

Chapter 2

Thermal Convection with LTNE

2.1 Stability and Symmetry

The object of this chapter is to present the equations of thermal convection with local thermal non-equilibrium (LTNE) effects and analyse stability properties of their solutions. The linear operator which arises in the LTNE equations often belongs to a special class of operators, known as symmetric operators, and this class of linear operator has special mathematical properties in the context of stability. Before we briefly discuss an important general nonlinear stability result we include an example of a symmetric linear operator which occurs in the classical theory of thermal convection in a fluid, a phenomenon usually referred to as Bénard convection.

2.1.1 Classical Bénard Convection

Let us consider a linear viscous fluid governed by equations (1.42). The fluid occupies the domain contained between the horizontal planes $z = 0$ and $z = d$ with $(x, y) \in \mathbb{R}^2$. The upper plane is held at a fixed constant temperature T_U , while the lower plane is held at a fixed constant temperature T_L , with $T_L > T_U$.

In the above situation equations (1.42) possess the motionless conduction solution

$$\bar{v}_i \equiv 0, \quad \bar{T} = \beta z + T_L, \quad (2.1)$$

where $\beta = (T_L - T_U)/d > 0$ and $\bar{p}(z)$ is determined up to a constant from equation (1.42)₁. To study the stability of solution (2.1) we introduce perturbations u_i, θ, π by $v_i = \bar{v}_i + u_i$, $T = \bar{T} + \theta$, $p = \bar{p} + \pi$ and we non-dimensionalize the resulting perturbation equations with the scalings of pressure $P = \rho_0 v U / d$, time $\mathcal{T} = d^2 / \nu$, length d , and velocity $U = \nu / d$. Introduce the Prandtl number $Pr = \nu / \kappa$ and the parameter $R = \sqrt{\alpha g \beta d^4 / \kappa \nu}$ where $Ra = R^2$ is the Rayleigh number. The

non-dimensional perturbation equations may then be shown to be

$$\begin{aligned}\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} &= -\frac{\partial \pi}{\partial x_i} + \Delta u_i + R\theta k_i, \\ \frac{\partial u_i}{\partial x_i} &= 0, \\ Pr \left(\frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_i} \right) &= Rw + \Delta \theta,\end{aligned}\tag{2.2}$$

for $(x, y) \in \mathbb{R}^2$ and $z \in (0, d)$, and where $w = u_3$. The boundary conditions to be satisfied are those of no slip on the velocity field together with constant temperatures on the upper and lower horizontal bounding planes of the fluid. Hence, we require the perturbation variables u_i and θ to be such that

$$u_i = 0, \quad \theta = 0, \quad \text{on } z = 0, 1,\tag{2.3}$$

together with u_i, π, θ satisfying a plane tiling periodicity in the (x, y) plane. This is consistent with the formation of hexagonal thermal convection cells in a horizontal strip of fluid.

Our intention here is not to give a full stability analysis of the steady solution (2.1) via equations (2.2). Instead, we wish to emphasize the special structure of system (2.2). Equations (2.2) may be written in the form

$$Au_t = Lu + N(u)$$

where $u = (u_1, u_2, u_3, \theta)$, A is the operator

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & Pr \end{pmatrix}$$

while L is the linear operator

$$L = \begin{pmatrix} \Delta & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 \\ 0 & 0 & \Delta & R \\ 0 & 0 & R & \Delta \end{pmatrix}$$

Let (\cdot, \cdot) denote the inner product on the Hilbert space $(L^2(V))^4$ where V is a periodic cell for the solution (u_i, θ, π) . Suppose $u = (u_1, u_2, u_3, \theta)$ and $v = (v_1, v_2, v_3, \phi)$ are functions which satisfy (2.2) and (2.3). Then it is straightforward to demonstrate that

$$(u, Lv) = (v, Lu).\tag{2.4}$$

Since (2.4) holds the operator L is symmetric. Furthermore, we may here show $(u, N(u)) = 0$, where $N(u)$ refers to the convective nonlinearities in (2.2). To be precise

$$\begin{aligned}
 (u, N(u)) &= \int_V u_i u_j u_{i,j} dV + Pr \int_V u_i \theta \theta_{,i} dV \\
 &= \frac{1}{2} \int_V u_j (u_i u_i)_{,j} dV + \frac{Pr}{2} \int_V u_i \theta_{,i}^2 dV \\
 &= -\frac{1}{2} \int_V u_{j,j} u_i u_i dV + \frac{1}{2} \oint_{\partial V} n_j u_j u_i u_i dS \\
 &\quad - \frac{Pr}{2} \int_V u_{i,i} \theta^2 dV + \frac{Pr}{2} \oint_{\partial V} n_i u_i \theta^2 dS \\
 &= 0
 \end{aligned}$$

where we have integrated by parts and used the boundary conditions (2.3). In this case since L is symmetric one has the very important result that the linear instability boundary governed by equations (2.2) is exactly the same as the global nonlinear stability boundary governed by the same equations. This result follows from theorem 2 of [150].

Our purpose here is to present a simple example showing that the case of a symmetric linear operator is a very special one. However, the class of symmetric linear operators is one which occurs frequently in thermal convection in a porous medium taking into account LTNE effects.

We now briefly consider the question of linear instability versus nonlinear stability in a general setting before embarking on describing various thermal convection problems in porous media in the presence of LTNE effects.

2.1.2 Symmetric Operators

In general, the equations governing problems in hydrodynamic stability (including those in porous media) are typically of the form

$$Au_t = L_S u + L_A u + N(u), \quad (2.5)$$

where u is a Hilbert space valued function, u_t is its time derivative, A is a bounded linear operator (typically a matrix with constant entries), $L = L_S + L_A$ is an unbounded, sectorial linear operator, and $N(u)$ represents the nonlinear terms. The operator L_S is the symmetric part of L while L_A denotes the anti-symmetric part. (Very roughly, a sectorial operator is one where the eigenvalues all lie in a sector in the complex plane. Detailed accounts of sectorial operators may be found in the books by [170] and by [484], chapter 2, and a very readable account may be found in [252]. In this book the linear operators all consist of terms like the identity or the Laplacian operator and under the boundary conditions we employ they are sectorial.)

The classical theory of linear instability writes

$$u = e^{\sigma t} \phi$$

and discards the $N(u)$ term in (2.5). One is then faced with solving the eigenvalue problem

$$\sigma A \phi = L_S \phi + L_A \phi, \quad (2.6)$$

where σ is the eigenvalue and ϕ the eigenfunction.

It is important to note that equation (2.5) involves both the skew-symmetric operator L_A and the symmetric operator L_S . In general, σ is complex, and one looks for the eigenvalue with largest real part to become positive for instability.

A classical nonlinear energy stability analysis, on the other hand, commences by forming the inner product of u with (2.5). If (\cdot, \cdot) denotes the inner product on the Hilbert space in question then one finds

$$\frac{d}{dt} \frac{1}{2} (u, Au) = (u, L_S u) + (u, N(u)) \quad (2.7)$$

since $(u, L_A u) = 0$. Nonlinear energy stability follows from (2.7) and it is very important to note that in this way the nonlinear stability boundary does not involve the skew part of L , L_A . Thus, one may expect, in general, that the linear instability and nonlinear stability boundaries are very different. Details of how nonlinear stability follows from (2.7) may be found in section 4.3 of [414], or from the paper of [150].

In fact, the reason why the nonlinear energy stability analyses discussed in the present chapter give optimal results is due to the fact that the associated operator L is symmetric.

There are two fundamental problems arising from (2.7) when one is faced with deriving *unconditional* nonlinear stability results. These are

- (a) the effect of L_A on the nonlinear stability boundary;
- (b) what does one do when $(u, N(u)) \not\geq 0$?

When the operator L is far from symmetric traditional energy stability arguments can break down completely, or yield very poor results for certain classes of problem. For example, in parallel shear flows progress is very difficult, as explained in chapter 8 of [410]. In this regard though, an interested reader may wish to consider the articles of [113, 201] and [202]. Certain classes of viscoelastic flows prove severely problematic to tackle via energy methods, as is shown in the interesting paper of [114].

Due to the failure of the classical energy method to yield sharp, or at least useful, nonlinear stability thresholds in problems such as shear flows, much research effort has recently been directed toward this area and a variety of novel approaches involving clever choices of Lyapunov functional have been suggested, cf. [68–70, 74, 131, 132, 182, 250, 271, 298, 299, 336, 342, 372, 374].

2.2 Darcy Theory

We begin with a description of the problem of a horizontal layer of porous material saturated with an incompressible fluid and heated below, allowing the fluid and solid temperatures to be different, LTNE. Our first step is to employ Darcy theory, see sections 1.3 and 1.8.3 but additionally utilize the LTNE equations for the temperature fields T^f and T^s as given in equations (1.83). Thus, we begin with the problem considered by [27]. The paper of [27] deals with the onset of thermal convection when the porous medium is modelled using Darcy's law. It is worth noting that [363] is a useful article which lucidly shows how one may compare convection in theories of Darcy and of Brinkman with an average temperature. His asymptotic estimates are very useful, and he also shows how the Forchheimer theory enters the picture.

In figure 2.1 we show a possible scenario for LTNE in a porous medium composed of spherical beads. Note that spherical beads are often used in a laboratory to perform experiments of thermal convection in a fluid saturated porous material, cf. [86]. For a real LTNE porous material we would expect to have many more beads than in figure 2.1 and the beads would be very small in keeping with a microfluidic setting. The continuum approximation assumes T_f and T_s are both defined at all points \mathbf{x} for time t and we likewise have a seepage velocity $v_i = \varepsilon V_i$ at all points \mathbf{x} . The seepage velocity is defined in section 1.8.1. For the configuration of figure 2.1 if we have a similar picture in the direction orthogonal to that of the figure then we have a three-dimensional box containing 320 beads, i.e. 8×5 in each orthogonal projection direction. If the beads have diameters D then the porosity may be calculated as $\varepsilon = 320D^3(1 - 4\pi/24)/320D^3 = 1 - \pi/6 \approx 0.4767$. The beads in figure 2.1 are not positioned in a close packing format. For a close packing format the porosity would be smaller, the lowest value of the porosity being $\varepsilon = 0.2595$.

Consider now a layer of porous material saturated with fluid and contained between the planes $z = 0$ and $z = d$. The temperatures of the solid, T_s , and fluid, T_f , are maintained at constants on the planes $z = 0$ and $z = d$ with

$$T_s = T_f = T_L, \quad z = 0; \quad T_s = T_f = T_U, \quad z = d; \quad (2.8)$$

where $T_L > T_U$. (If the layer is heated above, i.e. $T_U \geq T_L$, then one may demonstrate global nonlinear stability always holds.) The equations are formed by essentially a combination of equations (1.88) and (1.83), cf. Banu and Rees [27], and are

$$v_i = -\frac{K}{\mu} p_{,i} + \frac{\rho_f g \alpha K}{\mu} T_f k_i, \quad (2.9)$$

$$v_{i,i} = 0, \quad (2.10)$$

$$\varepsilon(\rho c)_f T_{,t}^f + (\rho c)_f v_i T_{,i}^f = \varepsilon k_f \Delta T_f + h(T_s - T_f), \quad (2.11)$$

$$(1 - \varepsilon)(\rho c)_s T_{,t}^s = (1 - \varepsilon)k_s \Delta T_s - h(T_s - T_f). \quad (2.12)$$

Note that in deriving equation (2.9) one makes the assumption that the density in the body force term in equation (1.88)₁ is linear in the fluid temperature T^f , i.e.

$$\rho = \rho_f [1 - \alpha(T^f - T_0)] \quad (2.13)$$

for a reference temperature T_0 . Equations (2.9)–(2.12) hold in the domain $\mathbb{R}^2 \times \{z \in (0, d)\} \times \{t > 0\}$, $\mathbf{k} = (0, 0, 1)$, and Δ is the three-dimensional Laplacian. The variables v_i, p, T_f and T_s are the velocity, pressure and fluid and solid temperatures, respectively. The constants $K, \mu, g, \alpha, \varepsilon, \rho_\alpha, c_\alpha, k_\alpha$ ($\alpha = f, s$), appearing in equations (2.9)–(2.13), are permeability, dynamic viscosity, gravity, thermal expansion coefficient, porosity, density, specific heat, thermal diffusion coefficient (where $\alpha = f, s$, denotes fluid or solid), $(\rho c)_\alpha = \rho_\alpha c_\alpha$, $\alpha = f, s$, and h is an interaction coefficient.

The steady solution whose stability is under investigation is

$$\begin{aligned} \bar{\mathbf{v}} &\equiv 0, \\ \bar{T}_f &= \bar{T}_s = -\beta z + T_L, \quad z \in (0, d), \end{aligned} \quad (2.14)$$

where β is the temperature gradient given by

$$\beta = \frac{T_L - T_U}{d} \quad (2.15)$$

and the steady pressure $\bar{p}(z)$ is a quadratic function determined from (2.9) (up to an arbitrary constant which defines the pressure scale).

2.2.1 Linear Instability

To investigate stability we introduce perturbations u_i, π, θ, ϕ to $\bar{v}_i, \bar{p}, \bar{T}_f$ and \bar{T}_s by

$$v_i = u_i + \bar{v}_i, \quad p = \pi + \bar{p}, \quad T_f = \theta + \bar{T}_f, \quad T_s = \phi + \bar{T}_s. \quad (2.16)$$

The perturbation equations are derived from (2.9) to (2.12) and are non-dimensionalized with velocity, pressure, temperature, time and length scales of $U = \varepsilon k_f / (\rho c)_f d$, $P = \mu d U / K$, $T^\sharp = U d \sqrt{\mu \beta c_f / \varepsilon k_f g \alpha K}$, $\mathcal{T} = (\rho c)_f d^2 / k_f$, $L = d$. The Rayleigh number Ra is defined by

$$Ra = R^2 = d^2 \rho_f^2 \sqrt{\frac{\beta c_f g \alpha K}{\varepsilon k_f \mu}}.$$

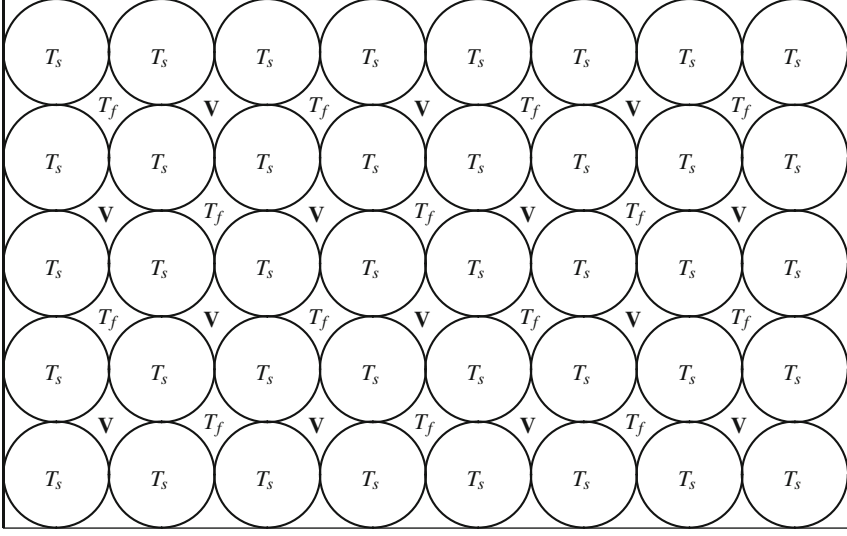


Fig. 2.1 An example of a local thermal non-equilibrium porous medium composed of spherical beads. The solid is fixed so the velocity is zero in the skeleton. In the fluid the velocity is the pore average velocity \mathbf{V} , cf. section 1.8.1. This is a horizontal projection from the side of a three-dimensional body comprised of 320 beads.

In addition $H = hd^2/\epsilon k_f$ and $\gamma = \epsilon k_f/(1 - \epsilon)k_s$ are the non-dimensional coefficients introduced by [27]. It then follows that the non-dimensional perturbation equations have form

$$u_i = -\pi_{,i} + R\theta k_i, \quad (2.17)$$

$$u_{i,i} = 0, \quad (2.18)$$

$$\theta_{,t} + u_i \theta_{,i} = Rw + \Delta \theta + H(\phi - \theta), \quad (2.19)$$

$$A\phi_{,t} = \Delta \phi - H\gamma(\phi - \theta), \quad (2.20)$$

where these equations hold on $\mathbb{R}^2 \times \{z \in (0, 1)\} \times \{t > 0\}$, $w = u_3$, and $A = \rho_s c_s k_f / k_s \rho_f c_f$ is a non-dimensional thermal inertia coefficient. Observe that equations (2.17)–(2.20) are still nonlinear due to the presence of the $u_i \theta_{,i}$ term in (2.19). Also, the form of the equations is different to that of [27] because we employ R rather than Ra (where $Ra = R^2$), although equations (2.17)–(2.20) are easily transformed to the equations of [27] and they are equivalent to them.

The boundary conditions to be satisfied are

$$u_i n_i = 0, \quad \theta = 0, \quad \phi = 0, \quad \text{on } z = 0, 1, \quad (2.21)$$

where n_i denotes the unit outward normal, together with u_i, π, θ, ϕ satisfying a plane tiling periodicity in x, y . Such forms are discussed in e.g. [85], p. 43, [414], p. 51, where especially the hexagonal planform of [91] is described.

One linearizes (2.19) and then we put $u_i = e^{\sigma t} u_i(\mathbf{x})$ with a similar representation for π, θ and ϕ to derive from (2.17) to (2.20) the linearized instability equations

$$\begin{aligned} u_i &= -\pi_{,i} + R\theta k_i, \\ u_{i,i} &= 0, \\ \sigma\theta &= R w + \Delta\theta + H(\phi - \theta), \\ \sigma A\phi &= \Delta\phi - H\gamma(\phi - \theta). \end{aligned} \quad (2.22)$$

If we put $u = (u_1, u_2, u_3, \theta, \phi)^T$ and consider the linear operator, L , as defined in the abstract equation (2.5), from equations (2.22) but with (2.22)₄ multiplied by γ^{-1} , we see that

$$L = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & R & 0 \\ 0 & 0 & R & \Delta - H & H \\ 0 & 0 & 0 & H & \frac{1}{\gamma}\Delta - H \end{pmatrix} \quad (2.23)$$

In this form the symmetry of L , taking account of boundary conditions (2.21) is evident.

Exchange of stabilities follows immediately from the symmetry of L , although it is easily proved directly from (2.22). To see this let V be a period cell for the solution and then multiply (2.22)₁ by u_i^* , (2.22)₃ by θ^* , (2.22)₄ by ϕ^*/γ where the $*$ denotes complex conjugate. The resulting equations are integrated over V and added, using the boundary conditions (2.21) to see that

$$\begin{aligned} \frac{\sigma}{2} \left(\|\theta\|^2 + \frac{A}{\gamma} \|\phi\|^2 \right) &= -\|\mathbf{u}\|^2 - \|\nabla\theta\|^2 - \frac{1}{\gamma} \|\nabla\phi\|^2 \\ &\quad - H\|\theta - \phi\|^2 + R[(\theta, w^*) + (w, \theta^*)], \end{aligned} \quad (2.24)$$

where $\|\cdot\|$ and (\cdot, \cdot) momentarily denote the norm and inner product on the complex Hilbert space $L^2(V)$. Put $\sigma = \sigma_r + i\sigma_1$ and then take the imaginary part of (2.24) to obtain

$$\frac{\sigma_1}{2} \left(\|\theta\|^2 + \frac{A}{\gamma} \|\phi\|^2 \right) = 0.$$

For a non-zero solution we must have $\sigma_1 = 0$ and so exchange of stabilities holds, as observed by [27].

One may now set $\sigma = 0$ in (2.22) and solve these equations for the Rayleigh number Ra . [27] show that

$$Ra = \frac{\Lambda^2}{a^2} \left(\frac{\Lambda + H(1 + \gamma)}{\Lambda + \gamma H} \right) \quad (2.25)$$

where a is the wavenumber and $\Lambda = \pi^2 + a^2$. [27] minimize Ra in (2.25) over the wavenumber to find the critical Rayleigh number for instability, Ra_c , for many values of γ and H . They find that increasing γ and H increases Ra_c and so stabilizes the

solution. In addition they provide many useful asymptotic results for Ra_c for small H and γ .

At this point it is worth pointing out that values of the parameter γ are certainly available for real materials. However, values for the thermal interaction coefficient, H , are more elusive. In this regard the papers of [364, 365] are extremely important. He presents very interesting analyses where he produces a possible way to calculate H for a variety of porous media. The porous media he uses consist of one-dimensional stripes of fluid between the solid skeleton, randomly striped one-dimensional porous media, two-dimensional porous media where the fluid occupies a checkerboard pattern, box type configurations, and random networks. The calculations of [364, 365] indicate there is a strong correlation between the porosity and the thermal conductivities of the fluid and solid components, but there is also a major effect due to a geometrical factor. This is a very interesting calculation and will be very useful when dealing with a known geometrical pattern of porous media and given solid and fluid components.

2.2.2 Nonlinear Stability

[416] demonstrated that the results of [27] are optimal in that their linearized instability results yield exactly the same Rayleigh number threshold as one obtains with a global (for all initial data) nonlinear stability analysis. This means that the results of [27] are particularly useful because they show that the linearized theory has captured the physics of the onset of convection. One may deduce the equivalence between the linear instability boundary and the nonlinear stability one by writing the problem (2.17)–(2.21) as an abstract system of partial differential equations in a Hilbert space and then verifying that appropriate conditions hold, namely that L given by (2.23) is symmetric and $(u, N(u)) \geq 0$, where N denotes the nonlinear terms as indicated in (2.5), see also (2.7). In fact, for equations (2.17)–(2.21), it is straightforward to show $(u, N(u)) = 0$.

It is instructive to include here a direct proof of the equivalence of the linear and nonlinear stability boundaries, as is done in [416]. Let V be a three-dimensional period cell for the solution to (2.17)–(2.21) and let (\cdot, \cdot) and $\|\cdot\|$ denote the inner product and norm on $L^2(V)$. The idea is to construct “energy identities” by multiplying (2.17) by u_i , (2.19) by θ , and (2.20) by ϕ/γ to obtain after integration by parts and use of (2.18),

$$0 = -\|\mathbf{u}\|^2 + R(\theta, w), \quad (2.26)$$

and

$$\frac{d}{dt} \frac{1}{2} \|\theta\|^2 = R(w, \theta) - \|\nabla \theta\|^2 - H(\theta, \theta - \phi), \quad (2.27)$$

and

$$\frac{d}{dt} \frac{A}{2\gamma} \|\phi\|^2 = -\frac{1}{\gamma} \|\nabla \phi\|^2 - H(\phi, \phi - \theta). \quad (2.28)$$

Define an energy function E , an indefinite production term I , and the dissipation D by

$$\begin{aligned} E(t) &= \frac{1}{2} \|\theta\|^2 + \frac{A}{2\gamma} \|\phi\|^2, \\ I &= 2(\theta, w), \\ D &= \|\mathbf{u}\|^2 + \|\nabla \theta\|^2 + \frac{1}{\gamma} \|\nabla \phi\|^2 + H \|\theta - \phi\|^2. \end{aligned} \quad (2.29)$$

Now add equations (2.26), (2.27) and (2.28) to arrive at the energy identity

$$\frac{dE}{dt} = RI - D. \quad (2.30)$$

From this equation one may deduce

$$\frac{dE}{dt} \leq -D \left(1 - \frac{R}{R_E}\right), \quad (2.31)$$

where the number R_E is defined by the relation

$$R_E^{-1} = \max_{\mathcal{H}} \frac{I}{D} \quad (2.32)$$

with \mathcal{H} being the space of admissible solutions, namely, $\mathcal{H} = \{(\mathbf{u}, \theta, \phi) \mid u_i \in L^2(V), \theta, \phi \in H^1(V), u_{i,i} = 0, u_i, \theta, \phi, \pi \text{ are periodic over a plane tiling domain in } x \text{ and } y\}$. By use of Poincaré's inequality we may show $D \geq 2\pi^2 \zeta_1 E$, where $\zeta_1 = \min\{1, A^{-1}\}$. Then for $R < R_E$, put $c = 1 - R/R_E > 0$, and from (2.31) we may deduce $dE/dt \leq -kE$, where $k = 2\pi^2 \zeta_1 c$. This inequality integrates to see that

$$E(t) \leq \exp(-kt)E(0). \quad (2.33)$$

From inequality (2.33) it follows that $E \rightarrow 0$ at least exponentially in time.

The exponential decay of E guarantees exponential decay of θ and ϕ (in $L^2(V)$ norm). To obtain decay of \mathbf{u} we observe that from (2.26) we may deduce

$$\begin{aligned} \|\mathbf{u}\|^2 &= R(\theta, w) \\ &\leq \frac{R^2}{2} \|\theta\|^2 + \frac{1}{2} \|w\|^2, \end{aligned}$$

where in the last line the arithmetic-geometric mean inequality has been employed. In this manner we see that

$$\|\mathbf{u}\|^2 \leq R^2 \|\theta\|^2. \quad (2.34)$$

This shows that the condition $R < R_E$ also guarantees exponential decay of $\|\mathbf{u}\|$.

The value of R_E thus represents a global (i.e. for all initial data) nonlinear stability threshold. The number R_E is calculated from the Euler-Lagrange equations which

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