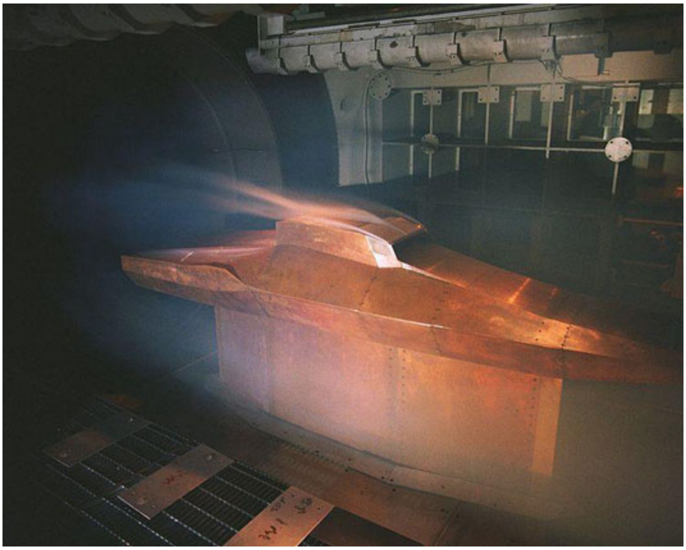


Fig. 2.3 A full-scale X-43 wind tunnel test. The test is designed to have dynamic similitude with the real application to ensure valid results

Similitude

Similitude is similarity of behavior for different systems with equal similarity parameters

Prototype	Model
(Real world)	(Physical/Analytical/Numerical. . ..Experiments)

For similitude we require that the similarity parameters (SPs) (e.g. angles, length ratios, velocity ratios, etc) are equal for the model and the real world.

Similitude	Similarity parameters (SPs)
Geometric similitude	Length ratios, angles
Kinematic similitude	Displacement ratios, velocity ratios
Dynamic similitude	Force ratios, stress ratios, pressure ratios
$\vdots$	
Internal constitution similitude	ρ, υ
Boundary conditions similitude	

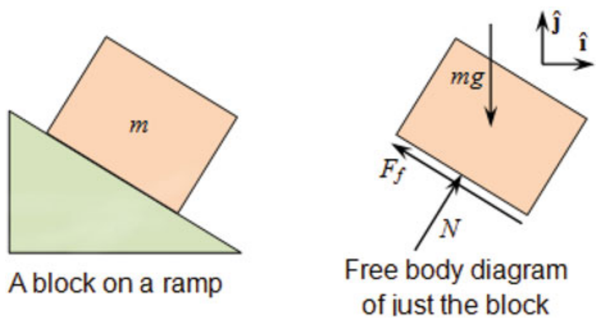
Example 2.1 Two similar triangles have equal angles or equal length ratios. In this case, the two triangles have *geometric similitude*.

Example 2.2 For the flow around a model ship to be similar to the flow around the prototype ship, both model and prototype need to have equal angles and equal length and **force ratios**. In this case, the model and the prototype have geometric and *dynamic similitude*.

As we said, the laboratory systems are usually thought of as *models* and are used to study the phenomenon of interest under carefully controlled conditions. From these model studies, empirical formulations can be developed, or specific predictions of one or more characteristics of some other similar system can be made. In order to achieve such goal, it is necessary to establish the relationship between the laboratory model and the *other* system. We present the following example from Wikipedia that shows such process.

Example 2.3 Consider a submarine modeled at 1/40th scale. The application operates in sea water at 0.5 °C, moving at 5 m/s. The model will be tested in fresh water at 20 °C. Find the power required for the submarine to operate at the stated speed. A free body diagram is constructed and the relevant relationships of force and velocity are formulated using techniques from continuum mechanics.

Free Body Diagram
A free body diagram, also called a force diagram, is a pictorial representation often used by physicists and engineers to analyze the forces acting on a body of interest. A free body diagram shows all forces of all types acting on this body. Drawing such, a diagram can aid in solving for the unknown forces or the equations of motion of the body. Creating a free body diagram can make it easier to understand the forces, and torques or moments, in relation to one another and suggest the proper concepts to apply in order to find the solution to a problem. The diagrams are also used as a conceptual device to help identify the internal forces—for example, shear forces and bending moments in beams—which are developed within structures



Block on a ramp (left) and corresponding free body diagram of just the block (right).

The variables, which describe the system, are:

Variable	Application	Scaled mode	Units
L (diameter of submarine)	1	1/40	(m)
V (speed)	5	calculate	(m/s)
ρ (density)	1028	998	(kg/m^3)
μ (dynamic viscosity)	1.88×10^{-3}	1.00×10^{-3}	Pa · s (N s/m^2)
F (force)	Calculate	To be measured	N (kg m/s^2)

Solution This example has five independent variables and three fundamental units. The fundamental units are: meter, kilogram, second.

Invoking the *Buckingham* π *theorem* shows that the system can be described with two dimensionless numbers and one independent variable.

Dimensional analysis is used to re-arrange the units to form the *Reynolds number* (Re) and pressure coefficient (C_P). These dimensionless numbers account for all the variables listed above except F, which will be the test measurement. Since the dimensionless parameters will stay constant for both the test and the real application, they will be used to formulate scaling laws for the test.

2.1.4 Scaling Laws

$$\begin{aligned} \mathrm{Re} &= \left(\frac{\rho V L}{\mu}\right) & V_{model} &= V_{application} \times \left(\frac{\rho_a}{\rho_m}\right) \times \left(\frac{\mu_m}{\mu_a}\right) \\ C_p &= \left(\frac{2\Delta P}{\rho V^2}\right), F = \Delta P L^2 & F_{application} &= F_{model} \times \left(\frac{\rho_a}{\rho_m}\right) \times \left(\frac{V_a}{V_m}\right)^2 \times \left(\frac{L_a}{L_m}\right)^2 \end{aligned} \tag{2.6}$$

This gives a required test velocity of:

$$V_{model} = V_{application} \times 2.19$$

The force measured from the model at that velocity is then scaled to find the force that can be expected for the real application:

$$F_{application} = F_{model} \times 21.9$$

The power P in watts required by the submarine is then:

$$P[W] = F_{application} \times V_{application} = F_{model}[\mathrm{N}] \times 17.2\, m/s$$

Note that even though the model is scaled smaller, the water velocity needs to be increased for testing. This remarkable result shows how similitude in nature is often counterintuitive.

2.1.5 Magnitudes of Forces

In a system, involving flow of fluid, different forces due to different causes may act on a fluid element. These forces are as follows:

Viscous Force (due to viscosity)	$\overrightarrow{F}_{\upsilon}$
Pressure Force (due to different in pressure)	$\overrightarrow{F}_P$
Gravity Force (due to gravitational attraction)	$\overrightarrow{F}_g$
Capillary Force (due to surface tension)	$\overrightarrow{F}_c$
Compressibility Force (due to elasticity)	$\overrightarrow{F}_e$

According to Newton's law, the resultant $\overrightarrow{F}_R$ of all these forces, will cause the acceleration of a fluid element. Hence $\overrightarrow{F}_R$ can be written as form of Eq. 2.7 below;

$$\overrightarrow{F}_R = \overrightarrow{F}_{\upsilon} + \overrightarrow{F}_{p} + \overrightarrow{F}_g + \overrightarrow{F}_c + \overrightarrow{F}_e \tag{2.7}$$

Moreover, the Inertia Force $\overrightarrow{F}_i$ is defined as equal and opposite to the resultant Accelerating Force $\overrightarrow{F}_R$, so we can write

$$\overrightarrow{F}_i = -\overrightarrow{F}_R \tag{2.8}$$

Therefore, Eq. 2.10 can be expressed as;

$$\overrightarrow{F}_{\upsilon} + \overrightarrow{F}_{p} + \overrightarrow{F}_g + \overrightarrow{F}_c + \overrightarrow{F}_e + \overrightarrow{F}_i = 0 \tag{2.9}$$

For dynamic similarity, the magnitude ratios of these forces have to be same for both the prototype and the model. The Inertia Force $\overrightarrow{F}_i$ is usually taken as the common one to describe the ratios as (or putting in other form we equate the nondimensionalised forces in the two systems)

$$\frac{\left|\overrightarrow{F}_{\upsilon}\right|}{\left|\overrightarrow{F}_i\right|}, \frac{\left|\overrightarrow{F}_P\right|}{\left|\overrightarrow{F}_i\right|}, \frac{\left|\overrightarrow{F}_g\right|}{\left|\overrightarrow{F}_i\right|}, \frac{\left|\overrightarrow{F}_c\right|}{\left|\overrightarrow{F}_i\right|}, \frac{\left|\overrightarrow{F}_e\right|}{\left|\overrightarrow{F}_i\right|} \tag{2.10}$$

This is in fact the required conditions for complete similitude that are developed from Newton's second law of motion, which mathematically can be define in one dimensional form as $\sum F_x = Ma_x$. Basically what this tells us that the forces such as Viscosity, Pressure, Gravity, Surface Tension, Elasticity and etc acting may be any one or combination of all or several of them, such that the following relation between *Model* (m) and *Prototype* (p) will stand;

$$\frac{\sum \text{forces (Viscous + Pressure + Gravity + Surface Tension + Elasticity)}_m}{\sum \text{forces (Viscous + Pressure + Gravity + Surface Tension + Elasticity)}_p} = \frac{M_m a_m}{M_p a_p}$$

2.2 Magnitudes of Different Forces

Most engineers and fluid dynamists in general are concern about forces such as gravity, viscosity and elasticity and their effects that predominantly govern their design models, but necessarily simultaneously. A fluid motion, under all such forces is characterized by

1. Hydrodynamic parameters like pressure, velocity and acceleration due to gravity,
2. Rheological and other physical properties of the fluid involved, and
3. Geometrical dimensions of the system.

It is important to express the magnitudes of different forces in terms of these parameters, to know the extent of their influences on the different forces acting on a fluid element in the course of its flow.

2.2.1 Inertia Forces

The Inertia Forces of $\vec{F}_i$ can be categorized as following;

- The inertia force acting on a fluid element is equal in magnitude to the mass of the element multiplied by its acceleration.
- The mass of a fluid element is proportional to ρL^3 where, ρ is the density of fluid and L is the characteristic geometrical dimension of the system.
- The acceleration of a fluid element in any direction is the rate at which its velocity in that direction changes with time and is therefore proportional in magnitude to some characteristic velocity V divided by some specified interval of time T. The time interval T is proportional to the characteristic length L divided by the characteristic velocity V, so that the acceleration becomes proportional to V^2/L.

Thus the magnitude of Intertie Force is proportional to;

$$\frac{\rho L^3 V^2}{L} = \rho L^2 V^2 \tag{2.11}$$

alternatively, this can be written as;

$$\left|\vec{F}_i\right| \propto \rho L^2 V^2 \tag{2.12}$$

The general law of dynamic similarity between model and prototype that is referred to as the *Newton Equation* can be expressed as *The Inertia Force Ration* and is developed into the following form:

$$\begin{cases} F_r = \dfrac{\text{force}_{\text{model}}}{\text{force}_{\text{prototype}}} = \dfrac{M_m a_m}{M_p a_p} = \dfrac{\rho_m L_m^3}{\rho_p L_p^3} \times \dfrac{L_r}{L_r^2} = \rho_r L_r^2 \left(\dfrac{L_r}{T_r}\right)^2 \\ F_r = \rho_r L_r^2 V_r^2 = \rho_r A_r V_r^2 \end{cases} \tag{2.13}$$

2.2.2 *Viscous Forces*

The Viscous Force $\overrightarrow{F_v}$ arises from shear stress in a flow of fluid. Therefore, we can write

Magnitude of viscous force $\overrightarrow{F_v}$= (**shear stress**)

× (**surface area over which the shear stress acts**)

Remember that, (shear stress) $\tau = \mu$ (viscosity) × (rate of shear strain)

where, rate of shear strain is proportional to velocity gradient V/L and surface area L^2. Hence

$$\left|\overrightarrow{F_v}\right| \propto \mu \frac{V}{L} L^2 = \mu V L \tag{2.14}$$

The *Viscous Force Ratio* also is known *Reynolds Number* is obtained from the following relationship:

$$\frac{Ma}{\tau A} = \frac{Ma}{\mu\left(\frac{dV}{dy}\right) A} = \frac{\rho L^2 V^2}{\mu\left(\frac{V}{L}\right) L^2} = \frac{\rho V L}{\mu} \tag{2.15}$$

2.2.3 *Pressure Forces*

The Pressure Force $\overrightarrow{F_p}$ arises due to the difference of pressure in a flow field. Hence, it can be written as

$$\left|\overrightarrow{F_p}\right| \propto \Delta p L^2 \tag{2.16}$$

where, $\triangle p$ is some characteristic pressure difference in the flow.

The *Pressure Ratio* that is also known as *Euler Number* using the quantity of $T = L/V$ provides the following relationship:

$$\frac{Ma}{pA} = \frac{\rho L^3 \times L/T^2}{pL^2} = \frac{\rho L^4 (V^2/L^2)}{pL^2} = \frac{\rho V^2}{p} \tag{2.17}$$

2.2.4 Gravity Forces

The Gravity Force $\left|\overrightarrow{F_g}\right|$ on a fluid element is its weight. Hence,

$$\left|\overrightarrow{F_g}\right| \propto pL^3 g \tag{2.18}$$

where g is the acceleration due to gravity or weight per unit mass.

The *Froude Number* can be resulted from the analysis of *Gravity Force Ratio* and that is done as follows;

$$\frac{Ma}{Mg} = \frac{\rho L^2 V^2}{\rho L^3 g} = \frac{V^2}{Lg} \tag{2.19}$$

and Froude Number is the square root of this ratio and is written as $V/\sqrt{Lg}$.

2.2.5 Capillary or Surface Tension Forces

The Capillary Force $\overrightarrow{F_c}$ arises due to the existence of an interface between two fluids.

- The surface tension force acts tangential to a surface.
- It is equal to the coefficient of surface tension σ multiplied by the length of a linear element L on the surface perpendicular to which the force acts.

Therefore,

$$\left|\overrightarrow{F_c}\right| \propto \sigma L \tag{2.20}$$

The *Weber Number* can be obtained by calculating the *Surface Tension Ratio* as follows;

$$\frac{Ma}{\sigma L} = \frac{\rho L^2 V^2}{\sigma L} = \frac{\rho L V^2}{\sigma} \tag{2.21}$$

2.2.6 Compressibility or Elastic Forces

The Elastic Force $\overrightarrow{F_e}$ arises due to the compressibility of the fluid in course of its flow.

- For a given compression (a decrease in volume), the increase in pressure is proportional to the bulk modulus of elasticity E.
- This gives rise to a force known as the elastic force.

Table 2.1 Flight can be roughly classified in six categories

Regime	Subsonic	Transonic	Sonic	Supersonic	Hypersonic	High-hypersonic
Mach	< 1.0	0.8–1.2	1.0	1.2–5.0	5.0–10.0	> 10.0

Hence, for a given compression $\Delta p \propto E$, and as result we can write

$$\left|\overrightarrow{F_e}\right| \propto EL^2 \tag{2.22}$$

The flow of a fluid in practice does not involve all the forces simultaneously. Therefore, the pertinent dimensionless parameters for dynamic similarity are derived from the ratios of significant forces causing the flow.

The *Elasticity Force Ratio* analysis will get us the *Cauchy Number* and results in a ratio that is well known as *Mach Number*. Here is the calculation:

$$\frac{Ma}{EA} = \frac{\rho L^2 V^2}{EL^2} = \frac{\rho V^2}{E}. \tag{2.23}$$

The square root of this ratio, ${}^{V}\!/\!{}_{\sqrt{E/\rho}}$ is well known Mach Number that is named after Austrian physicist and philosopher Ernst Mach. This ratio also can be written as following form and by now, we know that is a dimensionless number;

$$\begin{cases} M = \frac{V}{a} \\ M = \text{is the Mach number} \\ V = \text{is the velocity of the source relative to the medium} \\ a = \text{is the speed of sound in medium} \end{cases} \tag{2.24}$$

Where $a = \sqrt{E/\rho}$ is obvious.

Mach number varies by the composition of the surrounding medium and by local conditions, especially temperature and pressure. The Mach number is utilized to determine if a flow can be treated as an incompressible flow. If $M < 0.2 - 0.3$ and the flows are quasi-steady and isothermal, compressibility effects will be small and a simplified incompressible flow model can be used. In case of flying objects, they can be roughly classified in six categories as shown in Table 2.1, below:

Figure 2.4 below is an illustration US Navy F-18 Hornet fighter jet at the edge of braking the sound barrier by starting to depart speed of sound at Mach Number, and creating a vapor cone at Transonic Speed $M = 1$.

Now we can provide few examples here to expand on above definitions and for that, we turn to the book by R. Giles [15]. We are going to use some of the dimensional entities that given in Appendix F for most common know quantities in terms of Force (F), Mass (M), Length (L) and Time (T).

Example 2.4 For model and prototype, show that, when gravity and inertia are the only influences, the ratio of flow Q is equal to the ratio of the length dimension to the five-halves power.

Fig. 2.4 An F/A-18 Hornet creating a vapor cone at transonic speed just before reaching the speed of sound (7_July_1999)

Solution The ratio flow Q between model and prototype can be written as;

$$\text{Ratio flows:} \quad \frac{Q_m}{Q_p} = \frac{L_m^3/T_m}{L_p^3/T_p} = \frac{L_r^3}{T_r}$$

Now we have to establish the time ratio for the conditions influencing the flow by writing the expressions for the gravitation and inertia forces, as follows;

$$\text{Gravity:} \quad \frac{F_m}{F_p} = \frac{W_m}{W_p} = \frac{w_m}{w_p} \times \frac{L_m^3}{L_p^3} = w_r L_r^3$$

$$\text{Inertia:} \quad \frac{F_m}{F_p} = \frac{M_m a_m}{M_p a_m} = \frac{\rho_m}{\rho_p} \times \frac{L_m^3}{L_p^3} \times \frac{L_r}{T_r^2} = \rho_r L_r^3 \times \frac{L_r}{T_r^2}$$

Equating the force ratios, results in;

$$w_r L_r^3 = \rho_r L_r^3 \times \frac{L_r}{T_r^2}$$

Now if we solve for the time ratio, we obtain the following;

$$T_r^2 = L_r \times \frac{\rho_r}{w_r} = \frac{L_r}{g_r}$$

Recognizing that the value of g_r is unity and substitute this value in the flow ratio expression in above, we then get that;

$$Q_r = \frac{Q_m}{Q_p} = \frac{L_r^3}{L_r^{1/2}} = L_r^{5/2}$$

Example 2.5 Develop the Reynolds model law for time and velocity ratio for incompressible liquids.

Solution Assuming any other effects are negligible except for flow patterns that are subjected to inertia and viscous force only, then these forces for both model and prototype must be taken under consideration and we can write;

For inertia: $\frac{F_m}{F_p} = \rho_r L_r^3 \times \frac{L_r}{T_r^2}$ (from preceding example 2.5)

$$\text{For viscosity:} \begin{cases} \dfrac{F_m}{F_p} = \dfrac{\tau_m A_m}{\tau_p A_p} = \dfrac{\mu_m \left(\frac{dV}{dy}\right)_m A_m}{\mu_p \left(\frac{dV}{dy}\right)_p A_p} = \dfrac{\mu_m \left(\frac{L_m}{T_m} \times \frac{1}{L_m}\right)_m L_m^2}{\mu_p \left(\frac{L_p}{T_p} \times \frac{1}{L_p}\right)_p L_p^2} \\ \dfrac{F_m}{F_p} = \dfrac{\mu_m \left(\frac{L_m^2}{T_m}\right)}{\mu_p \left(\frac{L_p^2}{T_p}\right)} = \dfrac{\mu_r L_r^2}{T_r} \end{cases}$$

Equating the two force ratios, we obtain $\rho_r \frac{L_r^4}{T_r^2} = \frac{\mu_r L_r^2}{T_r}$ from which $T_r = \frac{\rho_r L_r^2}{\mu_r}$

Since $\nu = \frac{\mu}{\rho}$, we may write $T_r = \frac{L_r^2}{\nu_r}$ (1)

The velocity ratio, then is given by $V_r = \frac{L_r}{T_r} = \frac{L_r}{L_r^2} \nu_r = \frac{\nu_r}{L_r}$ (2)

Writing these ratio values in terms of model and prototype, we obtain from (2), that;

$$\frac{V_m}{V_p} = \frac{\nu_m}{\nu_p} \times \frac{L_p}{L_m}$$

Collecting all the term for model and prototype results in $\left({}^{V_m L_m}/_{\nu_m}\right) = \left({}^{V_p L_p}/_{\nu_p}\right)$ which the reader will recognize as: Reynolds number for model = Reynolds number for prototype.

The above two examples establishes another expression that we can called Time Ratios [15] and these ratios can be can be extended to the flow patterns governed by gravity, by viscosity, by surface tension and by elasticity and they respectively can be written as follows;

$$T_r = \sqrt{\frac{L_r}{g_r}} \qquad \text{(See Example 2.6)} \tag{2.25}$$

$$T_r = \frac{L_r^2}{\nu_r} \qquad \text{(See Example 2.6)} \tag{2.26}$$

$$T_r = \sqrt{L_r^3 \times \frac{\rho_r}{\sigma_r}} \tag{2.27}$$

$$T_r = \frac{L}{\sqrt{E_r/\rho_r}} \tag{2.28}$$

In order to classify what we said so far in preceding few sections in above about all the forces and their ratios that are normally used in fluid dynamics, we can summarize them as below, where we use Dynamic Similarity and each governance that fluids are using.

2.3 Dynamic Similarity of Flows governed by Viscous, Pressure and Inertia Forces

The criterion of dynamic similarity for the flows controlled by viscous, pressure and inertia forces are derived from the ratios of the representative magnitudes of these forces with the help of Eqs. 2.11, 2.12, 2.14 and 2.16 as follows:

$$\frac{\text{Viscous Force}}{\text{Inertia Force}} = \frac{\left|\vec{F}_{\upsilon}\right|}{\left|\vec{F}_{i}\right|} \propto \frac{\mu V L}{\rho V^{2} L^{2}} = \frac{\mu}{\rho V L} \tag{2.29a}$$

$$\frac{\text{Pressure Force}}{\text{Inertia Force}} = \frac{\left|\vec{F}_{p}\right|}{\left|\vec{F}_{i}\right|} \propto \frac{\Delta p L^{2}}{\rho V^{2} L^{2}} = \frac{\Delta p}{\rho V^{2}} \tag{2.29b}$$

As we can see and describe earlier, the term $\frac{\rho V L}{\mu}$ is known as *Reynolds* number, $\boldsymbol{Re}$ after the name of the scientist who first developed it and is thus proportional to the magnitude ratio of inertia force to viscous force (Reynolds number plays a vital role in the analysis of fluid flow) [Eq. 2.29a].

The term $\frac{\Delta p}{\rho V^{2}}$ is known as *Euler* number, $\boldsymbol{Eu}$ after the name of the scientist who first derived it [Eq. 2.29b]. This was seen in Eq. 2.17 as well.

The dimensionless terms $\boldsymbol{Re}$ and $\boldsymbol{Eu}$ represent the criteria of dynamic similarity for the flows, which are affected only by viscous, pressure and inertia forces. Such instances, for example, are

1. The full flow of fluid in a completely closed conduit,
2. Flow of air past a low-speed aircraft and
3. The flow of water past a submarine deeply submerged to produce no waves on the surface.

Hence, for a complete dynamic similarity to exist between the prototype and the model for this class of flows, the *Reynolds* number, $\boldsymbol{Re}$ and Euler number, $\boldsymbol{Eu}$ has to be same for the two (prototype and model). Thus

$$\frac{\rho_p V_p L_p}{\mu_p} = \frac{\rho_m V_m L_m}{\mu_m} \tag{2.29c}$$

$$\frac{\Delta p_p}{\rho_p V_p^2} = \frac{\Delta p_m}{\rho_m V_m^2} \tag{2.29d}$$

Where again, the suffix p and suffix m refer to the parameters for prototype and model respectively.

In practice, the pressure drop is the dependent variable, and hence it is compared for the two systems with the help of Eq. (2.29d), while the equality of Reynolds number (Eq. (2.29c)) along with the equalities of other parameters in relation to kinematic and geometric similarities are maintained.

- The characteristic geometrical dimension L and the reference velocity V in the expression of the Reynolds number may be any geometrical dimension and any velocity, which are significant in determining the pattern of flow.
- For internal flows through a closed duct, the hydraulic diameter of the duct D_h and the average flow velocity at a section are invariably used for L and V respectively.
- The hydraulic diameter D_h is defined as $D_h = 4A/P$ where A and P are the cross-sectional area and wetted perimeter respectively.

2.4 Dynamic Similarity of Flows with Gravity, Pressure and Inertia Forces

A flow of the type in which significant forces are gravity force, pressure force and inertia force, is found when a free surface is present.

Examples can be

1. The flow of a liquid in an open channel.
2. The wave motion caused by the passage of a ship through water.
3. The flows over weirs and spillways.

The condition for dynamic similarity of such flows requires

- The equality of the *Euler* number $\boldsymbol{Eu}$ (the magnitude ratio of pressure to inertia force), and
- The equality of the magnitude ratio of gravity to inertia force at corresponding points in the systems being compared.

Thus,

$$\frac{\text{GravityForce}}{\text{InertiaFroce}} = \frac{\left|\vec{F}_g\right|}{\left|\vec{F}_g\right|} \propto \frac{\rho L^3 g}{\rho V^2 L^2} = \frac{L_g}{V^2} \tag{2.29e}$$

- In practice, it is often convenient to use the square root of this ratio so to deal with the first power of the velocity.
- From a physical point of view, equality of $\sqrt{Lg}/V$ implies equality of $\sqrt{Lg}/V^2$ as regard to the concept of dynamic similarity.

The reciprocal of the term $\sqrt{Lg}/V$ as we know by now is known as *Froude* number (after William Froude who first suggested the use of this number in the study of naval architecture).

Hence *Froude* number, $Fr = V/\sqrt{Lg}$

Therefore, the primary requirement for dynamic similarity between the prototype and the model involving flow of fluid with gravity as the significant force, is the equality of *Froude* number, $\boldsymbol{Fr}$, i.e.,

$$\frac{(L_p g_p)^{1/2}}{V_p} = \frac{(L_m g_m)^{1/2}}{V_m} \tag{2.29f}$$

2.5 Dynamic Similarity of Flows with Surface Tension as the Dominant Force

Surface tension forces are important in certain classes of practical problems such as

1. Flows in which capillary waves appear
2. Flows of small jets and thin sheets of liquid injected by a nozzle in air
3. Flow of a thin sheet of liquid over a solid surface.

Here the significant parameter for dynamic similarity is the magnitude ratio of the surface tension force to the inertia force. This can be written as follow;

$$\frac{\left|\vec{F}_c\right|}{\left|\vec{F}_i\right|} \propto \frac{\sigma L}{\rho V^2 L^2} = \frac{\sigma}{\rho V^2 L} \tag{2.30a}$$

As we know by now from preceding section (See Eq. 2.18), the term $\frac{\sigma}{\rho V^2 L}$ is usually known as Weber number, $\boldsymbol{Wb}$ (after the German naval architect Moritz Weber who first suggested the use of this term as a relevant parameter).

Thus for dynamically similar flows $(Wb)_m = (Wb)_p$ then we can write;

$$\frac{\sigma_m}{\rho_m V_m^2 L_m} = \frac{\sigma_p}{\rho_p V_p^2 L_p} \tag{2.30b}$$

2.6 Dynamic Similarity of Flows with Elastic Force

When the compressibility of fluid in the course of its flow becomes significant, the elastic force along with the pressure and inertia forces has to be considered.

Therefore, the magnitude ratio of inertia to elastic force becomes a relevant parameter for dynamic similarity under this situation.

Thus we can write,

$$\frac{\text{Inertia Force}}{\text{Elastic Froce}} = \frac{\left|\vec{F}_i\right|}{\left|\vec{F}_e\right|} \propto \frac{\rho V^2 L^2}{E l^2} = \frac{\rho V^2}{E} \tag{2.31a}$$

The parameter $\rho V^2/E$ is known as *Cauchy Number* (after the French mathematician A.L. Cauchy)

If we consider the flow to be isentropic, then it can be written

$$\frac{\left|\vec{F}_i\right|}{\left|\vec{F}_e\right|} \propto \frac{\rho V^2}{E_S} \tag{2.31b}$$

where E_S is the isentropic bulk modulus of elasticity.

Thus for dynamically similar flows $(\text{Cauchy})_\text{m} = (\text{Cauchy})_\text{p}$

on the other hand, as result we can write

$$\frac{\rho V_m^2}{(E_S)_m} = \frac{\rho V_p^2}{(E_S)_p} \tag{2.31c}$$

where m and p are designation for Model and Prototype.

- The velocity with which a sound wave propagates through a fluid medium equals to $\sqrt{E_s/\rho}$.
- Hence, the term $\rho V^2/E_S$ can be written as V^2/a^2 where a is the acoustic velocity in the fluid medium.

The ratio $M_a = V/a$ is known as $\boldsymbol{MachNumber}$, M_a (after an Austrian physicist Earnest Mach)

It can easily been shown that the effects of compressibility become important when the Mach number exceeds 0.33.

The situation arises in the flow of air past high-speed aircraft, missiles, propellers and rotary compressors. In these cases, equality of Mach number is a condition for dynamic similarity. Therefore,

$$(Ma)_p = (Ma)_m \tag{2.31d}$$

alternatively, as result we can write

$$\frac{V_p}{a_p} = \frac{V_m}{a_m} \tag{2.31e}$$

2.7 Dimensional Analysis and Physical Similarity

The study of fluid mechanics is mainly based on experimental results. One technique used to minimize the number of experiments required is dimensional analysis.

Dimensional analysis does not provide a complete solution to a problem, but it reveals the mathematical relations among the variables involved. Table 2.2 shows a summary of the quantities, symbols, and dimensions used in fluid mechanics.

Many experiments in fluid mechanics are conducted on scale models rather than on prototypes. In all of these experiments, results taken from tests using the model are applied to the prototype. Physical similarity is a general proposition between the model and the prototype.

Table 2.2 Summary of the quantities, symbols, and dimensions

Dimensions		
Quantity	(M, L, T)	(F, L, T)
Acceleration	LT^{-2}	LT^{-2}
Area	L^2	L^2
Bulk modulus of elasticity	$ML^{-1}T^{-2}$	FL^{-2}
Density	ML^{-3}	FT^2L^{-4}
Discharge	L^3T^{-1}	L^3T^{-1}
Dynamic viscosity	$ML^{-1}T^{-1}$	FTL^{-2}
Force	MLT^{-2}	F
Gravity	LT^{-2}	LT^{-2}
Kinematic viscosity	L^2T^{-1}	L^2T^{-1}
Length	L	L
Mass	M	FT^2L^{-1}
Pressure	$ML^{-1}T^{-2}$	FL^{-2}
Specific weight	$ML^{-2}T^{-2}$	FL^{-3}
Tension	MT^{-2}	FL^{-1}
Time	T	T
Velocity	LT^{-1}	LT^{-1}
Angle	θ	θ
Angular velocity	T^{-1}	T^{-1}

2.7.1 Types of Physical Similarity

As it was mentioned in above, we may summarize them by saying, for any comparison between prototype and model to be valid, they must satisfy the following conditions of physical similarity and there exist three type of similarity that one should be aware of the in order to achieve such must conditions and they are as follows;

a. Geometric Similarity
This physical similarity requires, first, that the model and prototype have the same shape, and second, that their dimensions be related by a constant scale factor.

b. Kinematic Similarity
This physical similarity requires that the velocities be related in magnitude and direction by a constant scale factor.

c. Dynamic Similarity
This physical similarity requires that identical types of forces being related in magnitude and direction by a constant scale factor. This type of similarity is also known as *Kinetic Similarity.*

Table 2.3 Dimensional groups, in dynamic similarity

Name of ratio	Definition	Physical meaning
Reynolds number, Re	$\frac{VL\rho}{\mu}$	$\frac{\text{Inertia force}}{\text{Viscous force}}$
Froude number, Fr	$\frac{V}{(Lg)^{1/2}}$	$\frac{\text{Inertia force}}{\text{Gravity force}}$
Weber number, We	$\frac{\rho V_2 L}{\sigma}$	$\frac{\text{Inertia force}}{\text{Surface tension force}}$
Mach number, M	$\frac{V}{a}$	$\frac{\text{Inertia force}}{\text{Elastic force}}$
Euler number, Eu	$\frac{p}{\rho V^2}$	$\frac{\text{Pressure force}}{\text{Inertia force}}$

Some of the most important force ratios (Dimensional Groups), in dynamic similarity are listed in Table 2.3;

2.7.2 *The Application of Dynamic Similarity*

The Concept A physical problem may be characterized by a group of dimensionless similarity parameters or variables rather than by the original dimensional variables.

This gives a clue to the reduction in the number of parameters requiring separate consideration in an experimental investigation.

For an example, if the Reynolds number $\text{Re} = \rho V D_h/\mu$ is considered as the independent variable, in case of a flow of fluid through a closed duct of hydraulic diameter D_h, then a change in Re may be caused through a change in flow velocity V only. Thus a range of Re can be covered simply by the variation in V without varying other independent dimensional variables ρD_h and μ.

In fact, the variation in the Reynolds number physically implies the variation in any of the dimensional parameters defining it, though the change in **Re**, may be obtained through the variation in anyone parameter, say the velocity V.

A number of such dimensionless parameters in relation to dynamic similarity are shown in Table 2.3. Sometimes it becomes difficult to derive these parameters straight forward from an estimation of the representative order of magnitudes of the forces involved. An alternative method of determining these dimensionless parameters by a mathematical technique is known as dimensional analysis.

The Technique The requirement of dimensional homogeneity imposes conditions on the quantities involved in a physical problem, and these restrictions, placed in the form of an algebraic function by the requirement of dimensional homogeneity, play the central role in dimensional analysis.

There are two existing approaches:

- One due to Buckingham known as Buckingham's pi theorem
- Other due to Rayleigh known as Rayleigh's Indicial method

In our next few sections, we will see few examples of the dimensions of physical quantities

2.7.3 *Dimensions of Physical Quantities*

All physical quantities can be and are expressed by magnitudes and units.

For example, the velocity and acceleration of a fluid particle are 8 m/s and 10 m/s^2 respectively. Here the dimensions of velocity and acceleration are ms^{-1} and ms^{-2} respectively.

In **SI** (System International) units, the primary physical quantities which are assigned base dimensions are the mass, length, time, temperature, current and luminous intensity. Of these, the first four are used in fluid mechanics and they are symbolized as M (mass), L (length), T (time), and θ (temperature).

- Any physical quantity can be expressed in terms of these primary quantities by using the basic mathematical definition of the quantity.
- The resulting expression is known as the dimension of the quantity.

Let us consider some examples by utilizing Table 2.2:

1. Dimensional of Stress

Shear stress τ is defined as Force/Area. Again Force = Mass x Acceleration according to Newton's second law.

Dimensional of acceleration = Dimensional of velocity/Dimension of time

$$= \frac{\text{Dimension of Distance}}{(\text{Dimension of Time})^2} = \frac{L}{T^2}$$

Or dimension of area can be expressed as (Length)$^2 = L^2$. Hence, dimension of Shear Stress will be;

$$\tau = \left[{}^{(ML/T^2)}\Big/{}_{L^2}\right] = ML^{-1}T^2 \tag{2.32}$$

2. Dimensional of Viscosity

Consider Newtown's law for the definition of viscosity as;

$$\tau = \mu du/dy$$

or

$$\mu = \frac{\tau}{(du/dy)}$$

In this case the dimension of velocity gradient du/dy can be written as

Dimension of du/dy = (Dimension of u)/(Dimension of $y) = (L/T)/L = T^{-1}$

The dimension of Shear Stress τ is given in Eq. 2.22a, hence the dimension for μ will be as follow;

$$\mu = \frac{\text{Dimension of } \tau}{\text{Dimension of } du/dy} = \frac{ML^{-1}T^{-2}}{T^{-1}} = ML^{-1}T^{-2}$$

Dimensions of Various Physical Quantities in tabular format are presented in Appendix E of the book and readers can refer to them.

2.7.4 The Buckingham's Pi Theorem

This theorem was defined in Chap. 1 and basically states that the number of independent dimensionless groups used to describe a problem in which there are n variables and m dimensions is equal to $n - m$. Mathematically it can be represents as

$$f(\pi_1, \pi_2, \cdots, \pi_{n-m}) = 0$$

Where $\pi \equiv$ The dimensionless group. We will elaborate on this subject in Sect. 2.9.1 below.

2.7.5 Dimensional Analysis of a Problem

The dimensional analysis of a problem is performed in the following steps:

a. A list of parameters is selected.
b. The dimensionless π parameters are obtained by using the π theorem.
c. The dimensionless π parameters are obtained by using the π theorem.

Six steps are used to find the dimensionless parameters:

1. List all the variables involved.
2. Select a set of primary dimensions.
3. List the dimensions of all variables in terms of primary dimensions.
4. Select from the list of variables, a number of repeating variables equal to the number of primary dimensions, m, including all the primary dimensions.
5. Write the π parameters in terms of the unknown exponents and write the equations so that the sum of exponents is zero. Solve the equation to obtain n-m dimensionless group.
6. Check if each group obtained is dimensionless.

The relation among π parameters is determined experimentally.

2.8 Typical Applications

Similitude has been well documented for a large number of engineering problems and is the basis of many textbook formulas and dimensionless quantities. These formulas and quantities are easy to use without having to repeat the laborious task of dimensional analysis and formula derivation. Simplification of the formulas (by neglecting some aspects of similitude) is common, and needs to be reviewed by the engineer for each application.

Similitude can be used to predict the performance of a new design based on data from an existing, similar design. In this case, the model is the existing design.

Another use of similitude and models is in validation of computer simulations with the ultimate goal of eliminating the need for physical models altogether.

Another application of similitude is to replace the operating fluid with a different test fluid. Wind tunnels, for example, have trouble with air liquefying in certain conditions so helium is sometimes used. Other applications may operate in dangerous or expensive fluids so the testing is carried out in a more convenient substitute.

Some common applications of similitude and associated dimensionless numbers;

- **Incompressible flow**—Reynolds number, Pressure coefficient, (Froude number and Weber number for open channel hydraulics). See Example-3 above.
- **Compressible flows**—Reynolds number, Mach number, Prandtl number, Specific heat ratio
- **Flow excited vibration**—Strouhal number.
- **Centrifugal compressors**—Reynolds number, Mach number, Pressure coefficient, Velocity ratio.

These numbers were presented in Table 2.3.

2.9 Dimensional Analysis to Obtain Similarity Parameters

As we defined before dimensional analysis is a method by which we deduce information about a phenomenon from the single premise that phenomenon can be described by a dimensionally correct equation among certain variables. The generality of the method is both its strength and its weakness. "The result of a dimensional analysis of a problem is a reduction of the number of variables in the problem, thereby amplifying the information that is obtained from a few experiments.

2.9.1 Using Buckingham Pi theory to Obtain Similarity Parameters

Buckingham π (Pi) theorem can be summarized as follows;

- Reduce number of variables
- Derive dimensionally homogeneous relationship

In order to do just this we follow the following steps

1. Specify (all) the (say N) *relevant* variables (dependent or independent): $x_1, x_2, \ldots, x_n$, e.g. time, force, fluid density, distance....

   We want to relate the x_i's to each other $\Gamma(x_1, x_2, \ldots, x_N) = 0$

2. Identify (all) the (say P) relevant basic physical units ("dimensionless"), e.g. $M, L, T(P = 3)$, [temperature, charge,...]

3. Let $\pi = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N}$ be a dimensionless quantity formed from the x_i's. Suppose

$$x_i = C_i M^{m_i} L^{l_i} T^{t_i}, \qquad i = 1,2,\ldots,N$$

where the C_i are dimensionless constants. For example, if $x_i = KE = \frac{1}{2}MV^2 = \frac{1}{2}M^1L^2T^{-2}$ (kinetic energy), we have that $C_i = \frac{1}{2}, m_1 = 1, l_1 = 2, t_1 = -2$. Then

$$\pi = (C_1^{\alpha_1} C_1^{\alpha_2} \ldots C_N^{\alpha_N}) M^{\alpha_1 m_1 + \alpha_2 m_2 + \cdots \alpha_N m_N} L^{\alpha_1 l_1 + \alpha_2 l_2 + \cdots \alpha_N l_N} T^{\alpha_1 t_1 + \alpha_2 t_2 + \cdots \alpha_N t_N}$$

For π to be dimensionless, we require

$$P \left\{ \begin{array}{l} \overbrace{\alpha_i m_i = 0}^{N} \\ \alpha_i l_i = 0 \\ \underbrace{\alpha_i t_i = 0}_{\sum \text{notation}} \end{array} \right\} aP \times N \text{system of Linear Equations} \tag{1}$$

Since (1) is homogeneous, it always has a trivial solution,
$\alpha_i \equiv 0, \qquad i = 1,2,\ldots,N$ (i.e. π is constant) (i.e. π is constant)
There are two possibilities

a. (1) has no nontrivial solution (only solution is $\pi =$ constant, i.e. independent of x_i's), which implies that the N variable x_i, $i = 1,2,\ldots,N$ are *Dimensionally Independent* (DI), i.e. they are 'unrelated' and 'irrelevant' to the problem.
b. (1) has $J(J > 0)$ nontrivial solutions, $\pi_1, \pi_2, \ldots, \pi_J$. In general, $J < N$, in fact, $J = N - K$ where K is the *rank* or 'dimension' of the system of eq. (2.46).

2.9.2 Model Law

Instead of relating the N x_i's by $\Gamma(x_1, x_2, \ldots, x_N) = 0$, relate the J π's by

$$F(\pi_1, \pi_2, \ldots, \pi_J) = 0, \ where\ J = N - K < N$$

For similitude, we require

$$(\pi_{\text{model}})_j = (\pi_{\text{prototype}})_j \ where\ j = 1,2,\ldots,J$$

If 2 problems have all the same π's, they have similitude (in the π_j senses), so π's serve as similarity parameters.

Note:

- If π is dimensionless, so is $\pi \text{x} const, 1/\pi$, etc....
- If π_1, π_2 are dimensionless, so is $\pi_1 \times \pi_2, \frac{\pi_1}{\pi_2}, \pi_1^{const_1} \times \pi_2^{const_2}$, etc...

In general, we want the set (not unique) of independent π_j's, for e.g., π_1, π_2, π_3 or $\pi_1, \pi_1 \times \pi_2, \pi_3$, but not $\pi_1, \pi_2, \pi_1 \times \pi_2$.

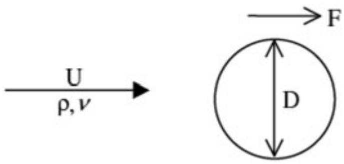


Fig. 2.5 Force on a smooth circle in steady incompressible fluid (no Gravity)

Note We will discuss the details of Modeling and Theory of Model in Sect. 2.15 of this chapter

Example 2.6 Force on a smooth circular cylinder in steady, incompressible flow with no gravity.

(Application of Buckingham's π Theory)

A Fluid Mechanician found that the relevant *dimensional* quantities required evaluating the force F on the cylinder from the fluid are: the diameter of the cylinder D, the fluid velocity U, the fluid density ρ and the kinematic viscosity of the fluid ν. Evaluate the *nondimensional* independent parameters that describe this problem.

$$x_i : F, U, D, \rho, \nu \rightarrow N = 5$$

$$x_i : c_i M^{m_i} L^{l_i} T^{t_i} \rightarrow P = 3$$

Now using Table 2.2 and identify dimensions of F, U, D, ρ, ν, we can write

$$P = 3 \left\{ \begin{array}{c} \overbrace{\begin{array}{cccccc} & F & U & D & \rho & \upsilon \\ m_i & 1 & 0 & 0 & 1 & 0 \\ l_i & 1 & 1 & 1 & -3 & 2 \\ t_i & -2 & -1 & 0 & 0 & -1 \end{array}}^{N=5} \end{array} \right.$$

$$\pi = F^{\alpha_1} U^{\alpha_2} D^{\alpha_3} \rho^{\alpha_4} \nu^{\alpha_5}$$

For π to be nondimensional, the set of equations

$$\alpha_i m_i = 0$$

$$\alpha_i l_i = 0$$

$$\alpha_i t_i = 0$$

has to be satisfied. The system of equations above after we substitute the values for the m_{i}'s, l_{i}'s and t_{i}'s assume the form:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & -3 & 0 \\ -2 & -1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The rank of this system is $K = 3$, so we have $j = 2$ nontrivial solutions. Two families of solutions for α_1 for each fixed pair of (α_4, α_5), exists a unique solution for $(\alpha_1, \alpha_2, \alpha_3)$. We consider the pairs $(\alpha_4 = 1, \alpha_5 = 0)$ and $(\alpha_4 = 0, \alpha_5 = 1)$, all other cases are linear combinations of these two.

1. Pair ($\alpha_4 = 1$ and $\alpha_5 = 0$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$$

which has solution

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

$$\therefore \pi_1 = F^{\alpha_1} U^{\alpha_2} D^{\alpha_3} \rho^{\alpha_4} \nu^{\alpha_3} = \frac{\rho U^2 D^2}{F}$$

Conventionally, $\pi_1 \rightarrow 2\pi_1^{-1}$ and $\therefore \pi_1 = \frac{\rho U^2 D^2}{F} \equiv C_d$, which is the Drag coefficient.

2. Pair $\alpha_4 = 0$ and $\alpha_5 = 1$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$$

which has solution

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

Table 2.4 Dimensions of some fluid properties

Quantities		Dimensions (MLT)
Angle	θ	None ($M^0L^0T^0$)
Length	L	L
Area	A	L^2
Volume	$\forall$	L^3
Time	t	T
Velocity	V	LT^{-1}
Acceleration	$\dot{V}$	LT^{-2}
Angular velocity	ω	$T^{-}1$
Density	ρ	ML^{-3}
Momentum	L	MLT^{-1}
Volume flow rate	Q	L^3T^{-1}
Mass flow rate	Q	MT^{-1}
Pressure	p	$ML^{-1}T^{-2}$
Stress	τ	$ML^{-1}T^{-2}$
Surface tension	σ	MT^{-2}
Force	F	MLT^{-2}
Moment	M	ML^2T^{-2}
Energy	E	ML^2T^{-2}
Power	P	ML^2T^{-3}
Dynamic viscosity	μ	$ML^{-1}\mathrm{T}^{-1}$
Kinematic viscosity	ν	L^2T^{-1}

$$\therefore \pi_2 = F^{\alpha_1} U^{\alpha_2} D^{\alpha_3} \rho^{\alpha_4} \nu^{\alpha_3} = \frac{\nu}{UD}$$

Conventionally, $\pi_2 \rightarrow \pi_2^{-1}, \therefore \pi_2 = \frac{UD}{\nu} \equiv \mathrm{Re}$, which is the Reynolds number.

Therefore, we can write the following equivalent expressions for the *nondimensional* independent parameters that describe this problem:

$$F(\pi_1, \pi_2) = 0 \qquad \text{or} \qquad \pi_1 = f(\pi_2)$$

$$F(C_d, \mathrm{Re}) = 0 \qquad \text{or} \qquad C_d = f(\mathrm{Re})$$

$$F\left(\frac{F}{1/2\rho U^2 D^2}, \frac{UD}{\nu}\right) = 0 \qquad \text{or} \qquad \frac{F}{1/2\rho U^2 D^2} = f\left(\frac{UD}{\nu}\right)$$

again dimensions of some fluid properties.

2.10 Application of Dimensional Analysis and Similarity in Fluid Dynamics

In this section, we will demonstrate few examples in order to provide more approach that is practical and make it easy for the readers to have better understanding of what it all about. The following sub-sections will be elaborating on dimensional analysis, similarity, and relationship between them.

2.10.1 Dimensional Analysis

Consider steady flow of an incompressible Newton fluid through long, smooth-walled, horizontal, circular pipe.

We are interested in the pressure drop per unit length that developed along the pipe because of friction

This problem cannot generally be solved analytically without the use of experimental data.

Before running experiment decide on the factors, or variables, that effects the pressure drop per unit length, $\triangle p_l$ Those are the pipe diameter, D, fluid density, ρ, fluid viscosity, μ, and mean velocity, V, This relationships can be expressed as

$$\triangle p_l = f(D, \rho, \mu, V)$$

The objective of experiment is to determine the nature of this function.

1. During experiment change one of the variables, while holding all others constant, and measure the corresponding pressure drop.
2. Experimental results can be represented *graphically*.
3. Another way is to use ***dimensional parameters***.

$$\frac{D\triangle p_l}{\rho V^2} = \phi\left(\frac{\rho VD}{\mu}\right)\begin{cases}\text{Only one dependent and independent variables}\\ \text{Easy to set up experiments to determine dependency}\\ \text{Easy to present results (one graph)}\end{cases}$$

Number of variables reduced from five to two.

Number and complexity of experiments also reduced.

Results of experiments are represented by a single curve.

Two groups in the above equation are *dimensionless* (Fig. 2.6).

Thus, results presented in the form of the graph are independent of the system of units used.

This type of the analysis is called ***dimensional analysis.***

Dimensional analysis is a method for reducing the number and complexity of experimental variables, which affect a given physical phenomenon.

Dimensional analysis is based on the ***Buckingham Pi theorem***.

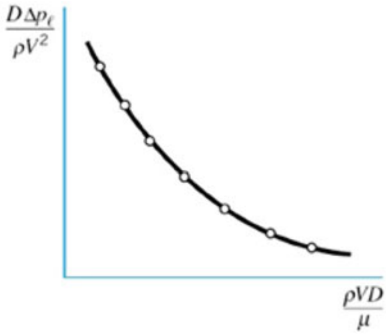


Fig. 2.6 An illustrative plot of pressure drop data using dimensionless parameters

2.11 Further Discussion of Buckingham Pi Theorem

How many dimensionless products are required to replace the original list of variables?

Buckingham Pi theorem states:

If an equation involving k variables is dimensionally homogeneous, it can be reduced to a relationship among $k - r$ independent dimensionless products, where r is the minimum number of references dimensions required to describe the variables.

Dimensionless products are referred to as "Pi terms" and denoted by letter Π.

Pi Theorem is based on the ***principle of dimensional homogeneity***, which states:

If an equation truly expresses a proper relationship between variables in a physical process, it will be ***dimensionally homogeneous***; i.e. each of its additive terms will have the same dimensions.

Thus, dimensionally homogeneous equation involving k variables

$$\mu_1 = f(u_2, u_3, \cdots, u_k)$$

can be written in dimensionless form

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \cdots, \Pi_{k-r})$$

2.12 Determination of Pi Terms

There are several methods can be employed to form the dimensionless product, or Pi terms, that arise in a dimensional analysis. Ultimately we are looking for a method that will allow us to systematically form the Pi terms so that we are certain that they are dimensionless and independent and they have the right number. In order to determine the Pi theorem a *method of repeating variables* are developed. More details of this method can be found in book by Munson et al. [9].

Pi theorem also can be formed by *inspection* that will be discussed in Sect. 2.8 of this chapter. Regardless of the specific method that can be used for dimensional analysis, there are certain aspects of this important engineering tool that must seem a little baffling and mysterious to the readers. In this section, we will try to elaborate on some of these issues. Therefore, in summary we will discuss two following methods to determine the Pi terms as follows;

1. Method of Repeating Variables
2. Method of Inspection

Each of these methods is presented along with few examples in proceeding section of this chapter.

2.12.1 Method of Repeating Variables

It will be helpful to break the repeating variable method down into series of distinct steps that can followed for any given problem under consideration and we show some examples to clarify these steps that are as follows;

Step 1: List all the variables that are involved in the problem.
Step 2: Express each of the variables in terms of basic dimensions.
Step 3: Determine the required number of terms.
Step 4: Select a number of repeating variables.
Step 5: From a Pi term by multiplying one of the nonrepeating variables by the product of repeating variables, each raised to an exponent that will make the combination dimensionless.
Step 6: Repeat Step 5 for each of the remaining nonrepeating variables.
Step 7: Check all the resulting Pi terms to make sure they are dimensionless.
Step 8: Express the final form as a relationship among the Pi terms, and think about what it means.

We will use some of examples from Munson et al. [9] books that is given in Chap. 7.

Considering the problem that was presented initially in Sect. 2.5.1 of this chapter in above we will illustrate these various steps, which were concerned with the steady flow of an incompressible Newton fluid through a long, smooth-walled, horizontal circular pipe. This problem was dealing with steady flow of an incompressible Newton fluid through a long smooth-walled, horizontal, circular pipe and we borrowed it from Munson [7], book. We are interested in the pressure drop per unit length, $\triangle p_l$, along the pipe as shown in Fig. 2.7 below. First (Step 1) we must list all the pertinent variables that are involved based on the experiment's knowledge of the problem. In this problem, we assume that;

$$\triangle p_l = f(D, \rho, \mu, V)$$

where

$\triangle p_l$ = Pressure drop per unit length [(lb/ft^2)/ft = lb/ft^3 or N/m^3].

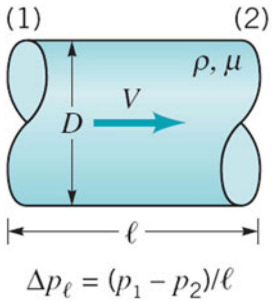


Fig. 2.7 Horizontal circular pipe

D= Pipe Diameter [ft or m].
ρ= Fluid Density [ft or m].
μ= Fluid Viscosity [slugs/ft^3 or slugs/m^3]
V= Fluid Mean Velocity [ft/s or m/s].

Step 2 we express all the variables in terms of basic dimensions, using F, L, and T as follows;

$$\Delta p_\ell \doteq FL^{-3}$$
$$D \doteq L$$
$$\rho \doteq FL^{-4}T^2$$
$$\mu \doteq FL^{-2}T$$
$$V \doteq LT^{-1}$$

Note that we could also use metric system for dimensional units (i.e. M, L and T) if so need be, where the result at the would be the same from dimensional analysis point of view. Also note for density, which is mass per unit volume (ML^{-3}), we implement the relationship of $F = MLT^{-2}$ to express the density in terms of F, L and T but we should not mix the dimensional units though. Either F, L, and T (US units) or M, L and T (SI units) should be used.

Step 3 requires usage of Pi theorem in order to identify the required number of Pi terms. An inspection of dimensions of variables from Step 2 reveals that all three basic dimensions are required to describe the variables. Since there are five ($k = 5$) variables including to count the dependent variable, Δp_l and three required reference dimensions ($r = 3$), then according to the Pi theorem there will be ($k - r = 5 - 3 = 2$), or two Pi terms required.

Step 4 now requires to form the Pi terms using the repeating variables, which we need to be selected from the list D, ρ, μ, and V. Since three reference dimensions are required, we will need to select three repeating variables. In general, dimensionally, the simplest repeating variables are chosen. For example if one of the variables has the dimension of a length, choose that as one of the repeating variables. In the example provided by Munson et al. [9], we use D, V, and ρ as repeating variables. Remember that these variables are dimensionally independent, since D is a length,

V involves both length and time, and ρ involves force, length, and time. This means that we cannot form a dimensionless product from this set.

Now proceeding Step 5 to form the two Pi. Typically, we would start with dependent variable and combine it with the repeating variables to form the first Pi term; that is

$$\Pi_1 = \Delta p_l D^a V^b \rho^c$$

Since this combination has to be dimensionless, it follows that

$$(FL^{-3})(L)^a(LT^{-1})^b(FL^{-4}T^2) = F^0L^0T^0$$

The exponents, a, b, and c must be determined such that the resulting exponent for each of the basic dimensions- F, L, and T -must be zero, so that the resulting combination is dimensionless.

Thus we have;

$$\begin{aligned} 1 + c &= 0 \quad (\text{for } F) \\ -3 + a + b - 4c &= 0 \quad (\text{for } L) \\ -b + 2c &= 0 \quad (\text{for } T) \end{aligned}$$

The result solution of this algebraic set of equation will provide the desired answer for a, b, and c as follow;

$$\begin{aligned} a &= 1 \\ b &= -2 \\ c &= -1 \end{aligned}$$

Following that we get;

$$\Pi_1 = \frac{\Delta p_l D}{\rho V^2}$$

The process is now repeated for remaining nonrepeating variables (Step 6). In this example there is only one additional variable (μ) so that

$$\Pi_2 = \mu D^a V^b \rho^c$$

or

$$(FL^{-2}T)(L)^a(LT^{-1})^b(FL^{-4}T^2)^c = F^0L^0T^0$$

and therefore

$$\begin{aligned} 1 + c &= 0 \quad (\text{for } F) \\ -2 + a + b - 4c &= 0 \quad (\text{for } L) \\ 1 - b + 2c &= 0 \quad (\text{for } T) \end{aligned}$$

Solving this set of equations, it follows that

$$a = -1$$
$$b = -1$$
$$c = -1$$

so that

$$\Pi_2 = \frac{\mu}{DV\rho}$$

Note that we end up with the correct number of Pi terms as determined from Step3. Next step which is Step 7 is to make sure the Pi terms are actually dimensionless so we will check using FLT and MLT. Thus, we have

$$\Pi_1 = \frac{\Delta p_\ell D}{\rho V^2} = \frac{(FL^{-3})(L)}{(FL^{-4}T^{-2})(LT^{-1})^2} = F^0L^0T^0$$
$$\Pi_2 = \frac{\mu}{DV\rho} = \frac{(FL^{-2}T)}{(L)(LT^{-1})(FL^{-4}T^{-2})} = F^0L^0T^0$$

or alternatively;

$$\Pi_1 = \frac{\Delta p_\ell D}{\rho V^2} = \frac{(ML^{-2}T^{-2})(L)}{(ML^{-3})(LT^{-1})^2} = M^0L^0T^0$$
$$\Pi_2 = \frac{\mu}{DV\rho} = \frac{(ML^{-1}T^{-1})}{(L)(LT^{-1})(ML^{-3})} = M^0L^0T^0$$

Finally, in Step 8, we can express the result of the dimensionless analysis as;

$$\frac{\Delta p_l D}{\rho V^2} = \tilde{\phi}\left(\frac{\mu}{DV\rho}\right)$$

This result indicates that this problem can be studied in terms of these two Pi terms, rather than the original five variables we started with. The eight steps carried out to obtain this result are summarized by the Fig. 2.8 below along with Fig. 2.7 in support of this example. Dimensional analysis will *not* provide the form of the function $\tilde{\phi}$. This can only be obtained from a suitable set of experiments. If desired, the Pi terms can be rearranged; that is, the reciprocal $\mu/DV\rho$ could be used, and of course the order in which the variables can be changed. Thus for example, Π_2 could be expressed as;

$$\Pi_2 = \frac{DV\rho}{\mu}$$

And the relationship between Π_1 and Π_2 could be expressed as

$$\frac{\Delta p_l D}{\rho V^2} = \phi\left(\frac{\mu}{DV\rho}\right)$$

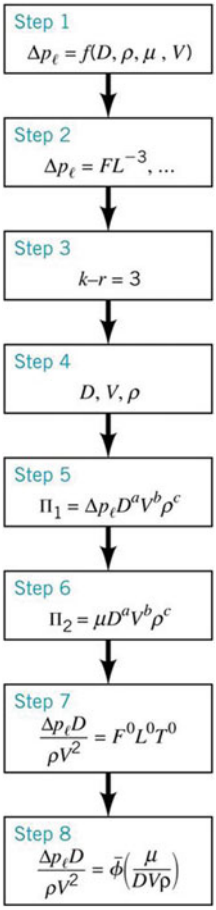


Fig. 2.8 Eight steps figures [9]

as shown by the Fig. 2.5. The dimensionless product $DV\rho/\mu$ is very famous one in fluid mechanics and it is known as—the Reynolds number.

To summarize, the steps to be followed in performing a dimensional analysis using the method of repeating variables refer to beginning of this section. More example of this kind is presented by Munson et al. [9] in Chap. 7.

Example 2.7 Method of Repeating Variables

A thin rectangular plate having a width w and a height h is located so that it is normal to a moving stream of fluid. Assume that the drag, D, that the fluid exerts on the plate is a function of w and h, the fluid viscosity, μ, and ρ, respectively, and the velocity, V, of the fluid approaching the plate. Determine a suitable set of Pi terms to study this problem experimentally.

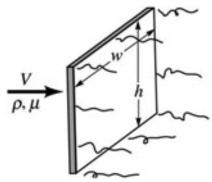


Fig. 2.9 Thin plate with normal flow

Solution From the statement of the problem the drag force is given by

$$D = f(w, h, \mu, \rho, V)$$

Step 1: List all the dimensional variables involved, D, w, h, μ, ρ and V, then $k = 6$ dimensional parameters are identified.
Step 2: Select primary dimensions M, L and T expresses each of the variables in terms of basic dimensions ($r = 3$).

$$D \doteq MLT^{-2}, w \doteq L, h \doteq L$$
$$\mu \doteq ML^{-1}T^{-1}, \rho \doteq ML^{-3}, V \doteq LT^{-1}$$

Step 3: Determine the required number of Pi terms $k - r = 6 - 3 = 3$.
Step 4: Select repeating variables, namely w, V, and ρ.
Step 5: Starting with the dependent variable D, the first Pi term can be formed to present dimensionless group or dimensionless products.

$$\Pi_1 = DW^a V^b \rho^c \quad \text{or} \quad (MLT^{-2})(L)^a(LT^{-1})^b(ML^{-3})^c = M^0L^0T^0$$

Then it follows that;

$$\begin{aligned} M:& \quad 1 + c = 0 & a = -2 \\ L:& \quad 1 + a + b - 3c = 0 & b = -2 \\ T:& \quad -2 - b = 0 & c = -1 \end{aligned}$$

Therefore;

$$\Pi_1 = \frac{D}{w^2V^2\rho}$$

Step 6: Next the procedure is repeated with the second nonrepeating variable, h, so that

$$\Pi_2 = hw^a V^b \rho^c \qquad \text{or} \qquad (L)(L)^a(LT^{-1})^b(ML^{-3})^c = M^0L^0T^0$$

Then, it follows that;

$$\begin{aligned} M:& \quad c = 0 & a = -1 \\ L:& \quad 1 + a + b - 3c = 0 & b = 0 \\ T:& \quad b = 0 & c = 0 \end{aligned}$$

Therefore;

$$\Pi_2 = \frac{h}{w}$$

Check all the resulting Pi terms to make sure they are dimensionless. So the remaining nonrepeating variable is μ so that;

$$\Pi_3 = \mu w^a V^b \rho^c \qquad \text{or} \qquad (ML^{-1}T^{-1})(L)^a(LT^{-1})^b(ML^{-3})^c = M^0L^0T^0$$

Then, it follows that;

$$\begin{array}{lrl} M: & 1 + c = 0 & a = -1 \\ L: & -1 + a + b - 3c = 0 & b = -1 \\ T: & -1 + b = 0 & c = -1 \end{array}$$

Therefore;

$$\Pi_3 = \frac{\mu}{wV\rho}$$

Step 7: Now if we check all three required Pi terms we can see they all dimensionless, and to do so we use F, L and T, which will also verify the correctness of the original dimensions used for the variables. Thus we have;

$$\Pi_1 = \frac{D}{w^2V^2\rho} \doteq \frac{(F)}{(L)^2(LT^{-1})^2(FL^{-4}T^2)} \doteq F^0L^0T^0$$

$$\Pi_2 = \frac{h}{w} \doteq \frac{(L)}{(L)} \doteq F^0L^0T^0$$

$$\Pi_3 = \frac{\mu}{wV\rho} \doteq \frac{(FL^{-2}T)}{(L)(LT^{-1})(FL^{-4}T^2)} \doteq F^0L^0T^0$$

Step 8: Express the final form as a relationship among the Pi terms, we have;

$$\Pi_1 = \bar{\phi}(\Pi_1, \Pi_3) \qquad \text{or} \qquad \frac{D}{w^2V^2\rho} = \bar{\phi}\left(\frac{h}{w}, \frac{\mu}{wV\rho}\right)$$

Since at this stage in the analysis the number of the function $\bar{\phi}$ is unknown, we could rearrange the Pi terms if we desire. For example, we could express the final result in the following form

$$\frac{D}{w^2\rho V^2} = \phi\left(\frac{h}{w}, \frac{\rho V w}{\mu}\right)$$

Which would be more conventional, since the ratio of the plate width to height, w/h, is called the aspect ratio, and $\rho V w/\mu$ is the Reynolds number.

2.12.2 Selection of Variables

Most of engineering problems involve certain simplifying assumptions that have an influence on the variables to be considered. Typically, we wish to keep the problem as simple as possible even if some accuracy is sacrificed, but suitable balance simplicity and accuracy should be a desirable goal. How accurate the solution must be depends on objective of the study. This means, we may be only concerned with general trends and, therefore, some variables that are thought to have only a minor influence in the problem may be neglected for simplicity. We can briefly summarize *selection of variables* in the following steps [9];

- One of the most important, and difficult, steps in applying dimensional analysis to any given problem is the selection of the variables that are involved.
- There is no simple procedure whereby the variable can be easily identified. Generally, one must rely on a good understanding of the phenomenon involved and the governing physical laws.
- If extraneous variables are included, then too many Pi terms appear in the final solution, and it may be difficult, time consuming, and expensive to eliminate these experimentally.
- If important variables are omitted, then an incorrect result will be obtained; and again, this may prove to be costly and difficult to ascertain.
- Most engineering problems involve certain simplifying assumptions that have an influence on the variables to be considered.
- Usually we wish to keep the problems as simple as possible, perhaps even if some accuracy is sacrificed.
- A suitable balance between simplicity and accuracy is an desirable goal.
- Variables can be classified into three general group:
  - Geometry: lengths and angles.
  - Material Properties: relate the external effects and the responses.
  - External Effects: produce, or tend to produce, a change in the system and these external effects are elements, such as force, pressure, velocity, or gravity.
  - Points should be considered in the selection of variables:
  - Clearly define the problem. What is the main variable of interest?
  - Consider the basic laws that govern the phenomenon.
  - Start the variable selection process by grouping the variables into three broad classes.
  - Consider other variables that may not fall into one the three categories. For example, time and time dependent variables.
  - Be sure to include all quantities that may be held constant (e.g., g).
  - Make sure that all variables are independent. Look for relationships among subsets of the variables.

2.12.3 Determination of Reference Dimensions

While it is very important to know how many reference dimensions are required to describe the variable in any given problem that dimensional analysis method is used to solve it, also as equally important to reduce the number of Pi terms to a minimum, so this way we reduce the number of variables to minimum. The methodology of dimensional analysis–the scaling techniques and their philosophical implications as well as its approach–is based in geometry, so visual displays will be emphasized. Theoretical perspectives on methods, scaling, and dimensions will be presented in the following sections particularly in Sect. 2.14 of this chapter.

The methods presented are about a variety of techniques for scaling, broadly defined, including unfolding analysis, proximity scaling, Guttman scaling, cluster analysis, factor analysis, and multidimensional scaling. These methods provide means for;

- Data reduction (reducing a large number of variables into a smaller set of composites),
- Examining dimensionality (representing the data in terms of the smallest possible number of unobserved underlying factors), and
- Measurement (scoring cases on the underlying dimensions and using those scores in further analysis).

As we have seen in the preceding example, F, L, and T appear to be a convenient set of basic dimensions for characterizing fluid-mechanical quantities of steady flow of an incompressible Newton fluid through a long, smooth-walled, horizontal circular pipe. In addition, we notice from same examples that M, L and T will work as well, so there is nothing unique and fundamental about neither of sets of dimensions. In fact any set of measurable quantities could be used as basic dimensions provided that the selected combination can be used to describe all secondary quantities. On the other hand, the use of FLT or MLT as basic and "fundamental" dimensions are the simplest ones, and these dimensions can be used to describe the fluid-mechanical phenomena of a problem such as preceding example [9].

In summary in order to achieve a way of dealing with a problem in using the determination of Reference Dimensions, we can write;

- When to determine the number of pi terms, it is important to know how many reference dimensions are required to describe the variables.
- In fluid mechanics, the required number of reference dimensions is three, but in some problems only one or two are required.
- In some problems, we occasionally find the number of reference dimensions needed to describe all variables is smaller than the number of basic dimensions and it is illustrated in following examples.

Additionally, we sometime encounter the number of reference dimensions that needed to describe all variables is smaller than the number of basic dimensions. This point can be shown by another example from Munson [9] book as well as other books such as Huntley [10] and Isaacson and Isaacson [11].

Fig. 2.10 Thin can of paint

Example 2.8 An open, cylindrical paint can having a diameter D is filled to a depth h with paint having a specified weight γ. The vertical deflection δ, of the center of the bottom is a function of D, h, d, γ, and E, where d is the thickness of the bottom and E is the modulus of elasticity of the bottom materials. Determine the functional relationship between the vertical deflection, δ, and the independent variables using dimensional analysis.

Solution From the statement of the problem we can write;

$\delta = f(D, h, d, \gamma, E)$

in addition, dimensions of the variables are

$$\begin{cases} \delta = L & & d = L \\ D = L & \text{and} & \gamma = FL^{-3} = ML^{-2}T^{-2} \\ h = L & & E = FL^{-2} = ML^{-1}T^{-2} \end{cases}$$

where the dimensions have been expressed in terms of both the FLT and MLT systems.

We now apply the Pi theorem to determine the required number of Pi terms. First, let us use F, L and T as our system of basic dimensions. There are six variables and two reference dimensions namely F and L required so that four Pi terms (Step Three, $6 - 2 = 4$) are needed. For repeating variables, we can select D and γ so that (Fig. 2.10);

$$\Pi_1 = \delta D^a \gamma^b \qquad \text{or} \qquad (L)(L)^a (FL^{-3})^b = F^0 L^0$$

and $\begin{matrix} L: 1 + a - 3b = 0 & \Rightarrow a = -1 \\ F: \qquad b = 0 & \Rightarrow b = 0 \end{matrix}$ then $\Pi_1 = \frac{\delta}{D}$

Similarly, $\Pi_2 = hD^a\gamma^b$ follows $\begin{matrix} a = -1 \\ b = 0 \end{matrix}$ $\qquad \Pi_2 = \frac{h}{D}$

The remaining two Pi terms can be found using the same process and obtain;

$$\Pi_3 = \frac{d}{D} \qquad \Pi_4 = \frac{E}{D\gamma}$$

In summary for F, L, and T Pi terms = 6 − 2 = 4 and considering that D and γ are selected as repeating variables, we have;

$$\begin{matrix} \Pi_1 = \delta D^a\gamma^b & \Pi_2 = hD^a\gamma^b \\ \Pi_3 = dD^a\gamma^b & \Pi_4 = ED^a\gamma^b \end{matrix} \Rightarrow \begin{matrix} \Pi_1 = \dfrac{\delta}{D} & \Pi_2 = \dfrac{h}{D} \\ \Pi_3 = \dfrac{d}{D} & \Pi_4 = \dfrac{E}{D\gamma} \end{matrix} \Rightarrow \frac{\delta}{D} = \phi\left(\frac{h}{D}, \frac{d}{D}, \frac{E}{D\gamma}\right)$$

For M, L, T system, Pi terms = 6 − 3 = 3 and a closer look at dimensions of the variables listed reveal that only two reference dimensions, L and MT^{-2} are required. This is an example of the situation in which the number of reference dimensions differs from the number of basic dimensions and it does happen more often and can be detected by looking at the dimensions of the variables regardless of the systems that are used. Once the number of reference dimensions has been determined, we can proceed as before. Since the number of repeating variables must equal the number of reference dimensions, it follows that two reference dimensions are still required and we could use D and γ as repeating variables. The Pi terms would be determined in the same manner. For example, the Pi term containing E would be developed as

$$\Pi_4 = ED^a\gamma^b \qquad \text{or} \qquad (ML^{-1}T^{-2})(L)^a(ML^{-2}T^{-2})^b = (MT^{-2})^0L^0$$

$$\begin{aligned} 1 + b &= 0 \quad (\text{for } ML^{-2}) \\ -1 + a - 2b &= 0 \quad (\text{for } L) \end{aligned}$$

in addition, this result in, $a = -1$, $b = -1$, so that $\Pi_4 = \frac{E}{D\gamma}$ which is the same as Π_4 obtained in the FLT system. The other Pi terms would be the same, and the final result is the same, that is [9],

$$\frac{\delta}{D} = \phi\left(\frac{h}{D}, \frac{d}{D}, \frac{E}{D\gamma}\right)$$

Example 2.9 Determine the reference dimension of the following fluid equation, where the variables are related to the flow under consideration.

$$\Delta h = f(D, \gamma, \sigma)$$

Solution

MLT SYSTEM				*FLT SYSTEM*			
Δh	D	γ	σ	Δh	D	γ	σ
L	L	$\frac{M}{L^2T^2}$	$\frac{M}{T^2}$	L	L	$\frac{F}{L^3}$	$\frac{F}{L}$

$Pi\ Term = 4 - 3 = 1$ (MLT SYSTEM)

$Pi\ Term = 4 - 2 = 2$ (FLT SYSTEM)

Set Dimensional Matrix

MLT SYSTEM

	Δh	D	γ	σ
M	0	0	1	1
L	0	0	-2	0
T	0	0	-2	-2

FLT SYSTEM

	Δh	D	γ	σ
F	0	0	1	1
L	0	0	-3	-1
T	0	0	0	0

$$Rank = 2\ Pi\ Term = 4 - 2 = 2 \qquad \begin{cases} \Pi_1 = \dfrac{\Delta h}{D} \\ \Pi_2 = \dfrac{\sigma}{D^2\gamma} \Rightarrow \dfrac{\Delta h}{D} = \Phi\left(\dfrac{\sigma}{D^2\gamma}\right) \end{cases}$$

Example 2.10 Consider pressure drop in a tube of length ℓ, hydraulic diameter d, surface roughness $\in$, with fluid of density ρ and viscosity μ moving with average velocity v.

Solution This can be expressed as.

$$f(\Delta P, U, d, \ell, \in, \rho, \mu) = 0$$

Now $m = 7$ since the phenomenon involves 7 independent parameters.

We select ρ, U, d as repeating variables, so that all 3 dimensions are represented.

Now 4π which is $\{(7-3)\}$ parameters are determined as

$$\Pi_1 = \rho^{a_1} U^{b_1} d^{c_1} \Delta p$$

$$\Pi_1 = \rho^{a_2} U^{b_2} d^{c_2} \mu$$

$$\Pi_1 = \rho^{a_3} U^{b_3} d^{c_3} \in$$

$$\Pi_1 = \rho^{a_4} U^{b_4} d^{c_4} \ell$$

Now basic units are as follows;

$$\rho \to ML^{-3}$$
$$U \to LT^{-1}$$
$$d \to L$$
$$\Delta p \to ML^{-1}T^{-2}$$
$$\mu \to ML^{-1}T^{-1}$$
$$\in \to L$$
$$\ell \to L$$

All Π parameters $\to M^0L^0T^0$, then the above four equations yield in the following relations;

$$a_1 = -1; \quad b_1 = -2; \quad c_1 = 0$$
$$a_2 = -1; \quad b_2 = -1; \quad c_2 = -1$$
$$a_3 = 0; \quad b_3 = 0; \quad c_3 = -1$$
$$a_4 = 0; \quad b_4 = 0; \quad c_4 = -1$$

Thus writing $\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4)$

Implies that $\frac{\rho U d}{\mu} = \mathrm{Re}$ (Reynolds Number)

Therefore $\frac{\Delta p}{\rho U^2} = f(\mathrm{Re}, {}^{\in}\!/_{d}, {}^{L}\!/_{d})$

where the ratio of ${}^{\in}\!/_{d}$ is called relative roughness. Similarly, other sets of Π parameters can be chosen to describe the phenomena. Thus, though it does not give the actual relationship, but it puts the data in a compact form.

2.12.4 Uniqueness of Pi Terms

The Pi terms obtained depend on the somewhat arbitrary selection of repeating variables. For example, in the problem of studying the pressures drop in a pipe. This is a reflection on the process that is used to determine Pi terms by the method of repeating variables. The following simple example about pressure drop in a pipe can explain the situation as follow;

$$\Delta p_\ell = f(D, \rho, \mu, V)$$

Selecting D, V, and ρ as repeating variables, we have; $\frac{\Delta p_\ell}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$

Selecting D, V, and μ as repeating variables, we have; $\frac{\Delta p_\ell D^2}{V\mu} = \phi\left(\frac{\rho V D}{\mu}\right)$

Both are correct, and both would lead to the same final equation for the pressure drop. There is not a unique set of Pi terms, which arises from a dimensional analysis.

The functions ϕ_1 and ϕ_2 are will be different because the dependent Pi terms are different for the two relationships. However, the required *number* of Pi terms is fixed, and once a correct set is, determined, all other possible sets can be developed from this set by combinations of products of powers of the original set. Thus, if we have a problem involving, say, three Pi terms [9],

$$\Pi_1 = \phi(\Pi_2, \Pi_3)$$

By now combining the Pi terms, we can always for a new set of Pi terms as below;

$$\Pi'_2 = \Pi_2^a \Pi_3^b$$

Where a and b are arbitrary exponents. Then the relationship could be expressed as;

$$\left\{ \underbrace{\Pi_1 = \phi_1\left(\Pi'_2, \Pi_3\right) = \phi_2\left(\Pi_2, \Pi'_2\right)}_{\text{All are correct}} \right.$$

Note that, the required number of Pi terms cannot be reduced by this manipulation; only the form of the Pi is altered. By using this technique, we see that the Pi terms in the following form could be obtained from those first set in above and that is where we multiply Π_1 in first set by Π_2, so that;

$$\begin{aligned} \frac{\Delta p_\ell D^2}{V\mu} &= \phi_1\left(\frac{\rho V D}{\mu}\right) \qquad \Rightarrow \qquad \frac{\Delta p_\ell D}{\rho V^2} \times \left(\frac{\rho V D}{\mu}\right) \\ &= \left(\frac{\Delta p_\ell D^2}{V\mu}\right) \qquad\qquad \frac{\Delta p_\ell D^2}{V\mu} = \phi\left(\frac{\rho V D}{\mu}\right) \end{aligned}$$

which is Π_1 of second set.

2.13 Determination of Pi Terms by Inspection

Finding Pi terms using the method of repeating variables was well defined in Sect. 2.7.1 of this chapter along with few examples that were presented. This method was providing a step-by-step procedure that if it is performing correctly, then it will provide a correct and complete Pi terms. Beautify of repeating variables method is its simplicity and straightforward approach, although rather tedious process, in particular when for a problem with large number of variables is under consideration. Since the only restrictions placed on the Pi terms are that they be

1. Correct in number.
2. Dimensionless, and
3. Independent

In this section, we present possibility of forming Pi terms using a method know as inspection method with resorting to the more formal procedure as we saw in determination of Pi terms using repeating variables method.

To illustrate determination of Pi terms by inspection, which is the method most intelligent and educated engineers and scientist do, we again consider the pressure drop per unit length along a smooth pipe problem. In order to solve this problem, regardless of the techniques that can be applied the starting point is the same as before. We are again seeking an approach to determine the variables, which in this case are;

$$\Delta p_\ell = f(D, \rho, \mu, V)$$

Next, the dimensions of the variables are listed

$$\begin{aligned} \Delta p_\ell &\doteq FL^{-3} \\ D &\doteq L \\ \rho &\doteq FL^{-4}T^2 \\ \mu &\doteq FL^{-2}T \\ D &\doteq LT^{-1} \end{aligned}$$

As result of the number of reference, dimensions determined (Also see Example-2 of Sect. 2.7.3). Now Pi theorem will tell you how many Pi terms are required. Since there are five variables and three reference dimensions, two Pi terms are needed ($k = n - r = 5 - 3 = 2$). We always let Π_1 to contain the dependent variable, which in this example is Δp_ℓ, which has the dimension of FL^{-3} in FLT system. We need to combine it with other variables so that a nondimensional product will result and the first possibility is as follow, which is associated with ρ to start with;

$$\frac{\Delta p_\ell}{\rho} = \frac{(FL^{-3})}{(FL^{-4}T^2)} = \frac{L}{T^2} \qquad \text{(cancels } F\text{)}$$

Although the dependency on F has been eliminated, but the ratio of $\Delta p_\ell/\rho$ is not dimensionless. To eliminate the T dependency, we divide by V^2, so we have;

$$\left(\frac{\Delta p_\ell}{\rho}\right)\frac{1}{V^2} = \left(\frac{L}{T^2}\right)\frac{1}{(LT^{-1})} = \frac{1}{L} \qquad \text{(cancels } T\text{)}$$

Finally to make the combination dimensionless we multiply by D so that;

$$\left(\frac{\Delta p_\ell}{\rho}\right)D = \left(\frac{1}{L}\right)(L) = L^0 \qquad \text{(cancel } L\text{)}$$

Thus

$$\Pi_1 = \frac{\Delta p_\ell D}{\rho V^2}$$

Next is to form the second Pi term Π_2 by selecting the variable that was not used in first step where Π_1 is involved, which in this case is μ. We combine μ with the other variables to make a dimensionless combinations, but do not use Δp_ℓ in Π_2, since we want the dependent variable to appear only in Π_1. For example, divide μ and ρ in order to eliminate *F*, then by *V* to eliminate *T*, and finally by *D* to eliminate *L*, thus we obtain Π_2 in the following form;

$$\Pi_2 = \frac{\mu}{\rho V D} = \frac{(FL^{-2}T)}{(FL^{-4}T^2)(LT^{-1})(L)} = F^0 L^0 T^0$$

and, therefore

$$\frac{\Delta p_\ell D}{\rho V^2} = \phi\left(\frac{\mu}{\rho V D}\right)$$

Which is the same result that we obtained by using method of repeating variables.

As additional concern, when we forming Pi term using determination by inspection method, is to make certain that the Pi terms are all independent. For example in pipe flow problem in Sect. 2.5.1 of this Chapter, Π_2 contains μ, which does not appear in Π_1, and therefore these two Pi terms are independent of each other. In a more general case, a Pi term would not be independent of the others in a given problem if it can be formed by some combination of the others. For example, if Π_2 can be formed by a combination of let say, Π_3, Π_4, and Π_5 such as

$$\Pi_2 = \frac{\Pi_3^2 \Pi_4}{\Pi_5}$$

then Π_2 is not an independent Pi term. We can ensure that each Pi term is independent of those preceding it by incorporating a new variable in each Pi term. Note that forming Pi terms by method of inspection is essentially equivalent to the repeating variable method, is less structured [9].

2.14 Common Dimensionless Groups

In this section, we present a list of variables that commonly arise in dimensional analysis and in particular, fluid mechanics problems following with more details of these dimensionless numbers. The list is not by any means a conclusive and complete but a broad range of variables that likely to be found in a typical fluid mechanic or fluid dynamic problem.

It is also possible to provide a physical interpretation to the dimensionless groups, which can be helpful in assessing their influence in a particular application. When combinations of these variables are present, it is standard practice to combine them into some common dimensionless group (Pi terms) given in Table 2.5. These combinations appear so frequently that special names are associated with them, as indicated in the table.

Table 2.5 Some Common Variables and Dimensionless Group in Fluid Mechanics [9]

Dimensionless groups	Name	Interpretation (Index of force kalin indicated)	Types of applications
$\frac{\rho V l}{\mu}$	Reynolds number, Re	$\frac{\text{inertia force}}{\text{viscous force}}$	Generally of importance in all types of fluid dynamics problems
$\frac{V}{\sqrt{gl}}$	Froude number, Fr	$\frac{\text{Inertia force}}{\text{gravitational force}}$	Flow with a free surface
$\frac{p}{\rho V^2}$	Euler number, Eu	$\frac{\text{Pressure force}}{\text{Inertia force}}$	Problems in which pressure, or pressure differences, are of interest
$\frac{\rho V^2}{E_v}$	Cauchy number,[a] Ca	$\frac{\text{Inertia force}}{\text{Compressibility force}}$	Flows in which the compressibility of the fluid is important
$\frac{V}{c}$	Mach number,[a] Ma	$\frac{\text{Inertia force}}{\text{Compressibility force}}$	Flows in which the compressibility of the fluid in important
$\frac{\omega l}{V}$	Strouhal number, St	$\frac{\text{Inertia (local) force}}{\text{Inertia (convective) force}}$	Unsteady flow with a characteristic frequency of oscillation
$\frac{\rho V^2 l}{\sigma}$	Weber number, We	$\frac{\text{Inertia force}}{\text{Surface tension force}}$	Problems in which surface tension is important

[a] Thc Cauchy number and the Mach number are related and either can be used as an index of the relative effec ts of inertia and com- p-ressibihty. See accompanying discussion

Variables: Acceleration of gravity, g; Hulk modulus, E_v; Characteristic length, l; Density, ρ; Frequency of oscillaling flow, ω; Pressore, p (or Δp); Speed of sound, c; Surface tension, σ; Velocity, V; Viscosity, μ

It is also often possible to provide a physical interpretation to the dimensionless groups, which can be helpful in assessing their influence in particular, applications. Brief description of each number is given as follow;

1. Froude Number

$$\mathrm{Fr} = \frac{V}{\sqrt{g\ell}} \gg \mathrm{Fr}^2 = \frac{V^2}{g\ell} = \frac{\rho V \ell^2}{\rho g \ell^3}$$

a. In honor of William Froude (1810 ~ 1879), a British civil engineer, mathematician, and naval architect who pioneered the use of towing tanks for the study of ship design
b. Froude number is the ratio of the forces due to the acceleration of a fluid particle (inertial force) to the force due to gravity (gravity forces).
c. Froude number is significant for flows with free surface effects.
d. Froude number less than unity indicate subcritical flow and values greater than unity indicate supercritical flow.

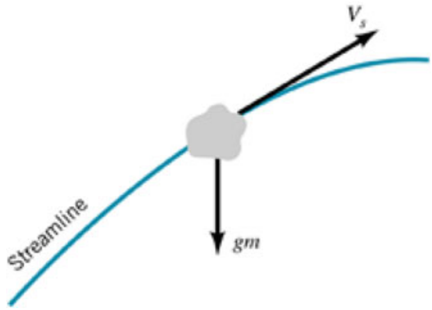


Fig. 2.11 The force of gravity acting on a fluid particle moving along a streamline [9]

Example 2.11: Froude Number The magnitude of the component of inertia force F_1 along the streamline can be expressed as $F_1 = a_s m$, where a_s is the magnitude of the acceleration along the streamline for a particle having a mass m. From study of particle motion along a curved path [9], we know that (See Fig. 2.9);

$$a_s = \frac{dV_s}{dt} = V_s \frac{dV_s}{ds}$$

where s is measured along the streamline. If we write the velocity, V, and length, s, in dimensionless form;

$$V_s^* = \frac{V_s}{V} \qquad s^* = \frac{s}{\ell}$$

where V and ℓ represent some characteristic velocity and length, respectively, then

$$a_s = \frac{V^2}{\ell} V_s^* \frac{dV_s^*}{ds^*} \qquad \text{and} \qquad F_I = \frac{V^2}{\ell} V_s^* \frac{dV_s^*}{ds^*} m$$

The magnitude of the weight of the particle, F_G, is given by $F_G = gm$, so the the ratio of the inertia to the gravitational force is given such that, the force ratio F_I/F_G is proportional to $V^2/g\ell$, and the square root of this ratio, $V/\sqrt{g\ell}$, is called the *Froude number.*

$$\mathrm{Fr} = \frac{F_I}{F_G} = \frac{V^2}{g\ell} V_s^* \frac{dV_s^*}{ds^*} \equiv \frac{V^2}{g\ell} \equiv \frac{V}{\sqrt{g\ell}}$$

And rest of dimensionless numbers again are mentioned here to refresh memory of the readers and remind them that you will be seeing these numbers over and over in various application of fluid mechanics, fluid dynamics, etc.

2. Reynolds Number

$$\mathrm{Re} = \frac{\rho V \ell}{\mu} = \frac{V \ell}{\upsilon}$$

a. In honor of Osborne Reynolds (1842 ~ 1912), the British engineer who first demonstrated that this combination of variables could be used as a criterion to distinguish between laminar and turbulent flow.
b. The Reynolds number is a measure of the ration of the inertia forces to viscous forces.
c. If the Reynolds number is small (Re < < 1), this is an indication that the viscous forces are dominant in the problem, and it may be possible to neglect the inertial effects; that is, the density of the fluid will not be an important variable.
d. Flows with very small Reynolds numbers are commonly referred to as "*creeping flows*".
e. For large Reynolds number flow, the viscous effects are small relative to inertial effects and for these cases, it may be possible to neglect the effect of viscosity and consider the problem as one involving a "nonviscous" fluid.
f. Flows with "large" Reynolds number generally are turbulent. Flows in which the inertia forces are "small" compared with the viscous forces are characteristically laminar flows.

3. Euler Number

$$\mathrm{Eu} = \frac{p}{\rho V^2} \equiv \frac{\Delta p}{\rho V^2}$$

a. In honor of Leonhard Euler (1707 ~ 1783), a famous Swiss mathematician who pioneered work on the relationship between pressure and flow
b. Euler's number is the ratio of pressure force to inertia forces. It is often called the *pressure coefficient*, C_p.

4. Cavitations Number

$$\mathrm{Ca} = \frac{p_r - p_v}{\frac{1}{2}\rho V^2}$$

a. For problems in which cavitations is of concern, the dimensionless group $(p_r - p_v)/\frac{1}{2}\rho V^2$ is commonly used, where p_v is the vapor pressure and p_r is some reference pressure.
b. The cavitations number is used in the study of cavitations phenomena.
c. The smaller the cavitations number, the more likely cavitations is to occur.

5. Cauchy Number

$$\mathrm{Ca} = \frac{\rho V^2}{E_v}$$

a. The Cauchy number is named in honor of Augustine Louis deCauchy (1789~1857), a French engineer, mathematician, and hydrodynamics.

6. Mach Number

$$\mathrm{Ma} = \frac{V}{c} = \frac{V}{\sqrt{\frac{E_\upsilon}{\rho}}} = V\sqrt{\frac{\rho}{E_\upsilon}}\mathrm{Ma}^2 = \frac{\rho V^2}{E_\upsilon} = \mathrm{Ca}$$

a. The Cauchy number is named in honor of Ernst Mach (1838 ~ 1916) an Austrian physicist and philosopher.
b. Both Cauchy and Mach number may be used in problems in which fluid compressibility is important.
c. Both numbers can be interpreted as representing an index of the ration of inertial force to compressibility force, where V is the flow speed and c is the local sonic speed.
d. Mach number is a key parameter that characterizes compressibility effects in a flow.
e. When the Mach number is relatively small (say, less than 0.3), the inertial forces induced by the fluid motion are not sufficiently large to cause a significant change in the fluid density, and in this case the compressibility of the fluid can be neglected.
f. For truly incompressible flow, $c = \infty$ so that $M = 0$.

7. Strouhal Number

$$\mathrm{St} = \frac{\omega \ell}{V}$$

a. In honor of Vincenz Strouhal (1850 ~ 1922), who used this parameter in his study of "singing wires." The most dramatic evidence of this phenomenon occurred in 1940 with the collapse of the Tacoma Narrow bridges. The shedding frequency of the vortices coincided with the natural frequency of the bridge, thereby setting up a resonant condition that eventually led to the collapse of the bridge.
b. This parameter is important in unsteady, oscillating flow problems in which the frequency of the oscillation is ω.
c. This parameter represents a measure of the ration of inertial force due to the unsteadiness of the flow (local acceleration) to the inertial forces due to change in velocity from point to point in the flow field (convective acceleration). This type of unsteady flow may develop when a fluid flows past a solid body (such as a wire or cable) placed in the moving stream.

For example, in a certain Reynolds number range, a periodic flow will develop downstream from a cylinder placed in a moving stream due to a regular pattern of vortices that are shed from the body. See Fig. 2.12 below;

8. Weber Number

$$\mathrm{We} = \frac{\rho V^2 \ell}{\sigma}$$

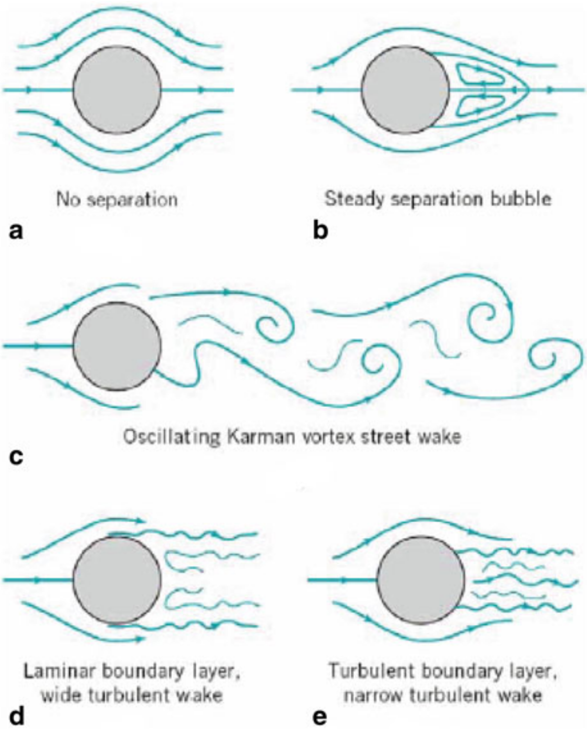


Fig. 2.12 Different condition flow based on reynolds number

a. Named after Moritz Weber (1871 – 1951), a German professor of naval mechanics who was instrumental in formalizing the general use of common dimensionless groups as a basis for similitude studies.
b. Weber number is important in problem in which there is an interface between two fluids. In this situation, the surface tension may play an important role in the phenomenon of interest.
c. Weber number is the ratio of inertia forces to surface tension forces.
d. Common examples of problems in which Weber number may be important include the flow of thin film of liquid, or the formation of droplets or bubbles.
e. The flow of water in a river is not affected significantly by surface tension, since inertial and gravitational effects are dominant ($\boldsymbol{We} >> 1$).

2.15 Rayleigh's Indicial Method

This alternative method is also based on the fundamental principle of dimensional homogeneity of physical variables involved in a problem.

Procedure:

1. The dependent variable is identified and expressed as a product of all the independent variables raised to an unknown integer exponent.
2. Equating the indices of n fundamental dimensions of the variables involved, n independent equations are obtained.
3. These n equations are solved to obtain the dimensionless groups.

Example 2.12 Let us illustrated this method by solving the pipe flow problem that initially, was introduced in Sect. 2.5.1 and was solved with *Method of Repeating Variables* in Sect. 1.12.1 in above.
Step 1: Here the dependent variable $\Delta p/\ell$ can be written as;

$$\frac{\Delta p}{\ell} = AV^a D_h^b \rho^c \mu^d \qquad \text{(where, } A \text{ is a dimensionless constant)}$$

Step 2: Inserting the dimensions of each variable in the above equation, using Appendix E, we obtain,

$$ML^{-2}T^{-2} = A(LT^{-1})^a (L)^b (ML^{-3})^c (ML^{-1}T^{-1})^d$$

Equating the indices of *M*, *L*, and *T* on both sides, we get;

$$\begin{cases} c + d = 1 \\ a + b - 3c - d = -2 \\ -a - d = -2 \end{cases}$$

Step 3: There are three equations and four unknowns. Solving these equations in terms of the unknown *d*, we have;

$$\begin{cases} a = 2 - d \\ b = -d - 1 \\ c = 1 - d \end{cases}$$

Hence, we can write the original equation in above in following format as;

$$\frac{\Delta p}{\ell} = AV^{2-d} D_h^{-d-1} \rho^{1-d} \mu^d$$

or $$\frac{\Delta p}{\ell} = \frac{AV^2 \rho}{D_h} \left(\frac{\mu}{VD_h \rho} \right)^d$$

and $$\frac{\Delta p D_h}{\ell \rho V^2} = A \left(\frac{\mu}{VD_h \rho} \right)^d$$

Therefore, we see that there are two independent dimensionless terms of the problem, namely, Both Buckingham's methods and Rayleigh's method of dimensional analysis determine only the relevant independent dimensionless parameters of a problem, but not the exact relationship between them.

For example, the numerical values of A and d can never be known from dimensional analysis. They will be found out from experiments.

If the system of equations is solved for the unknown c, it results,

$$\frac{\Delta p D_h^2}{\ell V \mu} = A\left(\frac{V D_h \rho}{\mu}\right)^c$$

Therefore, different interdependent sets of dimensionless terms are obtained with the change of unknown indices in terms of which the set of indicial equations are solved. This is similar to the situations arising with different possible choices of repeating variables in Buckingham's Pi theorem.

To illustrate Rayleigh's Method further, we present the following example, using the MLT system.

Example 2.13 We are interested in the Drag D, which is a force, on a ship. We want to know, what exactly is the drag function?

Solution In order to solve this problem, the proper variables that solution is depend on need to be chosen correctly, though selection of such variables depends largely on one's experience in the topic of fluid mechanics and dynamic. It is known that drag depends on the following relation;

$$D = f(\ell, \rho, \mu, V, g) \tag{i}$$

Where, f is some function and other variables with bracket can be defined as follows;

Quantity	Symbol	Dimension
Size (Length)	ℓ	L
Dynamic Viscosity	μ	$ML^{-1}T^{-1}$
Density	ρ	ML^{-3}
Velocity	V	LT^{-1}
Gravity	g	LT^{-2}

Using Rayleigh Method, we assume that,

$$D = C\ell^a \rho^b \mu^c V^d g^e \tag{ii}$$

Where C is a dimensionless constant, and a, b, c, d and e are exponents, whose values are need to be determined. Looking at relation (ii) in above we know that the dimension on left hand side of (ii) is forced to be equal to those on right hand side and as one can see, here we use only three independent dimensions for the variables on the right hand side, namely M, L, and T.

Step1: Setting up the equation

We write the Eq. (i) in terms of dimensions only and replacing the quantities with their respective units given in above table, then we have;

$$\frac{ML}{T^2} = (L)^a \left(\frac{M}{L^3}\right)^b \left(\frac{M}{LT}\right)^c \left(\frac{L}{T}\right)^d \left(\frac{L}{T^2}\right)^e \tag{iii}$$

Left hand side dimension is determined based on what is observed on right hand side in term of M, L, and T, therefore the exponents of right and left hand side should the same, namely $M^1 L^1 T^{-2}$

Step2: Solving for the exponents

Equating the exponents on both side to each other in terms of their respective fundamental units:

$$\begin{cases} M : 1 & = b + c & \text{since } M^1 = M^b M^c \\ L : 1 & = a - 3b - c + d + e & \text{since } L^1 = L^a L^{-3b} L^{-c} L^d L^e \\ T : -2 & = -c - d - 2e & \text{since } T^{-2} = T^{-c} T^{-d} T^{-2e} \end{cases} \tag{iv}$$

As it can be seen there are three equation in (v), there exists 5 unknown variables. This means that a complete solution cannot be obtained. Thus, we choose to solve a, b and d in terms of c and e. Note that these choices are based on experience and there is no set rule for it. Therefore, we have:

$$\begin{cases} \text{From } \; M : b = 1 - c & \text{(v)} \\ \text{From } \; T : d = 2 - c - 2e & \text{(vi)} \\ \text{From } \; L : a = 1 + 3b + c - d - e & \text{(vii)} \end{cases}$$

Solving (vi), (vii) and (viii) simultaneously, we obtain

$$\{a = 2 - c + e$$

Substituting the exponents back into the original Eq. (i), we obtain:

$$D = C\ell^{2+e-c} \rho^{1-c} \mu^c V^{2-c-2e} g^e$$

Collecting like exponents together, we get:

$$D = C\left(\frac{V^2}{\ell g}\right)^{-c} \left(\frac{V\ell\rho}{\mu}\right)^{-e} \rho \ell^2 V^2 \tag{viii}$$

which means that:

$$D = C\ell^2 \ell^e \ell^c \rho \rho^{-c} \mu^c V^2 V^{-c} V^{-2e} g^e \tag{ix}$$

For the different exponents;

Terms with exponent of 1: $C\rho$

Terms with exponent of 2: $\ell^2 V^2$

$$\text{Terms with exponent of e}: \ell^e V^{-2e} g^e = \left(\frac{\ell g}{V^2}\right)^e = \left(\frac{V^2}{\ell g}\right)^{-e} \tag{x}$$

$$\text{Terms with exponent of c}: \ell^{-c} \rho^{-c} \mu^c V^{-c} = \left(\frac{\ell \rho V}{\mu}\right)^{-c} \tag{xi}$$

The right sides of (xi) and (xii) are known as the dimensionless groups.

Step3: Determining the dimensionless groups

Note that e and c are unknown. Consider the following cases:

If e = 1 then (xi) becomes $\left(\frac{\ell g}{V^2}\right)$

If e = −1 then (xi) becomes $\left(\frac{V^2}{\ell g}\right)$

If c = 1 then (xii) becomes $\left(\frac{\mu}{\ell \rho V}\right)$

If c = −1 then (xii) becomes $\left(\frac{\ell \rho V}{\mu}\right) = \left(\frac{\ell V}{\nu}\right)$

Where $\nu = \mu/\rho$ which is known as the kinematic viscosity of the fluid. We can continue doing same thing with other exponents. It turns out that:

$\textit{Reynolds Number} \equiv \frac{V\ell}{\nu} = N_R = Re$

$\textit{Froude Number} \equiv \left(\frac{V^2}{\ell g}\right)^{\frac{1}{2}} = \frac{V}{\sqrt{\ell g}} = N_F = Fr$

Where N_R or Re and N_F or Fr are the usual notation for the *Reynolds* and *Froude Numbers* respectively and we know the form previous sections at the beginning of this chapter.

Now choosing the exponents of − 1 for c and − 1/2 for e, which results in the *Reynolds* and *Froude Numbers* respectively, we obtain:

$$D = g(Fr, Re)\rho \ell^2 V^2 \tag{xiii}$$

Where $g(Fr, Re)$ is a dimensionless function and above relation (xiii), can be written in form of:

$$\frac{D}{\rho \ell^2 V^2} = g(Fr, Re) \tag{xiv}$$

Which is a dimensionless quantity, and a function of only 2 variables instead of 5 that we saw at the beginning of this example. This dimensionless quantity turns out to be the drag coefficient, C_D.

$$C_D = \frac{D}{\rho \ell^2 V^2}$$

Note that the Rayleigh Method has limitations because of the premise that an exponential relationship exists between the variables. So in today's Dimensional Analysis, although there are other methods, but The *Buckingham π Theorem Method* is recommended.

2.16 The Purposes and Usefulness of Dimensional Analysis

Note: This section was part of lecture that was given by Professor John M. Cimbala at Penn State University in 2005.

As part of description of dimensional analysis as part of opening of this book and through chapter so far, we have mentioned that every quantitative statement about an objective magnitude is necessarily composed of two parts or factors as follows;

1. A number
2. A statement of the unit of measurement

The number is the mathematical ratio of the magnitude to that of the specific limit. Similarly, the ultimate end of all applied mathematics is the numerical evaluation, by the working of an arithmetical sum, of the magnitude of some physical quantity, which is inferred from the known magnitudes of others. The answer to the sum of all will be meaningful and intelligible only if the unit applicable to the calculated number is known with certainty. Thus, the number 4.2 may be part of the statement of a length, but this number is meaningless until is known whether it refers to centimeters, feet, yards or miles, as the case may be. Similarly the answer to a calculation of an amount of energy may give a numerical answer 4.2, but this is unintelligible until it is known whether it refers to ergs, foot-lbs., or kilowatt-hours.

By definition, a similarity number N is a nondimensional product of dimensional variables u_n (See Sect. 2.12) of the form;

$$N = u_1^{\alpha} u_2^{\beta} \cdots u_n^{n} \qquad [N] = 1$$

This number generally symbolized by capital N.

Reference similarity numbers contain only reference variables, while *Independent* similarity numbers contain only independent variables. A similarity number is dependent if it contains one or more dependent variables. Usually, it is not desirable to have more than one dependent variable in a single similarity number [4].

Any physical quantity can be completely defined by a number and any arbitrary chosen unit, if the unit is exactly specified. As we said before and Table 2.3 or 2.4 are shown, a collection of units for the measurement of physical quantities is known as a system of units, and, in such a system, the various units may be either arbitrarily defined, or they may be made depend in a simple way on other units.

This is the base for dimensional analysis and similarity where upon them, scientists undertaken the first complete investigation of the application of similarity techniques in the solution of partial differential equations. Such solutions are obtained with transformations that reduce a system of partial differential equations to one of ordinary differential equations. See Chap. 4 of this book on this particular techniques and approach of such nonlinear partial differential equations.

Dimensional analysis is a very powerful tool, not just in fluid mechanics, but also in many disciplines. It provides a way to plan and carry out experiments, and enables one to scale up results from model to prototype. Here we present an example of usefulness of dimensional analysis of prototyping design of an airplane wing and

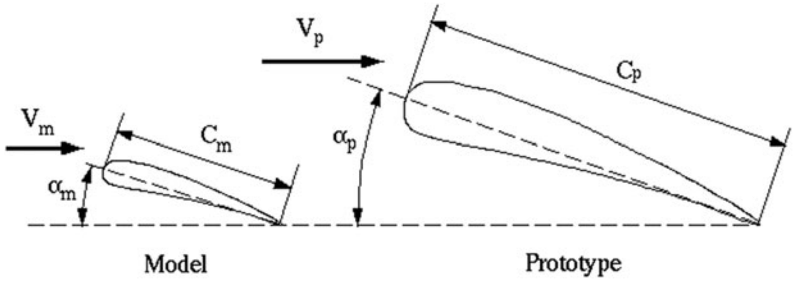


Fig. 2.13 Airplane wing design shape

how to take a model to a prototype stage and then to full construction of such wing in mass production by aerospace industries (Fig. 2.11).

- Consider, for example, the design of an airplane wing. The full-size wing, or prototype, has some chord length, C_p, operates at speed V_p, and generates a lift force, L_p, which varies with angle of attack. In addition, the fluid properties of importance to this flow are the density and viscosity. After the preliminary design, it is usually necessary to perform experiments to verify and fine-tune the design. To save both time and money, these tests are usually conducted with a smaller scale model in a wind tunnel or water tunnel. In the sketch above, a geometrically similar model is constructed. In this case, the model is smaller than the prototype. In some cases the opposite are true; i.e. it may be prudent to build a large model of some small prototype in order to perform more accurate experimental analysis.
- The goal of the experimental tests is to find a relationship between the dependent variable (in this case the wing's lift) and the independent variables in the problem (in this case the velocity, the wing's angle of attack, chord length, and the density and viscosity of the fluid). Note that here we are neglecting the speed of sound, which is only important at very high speeds. The functional relationship can be stated as follows:

$$L = \text{function of } (V, \alpha, c, \rho, \mu)$$

- There is a wrong way and a right way to conduct the experiments. The wrong way is to try to analyze the dependence of lift on each of the five independent variables separately. In other words, run the tests at many velocities (to see the effect of velocity on lift), and many angles of attack (to see the effect of angle of attack on lift), with many different model sizes (to see the effect of chord length on lift), and in many different fluids (to see the effect of viscosity and density on lift). This would take an enormous amount of time and resources, and it would be very difficult to summarize the results succinctly.
- The *right way* to do the experiments is to first perform a dimensional analysis of the above functional relationship, which leads to a revised form of the relationship in terms of nondimensional parameters or nondimensional groups. In this particular problem, dimensional analysis yields

$$\frac{L}{\rho V^2 c^2} = \text{function of } \left(\frac{\rho Vc}{\mu}, \alpha \right)$$

which is much simpler than the original functional relationship. In particular, instead of a dependent variable as a function of five independent variables, the problem has been reduced to one dependent parameter as a function of only two independent parameters. Furthermore, each of these three parameters is dimensionless, which makes them completely independent of the unit system used in the measurements.

- The parameter on the left is a kind of lift coefficient (the actual lift coefficient has a factor of 2 thrown in for convenience), while the first independent parameter on the right is called the **Reynolds number**. Angle of attack is already dimensionless, so it is a dimensionless group by itself.
- One plot is enough to completely describe the above functional relationship. In particular, lift coefficient is plotted versus angle of attack, and several curves are plotted at constant Reynolds number. This single plot is then valid for any size wing, in any Newtonian incompressible fluid, and at any speed. When experiments are conducted after performing the dimensional analysis, it is realized that only one wind tunnel model needs to be made, and only one fluid needs to be used (that fluid can be air or water or any other Newtonian incompressible fluid)! The wind tunnel or water tunnel test needs to consist of simply measuring lift as a function of velocity and angle of attack. Results of the experiment are plotted nondimensionally as indicated above.

Dynamically Similarity

The principle of ***dynamic similarity*** can be stated as follows:

- If the model and prototype are geometrically similar (i.e. the model is a perfect scale replica of the prototype), and if each independent dimensionless parameter for the model is equal to the corresponding independent dimensionless parameter of the prototype, then the *dependent* dimensionless parameter for the prototype will be equal to the corresponding *dependent* dimensionless parameter for the model.
- Consider the airplane wing example above. In this case, the two independent dimensionless parameters (those on the right hand side) are Reynolds number and angle of attack. The dependent parameter is the lift coefficient. The model wing in the wind tunnel must obviously be set at the same angle of attack as the desired angle of attack of the prototype. In order to achieve dynamic similarity, the Reynolds number of the model must also equal that of the prototype. Then, dynamic similarity assures us that the lift coefficient of the prototype will equal that of the model. Mathematically, we can solve for the wind tunnel velocity, V_m, required to match Reynolds number, and we can scale up the lift measurement

from the wind tunnel tests to the full scale prototype as follows:

$$\alpha_m = \alpha_p$$

$$Re_m = \frac{\rho_m V_m c_m}{\mu_m} = Re_p = \frac{\rho_p V_p c_p}{\mu_p}$$

$$V_m = \frac{\rho_p}{\rho_m}\frac{c_p}{c_m}\frac{\mu_m}{\mu_p}V_p$$

$$\frac{L_p}{\rho_p V_p^2 c_p^2} = \frac{L_m}{\rho_m V_m^2 c_m^2}$$

$$L_p = \frac{\rho_p}{\rho_m}\left(\frac{V_p}{V_m}\right)^2\left(\frac{c_p}{c_m}\right)^2 L_m$$

In this manner, we can set the wind tunnel speed properly to match Reynolds number. Then, after measuring the lift on the model wing, L_m, we can properly scale (using the last equation above) to predict the lift, L_p, on the prototype.

The Buckingham Pi Technique

- The ***Buckingham Pi technique*** is a formal "cookbook" recipe for determining the dimensionless parameters formed by a list of variables. There are six steps, which are outlined below, followed by a couple of example problems. Other examples can be found in the textbook and homework problems.
- ***Step*** 1. All the variables are listed and counted—The total number of variables is assigned to variable n. Note: The dependent variable as well as all the independent variables must be included in n, even if they are dimensionless (angles, for example, are already dimensionless, but still get counted in this first step).
- ***Step*** 2. The primary dimensions of each of the n variables are listed. As discussed in the text, either the force-length-time-temperature set or the mass-length-time-temperature set of primary dimensions can be used. In this course, only the latter will be used. Tables 2.3 and 2.4 in the text provides the dimensions of most of the variables needed in fluid mechanics, and is useful in this step.
- ***Step*** 3. The number of repeating variables, j, is found, where j is usually the number of primary dimensions represented in the problem. There are more formal mathematical ways to find j, but in most problems it is sufficient to simply count the number of primary dimensions available from all the original variables. For example, if mass, length, and time each appear in at least one variable, j is set to 3. As the Buckingham Pi technique progresses, it sometimes becomes clear that things just are not working out. In such cases, j should be reduced by 1 and Steps 4 through 6 should be repeated. Once j is found, the number of dimensionless parameters (or "Pi" groups) expected is $k = n - j$, where k is the number of Pi groups. This equation relating k to n and j is part of the Buckingham Pi Theorem.
- ***Step*** 4. A total of j "repeating variables" are chosen, which will be used to generate the Pi groups. It is somewhat arbitrary which variable to pick here, especially when n is large. The main thing that should be kept in mind is that these repeating variables may appear in each of the Pi groups. Therefore, it is important which variables are chosen. Some rules are helpful:

- The dependent variable should not be picked as a repeating variable. Otherwise, it will appear in more than one Pi, which will lead to an implicit expression in Step 6 below.
- The repeating variables must not be able to form a Pi group all by themselves. Otherwise, the procedure in Step 5 will be fruitless.
- Each of the primary dimensions in the problem must be represented. For example, if mass, length, and time appear in the original n variables, these three primary dimensions must also each appear at least once in the repeating variables.
- Variables, which are already dimensionless (such as angles), should not be picked. Such variables are already dimensionless Pi groups, and cannot contribute to formulating the remaining Pi groups.
- Two variables with the same dimensions or with dimensions differing by only an exponent should never be picked. For example, if some area and some length are among the list of variables, the length should be chosen as a repeating variable. It would be incorrect to also select the area as a repeating variable, since its dimensions are simply the square of the length, and can contribute nothing additional to the formulation of the Pi groups.
- Variables with very basic dimensions and/or variables that are "common" should be picked as repeating variables. This is perhaps the most difficult aspect of dimensional analysis, especially for the beginning student. After much practice, it becomes more or less obvious which variables to pick. For example, if there is a length, that length should be picked as a repeating variable since it is very basic and desirable in the Pi groups. Likewise, some velocity, mass, time, or density are also good choices. In most fluid flow problems, other flow properties like viscosity or surface tension should not be chosen if there are also more "basic" variables to choose from, such as a length, velocity, time, mass, or density. Why? Because it is usually not desirable to have viscosity or surface, tension appears in each of the Pi groups.

• ***Step*5**. The Pi groups are formulated by multiplying each of the remaining variables (those that were not chosen as repeating variables) in turn by the repeating variables, each in turn raised to some unknown exponent. The exponents are found algebraically by forcing the Pi to be dimensionless. The convention is to form the first Pi using the dependent variable. Note that Pi groups can be "adjusted" after they are formed in order to agree with the dimensionless groups commonly used in the literature. For example, a Pi can be raised to any exponent, including -1, which yields the inverse of the Pi. In addition, the Pi group can be multiplied by any dimensionless constant without altering its dimensions. (Often, factors of 2 or 1/2 are included in the standard Pi groups.) Table 2.4 in the text lists many of the common dimensionless groups used in Fluid Mechanics and in addition, Appendix E provides more information. The Pi groups generated in this step should be adjusted, if necessary, and named according to this table.

• ***Step*6**. The Pi groups are written in final functional form, typically as the first Pi as a function of the remaining Pi groups. If only one Pi is found, it must be a constant, since it is a function of nothing else.

Example ***Lift on a wing in incompressible flow***
Consider the case of incompressible flow over an airplane wing, as discussed in the previous lecture. Wing lift is known to depend on flow speed, angle of attack, chord length of the wing, and density and viscosity of the fluid. Let us examine this problem with the Buckingham Pi technique of dimensional analysis, following the steps outlined above:

- ***Step***1. $n =$ number of variables in the problem, which is 6 here. $n = 6$.

$$\text{L} = \text{function of } (\text{V},\alpha,\text{c},\rho,\mu)$$

- ***Step***2. List dimensions of each variable:

Variable	Description	Dimensions
L	Lift force	$M(L)(T^{-2})$
V	Velocity	$L(T^{-1})$
c	Chord length	L
ρ	Density	$M(L^{-3})$
μ	Viscosity	$M(L^{-1})(T^{-1})$
α	Angle of attack	1 (dimensionless)

- ***Step***3. Find j. Here, try first setting j = number of primary dimensions in the problem. From the above table, mass, length, and time are the only primary dimensions represented by the set of original variables. Thus, set j = 3. This yields k = n − j = 6 − 3 = 3. i.e., we expect three Pi's from the dimensional analysis.
- ***Step***4. Choose j repeating variables. Here we need to pick 3 repeating variables. Lift force is not a good choice since it is the dependent variable in our problem setup. Angle of attack is not allowed since it is already dimensionless. (Note that angle of attack will be shown to be a dimensionless Pi all by itself) Out of the remaining four, viscosity is the least "basic" or "desirable" variable to be repeated in the entire Pi groups. The best choice here is thus density, velocity, and chord length.
- ***Step***5. Construct the Pi groups. Let's pick the lift force first since it is the dependent variable:

$$\prod\nolimits_1 = LV^a c^b \rho^c$$

$$\{M^0 L^0 T^0\} = \left\{\left(\frac{ML}{T^2}\right)\left(\frac{L}{T}\right)^a (L)^b \left(\frac{M}{L^3}\right)^c\right\}$$

Equating exponents of mass: 0 = 1 + c, or $c = -1$.
Equating exponent of time: 0 = − 2− a, or $a = -2$.
Equating exponents of length: 0 = 1 + a + b − 3c, or $b = -2$.

Thus

$$\Pi_1 = \frac{\mathrm{L}}{\rho \mathrm{V}^2 \mathrm{c}^2}$$

Likewise, construct the second Pi group using viscosity and the repeating variables:

$$\Pi_2 = \mu V^e c^f \rho^g$$

$$\{M^0 L^0 T^0\} = \left\{\left(\frac{M}{LT}\right)\left(\frac{L}{T}\right)^e (L)^f \left(\frac{M}{L^3}\right)^g\right\}$$

Equating exponents of mass: $0 = 1 + \mathrm{g}$, or $g = -1$.
Equating exponents of time: $0 = -1 - \mathrm{e}$, or $e = -1$.
Equating exponents of length: $0 = -1 + \mathrm{e} + \mathrm{f} - 3\mathrm{g}$, or $f = -1$.
Thus,

$$\Pi_2 = \frac{\mu}{\rho \mathrm{Vc}} \text{on the other hand, better to say} : \Pi_2 = \frac{\rho \mathrm{Vc}}{\mu}$$

Note that this Pi group has been inverted in order to match the most well known dimensionless group in Fluid Mechanics, the Reynolds number. It would not be mathematically *incorrect* to leave it "upside down," but it is, shall we say, not "socially acceptable" to do so.

- ***Step*6**. Write the final functional relationship:

$$\frac{\mathrm{L}}{\rho \mathrm{V}^2 \mathrm{c}^2} = \text{function of } \left(\frac{\rho \mathrm{Vc}}{\mu}, \alpha\right)$$

Notice that instead of a dependent variable as a function of five independent variables, the problem has been reduced to one dependent parameter as a function of only two independent parameters. The dependent Pi group on the left hand side is a lift coefficient (which normally has a factor of 2 thrown in for convenience), while the first independent parameter on the right is the Reynolds number, as discussed above.

- Recall the principle of dynamic similarity. In this example, if a geometrically scaled model wing is built, and that wing is tested at some angle of attack and at some Reynolds number, the measured lift coefficient is guaranteed to equal that of the full-scale prototype if operated at the same Reynolds number and the same angle of attack. This is the case even if vastly different fluids are used (air and water for example).

Example ***Dimensional analysis of a soap bubble***

Consider a soap bubble. It is known that the pressure inside the bubble must be greater than that outside, and that surface tension acts like a "skin" to support this pressure difference. The pressure difference is then a function of surface tension and bubble radius. No other variables are important in this problem. Let us examine this problem with the Buckingham Pi technique of dimensional analysis, following the steps outlined above:

- ***Step*1**. $n =$ number of variables in the problem, which is 3 here. $n = 3$.

$$\Delta P = \text{function of } (\Upsilon, R)$$

- ***Step*2**. List dimensions of each variable:

Variable	Description	Dimension
ΔP	Pressure difference	$M(L^{-1})(T^{-2})$
Υ	Surface tension	$M(T^{-2})$
R	Bubble radius	L

- ***Step*3**. Find j. Here, try first setting j = number of primary dimensions in the problem. From the above table, mass, length, and time are the only primary dimensions represented by the set of original variables. Thus, set j = 3. This yields k = n − j = 3 − 3 = 0. i.e., we expect zero Pi's from the dimensional analysis. This makes no sense. When this happens, one of two reasons exists: either we do not have enough variables in the original problem statement (not enough physics is represented by the list of variables), or j is wrong. Here, the latter is the case, and we must reduce j by 1 before continuing. Set j = 2, which yields k = n − j = 3 − 2 = 1. i.e., we expect one Pi from the dimensional analysis.
- ***Step*4**. Choose j repeating variables. Here we need to pick 2 repeating variables. Pressure difference is not a good choice since it is the dependent variable in our problem setup. The best choice here is thus surface tension and bubble radius.
- ***Step*5**. Construct the Pi groups. Here there is only one, and it is found by combining the remaining variable with the two repeating variables to form a Pi group, as follows:

$$\Pi_1 = \Delta P \Upsilon^a, R^b$$
$$\{M^0 L^0 t^0\} = \left\{\left(\frac{M}{Lt^2}\right)\left(\frac{M}{t^2}\right)^a (L)^b\right\}$$

Equating exponents of mass: $0 = 1 + a$, or $a = --1$.

Equating exponents of time: $0 = -1 + b$, or $b = 1$.

Equating exponents of length: $0 = -2 - 2a$, or $a = -1$.

Fortunately here, the first and third equation yield the same value of exponent a. If they did not, we would suspect either an algebra mistake or a nonphysical setup of the problem. Our result is:

$$\Pi_1 = \frac{\Delta P R}{\Upsilon}$$

- ***Step*6.** Write the final functional relationship:

$$\Pi_1 = \frac{\Delta \mathrm{PR}}{\Upsilon} = \text{constant} \qquad \text{or} \qquad \Delta \mathrm{P} = \frac{\text{constant} \cdot \Upsilon}{\mathrm{R}},$$

Notice that instead of a dependent variable as a function of two independent variables, the problem has been reduced to one dependent parameter as a function of nothing. In cases like this where, there is only one Pi group, that Pi must be a constant. (If it is not a function of anything else, it must be a constant!)

- This is an excellent example of the power of dimensional analysis. Here we have obtained a functional relationship between pressure, radius, and surface tension to within a constant of proportionality without knowing any physics about the problem! Dimensional analysis cannot provide the constant, but it can provide information about how one variable depends on others. For example, our result shows that if the bubble radius is reduced by a factor of 2, the pressure difference will increase by a factor of 2. It also shows that pressure difference is linearly proportional to surface tension.

2.17 Development of Prediction Equations

The accuracy of prediction equations in any field of science and engineering cannot be ignored and a reliable such equations are tools along with experiment and data collection from a model behavior and observation make it possible to design prototypes with confidence. There exist two general methods available in order to develop prediction equations.

1. Method number one is based on establishing a careful observation and measurement of pertinent variables which upon that the quantity to be predicted. Other name for this method also known as the *experimental method*.
2. Method number two consists in applying those natural laws, which are pertinent to the problem in hand in order to develop relationships among the significant variables. Other name for this method also known as the *analytical method*.

The natural laws utilized in this method are simply generalizations of reliable information, which has been assembled through observation and measurement. However, each involves analysis, and each is dependent upon experimental finding. We show in Sect. 2.18 the general form of the equation for any phenomenon that maybe determined by dimensional analysis. It is essential that we analyze and resort the experimental finding in order to complete a solution and determine a prediction equation. If the readers need to learn more detailed about this subject, they should refer themselves to a book written by Murphy [5].

2.18 Similarity and Similar System

As we said, the concept of similarity extends way beyond just geometry. In fact, the similarity extends too many characteristic, such that might be specified that mass distribution in a model be similar to its prototype as an example. This means that the ratio of masses of homogeneous parts shall be a constant that does not dependent on the choice of the parts. For instance if we look at design of airplane wing-flutter model, is a restricted sense, this condition must be satisfied. The condition of similarity of mass distribution is not applied to all details of the structure, but it is required that the ratio of masses of segments of the wing and the model that are included between homologous cross sections shall be a constant. In other words, the mass statement expresses the condition that the span wise distribution of mass of the model is similar to that of the prototype. Furthermore, the distributions of mass must be similar, to the extent that the centers of mass of homologous segments of the wing and the model are homologous points, and the mass moments of inertia of homologous segments of the wing and the model, with respect to the axis of twist, have a constant ratio. In simple form of stating, this condition is that the span wise distributions of mass moments of inertia shall be similar.

To define similarity and similar system we assume a set of quantities $(a_1, \cdots, a_k)$ is said to have *independent dimensions* if none of these quantities have dimensions, which can be represented as a product of powers of the dimensions of the remaining quantities. As an example, the density ($[\rho] = ML^{-1}$), the velocity ($[\upsilon] = LT^{-1}$), and the force ($[F] = MLT^{-2}$) have independent dimensions, so that there is no product of powers of these quantities which is dimensionless [See the footnote below]. On the other hand, the density, velocity, and pressure ($[p] = ML^{-1}T^{-2}$) are not independent, for we can write $[p] = [\rho][\upsilon]^2$; i.e., $p/(\rho\upsilon^2)$ is a dimensionless quantity.

Footnote
Prove this formally by writing $[F]^a[\upsilon]^b[\rho]^c = 1$, and then show that the only solution is $a = b = c = 0$. Alternatively, show that it is impossible to write $[F] = [\rho]^a[\upsilon]^b$ for any a, b.

Now suppose we have a relationship between a quantity which is being determined in some experiment (which we will refer to as the *governed parameter*), and a set of quantities $(a_1, \cdots, a_n)$ which are under experimental control (the *governing parameters*). This relationship is of the general form

$$a = f(a_1, \cdots, a_k, a_{k+1}, \cdots, a_n) \tag{2.33}$$

where $(a_1, \cdots, a_k)$ have independent dimensions. This means that the dimensions of the governed parameter a are determined by the dimensions of $(a_1, \cdots, a_k)$, while all of the a's with $s > k$ have dimensions that can be written as products of powers of the dimensions of $(a_1, \cdots, a_k)$; e.g., $a_{k+1}/{a_1}^p \cdots {a_k}^r$, would be dimensionless, with $p, \cdots, r$ an appropriately chosen set of constants. With this set of definitions, it is possible to prove that Eq. 2.1 can be written as;

$$a = {a_1}^p \cdots {a_k}^r \Phi\left(\frac{a_{k+1}}{{a_1}^{p_{k+1}} \cdots {a_k}^{r_{k+1}}}, \cdots, \frac{a_n}{{a_1}^{p_n} \cdots {a_k}^{r_n}}\right) \tag{2.34}$$

with Φ some function of dimensionless quantities only. The great simplification is that while the function f in Eq. 2.1 was a function of n variables, the function Φ in Eq. 2.2 is only a function of $n - k$ variables. Eq. 2.2 is a mathematical statement of Buckingham's $\prod$-Theorem; Φ—we need a real theory for that.

As a simple example of how this works, let us return to the pendulum, but this time we will assume that the mass can be distributed, so that we relax the condition of the mass being concentrated at a point. The governed parameter is the frequency ω; the governing parameters are g, l (which we can interpret as the distance between the pivot point and the center of mass), m, and the moment of inertia about the pivot point, I (See Fig. B.1, in Appendix B). Since $[I] = ML^2$, the set (g, m, l, I) is not independent; we can choose as our independent parameters (g, m, l) as before, with I/ml^2 a dimensionless parameter. In the notation developed above, $n = 4$ and $k = 3$. Therefore, dimensional analysis tells us that

$$\omega = \sqrt{\frac{g}{l}\Phi\left(\frac{I}{ml^2}\right)} \tag{2.35}$$

with Φ some function, which cannot be determined from dimensional analysis alone; we need a *theory* in order to determine it.

2.18.1 Similarity, Modeling, and Estimating

As we said in Sect. 1.12 of Chap. 1, the notion of *similarity* is familiar from geometry. Two triangles are said to be similar if all of their angles are equal, even if the sides of the two triangles are of different lengths. The two triangles have the same shape; the larger one is simply a scaled up version of the smaller one. This notion can be generalized to include physical phenomena. This is important when *Modeling* physical phenomena–for instance, testing a prototype of a plane with a scale model in a wind tunnel. The design of the model is dictated by dimensional analysis. Details of mathematics were presented in Chap. 1, Sect. 1.11.

In summary, similarity is an extension of geometrical similarity and by definition; two systems are similar if their corresponding variables are proportional at corresponding locations and times [3]. In preceding chapter, sections, the mathematical, requirements, and results of similarity from principle equations were clarified and physical meanings of this definition was presented.

The main advantages of similarity in applications is the flexibility of its requirements (See Sect. 2.1 of this Chapter) while maintaining a relationship between systems (i.e. *Model* and *Prototype*). For example, if the sides of two similar triangles are a, b, c and x, y and z, then

$$\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$$

In this equation, if a, b, c and x are specified, y and z may be calculated. Using such concept, the dimension of second similar triangle may be calculated from those of the first one, even though they are not identical. In general, similar systems may be "*Physically*" similar or "*Analogous*" and in many cases, the requirements of the similarity allow flexibility in the character as well as the size of similar system [4]. Both physical and analogous similarity can be summarized as follow;

- **Physical Similarity:** are the systems that have same physical qualities, for example, two fluid systems.
- **Analogous Similarity:** Analogues systems, or simply analogs, have different physical qualities, as in mechanical and electrical systems and the concept has yields many valuable results, providing that we simplify assumptions that are necessary before systems are analogous.

Note, the requirements and results of similarity are derived from principle equations, initial, and boundary conditions, from which a system is defined. For a specified system, the first step is to transform the principle equations to a nondimensional form involving nondimensional variables and non dimensional constant coefficients, which are known as *Reference Similarity Numbers* (See Sect. 2.8 and 2.16.1 of this Chapter as well).

Per description that is given by Skoglund [4];

- The first requirement of similarity of two systems (i.e. model and prototype) is that their nondimensional principle equations be the same. Mathematically speaking, the nondimensional solutions of a set of nondimensional principle equations will be the same if a sufficient number of nondimensional initial and boundary conditions are the same. In a particular case, if the dimensional initial and boundary conditions are sufficient to define the system, they will be sufficient for similarity when transformed to a nondimensional form.
- The second requirement of similarity is the equality of the nondimensional independent variables, which constitute the nondimensional initial and boundary conditions.

The result of similarity is that the nondimensional variables of similar systems are equal. This equality implies proportionality of the same variable in similar systems, in accordance with the definition of similarity.

2.19 Dissimilarity and Dissimilar System

There are not any well-defined methods to analyze dissimilar systems. They depend largely on the problem and the ingenuity of the designer and analyst. In general, they require additional theory or experimental results to consider for distortions. Finding a complete similarity of a model to its prototype neither from return on investment point of view nor from practical perspective might not be there, but proper design, testing, and analysis, valuable information may be obtained from dissimilar systems. Most important example of such analysis will take place in the design of ships and airplanes as well as in modifications of waterways. Minor distortions may take place where a complete similarity was assumed and greater precision took place. The extrapolation of the results of a dissimilar system to a prototype may be uncertain, but it may be the best available. Some cases have been oversimplified to emphasize methods, but the methods themselves may be applied to problems that are more complicated. For any new applications, the possibilities of if not complete but near complete similarity and similar method should be taken under consideration first. When a satisfactory method of extrapolations has been devised, the design of the dissimilar system may be completed, and its fabrication and testing may be started [4].

2.20 Scaling and Scaling Process

A scale model is a physical model, a representation or copy of an object that is larger or smaller than the actual size of the object, which seeks to maintain the relative proportions (the scale factor) of the physical size of the original object. Very often, the scale model is smaller than the original and used as a guide to making the object in full size. Scale models are built or collected for many reasons, including to reduce the time and cost of solving technical problems and building the prototype, improving designs and production techniques, promoting sales in marketing filed, and solving operating problems. However their main advantage is;

1. Reducing costs of production and operation
2. More precise measurements under a wider range of controlled conditions
3. It permits a cut and tries procedure of design at a reasonable cost

In complicated problems, the model investigation may provide only a part of a solution, and it must be integrated with other techniques to complete the solution.

Professional model makers often create models for many professions:

- Engineers who require scale models to test the likely performance of a particular design at an early stage of development without incurring the full expense of a full-sized prototype.
- Architects who require architectural models to evaluate and sell the look of a new construction before it is built.

- Filmmakers who require scale models of objects or sets that cannot be built in full size.
- Sales men who require scale models to promote new products such as heavy equipment and automobiles and other vehicles.

Hobbyists or amateur model makers make die-cast models, injection molded, model railroads, remote control vehicles, war gaming and fantasy collectibles, model ships and ships in bottles for their own enjoyment. Scale models can also be objects of art, either being created by artists or being rediscovered and transformed into art by artists.

A scale factor is a number, which scales, or multiplies, some quantity. In the equation $y = Cx, C$ is the scale factor for x. C is also the coefficient of x, and may be called the constant of proportionality of y to x. For example, doubling distances corresponds to a scale factor of 2 for distance, while cutting a cake in half results in pieces with a scale factor of ½. The basic equation for it is image over pre-image.

Scale is a technique of relating the prototype and model variables at corresponding points. From Eq. 1.51, a scaling factor k_u can be defined as;

$$k_u = \frac{u(\chi_i, \tau)_p}{u(\chi_i, \tau)_m} = \frac{u_{op}}{u_{om}} = \text{constant}$$

where p and m subscriptions standing for model and prototype and rest of variables are define as before in Sect. 1.14.1 of Chapter one and are presented here again;

u = dimensional variable.
n = nondimensional variable.
χ_i = nondimensional coordinate.
τ = nondimensional time.

This indicates the ratio of the corresponding dimensional variables of prototype and model at corresponding point is constant. Although scaling factors obscure the basis of similarity, but they are commonly used in the design and interpretation of models.

Note that $n_u(\chi_i, \tau)_m = n_u(\chi_i, \tau)_p$ includes geometrical similarity requirements (See beginning of this Chapter and Sect. 2.2.1 as well proceeding Sect. 2.15) and the complete general requirements of similarity calls for;

1. Geometrical similarity
2. $N_{om} = N_{op}$
3. $n_u(\chi_i, \tau)_m = n_u(\chi_i, \tau)_p$

These requirements of similarity are directly applicable to model design [4].

The dimensional approach in support of scaling and modeling process, is based in geometry, so visual displays will be emphasized. Theoretical perspectives on methods, scaling, and dimensions will be presented. The methods presented are a variety of techniques for scaling, broadly defined, including unfolding analysis, proximity scaling, Guttman scaling, cluster analysis, factor analysis, and multidimensional scaling. The terms "scaling" and "dimensional analysis" refer to a wide variety of research strategies and procedures. The common element among them is that they all seek to provide quantitative and/or geometric representations of the internal structure in a set of data. Researchers apply these techniques for three main reasons:

- **Simple Data Reduction:** Reducing a large number of variables into a smaller set of composites, or summarizing a large set of variables with a smaller number of composite measures.
- **Examining Dimensionality:** Representing the data in terms of the smallest possible number of unobserved underlying factors, to testing the underlying sources of variation in a dataset and
- **Measurement:** Scoring cases on the underlying dimensions and using those scores in further analysis or obtaining empirical representations of the underlying (and usually unobservable) dimensions, which can be employed as analytic variables in other statistical procedures.

On a less formal note, researchers will often find that dimensional analysis is very beneficial for "conceptualizing" the contents of their data. In addition, these techniques usually provide visual displays that are very useful for presenting analytical results to other people. Thus, for a variety of reasons, scaling and dimensional analysis are useful additions to the social scientist's "repertoire" of research strategies.

2.21 Modeling and Similitude

What is Model? Word of model is derived from Latin word of "***Modus***" means (*a measure*), the word of 'Model' implies a change of scale in the representation, and only later in its history did it acquire the meaning of type of design. In sense that we are seeking a different scale of thought or mode of understanding, we are using the word in its older meaning and have been used in a number of senses by both philosophers and scientists. The purpose for which a model is constructed should not be taken as granted but at any rate initially, needs to be made explicit. Model will be used for any complete and consistent set of mathematical equations, which is thought to correspond to some entity that is known as its prototype. The mathematical aspect of modeling is well described in Sect. 2.4.2 of this chapter.

What is Similitude? It is a fancy word representing a very simple idea. In addition, it is very, very popular in fluid mechanics. It is a methodology for identifying different problems being the same problem. For example, we want to know, what is the pressure distribution on the wing of a big, subsonic aircraft?. Can we use the data from any small tunnel (using a scaled model) to make design decisions on the big aircraft? The answer is yes, provided the 'desired answers' of these two problems can be shown to be 'similar.' Another example: you have data on the destructive power of a small atomic bomb; what would be the destructive power with a bomb with 10,000 times more energy ? A major tool in such endeavors is dimensional analysis. The whole idea of dimensional analysis is exquisitely simple: Let us use dimensionless variables! Let the desired answers (in dimensionless form) be expressed in terms of dimensionless parameters! Having agreed to this intuitively simple idea, the big question is: how many dimensionless parameters (including the desired answers) are there in the problem? This is the job of the next sections.

Once you have the relationship between the Pi terms (obtained by experiments on a model or by theoretical analysis), you can used them to make predictions!. Here, there are some subtle but very sensible tricks. What if the value of some of the Pi's in the real system are not exactly the same as the corresponding values of Pi's in the experiment (or the theory)? What if they are just a little bit off? (What do you think?) We will talk about this and try to clarify all these and find some answers for all these questions as well.

Engineering is design and analysis with attention paid to cost, risk, safety, etc. Analysis is a model that predicts performance. Performance must not exceed performance limits. Mathematical models are convenient. Physical, small-scale models are better for complex structure of prototype from model and eventually taking it to full production line. The most dependable models are full-scale prototype. Mathematical models are often written to describe prototype performance because it is empirical to perform a full-scale prototype study for every complex structure. The set of principles upon which a model can be related to the prototype for predicting prototype performance is called ***Similitude***. Similitude applies to all models—mathematical, small-scale and prototype.

There are three basic steps in achieving similitude [12];

1. Fundamental Variables (FVs) are all the variables that affect the phenomenon. All the FVs must be uniquely interdependent. However, no subset of FVs can be uniquely interdependent. For example, force, mass, and acceleration of gravity cannot all be used as fundamental variables in a more complex phenomenon, because force equals mass times acceleration. Therefore, the sub-set is uniquely interdependent. Only two of the three fundamental variables could be used in the phenomenon to be investigated.
2. Basic Dimensions (BDs) are the dimensions in which the FVs can be written. The basic dimensions for basic fluid mechanics problem are usually force F, distance L, and sometimes time T and temperature.
3. Pi terms are combinations of the FVs that meet the following three requirements;
   a. The number of Pi terms must be at least the number of FVs minus the number of BDs.
   b. The Pi terms must all be dimensionless.
   c. No subject of Pi terms can be interdependent.

This is ensured if each Pi term contains a fundamental variable not contained in any other Pi term.

As we said most aspect of Modeling and Models are widely used in fluid mechanics or aerodynamic type's problems. In particular, industries that deal with projects involving skyscrapers and bridges structures, aircraft, ships, submarines, rivers, harbors, dams, air and water pollutions, and so on are few that fall in mathematical modeling categories. Technically engineering modeling is better term than just modeling for identifying these problems solution in general context.

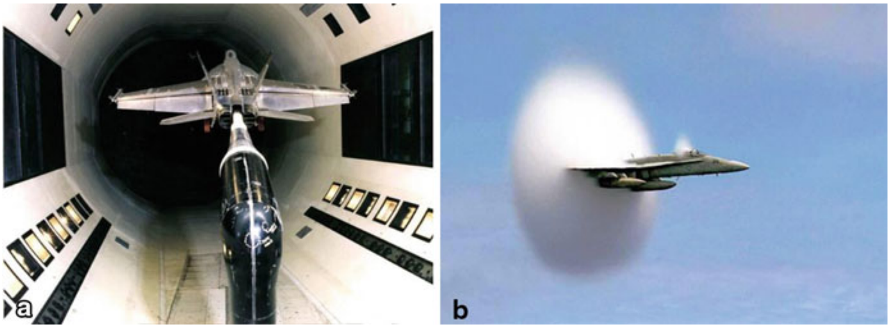


Fig. 2.14 F-18 model in wind tunnel (**a**) and prototype in flight (**b**)

Pi terms can be determined in many different way and two most important one that we will elaborate o it next few section are;

1. Repeating Variables
2. By Inspection

A *model* is a representation of a physical system that may be used to predict the behavior of the system in same desired respect. As result, in contrast, the physical system for which the predictions are to be made is called *prototype*. With help of mathematical or computer modeling we can conform these definitions. However, our interest is focused on physical modeling in engineering, where models resemble the prototype but are generally of different size and often operate under different conditions (pressure, velocities, different fluid, etc.). Figure 2.12a, 2.12b can illustrate such situation (Fig. 2.14).

Notice that the rear of the model is distorted relative to the actual F-18 design (Fig. 2.12a). The actual F-18 has twin-engine exhausts at the rear of the aircraft as shown in the small in-flight photo. To accurately predict the drag of the aircraft, corrections must be applied to the data from the airframe test to account for the nozzle exhaust. Often, a special model is used in which the aircraft is mounted by the wing tips and the after body and nozzles are correctly modeled. Here is a photo of an F-18 nozzle/after body model mounted in the NASA Langley 16-foot tunnel. Figure 2.12b is an impressive picture of an F-18 hornet in flight and the sound barrier breaking point.

As you can see from above picture that usually in most applications, model is smaller than prototype. With the successful development of a valid model, it is possible to predict the behavior of the prototype under a certain set of conditions. There is, of course, an inherent danger in the use of models in that predictions can be made that are in error and the error not detected until the prototype is found not to perform as predicted. Therefore, it is very important that the model be properly designed and tested and that the results are interpreted correctly. It is, imperative to develop the procedures for designing models so that the model and prototype will behave in a similar fashion. If we summaries models versus prototype we can write;

- Model: A model is a representation of a physical system that may be used to predict the behavior of the system in some desired respect. Mathematical or computer models may also conform to this definition; our interest will be in physical model.
- Prototype: The physical system for which the prediction are to be made.
- Models that resemble the prototype but are generally of a different size, may involve different fluid, and often operate under different conditions.
- Usually a model is smaller than the prototype.
- Occasionally, if the prototype is very small, it may be advantageous to have a model that is larger than the prototype so that it can be more easily studied. For example, large models have been used to study the motion of red blood cells.
- With the successful development of a valid model, it is possible to predict the behavior of the prototype under a certain set of conditions.
- There is an inherent danger in the use of models in that predictions can be made that are in error and the error not detected until the prototype is found not to perform as predicted.
- It is imperative that the model be properly designed and tested and that the results be interpreted correctly.

The correlation and conditions that must be met between similarity of model and prototype in order to ensure similarity of model and prototype can also be summarized and they are described in more details in preceding sections.

- **Geometric Similarity**.
  - Model and prototype have same physical shape.
  - Linear dimensions on model and prototype correspond within constant scale factors.
- **Kinematic Similarity**.
  - Velocities at corresponding points on model and prototype differ only by a constant scale factor.
- **Dynamic Similarity**.
  - Forces on model and prototype differ only by a constant scale factor.

In the following sections, we develop the procedures for designing models so that the model and prototype will behave in a similar fashion.

2.21.1 Benefits of Models

In this section, we introduce a realistic perspective for applications of models. In practice and real world where we are building, a prototype from ground up is so expensive and one should consider the return on investment of any complex system. For example a complex building a nuclear submarine, sophisticated fighter jets, or ship building and other areas such construction of damn or high-rise, the models should be supplemented by experience and studies of details of models of the prototype of interest. Modeling will make production of such complex prototype more cost effective while reduces production time. The purpose of this section is to introduce a realistic perspective for applications of model and its theory.

2.21.2 Theory of Models

So far, we have illustrated the fact that dimensional analysis is useful to an experimenter because to determine the behavior of the system, one needs only make measurements of the relatively few dimensionless parameters that characterize the system. There is another reason for the usefulness of dimensional analysis to scientists and engineers, which with that, one can shows the scale model that, will duplicate the behavior of the original system provided, that the governing dimensionless parameters have the same value in the two system. Note that we emphasize that in this discussion the word *model* does not refer to a mathematical model but is used in the familiar sense, which applies, for example, to a small replica of an aircraft such as F-18 in Fig. 2.11a. Mathematical modeling is discussed in Sect. 2.16.3 below. Process of mathematical modeling also has been described in Sect. 2.16 of this chapter as well.

The theory of models can be readily developed by using the principles of dimensional analysis. Since theoreticians and engineers must have an appreciation of the nature and limitation of experiment, we have devoted in both preceding and proceeding sections on discussion of scaling and scale model. As we said before, a scale model has the same shape as the system (prototype) of primary interest but is of a more convenient size for laboratory testing and experimental data gathering in order to predict the behavior of the system (prototype) and as result mass production of such system.

Per preceding section on theory of Buckingham Pi and determination of Pi terms in Sect. 2.6 of this chapter, we can show that for any given problem, a set of Pi terms can be identified as following formulation;

$$\Pi_1 = \phi\left(\Pi_2, \Pi_3, \cdots, \Pi_n\right) \tag{2.36}$$

Developing such equation and relationship the only requirement is knowledge of the general nature of the physical phenomenon, and the variables that is accounted for. Specific variables such as size of components, fluid properties, and so on are not needed to perform the dimensional analysis. Thus, Eq. 2.1 applies to any system that is governed by the same variables [9]. If the behavior of a particular system (prototype) is described by Eq. 2.36, then a similar relationship also can be written for a model of this prototype as formulated in Eq. 2.37 below;

$$\Pi_{1m} = \phi\left(\Pi_{2m}, \Pi_{3m}, \cdots, \Pi_{nm}\right) \tag{2.37}$$

In Eq. 2.6 the function form will be the same so long as the same phenomenon is involved in both the model and prototype. Variables, or Pi terms, without a subscript will refer to the prototype, whereas the subscript m will be used to designate the model variables or Pi terms. The Pi terms can be established so that $\prod_1$ contains the variable that is to be predicated from observations made on the model, so if the model is designed and operated under the following conditions,

$$\begin{aligned} \Pi_{2m} &= \Pi_2 \\ \Pi_{3m} &= \Pi_3 \\ &\vdots \\ \Pi_{nm} &= \Pi_n \end{aligned} \tag{2.38}$$

Under assumption that the form ϕ is the same for both model and prototype, it follows that

$$\Pi_1 = \Pi_{1m} \tag{2.39}$$

This means also

For prototype $\prod_1 = \phi\left(\prod_2, \prod_3, \cdots, \prod_n\right)$

For model $\prod_{1m} = \phi\left(\prod_{2m}, \prod_{3m}, \cdots, \prod_{nm}\right)$

Equation 2.39 is the desired *prediction equation* and indicates that the measured value of $\prod_{1m}$ obtained with the model will be equal to the corresponding $\prod_1$ for the prototype as long as the other Pi terms are equal [9]. The conditions specified by Eq. 2.36 provide the *model design conditions*, also called *similarity requirements* or *modeling laws*.

The prediction equation in general, is formulated by dividing the general equation for prototype by the general equation for the model. This is different form of Eq. 2.39 and we can write in form of Eq. 2.40 as below;

$$\frac{\text{Prototype}}{\text{Model}} = \frac{\prod_1}{\prod_{1m}} = \frac{\phi\left(\prod_2, \prod_3, \cdots, \prod_n\right)}{\phi\left(\prod_{2m}, \prod_{3m}, \cdots, \prod_{nm}\right)} \tag{2.40}$$

If all of design conditions are satisfied, the model may be considered a “true” model in that it will give accurate information concerning the behavior of the prototype, if all of the pertinent quantities are included in analysis from which the Pi terms are obtained. If not all conditions can be satisfied, the behavior of the model may be distorted with reference to the factors included in the corresponding Pi terms and the prediction equations may be affected mathematically. The distorted model is explained in Sect. 2.22 of this chapter with more details.

If the design conditions for a true model are all satisfied $\left(\prod_{im} = \prod_i\right)$, the functions are equal and the Eq. 2.8 is satisfied. However, if one design condition is violated, the functions may not be equal, whereupon Eq. 2.8 is not satisfied and indication is that;

$$\Pi_1 \neq \Pi_{1m} \tag{2.41}$$

In order to be able to deal with conditions that Eq. 2.38 imposes during the circumstances that design conditions is not satisfied (i.e. $\prod_{3m} \neq \prod_3$), the prediction factor δ (fudge factor) can be introduced that is derived by definition form Eq. 2.40 and will be shown in form of Eq. 2.42 below;

$$\delta = \frac{\prod_1}{\prod_{1m}} = \frac{\phi\left(\prod_2, \prod_3, \prod_4 \cdots, \prod_n\right)}{\phi\left(\prod_2, \prod_{3m}, \prod_4 \cdots, \prod_n\right)} \tag{2.42}$$

Hence, to evaluate δ, the ratio of the two functions must be analyzed and be evaluated. This kind of determination will involve either collection additional experimental data and evidences or knowledge of how $\prod_3$ influences the function and Pi terms. If, for example, it can be established that [5];

$$\phi(\Pi_2, \Pi_3, \Pi_4 \cdots, \Pi_n) = \frac{\phi\left(\prod_3\right) \phi\left(\prod_2, \prod_3, \prod_4 \cdots, \prod_n\right)}{\phi\left(\prod_2, \prod_{3m}, \prod_4 \cdots, \prod_n\right)} \tag{2.43}$$

It follows that;

$$\delta = \frac{\phi\left(\prod_3\right)}{\phi\left(\prod_{3m}\right)} \tag{2.44}$$

We can illustrate the following examples as procedure of the modeling theory.

Example 2.14 Considering the drag force on a sphere.

$$F = f(D, V, \rho, \mu) \rightarrow \frac{F}{\rho V^2 D^2} = f_1\left(\frac{\rho V D}{\mu}\right)$$

The prototype and the model must have the same phenomenon

$$\frac{F_m}{\rho_m {V_m}^2 {D_m}^2} = f_1\left(\frac{\rho_m V_m D_m}{\mu_m}\right) \quad \text{and} \quad \frac{F}{\rho V^2 D^2} = f_1\left(\frac{\rho V D}{\mu}\right) \text{ for prototype}$$

Design conditions

$$\left(\frac{\rho V D}{\mu}\right)_{\text{model}} = \left(\frac{\rho V D}{\mu}\right)_{\text{prototype}}$$

Then

$$\left(\frac{F}{\rho V^2 D^2}\right)_{\text{model}} = \left(\frac{F}{\rho V^2 D^2}\right)_{\text{Prototype}}$$

Example 2.15 Determining the drag force on a thin rectangular plate ($w \times h$ in size).

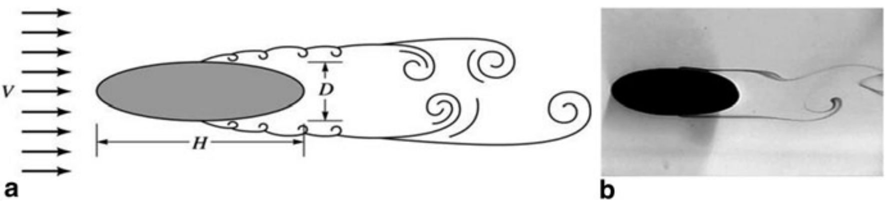


Fig. 2.15 **a** A long elliptical structure component. **b** Flow past an ellipse [9]

$$F = f(w, h, \mu, \rho, V) \rightarrow \frac{D}{w^2 \rho V^2} = \Phi\left(\frac{w}{h}, \frac{\rho V w}{\mu}\right)$$

The prototype and the model must have the same phenomenon

$$\frac{D_m}{{w_m}^2 \rho_m {V_m}^2} = \Phi\left(\frac{w_m}{h_m}, \frac{\rho_m V_m w_m}{\mu_m}\right) \qquad \text{and} \qquad \frac{D_m}{w^2 \rho V^2} = \Phi\left(\frac{w}{h}, \frac{\rho V w}{\mu}\right)$$

Design conditions

$$\frac{w_m}{h_m} = \frac{w}{h} \qquad \text{and} \qquad \frac{\rho_m V_m w_m}{\mu_m} = \frac{\rho V w}{\mu}$$

Then

$$\frac{D}{w^2 \rho V^2} = \frac{Dm}{{w_m}^2 \rho_m {V_m}^2} \gg D = \left(\frac{w}{w_m}\right)^2 \left(\frac{\rho}{\rho_m}\right) \left(\frac{V}{V_m}\right)^2 D_m$$

Example 2.15 Prediction of Prototype Performance from Model Data [9].

A long structural component of a bridge has an elliptical cross section shown in Fig. 2.13a, 2.13b. It is known that when a steady wind blows past this type of bluff body, vortices may develop on the downwind side that is shed in a regular fashion at some definite frequency. Since these vortices can create harmful periodic forces acting on the structure, it is important to determine the shedding frequency (Fig. 2.15).

For the specific structure of interest, $D = 0.1\text{m}, H = 0.3\text{m}$, and a representative wind velocity is 50 km/h. Standard air can be assumed. The shedding frequency is to be determined through the use of a small-scale model that is to be tested in a water tunnel. For the model $D_m = 20$ mm and the water temperature is 20 °C.

Determine the model dimension, H_m and the velocity at which the test should be performed. If the shedding frequency for the model is found to be 49.9 Hz, what is the corresponding frequency for the prototype?

Solution We expect the shedding frequency, ω, to depend on the lengths D and H, the approach velocity, V, and the fluid density ρ and viscosity, μ. Thus,

$$\omega = f(D, H, V, \rho, \mu)$$

Where dimensional equations are as follows;

$$\begin{aligned}
\omega &= T^{-1} \\
D &= L \\
H &= L \\
V &= LT^{1} \\
\rho &= ML^{-3} \\
\mu &= ML^{-1}T^{-1}
\end{aligned}$$

Since there are six variables and three reference dimensions (MLT), three Pi terms are required. Application of the Pi theorem yields;

$$\frac{\omega D}{V} = \phi\left(\frac{D}{H}\frac{\rho V D}{\mu}\right)$$

We recognize the Pi term on the left as the Struhal number, and the dimensional analysis indicates that the Strouhal number is a function of the geometric parameter, D/H, and the Reynolds number. Thus, to maintain similarity between model and prototype

$$\frac{D_m}{H_m} = \frac{D}{H}$$

and

$$\frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho V D}{\mu}$$

From the first similarity requirement

$$\begin{aligned}
H_m &= \frac{D_m}{D}H \\
&= \frac{(20 \times 10^{-3} m)}{(0.1\, m)}(0.3\, m)
\end{aligned}$$

$$H_m = 60 \times 10^{-3} m = 60\, mm \qquad \text{(Answer)}$$

The second similarity requirement indicates that the Reynolds number must be the same for model and prototype so that the model velocity must satisfy the condition

$$V_m = \frac{\mu_m}{\mu}\frac{\rho}{\rho_m}\frac{D}{D_m} \tag{2.45}$$

For air at standard conditions, $\mu = 1.79 \times 10^{-5} kg/ms, \rho = 1.23\, kg/m^3$, and for water at 20 °C, $\mu = 1.00 \times 10^{-3} kg/ms, \rho = 998\, kg/m^3$. The fluid velocity for prototype is;

$$V = \frac{(50 \times 10^3 m/hr)}{(3600 s/hr)} = 13.9\, m/s$$

The required velocity can now be calculated from Eq. (a) in above as;

$$V_m = \frac{[1.00 \times 10^{-3} kg/ms]}{[1.79 \times 10^{-5} kg/ms]} \frac{(1.23\, kg/m^3)}{(998\, kg/m^3)} \times \frac{(0.1\, m)}{(20 \times 10^{-3} m)} (13.9\, m/s)$$

$$V_m = 4.97\, m/s \qquad \text{(Answer)}$$

This reasonable velocity could be readily achieved in a water tunnel.

With the two similarity requirements satisfied, it follows that the Strouhal number for prototype and model will be the same so that

$$\frac{\omega D}{V} = \frac{\omega_m D_m}{V_m}$$

in addition, the predicted prototype vortex shedding frequency is

$$\begin{aligned} \omega &= \frac{V}{V_m} \frac{D_m}{D} \omega_m \\ &= \frac{(13.9\, m/s)}{(4.79\, m/s)} \frac{(20 \times 10^{-3} m)}{(0.1\, m)} (49.9\text{Hz}) \\ \omega &= 29.0\text{Hz} \qquad \text{(Answer)} \end{aligned}$$

Comments "This same model could also be used to predict the drag per unit length, D_ℓ(lb/ft or N/m^2), on the prototype, since the drag would depend on the same variables as those used for the frequency. Thus, the similarity requirements would be the same and with these requirements satisfied, it follows that the drag per unit length expressed in dimensionless form, such as $D_\ell / D\rho V^2$ would be equal in model and prototype. The measured drag per unit length on the model could then be related to the corresponding drag per unit length on the prototype through the relationship" [9].

$$D_\ell = \left(\frac{D}{D_m}\right)\left(\frac{\rho}{\rho_m}\right)\left(\frac{V}{V_m}\right) D_{\ell m}$$

As part of simplification of Dimensional Analysis, and Scaling, we can discuss another good example that can be discussed from a book by Lin and Segel [16]. They illustrate the usage of scale model, which one can show that scale model will replicate the behavior of the original set of system, providing that the governance of dimensionless parameter stay the same between model and prototype. We did touch upon this in preceding sections as well as giving more details in next section in next section. A scale model is primarily chosen to deal with convenience and has the same shape as the system of primary (prototype) interest but is size that is more convenient and stress is on model performance rather than size, which duplicates similar requirements as prototype and eventually production units. This also attributes to the proportionality so that the dimensionless parameters are identical in both prototype and model. Their ship design example provides a good example of such use, by assuming the interest lies in the amount of work per unit time and designated by P (for power) for a ship of length ℓ moving in straight line at a constant speed V. In this example, a primary work is needed to replace energy either that is wasted in making waves on the water surface or that is dissipated because of the viscosity of the water. So the general equation can be written as:

$$P = f(V, \ell, g, \rho, \nu) \tag{1}$$

where,

g = is the acceleration due to gravity.
ρ = is the density of the water, and
ν = is the kinematic viscosity of water.

By virtue of dimensional analysis and illustration of Example 2.13 and relationship of (xiv) in that example, one can conclude that:

$$\frac{P}{\rho \ell^2 V^2} = \phi(Fr, \mathrm{Re}) \tag{2}$$

whereas before;

$$Fr = V/(\ell g)^{1/2} = \text{Froude Number} \tag{3}$$

$$\mathrm{Re} = V\ell/\nu = \text{Reynolds Number} \tag{4}$$

As we know by now introduction of these two numbers associates the problem with wave resistance and viscous resistance of water in this example. This matter should not surprise us at all, since the Froude Number contains a measure of the force of gravity that is overcome by the water, while the Reynolds number contains a measure of fluid viscosity.

Now suppose one wants to constructs a scale model of the ship in example, of exactly the same shape but one-hundredth size. If we use subscribe m for model parameters, this means that $\ell_m = 10^{-2}\ell$. In order to deal with this boundary condition, the model experiment should be associated with the same Froude number as the full scale ship in this example, then one would have to move the model one-tenth as fast as the speed V. For then the Froude number can be written as:

$$Fr_m = \frac{V_m}{(\ell_m g)^{1/2}} = \frac{10^{-1} V}{(10^{-2}\ell g)^{1/2}} = Fr \tag{5}$$

Note that in this argument the word "*model*" does not refer to a mathematical model but is used in the familiar sense, which applies, for example, to a small replica of ship and gravity force g is the same value for both model and prototype products.

By the same arguments in above one needs to obtain the similar Reynolds number, by using a fluid whose *Kinematic Viscosity* v is one-thousandth that of water. Knowing that, there exists no such fluid (for example Air, has a kinematic viscosity which is about 15 times *greater* than that of water). Therefore, it turns out that one *cannot* make a model that reproduces the dimensionless parameters of the full-scale ship. As result of big value of Reynolds, number for full-scale ship motion does not help the situation at all. For example if we assume the actual ship has length of 100 m that moves at speed of 10 m/sec or about 20 miles/h, since kinematic viscosity for water is $\mathrm{v} = 10^{-2}$ cm^2/s, it turns out that Reynolds number should around 10^9. Given all the facts that is known by field of fluid mechanics one knows that, "*The flow characteristics do not simply flatten out when Reynolds Number becomes Small or Large*. These characteristics are subject to variations and for many geometrical shapes give big and unexpected '*kicks*' as Re passes through values which are significantly high". (The "kick" arise from phenomena such as the sudden transition from smooth laminar to eddying turbulent flow or from other qualitative changes in the flow pattern) [17]. In analyzing ship resistance, we easily can conclude, "kicks" when the Froude number changes as well. One important reason behind this matter is the sensitivity of the resistance to the nature of the interaction between bow and stern waves.

This example, where we are looking a ship design is a good indication of an area of dimensional analysis and modeling which a model design and experiment cannot provide a simple answers. Therefore, ship design is a state of art than a science, although significant contribution to such design comes from field of theoretical fluid mechanics [18, 19]. Expectedly, model experiment cannot provide accurate results, if important parameters are ignored. In other words, a possible source of error in model experiment comes from the fact that, the correctness of conclusion made by dimensional analysis and modeling requires the correctness of the basic assumption that the quantities on interest depend on only parameters that are listed. For example wave from a significantly small ship model, may be influenced by surface tension T. In this case one of the dimensionless parameters that need to be taken under consideration is *Bond Number* Bo $= \rho g \ell^2 / T$. Naturally, if the model does not possess the same Bond number as the full scale prototype, then errors will result if this surface tension T is significant in both experiment that contains model and prototype. Therefore, a good understanding of such circumstances, demands expression of all-important parameters. Some of the difficulties that we encountering are, when such understanding is missing and could be demonstrated by the scale-up problems. For example, one sees in chemical engineering to build new chemical plants, where geometrically model is similar to prototype plants and appropriately similar in many other ways as well, sometimes do not work properly, just because the pilot did not produce some dimensionless parameters correctly, which has a major influence on plant operation, unexpectedly. This fact arises from the angle that in chemical plants, the chemical reactions taking place in pilot plant cannot be analyzed in model experiments and that makes the design a complicated one. For more details, we encourage the readers to the book by Lin and Segel [17].

2.21.3 Models and Mathematical Modeling Techniques and Their Differences

As we said at the beginning of Sect. 2.20 of this chapter, in sense that we are seeking a different scale of thought or mode of understanding we are using the word in its older meaning. The idea of a change of scale, which inheres in the notion of model through its historical wording or etymology, can be interpreted in various ways. The term 'mathematical model' is used for any complete and consistent set of mathematical equations, which thought to correspond to some other entity known as prototype. The prototype may be a physical, biological, social, psychological or conceptual entity, or perhaps even another mathematical model.

Since we agree that, prototype is a physical or natural object; the mathematical model represents a change on the scale of abstraction. To reach this goal, certain particularities have been moved and simplifications made in obtaining the model. For this given reason, some folks and engineers seem to regard the model as less "real" than prototype [13].

Processes taking place in the real world are investigated with the use of appropriate mathematical tools and basing on *mathematical models* built for these processes. In order to achieve such goal, an ideal scientific should include a set of minimal axiom and associated elementary rules and notions that are taken without any proof on basis of which any problem can be solved with the use of formal, i. e., mathematical logic.

Different types of model call for different modes of evaluation and mathematical analysis. There cases that that failing a modeling process suggest that inadequacy of modeling due to difficulty of fitting data to compute and obscures the peculiar strengths and weakness of particular model (See Example-4 of Sect. 2.15 of this chapter). Moreover, it may be sensitive to uninteresting differences between the model and the experimental set-up or certain behavior of the prototype operating environments. Instead, scientist and engineers use a number of trial and error analysis and different statistics such as;

1. Trials before first avoidance.
2. Trails before last shock, etc.

This allows the match between the data and the learning sequences calculated from each model. The formulation of the equations of model is usually a matter of expressing the physical laws or conservation principles in appropriate symbols.

Models can only be approximate and can never claim to be fully adequate to the processes they describe and it is obvious that for the descriptions of phenomena and processes, the construction of mathematical models is essential importance. For mathematical model to be constructed, it is necessary that the most characteristic features of the process under analysis be identified and be taken into account [6]. An acceptable mathematical model should be relatively a simple one while it provides all necessary information about the process under consideration. While certain characteristic should be taken into account, others are considered to a certain extent only, while some others become negligible. This kind of approach is known as an *idealization process* and is responsible for the final success of the problem under investigation. In practical process, a problem under consideration from point of view of mathematical modeling should be simplified while all the constraints and assumption both physically and mathematically should make sense and be verifiable and experimentally be confirmed.

As far as *Mathematical Modeling* is concerned, most of often it is very desirable to be able to find and to describe the behavior of some system in practical world during their real life span, whether is this event should be mathematically modeled in case of physical, economic or even sociological in mathematical terms. Therefore we can use a phenomenon that is known to us by now as a ***mathematicalmodel*** where mathematical description and behavior of it in real-world is defined and constructed with certain specification in mind as part of its expected performance of the system. For instance if we are interested to have some understanding of ecosystem of the growth of animal populations buy studying in that system, or determine age of fossils by analyzing the decay of a radiative substances either in them or in the stratum in which it was discovered and this type of modeling in a system starts with identifying of the variables in that system that are responsible for changes in it. Note that in some cases we may not consider and incorporate all these variables into the model at first pass. In other words, we just interested to specify the **level of resolution** of the model, then as a second step effort, we make a set of reasonable assumption or defining certain hypotheses about system description that we interest to model it mathematically.

These sort of mathematical modeling may end up with formulation set of differential equation or equations as whole in the system that we need to solve. We are then successfully have modeled the behavior of the system if our analytical or numerical solution corresponds with result of our experimental data for the given system of interest and its expected behavior. On the other-hand if the prediction that are produced by result of a poor solution,, then we need to increase the level of resolution of the model we assumed at first place and revisit some of our assumption as well as our hypothesis where system goes through and in that case may need some changes within mechanisms of the system. We cans depict all these steps in the form of following diagram (Fig. 2.16);

New problem always required qualitatively new experiments since introduced simplification can frequently lead to conclusions that are not realized physically. Therefore, an idealization process requires certain ordering of its different elements, which is achieved by scaling and similarity in order to compare the elements with one another in given system to particular characteristic quantities chosen earlier. For example from dimensional analysis perspective, if one of the system dimension or elements is 1 cm in length, a natural question arises whether this quantity is smaller or bigger.

Depending on the problem, whether is *acrostic* or *microscopic* this dimension or quantity is negligible and can be decided based on the initial formulation of the problem under consideration. For example, when we are looking at traveling from earth to the moon1 cm scale can be treated as a negligibly small quantity. However, under hand if we deal with measuring in microscopic scale then this quantity plays and can be accounted for comparison purpose to model the problem mathematically.

Sometime certain elements are not considered and will not account for mathematical modeling for one problem under certain boundary condition and environment that problem is dealt with, yet same element can play a huge rule in solving same problem under different circumstances. Example of this encountering is air compressibility. In general the air is compressible and it is always necessary to take it into account for any moving object within it, but will not play a big role in a moving object in it with very small velocity and drag produced by it can be negligible. On the other hand, if the object's speed is very high, say close to speed of sound (supersonic or hypersonic) conditions, then the compressibility cannot be neglected and plays in any mathematical modeling of any related design such as flying supersonic air plane. In this case it is convenient to introduce a dimensionless quantity known as Mach number that is presented as $M = V/a$, which plays an important role in aerodynamics. When $M \ll 1$, an idealized mathematical model of incompressible gas can be applied, while for greater value of Mach number, air compressibility should be taking under consideration. A similar situation occurs when a mathematical model is constructed in the domains of science and technology where other characteristic dimensionless numbers are significant, usually built as combinations of three-dimensional quantities, i. e., length L, time T, and mass M. If F stands for force then the dimension of combination $(FT^2)/(ML)$ is equal to 1 and this indicates that one of the quantities F, T, L, and M may be chosen to be independent.

Using Pi theorem concept and dimensional analysis method should include reduction of all variables as first step for a process that is under consideration to dimensionless quantities. This can be achieved through assigning of all initial variables of the process to certain characteristic quantities corresponding to the variables in terms of the dimension or their combinations: length L, velocity V, friction coefficient μ, spring stiffness k, dynamic viscosity coefficient υ, and the like. This is the reason why application of a dimensional analysis theory to a problem of any branch of science or technology always leads to the determination of a set of characteristic dimensionless quantities or similarity parameters whose values characterize qualitatively the essence of examined processes, which is known as process similarity rules. The examples such as the numbers like Mach, Nusselt, Reynolds, Strouhal, Froude, Biot, together with many others, very big or very small, can be presented as many that are discussed in next section.

2.21.4 Examples of Modeling

The following examples-1 from book by Awrejcewicz and Krysko [6] provides an illustration of how a problem is reduced to its dimensionless form and a small parameter appears.

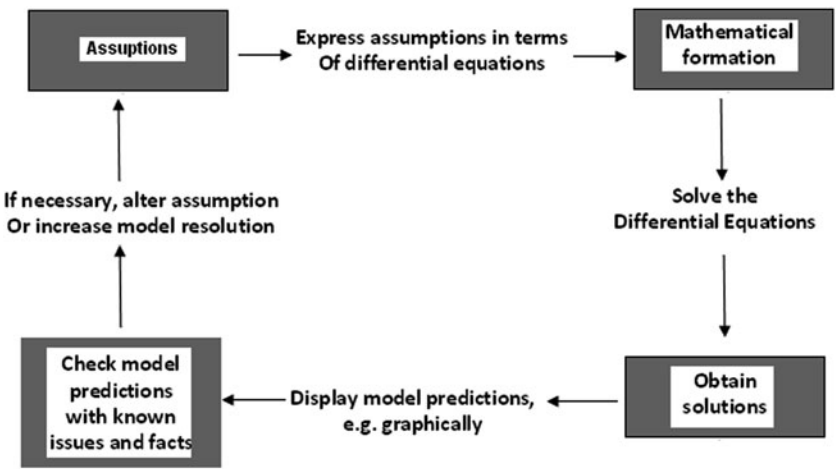
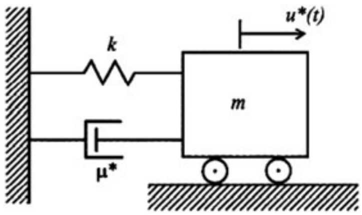


Fig. 2.16 Mathematical modeling steps illustration

Fig. 2.17 Free vibrations of one degree of freedom system



Example 2.15 Consider the motion of a rigid body of mass m attached to a frame by means of a mass-less linear of spring with stiffness coefficient μ^* in a medium characterized by viscous damping. See Fig. 2.17.

Solution This is a classic system of free vibrations with damping often referred to as *the problem of linear oscillator with damping*. From Newton's second law of dynamics, the equation like below

$$m\left(\frac{d^2u^*}{dt^{*2}}\right) + \mu^*\left(\frac{du^*}{dt^*}\right) + ku^* = 0 \tag{I}$$

is obtained, where dependent variables $u^*(t^*)$ describe displacement of the examined body from a certain initial position $u_0{}^*$, and time t^* is an independent variable. Assuming that an initial moment $t^* = 0$ the body is in an initial position a^* and the initial speed of its motion is zero, i. e.;

$$u_0{}^* = u^*(0) = a^* \qquad \frac{du^*}{dt^*(0)} = 0 \tag{II}$$

To reduce above two Eqs. (I) and (II) to a dimensionless form, characteristic values should be first assumed. To do that, we let $u_0{}^* - a$ be a characteristic length value, and $\omega_0 = \sqrt{k/m}$ the eigenfrequency of the considered body for $\mu^* = 0$ in the differential equation. From eq. (I), we obtain;

$$\frac{d^2u^*}{dt^{*2}} + 2h\left(\frac{du^*}{dt^*}\right) + {\omega_0}^2 u^* = 0 \tag{III}$$

The characteristic equation corresponding to it assumes the form of

$$\lambda^2 + 2h\lambda + \omega^2 = 0$$

in addition, it yields two roots $\lambda_{1,2} = -h \pm i\sqrt{\lambda}$, where $\lambda = \sqrt{\omega^2 - h^2}, 2h = \mu^*/m, i^2 = -1$.

In the case of $h = 0$, since the real part of the complex conjugated roots equals zero, the general solution of differential Eq. (III) has the form

$$u^*(t, c_1, c_2) = c_1 \cos\ \omega_0 t^* + c_2 \sin\ \omega_0 t^*$$

where c_1 and c_2 are arbitrary constant numbers. From the initial conditions, we determine $c_1 = a^*$, $c_2 = 0$ and

$$u^*(t^*) = a^* \cos \omega_0 t^*$$

which means that $\omega_0 = \sqrt{k/m}$ is the eigenfrequency, and a^* is the amplitude of vibrations.

Introducing dimensionless quantities t and $u(t)$ in accordance with the formulas;

$$t = \omega_0 t^* \qquad u = \frac{u^*}{{u_0}^*} \tag{IV}$$

then transforming Eqs. (I) and (II), and making use of both Eq. (IV) and the composite function differentiation rule, we arrive at

$$\frac{du^*}{dt^*} = \frac{du^*}{dt}\frac{dt}{dt^*} = \frac{d({u_0}^*u)}{dt}\frac{dt}{d(t/\omega_0)} = \omega_0 {u_0}^* \frac{du}{dt} \tag{V}$$

Analogically, we obtain

$$\frac{d^2u^*}{dt^{*2}} = {\omega_0}^* {u_0}^* \frac{d^2u}{dt^2} \tag{VI}$$

Substituting Eq. (IV) through (VI) into (I), (II), we have;

$$m{\omega_0}^2{u_0}^*\frac{d^2u}{dt^2} + \mu^*\omega_0{u_0}^*\frac{du}{dt} + k{u_0}^*u = 0$$

$$u^*(0) = {u_0}^*u(0) = a^* \qquad \omega_0{u_0}^*(0) = 0$$

which, finally gives

$$m{\omega_0}^2\frac{d^2u}{dt^2}+\mu^*\omega_0\frac{du}{dt}+ku=0$$

or

$$\frac{d^2u}{dt^2}+\frac{\mu^*}{m\omega_0}\frac{du}{dt}+k\frac{u}{m{\omega_0}^2}=0$$

$$u(0)=\frac{a^*}{{u_0}^*}.\frac{du}{dt(0)}=0$$

Introducing a dimensionless viscosity coefficient

$$\mu=h{\omega_0}^{-1} \tag{VII}$$

in addition, selecting for example ${u_0}^*=a^*$, we arrive at the following dimensionless form of a mathematical problem connected with *a damped linear oscillator analysis*:

$$\frac{d^2u}{dt^2}+2\mu\frac{du}{dt}+u=0 \tag{VIII}$$

$$u(0)=1 \qquad \frac{du}{dt(0)}=0 \tag{IX}$$

As we can see Eqs. (VIII) and (IX) define the initial problem that is known as *Cauchy Problem* for the linear differential Eq. (VIII) of the second order with constant coefficients (Cauchy problem is defined in Sect. 4.7 of Chap. 4). Its solution depends on one dimensionless parameter μ, being the ratio of the forces of resistance and inertia or, in other words, of damping and elasticity and it has the form;

$$\mu=e^{-\mu t}\cos\sqrt{{\omega_0}^2-\mu^2 t} \tag{X}$$

If μ is small, it is said that the oscillator is weakly damped. If $\mu=0$, then harmonic eigenfrequency $\omega_0=\sqrt{k/m}$ occurs, It is worth noticing that exploiting the solution to equation (VIII), (IX), and using (IV) and (VII) for calculations, it is possible to obtain a solution of the initial Eqs. (I), (II) for different sets of values m, k, μ^*, a^*. This is where the very essence of the transition to a dimensionless model before the solving procedure starts should be searched. The solution of problem (VIII) and (IX) is discussed in Chap. 4, Sect. 4.7.

Analogous procedure can be applied in transition to dimensionless quantities while analyzing other more complex process. For more discussion, reader should refer to reference 6 of this chapter.

2.21.5 The Modeling Process, Proportionality, and Geometric Similarity

A scale model has the same shape as the system of primary interest but is of a more convenient size. The object of the present remarks is to stress that model performance which closely mimics that of the given system (prototype) can only be expected if not only size, but also other attributes, are proportioned so that the important dimensionless parameters are identified in both system (prototype) and model. Suppose we want to understand some behavior or phenomenon in the real world. We may wish to make predictions about that behavior in the future and analyze the effects that various situations have on it.

For example, in the case of the administration of a drug to a patient, it is important to know the correct dosage and the time between doses to maintain a safe and effective level of the drug in the bloodstream. Constructing a good usage of model in the mathematical world to help us better understand real-world systems is a huge step toward cost effectiveness and time reduction of process. In order to link the two world that is depicted in Fig. 2.18 we need to consider what do we mean by real-world system?, and why we would be interested in constructing a mathematical model for a system in the first place [14].

The modeler is interested in understanding how a particular system works and what causes changes in the system, and how sensitive the system is to certain changes. He might be also interested in predicting what changes might occur and when they occur. How such information might be obtained. To show these entire questions in some sort of depiction we use the Fig. 2.19 that is presented by reference 15. An examination of Fig. 2.19 suggests an alternative way of reaching conclusions about the real world. First, we make specific observations about the behavior being studied and identify the factors that seem to be involved (in case of distortion that may be predicting factor defined in Sect. 2.16.2). Technically, we cannot identify all the involved factors, but by collecting more data based on observation and experiment, we can be close to real world and system behavior in that world. In addition, we sometime ignore certain factor that is involved with prototype during build of the model. For instance, we may ignore the humidity in city of New York while we are concern about any radiation leak from nuclear power plant nearby. We also need to identify relationships among the factors we have chosen to build or model upon. Having constructed a model, we then apply appropriate mathematical analysis leading to conclusions about the model.

In summary, we have the following rough modeling procedure [14]:

1. Through observation, identify the primary factors involved in the real-world behavior, possibly making simplifications.
2. Conjecture tentative relationships among the factors.
3. Apply mathematical analysis to the results model.
4. Interpret mathematical conclusions in terms of the real-world problem.

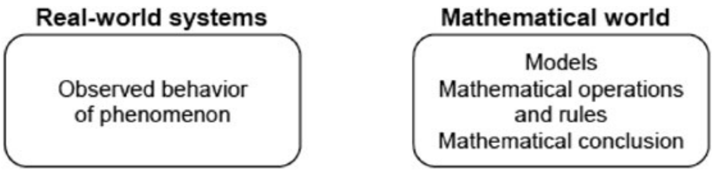


Fig. 2.18 The real and mathematical world

Fig. 2.19 Reaching conclusions about the behavior of real-world systems

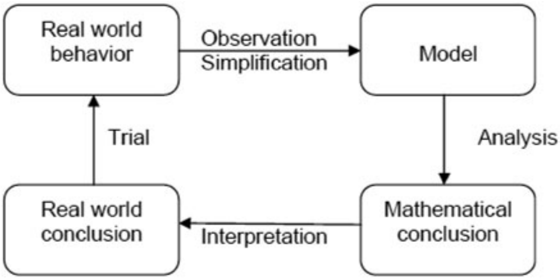
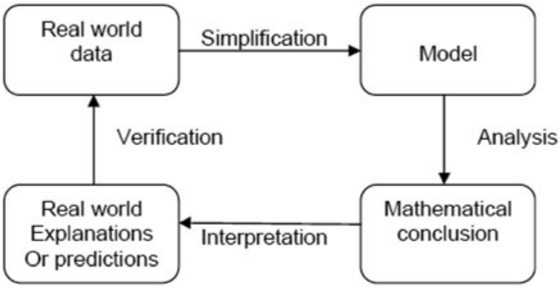


Fig. 2.20 The modeling process as a closed system



All these can also be depicted in Fig. 2.20 that is presented by Giordano et al. [14] book.

By the same talking, we can also show the mathematical modeling that supports the above process in flow diagram such as is the one presented in Fig. 2.19 and is copied from reference 15 of this chapter. The question of constructing a mathematical model, a variety of conditions can cause us to abandon hope of achieving any success. The mathematics involved may be so complex and intractable that there is little hope of analyzing or solving the model, thereby defeating its utility [14].

This complexity can occur, for example, when attempting to use a model given by a system of partial differential equations or a system of nonlinear algebraic equations. Alternatively, the problem may be so large in terms of the number of factors involved, that it is impossible to capture all the necessary information in a single mathematical model (See Fig. 2.21 below).

2.21.6 Structural Models

If we are interested in analyzing the behavior any structure under load, then we should consider the subject of structural model. The behavior is normally evaluated in terms of resistance to failure as part of design considerations by [5];

a. Excessive elastic deformation
b. Undesirable inelastic action
c. Fracture
d. Buckling
e. Crippling or localized bucking

The deflection and stress are functions of

a. the force and restraints applied to the structure,
b. the geometry of the structure,
c. the properties of the materials used in the structure

Acting forces upon each other is a function of the geometry of the cross section and properties of the material. For axial tension, the geometry maybe adequately described by the area, while the properties involved are the modules of elasticity, proportional limit or other measure of strength, and in some cases Poisson's ratio. We can go on with more details design consideration, which is beyond scope of this book but readers can refer to any structure and material science books in order to understand behavior shear and normal forces and so forth. Murphy [5] also provides some chapter allocated in this subject and structural modeling.

Few examples that can be named dealing with structural modeling include;

1. Beggs Deformeter Gage
2. Application of Beggs Deformeter Gage to Arches
3. Tower of George Washington Bridge
4. Hoover Dam Structure
5. Wind Forces on the Empire State Building
6. Earthquake Effects on Water Towers
7. etc

If one interested in more details of any of these examples look at reference 5 at the end of this chapter.

2.22 Distorted Models

As we quickly in Sect. 2.15 touched upon, it is impossible to satisfy all of the design conditions for a true model where Eq. 2.7 holds. More often, the proper materials may not be available, and details may impose excessive fabrication cost, or available equipment may impose limitations, which is precluding the possibility of meeting all of the design requirements. As result these conditions by definition we have a *Distorted Model*, which is the result of the model not satisfying all the design equations criteria such as what we have seen in Prediction Eq. 2.7 and 2.8. Usually several alternatives of design are possible, depending upon which of the requirements should be sacrificed and as result be ignored in the design process. Although it is very important and highly desirable to determine which of the possible designs is preferable, and what affects the distortion and Eq. 2.9 plays on prediction equation (Eq. 2.7 and 2.8). Murphy [5] best presents the description of Distorted Model in his book.

A distorted model is the one that it is so constructed through either ignorance, unavoidably or deliberately. This model is the one, which one of the Pi terms is not equal to corresponding Pi term in the prototype. One possible solution to overcome this problem is an introduction of some fudge factor (prediction factor) or an estimate of the error by distortion should be introduced and be formulated [5].

In some cases, the effects of distortion maybe predicted with accuracy compatible with ordinary design standards, while under other conditions it is difficult to come up with such formulation to estimate a reasonable effect of the distortion without constructing more than one model. Some types and consequences of distortion are considered in the following.

Distortion maybe classified based on the nature of the Pi term, which is discarded. The general equation for displacement in a statistical structure member, Eq. 2.45 is [5];

$$\frac{d}{l} = \phi_1\left(\frac{\lambda}{l}, \frac{R}{El^2}, \frac{S_p}{E}, \frac{S_p}{{S'}_p}, \frac{G}{E}\right) \tag{2.45}$$

Analyzing Eq. 2.14 will produce three types of distortion as follow

1. **Geometrical**, $\frac{\lambda_m}{l_m} \neq \frac{\lambda}{l}$, which also can be divided in two groups by itself and design of cross sections in a model truss on the basis of an area scale or a moment of inertia scale not compatible with the length scale is another common manifestation of geometrical distortion.
   a. **Distortion of configuration:** As example, representing a triangular panel in a truss by a rectangular panel in the model, leads to such obvious difficulties of analysis that it will not be considered further.
   b. **Distortion of dimension:**
      i. Length
      ii. Width
      iii. Depth

   By having all length scale, width scale, or height scale different from other two, but having all lengths to same scale, all depth to the same scale, and all widths to the same scale, is a common type of distortion.

2. **Loading,** $\frac{R_m}{E_m {l_m}^2} \neq \frac{R}{El^2}$

Usually involves the application of loads too large or too small to satisfy similitude requirements. The model and prototype may be geometrically similar before loading, but they are not geometrically similar after loading. If the proportional limit is not exceeded and if buckling does not occur, distortion of loading is not difficult to take into account in adjusting the prediction equation.

3. **Material,** $\frac{S_{pm}}{E_m} \neq \frac{S_p}{E}$, etc.

   Distortion of property of materials maybe considered in several categories:

   a. **Elastic behavior:** This behavior occurs if Poison's ratio is not identical in model and prototype. Under conditions of axial stress, this is relatively unimportant, but it may become significant under conditions of biaxial and triaxial stress.

   b. **Plastic behavior:** This behavior may be described as;

      i. Low proportional limit for model or prototype

      ii. Different stress-strain behavior in model and prototype above proportional limit

      iii. Different strain-time characteristic under sustained loading

Any of these may cause difficulty in analysis and is to be avoided if possible. All these distortion types are dealt with under the assumption that the structure is in equilibrium. If dynamic loading is involved, there are additional Pi terms in Prediction Eq. 2.8. These Pi terms impose restrictions of operating procedure rather than basic design of the model, but they serve to establish kinematic similarity of model and prototype, and if they are ignored, kinematic distortion will result.

If a test is described using dimensional analysis and associated Pi theory by its terms in form of Eq. 2-xx2 (See Murphy) [5], then requires establishing $\prod_1$ as a function of $\prod_3$ for two different values of one of the other Pi term.

$$\prod_1 = \frac{\Phi\left(\Pi_2, \Pi_{3m}, \Pi_{4m} \cdots, \Pi_{nm}\right) \Phi\left(\Pi_{2m}, \Pi_3, \Pi_{4m} \cdots, \Pi_{nm}\right) \Phi\left(\Pi_{2m}, \Pi_{3m}, \Pi_4 \cdots, \Pi_{nm}\right) \cdots \Phi\left(\Pi_{2m}, \Pi_{3m}, \Pi_{4m} \cdots, \Pi_n\right)}{\left[\Phi\left(\Pi_{2m}, \Pi_{3m}, \Pi_{4m} \cdots, \Pi_{nm}\right)\right]^{n-2}} \tag{2.46}$$

From the experimental point of view, this requires running two sets of test in which the degree of distortion is varied. The degree of distortion may be evaluated as a distortion factor α, defined as,

$$\alpha = \frac{\prod_{3m}}{\prod_3} \qquad \text{or} \qquad \Pi_{3m} = \alpha \Pi_3 \tag{2.47}$$

Thus $\alpha = 1$ indicates that modeling system has no distortion to be worried about. Therefore, if a series of test is to be made for any modeling investigation and validity of combining the component equations as a product, it is imperative and would desirable that α be near unity as much as possible. However, it is apparent that if α could not be made equal to unity, the model would be undistorted and the evaluation of the prediction factor δ would be unnecessary. There may be instances in which it is impossible to let α equal unity, but in which values greater than 1 and less than 1 are possible. This situation may arise where the distorted quantity involves properties of the material [5].

Murphy [5] argues that if the log plots of $\prod_1$ versus $\prod_3$ is a straight line and $\prod_3$ satisfies the requirements for combination as a product, it follows from Eq. 2.2 that;

$$\delta = \frac{\phi\left(\prod_3\right)}{\phi\left(\alpha\prod_3\right)} \tag{2.48a}$$

$$\delta = \frac{C{\prod_3}^m}{C\alpha^m{\prod_3}^m} \tag{2.48b}$$

$$\delta = \frac{1}{\alpha^m} = \alpha^{-m} \tag{2.48c}$$

in which m is the slop of the line in the log plot. Even if values of $\prod_1$ plotted against $\prod_3$ do not plot as a straight line, the value of $\prod_1$ corresponding to the correct value of $\prod_3$ for a true model, may be taken from the curve. Readers should refer to Murphy's book (Chap. 6) [5] for details and examples in order to have a better concept of distorted model.

2.23 Nondimensionalization

The numerical value of any quantity in a mathematical model is measured with respect to a system of units (for example, meters in a mechanical model, or dollars in a financial model). The units used to measure a quantity are arbitrary, and a change in the system of units (for example, to feet or yen, at a fixed exchange rate) cannot change the predictions of the model. A change in units leads to a rescaling of the quantities. Thus, the independence of the model from the system of units corresponds to a scaling invariance of the model. In cases when the zero point of a unit is arbitrary, we also obtain a translational invariance, but we will not consider translational invariance here.

Suppose that a model involves quantities $(a_1, a_2, \cdots, a_n)$, which may include dependent and independent variables as well as parameters. We denote the dimension of a quantity a by [a]. A fundamental system of units is a minimal set of independent units, which we denote symbolically by $(d_1, d_2, \cdots, d_r)$. Different fundamental system of units can be used, but given a fundamental system of units any other derived unit may be constructed uniquely as a product of powers of the fundamental units, so that

$$[a] = {d_1}^{\alpha_1}{d_2}^{\alpha_2}\cdots{d_r}^{\alpha_r} \tag{2.49}$$

for suitable exponents $(\alpha_1, \alpha_2, \cdots, \alpha_r)$.

Example 2.16 In mechanical problem, a fundamental set is $d_1 =$ mass, $d_2 =$ length, $d_3 =$, or $d1 = M$, $d_2 = L$, $d_3 = T$ for short notation. Then velocity $V = L/T$ and momentum $P = ML/T$ are derived units. We could use instead momentum P, velocity V, and time T as a fundamental system of units, when mass may introduce temperature (measured, for example, in degrees Kelvin) as another fundamental unit, and in problems involving electromagnetism, we may introduce current (measured, for example, in Amperes) as another fundamental unit.

The invariance of a model under the change in units $d_j \mapsto \lambda_j d_j$ implies that it is invariant under the scaling transformation

$$a_i \to \lambda_1{}^{\alpha_{1,i}} \lambda_2{}^{\alpha_{2,i}} \cdots \lambda_r{}^{\alpha_{r,i}} a_i \qquad i = 1, \cdots, n$$

for any $\lambda_1, \cdots, \lambda_r > 0$, where

$$[a_i] = d_1{}^{\alpha_{1,i}} d_2{}^{\alpha_{2,i}} \cdots d_r{}^{\alpha_{r,i}} \tag{2.50}$$

Thus, if

$$a = f(a_1, \cdots, a_n)$$

is any relation between quantities in the model with the dimensions in Eqs. 2.49 and 2.50, then f has the scaling property that

$$\lambda_1{}^{a_{1,i}} \lambda_2{}^{a_{2,i}} \cdots \lambda_r{}^{a_{r,i}} f(a_1, \cdots, a_n)$$
$$= f(\lambda_1{}^{a_{1,1}} \lambda_2{}^{a_{2,1}} \cdots \lambda_r{}^{a_{r,1}} a_1, \cdots, \lambda_1{}^{a_{1,n}} \lambda_2{}^{a_{2,n}} \cdots \lambda_r{}^{a_{r,n}} a_n)$$

A particular consequence of the invariance of a model under a change of units is that any two quantities, which are equal, must have the same dimensions. This fact is often useful in finding the dimension of some quantity.

Example 2.17 According to Newton's second law,

Force = rate of change of momentum with respect to time

Thus, if F denotes the dimension of force and P the dimension of momentum, then $F = P/T$. Since $P = MV = ML/T$, we conclude that $F = ML/T^2$ (or mass x acceleration).

Example 2.18 In fluid mechanics, the shear viscosity μ of a Newtonian fluid is the constant of proportionality that relates the viscous stress tensor T to the velocity gradient $\nabla\vec{u}$:

$$T = \frac{1}{2}\mu(\nabla\vec{u} + \nabla\vec{u}^T)$$

Stress has dimensions of force/area, so

$$[T] = \frac{ML}{T^2}\frac{1}{L^2} = \frac{M}{LT^2}$$

The velocity gradient $\nabla\vec{u}$ has dimensions of velocity/length, so

$$[\nabla\vec{u}] = \frac{L}{T}\frac{1}{L} = \frac{1}{T}$$

Equating dimensions, we find that

$$[\mu] = \frac{M}{LT}$$

We can also write $[\mu] = (M/L^3)(L^2/T)$. It follows that if ρ_0 is the density fluid, and $\mu = \rho_0 \nu$, then

$$[\nu] = \frac{L^2}{T}$$

Thus ν, which is called the kinematical viscosity, has the dimensions of diffusivity. Physically it is the diffusivity of momentum. For example, in time T, viscous effects lead to the diffusion of momentum over a length scale of the order $\sqrt{\nu T}$.

At 20 °C, the kinematic viscosity of water is approximately 1 $(\mathrm{mm})^2$/s. Thus, in one second, viscous effects diffuse the fluid momentum over a distance of the order 1 mm.

Scaling invariance implies that we can reduce the number of quantities appearing in the problem by the introduction of dimensionless variables. Suppose that $(a_1, \cdots, a_r)$ are a set of (nonzero) quantities whose dimensions form a fundamental system of units. We denote the remaining quantities in the model by $(b_1, \cdots, b_m)$, where $r + m = n$. Then for suitable exponents $(\beta_{1,i}, \cdots, \beta_{r,i})$, the quantity

$$\Pi_i = \frac{b_i}{{a_1}^{\beta_{1,i}} \cdots {a_r}^{\beta_{r,i}}}$$

is dimensionless, meaning that it is invariant under the scaling transformations induced by changes in units. Such dimensionless quantities can often be interpreted as the ratio of two quantities of the same dimension appearing in the problem (such as a ratio of lengths, times, diffusivities, and so on). Perturbation methods are typically applicable when one or more of these dimensionless quantities is small or large.

Any relationship of the form

$$b = f(a_1, \cdots, a_r, b_1, \cdots, b_m)$$

is equivalent to a relation as follow;

$$\prod = f\left(1, \cdots, 1, \prod_1 \cdots, \prod_m\right)$$

Thus, the introduction of dimensionless quantities reduces the number of variables in the problem by the number of fundamental units involved in the problem. In many cases, nondimensionalization leads to a reduction in the number of parameters in the problem to a minimal number of dimensionless parameters. In some cases, one may be able to use dimensional arguments to obtain the form of self-similar solutions.

Example 2.19 Consider the following IVP for the Green's function of the heat equation in $\mathbb{R}^d$:

$$u_t = \nu \Delta u$$
$$u(x,0) = E\delta(x)$$

Here δ is the delta-function. The dimensioned parameters in this problem are the diffusivity v and the energy E of the point source. The only length and times scales are those that come from the independent variables (x, t), so the solution is self-similar.

We have $[u] = \theta$ where θ denotes temperature, and since

$$\int_{\mathbb{R}^d} u(x,0)dx = E$$

We have $[E] = \theta L^d$. Dimensional analysis and the rotational invariance of the Laplace Δ imply that

$$u(x,t) = \frac{E}{(\nu t)^{d/2}} f\left(\frac{|x|}{\sqrt{\nu t}}\right)$$

Using this expression for $u(x, t)$ in the PDE, we get an ODE for $f(\xi)$,

$$f'' + \left(\frac{\xi}{2} + \frac{d-1}{\xi}\right) f' + \frac{d}{2} f = 0$$

We can rewrite this equation as a first-order ODE for $f' + \frac{\xi}{2} f$,

$$\left(f + \frac{\xi}{2} f\right)' + \frac{d-1}{\xi}\left(f' + \frac{\xi}{2} f\right) = 0$$

Solving this equation, we get

$$f' + \frac{\xi}{2} f = \frac{b}{\xi^{d-1}}$$

where b is a constant of integration. In order for f to be integrable, we must set $b = 0$. Then

$$u(x,t) = \frac{aE}{(\nu t)^{d/2}} \exp\left(-\frac{|x|^2}{4\nu t}\right)$$

Imposing the requirement that

$$\int_{\mathbb{R}^d} u(x,t)dx = E$$

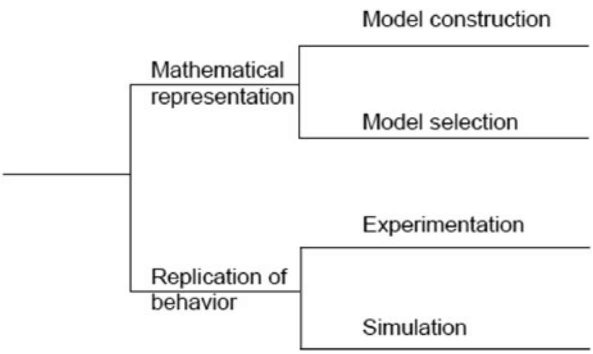
and using the standard integral

$$\int_{\mathbb{R}^d} \exp\left(-\frac{|x|^2}{4\nu t}\right) dx = (2\pi c)^{d/2}$$

we find that $a = (4\pi)^{-d/2}$, and

$$u(x,t) = \frac{E}{(4\pi \nu t)^{d/2}} \exp\left(-\frac{|x|^2}{4\nu t}\right)$$

Fig. 2.21 The nature of the model



Example 2.20 Consider a sphere of radius L moving through a fluid with constant speed U. A primary quantity of interest is the total drag force D exerted by the fluid on the sphere. We assume that the fluid is incompressible, which is a good approximation if the flow speed U is much less than the speed of sound in the fluid. The fluid properties are then determined by the density ρ_0 and the kinematic viscosity v. Hence,

$$D = f(U, L, \rho_0, v)$$

Thus, the dimensionless drag is given by;

$$\frac{D}{\rho_0 U^2 L^2} = F(\mathrm{Re})$$

is a function of the Reynolds's number

$$\mathrm{Re} = \frac{UL}{v}$$

The function F has a complicated dependence on Re that is difficult to compute explicitly. For example, F changes rapidly near Reynolds number for which the flow past the sphere becomes turbulent. Nevertheless, experimental measurements agree very well with the result of this dimensionless analysis. See Fig. 2.22 for example.

The equations of motion of the fluid are the incompressible Navier-Stokes equations,

$$u_t + u \cdot \nabla u + \nabla p = v \Delta u$$
$$\nabla \cdot u = 0$$

To nondimensionalize these equations with respect to (U, L, ρ), we introduce dimensionless variables

$$u^* = \frac{u}{U} \qquad p^* = \frac{p}{\rho U^2} \qquad x^* = \frac{x}{L} \qquad t^* = \frac{Ut}{L}$$

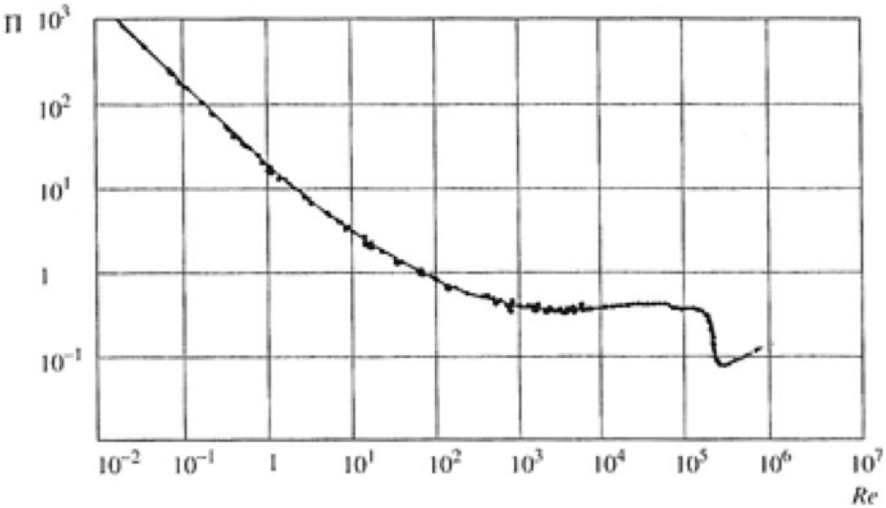


Fig. 2.22 The dimensionless drag force as a function of Reynolds Number. The data is from various experiments turn out to lie on a single curve 3

in addition, find that

$$u_{t*}{}^{*} + u^{*} \cdot \nabla^{*} u^{*} + \nabla^{*} p^{*} = \varepsilon \Delta^{*} u^{*}$$
$$\nabla^{*} \cdot u^{*} = 0$$

Here

$$\varepsilon = \frac{\nu}{UL} = \frac{1}{\mathrm{Re}}$$

The boundary conditions correspond to allow of speed 1 past a sphere of radius 1. Thus, assuming that no other parameters enter into the problem, the drag computed from the solution of these equations depends only on, as obtained from the dimensional analysis above.

Dimensional analysis leads to continuous scaling symmetries. These scaling symmetries are not the only continuous symmetries possessed by differential equations. The theory of Lie groups and Lie algebras provides a systematic method for computing all continuous symmetries of a given differential equation [8]. Lie originally introduces the notions of Lie groups and Lie algebras precisely for this purpose.

Example 2.21 The full group of symmetries of the one-dimensional heat equation

$$u_t = u_{xx}$$

is generated by the following transformations [8];

$$
\begin{aligned}
u(x,t) &\mapsto u(x-\alpha, t) \\
u(x,t) &\mapsto u(x, t-\beta) \\
u(x,t) &\mapsto \gamma u(x,t) \\
u(x,t) &\mapsto u(\delta x, \delta^2 t) \\
u(x,t) &\mapsto e^{-\varepsilon x+\varepsilon^2 t} u(x-2\varepsilon t, t) \\
u(x,t) &\mapsto \frac{1}{\sqrt{1+4\eta t}} \exp\left[\frac{-\eta x^2}{1+4\eta t}\right] u\left(\frac{x}{1+4\eta t}, \frac{t}{1+4\eta t}\right) \\
u(x,t) &\mapsto u(x,t) + \upsilon(x,t)
\end{aligned}
$$

where $(\alpha, \cdots, \eta)$ are constants, and $\upsilon(x,t)$ is an arbitrary solution of the heat equation. The scaling symmetries involving γ and δ can be deduced by dimensional arguments, but the symmetries involving ε and η cannot.

References

1. Barenblatt GI, Zel'dovich YB (1972) Self-similar solutions as intermediate asymptotic Annu Rev Fluid Mech 4:295–312
2. Barenblatt GI (1996) Scaling, self-similarity, and intermediate asymptotic. Cambridge University Press, Cambridge
3. Barenblatt GI (2003) Scaling. Cambridge University Press, Cambridge
4. Skoglund V (1967) Similitude: theory and applications. International Textbook Company
5. Murphy G (1950) Similitude in engineering. The Ronald Press Company, New York
6. Awrejcewicz J, Vadim A Krysko (2006) Introduction to asymptotic methods. Chapman & Hall/CRC, Taylor & Francis Group
7. Bluman GW, Cole JD (1943) Similarity methods for differential equations. Springer
8. Olver P (1991) Applications of lie groups to differential equations, 2nd edn. Springer-Verlag, New York
9. Munson BR, Young DF, Okiishi TH, WW Huebsch (2009) Fundamentals of fluid mechanics, 6th edn. Wiley (Chapter 7)
10. Huntley HE (1952) Dimensional analysis. Macdonald, London
11. Isaacson E. de St. Q, Isaacson M. de St. Q (1975) Dimensional methods in engineering and physics. Wiley, New York
12. Moser A, Folkman S (2008) Buried pipe Design, 3rd edn. McGraw-Hill Professional
13. Aris R (1994) Mathematical modeling techniques. Dover Publication, Inc., New York
14. Giordano FR, Fox WR, Horton SB, Weir MD (2008) A First Course in mathematical modeling, 4th edn. Brooks/Cole Cengage Learning.
15. Giles RV (1962) Fluid mechanics and hydraulics, 2nd edn. Schums's Outline Series, Published by McGraw Hill
16. Lin CC, Segel LA (1988) Mathematics applied to deterministic problems in the natural sciences SIMA by C. C. Lin and L. A. Segel Published by SIAM: Society for Industrial and Applied Mathematics

17. Lighthill MJ (1963) Introduction boundary layer theory (1988) Inside: Rosenhead L (ed) Laminar boundary layers. Oxford University Press, Oxford, 46–113. (Dover Publisher recently publishes this book in May (72–78))
18. Gadd G (1965) Understanding ship resistance mathematically J Inst Math Appl 4:43–57
19. Peregrine DH (1971) A ship's wave and its wake J Fluid Mech 49:353–60

http://www.springer.com/978-3-319-13475-8

Dimensional Analysis and Self-Similarity Methods for
Engineers and Scientists
Zohuri, B.
2015, XVI, 372 p. 103 illus., 35 illus. in color., Hardcover
ISBN: 978-3-319-13475-8