

Chapter 2

Robust Filtering of 2-D Uncertain State-Delayed Systems

2.1 Introduction

In this chapter, the robust $\mathcal{H}_\infty$ filtering problem is investigated for uncertain 2-D discrete systems with time-delay in its states. The mathematical model of the considered 2-D systems is established in terms of the well-known FMLSS model incorporating time delays. We aim at designing a full-order filter that guarantees the asymptotic stability of the filtering error system while keeping the prescribed $\mathcal{H}_\infty$ disturbance attenuation performance. By using a 2-D Lyapunov approach combining with the LMI technique, the existence conditions of the desired filters are first established, and the corresponding filter design problem is converted into a convex optimization one that can then be efficiently handled with help from available numerical software. Furthermore, the obtained results are extended to some more general cases where the system matrices also contain uncertain parameters. Most frequently used descriptions for the parameter uncertainties, including polytopic and norm-bounded characterizations, are taken into consideration within the unified LMI framework.

2.2 System Description and Preliminaries

Consider the 2-D state-delayed system described by the following FMLSS model with delays in the states:

$$\begin{aligned} x_{i+1,j+1} = & A_1 x_{i,j+1} + A_2 x_{i+1,j} + A_{d1} x_{i-d_1,j+1} + A_{d2} x_{i+1,j-d_2} \\ & + B_1 \omega_{i,j+1} + B_2 \omega_{i+1,j}, \end{aligned} \quad (2.1a)$$

$$y_{i,j} = C x_{i,j} + D \omega_{i,j}, \quad (2.1b)$$

$$z_{i,j} = E x_{i,j}, \quad (2.1c)$$

where $x_{i,j} \in \mathbf{R}^n$ is the state; $\omega_{i,j} \in \mathbf{R}^l$ is the disturbance input which belongs to $\ell_2 \{[0, \infty), [0, \infty)\}$; $y_{i,j} \in \mathbf{R}^m$ is the measured output; $z_{i,j} \in \mathbf{R}^p$ is the signal to be

estimated with $i, j \in \mathbf{Z}_+$; and d_1 and d_2 are constant positive integers representing delays along vertical and horizontal directions, respectively. $A_1, A_2, A_{d1}, A_{d2}, B_1, B_2, C, D$ and E are constant matrices with compatible dimensions. The boundary conditions are given by

$$\begin{cases} x_{\phi,j} = 0, & \forall j \geq 0, \phi = -d_1, -d_1 + 1, \dots, 0, \\ x_{i,\varphi} = 0, & \forall i \geq 0, \varphi = -d_2, -d_2 + 1, \dots, 0. \end{cases} \quad (2.2)$$

Throughout this chapter, the following assumptions are made.

Assumption 2.1 The boundary condition is assumed to satisfy

$$\lim_{N \rightarrow \infty} \sum_{k=0}^N (|x_{0,k}|^2 + |x_{k,0}|^2) < \infty. \quad (2.3)$$

The aim of the $\mathcal{H}_\infty$ filtering problem addressed in this chapter is to estimate the signal $z_{i,j}$ by a linear, full-order, dynamic filter of the structure described by

$$\hat{x}_{i+1,j+1} = A_{1f}\hat{x}_{i,j+1} + A_{2f}\hat{x}_{i+1,j} + B_{1f}y_{i,j+1} + B_{2f}y_{i+1,j}, \quad (2.4a)$$

$$\hat{z}_{i,j} = C_f\hat{x}_{i,j}, \quad (2.4b)$$

where $\hat{x}_{i,j} \in \mathbf{R}^n$ is the filter state vector, and $\hat{x}_{i,j} = 0$ ($i = 0$ or $j = 0$) denotes the initial condition; $A_{1f}, A_{2f}, B_{1f}, B_{2f}$ and C_f are appropriately dimensioned constant matrices to be determined.

Now, augmenting the system model in (2.1a–2.1c) to include the filter dynamics in (2.4a, 2.4b), the filtering error system is now given by

$$\begin{aligned} \xi_{i+1,j+1} &= \tilde{A}_1\xi_{i,j+1} + \tilde{A}_2\xi_{i+1,j} + \tilde{A}_{d1}\xi_{i-d_1,j+1} + \tilde{A}_{d2}\xi_{i+1,j-d_2} \\ &\quad + \tilde{B}_1\omega_{i,j+1} + \tilde{B}_2\omega_{i+1,j}, \end{aligned} \quad (2.5a)$$

$$e_{i,j} = \tilde{C}\xi_{i,j}, \quad (2.5b)$$

where $\xi_{i,j} \triangleq \begin{bmatrix} x_{i,j} \\ \hat{x}_{i,j} \end{bmatrix}$, $e_{i,j} \triangleq z_{i,j} - \hat{z}_{i,j}$ and

$$\begin{cases} \tilde{A}_1 \triangleq \begin{bmatrix} A_1 & 0 \\ B_{1f}C & A_{1f} \end{bmatrix}, \tilde{A}_{d1} \triangleq \begin{bmatrix} A_{d1} & 0 \\ 0 & 0 \end{bmatrix}, \tilde{B}_1 \triangleq \begin{bmatrix} B_1 \\ B_{1f}D \end{bmatrix}, \\ \tilde{A}_2 \triangleq \begin{bmatrix} A_2 & 0 \\ B_{2f}C & A_{2f} \end{bmatrix}, \tilde{A}_{d2} \triangleq \begin{bmatrix} A_{d2} & 0 \\ 0 & 0 \end{bmatrix}, \tilde{B}_2 \triangleq \begin{bmatrix} B_2 \\ B_{2f}D \end{bmatrix}, \\ \tilde{C} \triangleq [E \quad -C_f]. \end{cases} \quad (2.6)$$

Before problem formulating, we give the following definitions.

Definition 2.1 Given a scalar $\gamma > 0$ and constant weighting matrices $\tilde{P} > 0$, $\tilde{Q} > 0$, $\tilde{Q}_1 > 0$ and $\tilde{Q}_2 > 0$, the filtering error system in (2.5a, 2.5b) has an $\mathcal{H}_\infty$ performance if it is asymptotically stable and satisfies

$$\frac{\|\tilde{e}_{i,j}\|_2^2}{\left[\|\tilde{\omega}_{i,j}\|_2^2 + \sum_{j=0}^{\infty} \xi_{0,j}^T(0,1) \tilde{P} \xi_{0,j}(0,1) + \sum_{i=0}^{\infty} \xi_{i,0}^T(1,0) \tilde{Q} \xi_{i,0}(1,0) + \sum_{j=0}^{\infty} \sum_{\kappa=-d_1}^{-1} \xi_{0,j}^T(\kappa,1) \tilde{Q}_1 \xi_{0,j}(\kappa,1) + \sum_{i=0}^{\infty} \sum_{\kappa=-d_2}^{-1} \xi_{i,0}^T(1,\kappa) \tilde{Q}_2 \xi_{i,0}(1,\kappa) \right]} < \gamma^2, \quad (2.7)$$

where $\xi_{i,j}(\alpha, \beta) \triangleq \xi_{i+\alpha, j+\beta}$. In the case of the zero boundary conditions as in (2.2), the above $\mathcal{H}_\infty$ performance measure (2.7) reduces to

$$\|\tilde{e}_{i,j}\|_2 < \gamma \|\tilde{\omega}_{i,j}\|_2, \quad (\gamma > 0), \quad (2.8)$$

where $\tilde{e}_{i,j} \triangleq \begin{bmatrix} e_{i,j+1} \\ e_{i+1,j} \end{bmatrix}$, $\tilde{\omega}_{i,j} \triangleq \begin{bmatrix} \omega_{i,j+1} \\ \omega_{i+1,j} \end{bmatrix}$, and $\|\cdot\|_2$ is ℓ_2 norm defined by

$$\begin{aligned} \|\tilde{e}_{i,j}\|_2 &\triangleq \sqrt{\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left(e_{i,j+1}^T e_{i,j+1} + e_{i+1,j}^T e_{i+1,j} \right)}, \\ \|\tilde{\omega}_{i,j}\|_2 &\triangleq \sqrt{\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left(\omega_{i,j+1}^T \omega_{i,j+1} + \omega_{i+1,j}^T \omega_{i+1,j} \right)}. \end{aligned}$$

Definition 2.2 The filter in (2.4a, 2.4b) is said to be an $\mathcal{H}_\infty$ filter if the filtering error system in (2.5a, 2.5b) is asymptotically stable and satisfies $\mathcal{H}_\infty$ performance in (2.8) with zero boundary conditions as in (2.2).

Problem 2.3 The problem (objective) of this chapter is to find the matrices A_{1f} , A_{2f} , B_{1f} , B_{2f} and C_f of the $\mathcal{H}_\infty$ filter in the form of (2.4a, 2.4b) for the 2-D state-delayed system in (2.1a–2.1c), such that the filtering error system in (2.5a, 2.5b) is asymptotically stable with an $\mathcal{H}_\infty$ performance, that is, (2.8) is satisfied.

2.3 Main Results

2.3.1 Filtering Analysis

In this section, we shall analyze the stability and $\mathcal{H}_\infty$ performance for the filtering error system in (2.5a, 2.5b). The following lemma is essential in establishing the main results.

Lemma 2.4 *The filtering error system in (2.5a, 2.5b) with $\omega_{i,j} \equiv 0$ is asymptotically stable if there exist matrices $P > 0$, $Q > 0$, $Q_1 > 0$ and $Q_2 > 0$ such that the following LMI holds:*

$$\Psi \triangleq \begin{bmatrix} \tilde{A}_1^T \\ \tilde{A}_2^T \\ \tilde{A}_{d1}^T \\ \tilde{A}_{d2}^T \end{bmatrix} P \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 & \tilde{A}_{d1} & \tilde{A}_{d2} \end{bmatrix} - \begin{bmatrix} P - Q - Q_1 & 0 & 0 & 0 \\ * & Q - Q_2 & 0 & 0 \\ * & * & Q_1 & 0 \\ * & * & * & Q_2 \end{bmatrix} < 0. \quad (2.9)$$

The result can be obtained by employing the same techniques in the proof of Theorem 3 of [145].

Next, the following theorem provides a sufficient condition under which the filtering error system in (2.5a, 2.5b) is asymptotically stable and the performance constraint (2.8) is satisfied.

Theorem 2.5 *The filtering error system in (2.5a, 2.5b) is asymptotically stable with an $\mathcal{H}_\infty$ performance if there exist matrices $P > 0$, $Q > 0$, $Q_1 > 0$ and $Q_2 > 0$ such that*

$$\begin{bmatrix} -P & 0 & 0 & P\tilde{A}_1 & P\tilde{B}_1 & P\tilde{A}_{d1} & P\tilde{A}_2 & P\tilde{B}_2 & P\tilde{A}_{d2} \\ * & -I & 0 & \tilde{C} & 0 & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 & \tilde{C} & 0 & 0 \\ * & * & * & Q + Q_1 - P & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q_1 & 0 & 0 & 0 \\ * & * & * & * & * & * & Q_2 - Q & 0 & 0 \\ * & * & * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & * & * & -Q_2 \end{bmatrix} < 0. \quad (2.10)$$

Proof First, let us establish the asymptotic stability of the filtering error system with $\omega_{i,j} = 0$. Denote

$$\left\{ \begin{aligned} V_{i,j}^{(1)} &\triangleq \xi_{i,j}^T(1, 1)P\xi_{i,j}(1, 1) + \sum_{\kappa=-d_1}^{-1} \xi_{i+1,j}^T(\kappa, 1)Q_1\xi_{i+1,j}(\kappa, 1) \\ &\quad + \sum_{\kappa=-d_2}^{-1} \xi_{i,j+1}^T(1, \kappa)Q_2\xi_{i,j+1}(1, \kappa), \\ V_{i,j}^{(2)} &\triangleq \xi_{i,j}^T(0, 1)(P - Q)\xi_{i,j}(0, 1) + \sum_{\kappa=-d_1}^{-1} \xi_{i,j}^T(\kappa, 1)Q_1\xi_{i,j}(\kappa, 1), \\ V_{i,j}^{(3)} &\triangleq \xi_{i,j}^T(1, 0)Q\xi_{i,j}(1, 0) + \sum_{\kappa=-d_2}^{-1} \xi_{i,j}^T(1, \kappa)Q_2\xi_{i,j}(1, \kappa). \end{aligned} \right.$$

Consider the increment $\Delta V_{i,j}$ given by

$$\Delta V_{i,j} \triangleq V_{i,j}^{(1)} - V_{i,j}^{(2)} - V_{i,j}^{(3)},$$

then along the solution of the filtering error system with $\omega_{i,j} = 0$, we have

$$\begin{aligned} \Delta V_{i,j} &= \left[\tilde{A}_1 \xi_{i,j+1} + \tilde{A}_2 \xi_{i+1,j} + \tilde{A}_{d1} \xi_{i-d_1,j+1} + \tilde{A}_{d2} \xi_{i+1,j-d_2} \right]^T P \\ &\quad \times \left[\tilde{A}_1 \xi_{i,j+1} + \tilde{A}_2 \xi_{i+1,j} + \tilde{A}_{d1} \xi_{i-d_1,j+1} + \tilde{A}_{d2} \xi_{i+1,j-d_2} \right] \\ &\quad - \xi_{i,j+1}^T (P - Q - Q_1) \xi_{i,j+1} - \xi_{i+1,j}^T (Q - Q_2) \xi_{i+1,j} \\ &\quad - \xi_{i-d_1,j+1}^T Q_1 \xi_{i-d_1,j+1} - \xi_{i+1,j-d_2}^T Q_2 \xi_{i+1,j-d_2} \\ &\triangleq \eta_{i,j}^T \Psi \eta_{i,j}, \end{aligned}$$

where $\eta_{i,j} \triangleq \left[\xi_{i,j+1}^T \quad \xi_{i+1,j}^T \quad \xi_{i-d_1,j+1}^T \quad \xi_{i+1,j-d_2}^T \right]^T$ and Ψ is defined in (2.9). By Schur complement, LMI (2.10) implies $\Psi < 0$. It follows from Lemma 2.4 that the filtering error system in (2.5a) with $\omega_{i,j} = 0$ is asymptotically stable.

Now, to establish the $\mathcal{H}_\infty$ performance for the filtering error system, we introduce the following index:

$$\mathcal{J} \triangleq \Delta V_{i,j} + \tilde{e}_{i,j}^T \tilde{e}_{i,j} - \gamma^2 \tilde{\omega}_{i,j}^T \tilde{\omega}_{i,j}, \quad (2.11)$$

where

$$\begin{aligned} \Delta V_{i,j} &= \left[\tilde{A}_1 \xi_{i,j+1} + \tilde{A}_2 \xi_{i+1,j} + \tilde{A}_{d1} \xi_{i-d_1,j+1} + \tilde{A}_{d2} \xi_{i+1,j-d_2} + \tilde{B}_1 \omega_{i,j+1} \right. \\ &\quad \left. + \tilde{B}_2 \omega_{i+1,j} \right]^T P \left[\tilde{A}_1 \xi_{i,j+1} + \tilde{A}_2 \xi_{i+1,j} + \tilde{A}_{d1} \xi_{i-d_1,j+1} + \tilde{A}_{d2} \xi_{i+1,j-d_2} \right. \\ &\quad \left. + \tilde{B}_1 \omega_{i,j+1} + \tilde{B}_2 \omega_{i+1,j} \right] \\ &\quad - \xi_{i,j+1}^T (P - Q - Q_1) \xi_{i,j+1} - \xi_{i+1,j}^T (Q - Q_2) \xi_{i+1,j} \\ &\quad - \xi_{i-d_1,j+1}^T Q_1 \xi_{i-d_1,j+1} - \xi_{i+1,j-d_2}^T Q_2 \xi_{i+1,j-d_2}. \end{aligned}$$

Thus, according to the stability of the filtering error system, we have

$$\begin{aligned} \mathcal{J} &= \left[\tilde{A}_1 \xi_{i,j+1} + \tilde{A}_2 \xi_{i+1,j} + \tilde{A}_{d1} \xi_{i-d_1,j+1} + \tilde{A}_{d2} \xi_{i+1,j-d_2} \right]^T P \\ &\quad \times \left[\tilde{A}_1 \xi_{i,j+1} + \tilde{A}_2 \xi_{i+1,j} + \tilde{A}_{d1} \xi_{i-d_1,j+1} + \tilde{A}_{d2} \xi_{i+1,j-d_2} \right] \\ &\quad - \xi_{i,j+1}^T (P - Q - Q_1) \xi_{i,j+1} - \xi_{i+1,j}^T (Q - Q_2) \xi_{i+1,j} \\ &\quad - \xi_{i-d_1,j+1}^T Q_1 \xi_{i-d_1,j+1} - \xi_{i+1,j-d_2}^T Q_2 \xi_{i+1,j-d_2} \\ &\quad + \xi_{i,j+1}^T \tilde{C}^T \tilde{C} \xi_{i,j+1} + \xi_{i+1,j}^T \tilde{C}^T \tilde{C} \xi_{i+1,j} \end{aligned}$$

$$\begin{aligned}
& + \left[\tilde{A}_1 \xi_{i,j+1} + \tilde{A}_2 \xi_{i+1,j} + \tilde{A}_{d1} \xi_{i-d_1,j+1} + \tilde{A}_{d2} \xi_{i+1,j-d_2} \right]^T 2P \\
& \times \left[\tilde{B}_1 \omega_{i,j+1} + \tilde{B}_2 \omega_{i+1,j} \right] - \left\{ \gamma^2 \omega_{i,j+1}^T \omega_{i,j+1} + \gamma^2 \omega_{i+1,j}^T \omega_{i+1,j} \right. \\
& \left. - \left[\tilde{B}_1 \omega_{i,j+1} + \tilde{B}_2 \omega_{i+1,j} \right]^T P \left[\tilde{B}_1 \omega_{i,j+1} + \tilde{B}_2 \omega_{i+1,j} \right] \right\} \\
& \triangleq \eta_{i,j}^T \Pi \eta_{i,j} + 2\eta_{i,j}^T \Omega \tilde{\omega}_{i,j} - \tilde{\omega}_{i,j}^T \Phi \tilde{\omega}_{i,j} \\
& = \eta_{i,j}^T \left(\Pi + \Omega \Phi^{-1} \Omega^T \right) \eta_{i,j} - \left[\eta_{i,j}^T \Omega \Phi^{-1} \Omega^T \eta_{i,j} - 2\eta_{i,j}^T \Omega \tilde{\omega}_{i,j} + \tilde{\omega}_{i,j}^T \Phi \tilde{\omega}_{i,j} \right] \\
& \triangleq \eta_{i,j}^T \Sigma \eta_{i,j} - \mu_{i,j}^T \mu_{i,j}, \tag{2.12}
\end{aligned}$$

where $\eta_{i,j}$ is defined in (2.11) and

$$\left\{ \begin{aligned} \Sigma & \triangleq \Pi + \Omega \Phi^{-1} \Omega^T, \quad \mu_{i,j} \triangleq \Phi^{\frac{1}{2}} (\tilde{\omega}_{i,j} - \Phi^{-1} \Omega^T \eta_{i,j}), \\ \Pi & \triangleq \begin{bmatrix} \tilde{A}_1^T \\ \tilde{A}_2^T \\ \tilde{A}_{d1}^T \\ \tilde{A}_{d2}^T \end{bmatrix} P \begin{bmatrix} \tilde{A}_1^T \\ \tilde{A}_2^T \\ \tilde{A}_{d1}^T \\ \tilde{A}_{d2}^T \end{bmatrix}^T + \begin{bmatrix} Q + Q_1 - P + \tilde{C}^T \tilde{C} & 0 & 0 & 0 \\ * & Q_2 - Q + \tilde{C}^T \tilde{C} & 0 & 0 \\ * & * & -Q_1 & 0 \\ * & * & * & -Q_2 \end{bmatrix}, \\ \Omega & \triangleq \begin{bmatrix} \tilde{A}_1^T \\ \tilde{A}_2^T \\ \tilde{A}_{d1}^T \\ \tilde{A}_{d2}^T \end{bmatrix} P [\tilde{B}_1 \quad \tilde{B}_2], \quad \Phi \triangleq \gamma^2 I - \begin{bmatrix} \tilde{B}_1^T \\ \tilde{B}_2^T \end{bmatrix} P [\tilde{B}_1 \quad \tilde{B}_2]. \end{aligned} \right.$$

By Schur complement, LMI (2.10) implies $\Sigma < 0$. This together with (2.11) and (2.12) yields

$$\Delta V_{i,j} + \tilde{e}_{i,j}^T \tilde{e}_{i,j} - \gamma^2 \tilde{\omega}_{i,j}^T \tilde{\omega}_{i,j} < -\mu_{i,j}^T \mu_{i,j}. \tag{2.13}$$

Therefore, we can sum both sides of (2.13) to obtain

$$\begin{aligned}
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left(\Delta V_{i,j} + \tilde{e}_{i,j}^T \tilde{e}_{i,j} - \gamma^2 \tilde{\omega}_{i,j}^T \tilde{\omega}_{i,j} \right) & < - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mu_{i,j}^T \mu_{i,j} \\
& = - \|\mu_{i,j}\|_2^2. \tag{2.14}
\end{aligned}$$

For any integers $p > 0$ and $q > 0$, it follows from $\Delta V_{i,j} \triangleq V_{i,j}^{(1)} - V_{i,j}^{(2)} - V_{i,j}^{(3)}$ that

$$\sum_{i=0}^p \sum_{j=0}^q \Delta V_{i,j} = \sum_{j=0}^q \left[\xi_{p,j}^T (1, 1) (P - Q) \xi_{p,j} (1, 1) - \xi_{0,j}^T (0, 1) (P - Q) \xi_{0,j} (0, 1) \right]$$

$$\begin{aligned}
& + \sum_{i=0}^p \left[\xi_{i,q}^T(1, 1) Q \xi_{i,q}(1, 1) - \xi_{i,0}^T(1, 0) Q \xi_{i,0}(1, 0) \right] \\
& + \sum_{j=0}^q \left[\sum_{\kappa=-d_1}^{-1} \xi_{p+1,j}^T(\kappa, 1) Q_1 \xi_{p+1,j}(\kappa, 1) \right. \\
& \quad \left. - \sum_{\kappa=-d_1}^{-1} \xi_{0,j}^T(\kappa, 1) Q_1 \xi_{0,j}(\kappa, 1) \right] \\
& + \sum_{i=0}^p \left[\sum_{\kappa=-d_2}^{-1} \xi_{i,q+1}^T(1, \kappa) Q_2 \xi_{i,q+1}(1, \kappa) \right. \\
& \quad \left. - \sum_{\kappa=-d_2}^{-1} \xi_{i,0}^T(1, \kappa) Q_2 \xi_{i,0}(1, \kappa) \right]
\end{aligned}$$

which together with (2.14) implies that

$$\begin{aligned}
& \|\tilde{e}_{i,j}\|_2^2 - \gamma^2 \|\tilde{\omega}_{i,j}\|_2^2 + \|\mu_{i,j}\|_2^2 \\
& < \sum_{j=0}^{\infty} \xi_{0,j}^T(0, 1) (P - Q) \xi_{0,j}(0, 1) + \sum_{i=0}^{\infty} \xi_{i,0}^T(1, 0) Q \xi_{i,0}(1, 0) \\
& + \sum_{j=0}^{\infty} \sum_{\kappa=-d_1}^{-1} \xi_{0,j}^T(\kappa, 1) Q_1 \xi_{0,j}(\kappa, 1) + \sum_{i=0}^{\infty} \sum_{\kappa=-d_2}^{-1} \xi_{i,0}^T(1, \kappa) Q_2 \xi_{i,0}(1, \kappa),
\end{aligned}$$

thus,

$$\begin{aligned}
& \|\tilde{e}_{i,j}\|_2^2 + \|\mu_{i,j}\|_2^2 \\
& < \gamma^2 \left[\|\tilde{\omega}_{i,j}\|_2^2 + \sum_{j=0}^{\infty} \xi_{0,j}^T(0, 1) \tilde{P} \xi_{0,j}(0, 1) + \sum_{i=0}^{\infty} \xi_{i,0}^T(1, 0) \tilde{Q} \xi_{i,0}(1, 0) \right. \\
& \quad \left. + \sum_{j=0}^{\infty} \sum_{\kappa=-d_1}^{-1} \xi_{0,j}^T(\kappa, 1) \tilde{Q}_1 \xi_{0,j}(\kappa, 1) + \sum_{i=0}^{\infty} \sum_{\kappa=-d_2}^{-1} \xi_{i,0}^T(1, \kappa) \tilde{Q}_2 \xi_{i,0}(1, \kappa) \right],
\end{aligned}$$

where $P - Q < \gamma^2 \tilde{P}$, $Q < \gamma^2 \tilde{Q}$, $Q_1 < \gamma^2 \tilde{Q}_1$ and $Q_2 < \gamma^2 \tilde{Q}_2$.

Moreover, to establish the $\mathcal{H}_{\infty}$ performance, we show that there exists a scalar $\alpha > 0$ such that

$$\|\mu_{i,j}\|_2^2 \geq \alpha^2 \left[\|\tilde{\omega}_{i,j}\|_2^2 + \sum_{j=0}^{\infty} \xi_{0,j}^T(0, 1) \tilde{P} \xi_{0,j}(0, 1) + \sum_{i=0}^{\infty} \xi_{i,0}^T(1, 0) \tilde{Q} \xi_{i,0}(1, 0) \right]$$

$$\begin{aligned}
& + \sum_{j=0}^{\infty} \sum_{\kappa=-d_1}^{-1} \xi_{0,j}^T(\kappa, 1) \tilde{Q}_1 \xi_{0,j}(\kappa, 1) \\
& + \sum_{i=0}^{\infty} \sum_{\kappa=-d_2}^{-1} \xi_{i,0}^T(1, \kappa) \tilde{Q}_2 \xi_{i,0}(1, \kappa) \Bigg]. \quad (2.15)
\end{aligned}$$

Consider the inverse system of the filtering error system (2.5a, 2.5b):

$$\begin{aligned}
\xi_{i+1,j+1} &= \tilde{A}_1 \xi_{i,j+1} + \tilde{A}_2 \xi_{i+1,j} + \tilde{A}_{d1} \xi_{i-d_1,j+1} + \tilde{A}_{d2} \xi_{i+1,j-d_2} \\
&+ \tilde{B}_1 \omega(i, j+1) + \tilde{B}_2 \omega(i+1, j) \\
&\triangleq \mathcal{A} \eta_{i,j} + \mathcal{B} \tilde{\omega}_{i,j} \\
&= (\mathcal{A} + \mathcal{B} \Phi^{-1} \Omega^T) \eta_{i,j} + \mathcal{B} \Phi^{-\frac{1}{2}} \mu_{i,j}, \quad (2.16a)
\end{aligned}$$

$$\tilde{\omega}_{i,j} = \Phi^{-1} \Omega^T \eta_{i,j} + \Phi^{-\frac{1}{2}} \mu_{i,j}, \quad (2.16b)$$

where $\mathcal{A} \triangleq [\tilde{A}_1 \ \tilde{A}_2 \ \tilde{A}_{d1} \ \tilde{A}_{d2}]$, $\mathcal{B} \triangleq [\tilde{B}_1 \ \tilde{B}_2]$, and $\eta_{i,j}$, $\mu_{i,j}$ are defined before.

It can be verified from (2.10) that the system in (2.16a) is asymptotically stable, thus there exists a bounded $\beta > 0$ such that

$$\begin{aligned}
\|\tilde{\omega}_{i,j}\|_2^2 &+ \left[\sum_{j=0}^{\infty} \xi_{0,j}^T(0, 1) (\tilde{P} - \tilde{Q}) \xi_{0,j}(0, 1) + \sum_{i=0}^{\infty} \xi_{i,0}^T(1, 0) \tilde{Q} \xi_{i,0}(1, 0) \right. \\
&+ \sum_{j=0}^{\infty} \sum_{\kappa=-d_1}^{-1} \xi_{0,j}^T(\kappa, 1) \tilde{Q}_1 \xi_{0,j}(\kappa, 1) \\
&\left. + \sum_{i=0}^{\infty} \sum_{\kappa=-d_2}^{-1} \xi_{i,0}^T(1, \kappa) \tilde{Q}_2 \xi_{i,0}(1, \kappa) \right] \leq \beta^2 \|\mu_{i,j}\|_2^2.
\end{aligned}$$

This implies (2.15) with $\beta = \frac{1}{\alpha}$. With zero boundary conditions as in (2.2), we can easily obtain (2.8), hence the proof is completed. $\square$

For the delay-free case, that is, $\tilde{A}_{d1} = 0$ and $\tilde{A}_{d2} = 0$, according to the procedure of the proof of Theorem 2.5, it is clear that setting $Q_1 = 0$ and $Q_2 = 0$ in Theorem 2.5 would yield the following corollary.

Corollary 2.6 *The delay-free filtering error system in (2.5a, 2.5b) (with $\tilde{A}_{d1} = 0$ and $\tilde{A}_{d2} = 0$) is asymptotically stable with an $\mathcal{H}_\infty$ performance if there exist matrices $P > 0$ and $Q > 0$ such that*

$$\begin{bmatrix} -P & 0 & 0 & P\tilde{A}_1 & P\tilde{B}_1 & P\tilde{A}_2 & P\tilde{B}_2 \\ * & -I & 0 & \tilde{C} & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & \tilde{C} & 0 \\ * & * & * & Q - P & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & * & * & -Q & 0 \\ * & * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0.$$

Remark 2.7 It should be pointed out that the result in Corollary 2.6 is actually the main result in [80]. In other words, Theorem 2.5 in this chapter is an extension of the main result of [80]. $\diamond$

2.3.2 Filter Design

We are now ready to deal with the $\mathcal{H}_\infty$ filter design in the following theorem.

Theorem 2.8 Consider the 2-D state-delayed system in (2.1a–2.1c) and let $\gamma > 0$ be a prescribed constant scalar. There exists a filter in the form of (2.4a, 2.4b) such that the filtering error system in (2.5a, 2.5b) is asymptotically stable with an $\mathcal{H}_\infty$ performance if there exist matrices $\mathcal{U} > 0$, $\mathcal{V} > 0$, $\mathcal{Q}_1 > 0$, $\mathcal{Q}_3 > 0$, $\mathcal{Q}_{11} > 0$, $\mathcal{Q}_{13} > 0$, $\mathcal{Q}_{21} > 0$, $\mathcal{Q}_{23} > 0$, $\mathcal{Q}_2$, $\mathcal{Q}_{12}$, $\mathcal{Q}_{22}$, $\mathcal{A}_{1f}$, $\mathcal{A}_{2f}$, $\mathcal{B}_{1f}$, $\mathcal{B}_{2f}$ and $\mathcal{C}_f$ such that the following LMIs hold:

$$\begin{bmatrix} \Pi_{11} & 0 & 0 & \Pi_{14} & \Pi_{15} & \Pi_{16} & \Pi_{17} & \Pi_{18} & \Pi_{19} \\ * & -I & 0 & \Pi_{24} & 0 & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 & \Pi_{24} & 0 & 0 \\ * & * & * & \Pi_{44} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Pi_{66} & 0 & 0 & 0 \\ * & * & * & * & * & * & \Pi_{77} & 0 & 0 \\ * & * & * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & * & * & \Pi_{99} \end{bmatrix} < 0, \quad (2.17a)$$

$$\Pi_{11} - \Pi_{44} - \Pi_{66} < 0, \quad (2.17b)$$

$$\Pi_{66} < 0, \quad (2.17c)$$

$$\Pi_{99} < 0, \quad (2.17d)$$

where

$$\left\{ \begin{array}{l} \Pi_{11} \triangleq \begin{bmatrix} -\mathcal{U} & -\mathcal{V} \\ * & -\mathcal{V} \end{bmatrix}, \\ \Pi_{14} \triangleq \begin{bmatrix} \mathcal{U}A_1 + \mathcal{B}_{1f}C & \mathcal{A}_{1f} \\ \mathcal{V}A_1 + \mathcal{B}_{1f}C & \mathcal{A}_{1f} \end{bmatrix}, \\ \Pi_{24} \triangleq [E \quad -\mathcal{C}_f], \\ \Pi_{44} \triangleq \begin{bmatrix} \mathcal{Q}_1 + \mathcal{Q}_{11} - \mathcal{U} & \mathcal{Q}_2 + \mathcal{Q}_{12} - \mathcal{V} \\ * & \mathcal{Q}_3 + \mathcal{Q}_{13} - \mathcal{V} \end{bmatrix}, \\ \Pi_{15} \triangleq \begin{bmatrix} \mathcal{U}B_1 + \mathcal{B}_{1f}D \\ \mathcal{V}B_1 + \mathcal{B}_{1f}D \end{bmatrix}, \\ \Pi_{16} \triangleq \begin{bmatrix} \mathcal{U}A_{d1} & 0 \\ \mathcal{V}A_{d1} & 0 \end{bmatrix}, \\ \Pi_{66} \triangleq \begin{bmatrix} -\mathcal{Q}_{11} & -\mathcal{Q}_{12} \\ * & -\mathcal{Q}_{13} \end{bmatrix}, \\ \Pi_{17} \triangleq \begin{bmatrix} \mathcal{U}A_2 + \mathcal{B}_{2f}C & \mathcal{A}_{2f} \\ \mathcal{V}A_2 + \mathcal{B}_{2f}C & \mathcal{A}_{2f} \end{bmatrix}, \\ \Pi_{77} \triangleq \begin{bmatrix} \mathcal{Q}_{21} - \mathcal{Q}_1 & \mathcal{Q}_{22} - \mathcal{Q}_2 \\ * & \mathcal{Q}_{23} - \mathcal{Q}_3 \end{bmatrix}, \\ \Pi_{18} \triangleq \begin{bmatrix} \mathcal{U}B_2 + \mathcal{B}_{2f}D \\ \mathcal{V}B_2 + \mathcal{B}_{2f}D \end{bmatrix}, \\ \Pi_{19} \triangleq \begin{bmatrix} \mathcal{U}A_{d2} & 0 \\ \mathcal{V}A_{d2} & 0 \end{bmatrix}, \\ \Pi_{99} \triangleq \begin{bmatrix} -\mathcal{Q}_{21} & -\mathcal{Q}_{22} \\ * & -\mathcal{Q}_{23} \end{bmatrix}. \end{array} \right.$$

Moreover, the parameters of a desired $\mathcal{H}_\infty$ filter in the form of (2.4a, 2.4b) can be computed from

$$\begin{bmatrix} A_{1f} & B_{1f} \\ A_{2f} & B_{2f} \\ C_f & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{V}^{-1} & 0 & 0 \\ * & \mathcal{V}^{-1} & 0 \\ * & * & I \end{bmatrix} \begin{bmatrix} \mathcal{A}_{1f} & \mathcal{B}_{1f} \\ \mathcal{A}_{2f} & \mathcal{B}_{2f} \\ \mathcal{C}_f & 0 \end{bmatrix}. \quad (2.18)$$

Proof According to Theorem 2.5, if (2.10) holds P is nonsingular since $P > 0$. Now, partition P as

$$P \triangleq \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix}. \quad (2.19)$$

where P_1 , P_2 and P_3 are all $n \times n$ matrix variables. Without loss of generality, we assume that P_2 is nonsingular (if not, P_2 may be perturbed by a matrix ΔP_2 with sufficiently small norm such that $P_2 + \Delta P_2$ is nonsingular and satisfies (2.10)). Define the following matrices:

$$\left\{ \begin{array}{l} \Gamma \triangleq \begin{bmatrix} I & 0 \\ 0 & P_3^{-1} P_2^T \end{bmatrix}, \\ \mathcal{U} \triangleq P_1 > 0, \quad \mathcal{V} \triangleq P_2 P_3^{-1} P_2^T > 0, \\ \mathcal{Q} \triangleq \Gamma^T Q \Gamma \triangleq \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} > 0, \\ \mathcal{Q}_1 \triangleq \Gamma^T Q_1 \Gamma \triangleq \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{13} \end{bmatrix} > 0, \\ \mathcal{Q}_2 \triangleq \Gamma^T Q_2 \Gamma \triangleq \begin{bmatrix} Q_{21} & Q_{22} \\ * & Q_{23} \end{bmatrix} > 0, \end{array} \right. \quad (2.20)$$

and

$$\begin{bmatrix} \mathcal{A}_{1f} & \mathcal{B}_{1f} \\ \mathcal{A}_{2f} & \mathcal{B}_{2f} \\ \mathcal{C}_f & 0 \end{bmatrix} \triangleq \begin{bmatrix} P_2 & 0 & 0 \\ * & P_2 & 0 \\ * & * & I \end{bmatrix} \begin{bmatrix} A_{1f} & B_{1f} \\ A_{2f} & B_{2f} \\ C_f & 0 \end{bmatrix} \begin{bmatrix} P_3^{-1} P_2^T & 0 \\ 0 & I \end{bmatrix}. \quad (2.21)$$

Performing a congruence transformation to (2.10) by matrix $\text{diag}\{\Gamma, I, I, \Gamma, I, \Gamma, \Gamma, I, \Gamma\}$, it follows that

$$\begin{bmatrix} A_1 & 0 & 0 & A_{21} & A_{31} & A_{41} & A_{22} & A_{32} & A_{42} \\ * & -I & 0 & A_5 & 0 & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 & \tilde{C}\Gamma & 0 & 0 \\ * & * & * & \mathcal{Q} + \mathcal{Q}_1 + A_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\mathcal{Q}_1 & 0 & 0 & 0 \\ * & * & * & * & * & * & \mathcal{Q}_2 - \mathcal{Q} & 0 & 0 \\ * & * & * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & * & * & -\mathcal{Q}_2 \end{bmatrix} < 0, \quad (2.22)$$

in which

$$\left\{ \begin{array}{l} \Lambda_1 \triangleq \begin{bmatrix} -P_1 & -P_2 P_3^{-1} P_2^T \\ -P_2 P_3^{-T} P_2^T & -P_2 P_3^{-1} P_2^T \end{bmatrix}, \\ \Lambda_{2j} \triangleq \begin{bmatrix} P_1 A_j + P_2 B_{jf} C & P_2 A_{jf} P_3^{-1} P_2^T \\ P_2 P_3^{-T} P_2^T A_j + P_2 B_{jf} C & P_2 A_{jf} P_3^{-1} P_2^T \end{bmatrix}, \\ \Lambda_{3j} \triangleq \begin{bmatrix} P_1 B_j + P_2 B_{jf} D \\ P_2 P_3^{-T} P_2^T B_j + P_2 B_{jf} D \end{bmatrix}, \\ \Lambda_{4j} \triangleq \begin{bmatrix} P_1 A_{dj} & 0 \\ P_2 P_3^{-T} P_2^T A_{dj} & 0 \end{bmatrix}, \quad (j = 1, 2), \\ \Lambda_5 \triangleq \begin{bmatrix} E & -C_f P_3^{-1} P_2^T \end{bmatrix}. \end{array} \right. \quad (2.23)$$

Substituting (2.19)–(2.21) and (2.23) into (2.22), we can obtain (2.17a). Also, from (2.20), we can obtain (2.17b–2.17d).

On the other hand, (2.21) is equivalent to

$$\begin{bmatrix} A_{1f} & B_{1f} \\ A_{2f} & B_{2f} \\ C_f & 0 \end{bmatrix} = \begin{bmatrix} P_2^{-1} & 0 & 0 \\ * & P_2^{-1} & 0 \\ * & * & I \end{bmatrix} \begin{bmatrix} \mathcal{A}_{1f} & \mathcal{B}_{1f} \\ \mathcal{A}_{2f} & \mathcal{B}_{2f} \\ \mathcal{C}_f & 0 \end{bmatrix} \begin{bmatrix} P_2^{-T} P_3 & 0 \\ 0 & I \end{bmatrix}, \quad (2.24)$$

and it follows from (2.4a, 2.4b) that the transfer function of the filter can be described by

$$\mathbf{T}(z_1, z_2) = C_f (z_1 z_2 I - z_1 A_{1f} - z_2 A_{2f})^{-1} (z_1 B_{1f} + z_2 B_{2f}). \quad (2.25)$$

Substituting (2.24) into (2.25) results in

$$\begin{aligned} \mathbf{T}(z_1, z_2) &= C_f P_2^{-T} P_3 \left(z_1 z_2 I - z_1 P_2^{-1} \mathcal{A}_{1f} P_2^{-T} P_3 - z_2 P_2^{-1} \mathcal{A}_{2f} P_2^{-T} P_3 \right)^{-1} \\ &\quad \times \left(z_1 P_2^{-1} \mathcal{B}_{1f} + z_2 P_2^{-1} \mathcal{B}_{2f} \right) \\ &= C_f \left(z_1 z_2 I - z_1 \mathcal{V}^{-1} \mathcal{A}_{1f} - z_2 \mathcal{V}^{-1} \mathcal{A}_{2f} \right)^{-1} \\ &\quad \times \left(z_1 \mathcal{V}^{-1} \mathcal{B}_{1f} + z_2 \mathcal{V}^{-1} \mathcal{B}_{2f} \right). \end{aligned}$$

Then, the realization of the filter in (2.18) can be readily established, which completes the proof. $\square$

Remark 2.9 Note that Theorem 2.8 provides a sufficient condition for the solvability of $\mathcal{H}_\infty$ filter design problem for the 2-D state-delayed system. Since the obtained conditions are expressed by strict LMIs, the desired filter can be determined by solving the following convex optimization problem:

$$\min \delta \quad \text{subject to (2.17a–2.17d) (where } \delta \triangleq \gamma^2 \text{).} \quad (2.26)$$

If the above convex optimization problem has feasible solution, then the parameters of a desired $\mathcal{H}_\infty$ filter in the form of (2.4a, 2.4b) can be computed from (2.18). $\diamond$

2.4 Further Extensions

In this section, we further extend the results obtained so far to 2-D state-delayed systems with uncertain model data, that is, the uncertain parameters are present in the system matrices $A_1, A_2, A_{d1}, A_{d2}, B_1, B_2, C, D$ and E . In the following, we will consider two types of parameter uncertainties: polytopic uncertainty and norm-bounded uncertainty.

2.4.1 Polytopic Uncertain Case

Theorem 2.8 addresses the $\mathcal{H}_\infty$ filtering problem for the 2-D state-delayed system in (2.1a–2.1c) where the system matrices are all known. However, since LMIs (2.17a–2.17d) are affine in the system matrices, Theorem 2.8 can be directly used to solve the $\mathcal{H}_\infty$ filtering problem for the case where the system matrices are not exactly known but reside within a given polytope.

Assumption 2.2 The matrices $A_1, A_2, A_{d1}, A_{d2}, B_1, B_2, C, D$ and E of system (2.1a–2.1c) contain partially unknown parameters. Assume that $\Omega \triangleq (A_1, A_2, A_{d1}, A_{d2}, B_1, B_2, C, D, E) \in \chi$, where χ is a given convex bounded polyhedral domain described by s vertices:

$$\chi \triangleq \left\{ \chi(\lambda) \left| \chi(\lambda) = \sum_{i=1}^s \lambda_i \chi_i; \sum_{i=1}^s \lambda_i = 1, \lambda_i \geq 0 \right. \right\}$$

where $\chi_j \triangleq (A_{1j}, A_{2j}, A_{d1j}, A_{d2j}, B_{1j}, B_{2j}, C_j, D_j, E_j)$ denotes the j th vertex of the polytope χ .

We state the following theorem without proof, since the proof can be obtained along the same line of the derivation of Theorem 2.8.

Theorem 2.10 *Consider the uncertain 2-D state-delayed system in (2.1a–2.1c) with Assumption 2.2 and let $\gamma > 0$ be a prescribed constant scalar. Then there exists a robust filter in the form of (2.4a, 2.4b) such that the filtering error system in (2.5a, 2.5b) is robustly asymptotically stable with an $\mathcal{H}_\infty$ performance, if there exist matrices $\mathcal{U} > 0$, $\mathcal{V} > 0$, $\mathcal{Q}_{1j} > 0$, $\mathcal{Q}_{3j} > 0$, $\mathcal{Q}_{11j} > 0$, $\mathcal{Q}_{13j} > 0$, $\mathcal{Q}_{21j} > 0$, $\mathcal{Q}_{23j} > 0$, $\mathcal{Q}_{2j}$, $\mathcal{Q}_{12j}$, $\mathcal{Q}_{22j}$, $\mathcal{A}_{1f}$, $\mathcal{A}_{2f}$, $\mathcal{B}_{1f}$, $\mathcal{B}_{2f}$ and $\mathcal{C}_f$ such that, for $j = 1, 2, \dots, s$, the following LMIs hold:*

$$\begin{bmatrix} \Pi_{11} & 0 & 0 & \Pi_{14j} & \Pi_{15j} & \Pi_{16j} & \Pi_{17j} & \Pi_{18j} & \Pi_{19j} \\ * & -I & 0 & \Pi_{24j} & 0 & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 & \Pi_{24j} & 0 & 0 \\ * & * & * & \Pi_{44j} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Pi_{66j} & 0 & 0 & 0 \\ * & * & * & * & * & * & \Pi_{77j} & 0 & 0 \\ * & * & * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & * & * & \Pi_{99j} \end{bmatrix} < 0, \quad (2.27a)$$

$$\Pi_{11} - \Pi_{44j} - \Pi_{66j} < 0, \quad (2.27b)$$

$$\Pi_{66j} < 0, \quad (2.27c)$$

$$\Pi_{99j} < 0. \quad (2.27d)$$

Moreover, a robust $\mathcal{H}_\infty$ filter is given in (2.4a, 2.4b) with parameters can be computed from (2.18). The notions in (2.27a–2.27d) are given as follows.

$$\left\{ \begin{array}{l} \Pi_{14j} \triangleq \begin{bmatrix} \mathcal{U}A_{1j} + \mathcal{B}_{1f}C_j & \mathcal{A}_{1f} \\ \mathcal{V}A_{1j} + \mathcal{B}_{1f}C_j & \mathcal{A}_{1f} \end{bmatrix}, \\ \Pi_{24j} \triangleq \begin{bmatrix} E_j & -C_f \end{bmatrix}, \\ \Pi_{44j} \triangleq \begin{bmatrix} \mathcal{Q}_{1j} + \mathcal{Q}_{11j} - \mathcal{U} & \mathcal{Q}_{2j} + \mathcal{Q}_{12j} - \mathcal{V} \\ * & \mathcal{Q}_{3j} + \mathcal{Q}_{13j} - \mathcal{V} \end{bmatrix}, \\ \Pi_{15j} \triangleq \begin{bmatrix} \mathcal{U}B_{1j} + \mathcal{B}_{1f}D_j \\ \mathcal{V}B_{1j} + \mathcal{B}_{1f}D_j \end{bmatrix}, \\ \Pi_{16j} \triangleq \begin{bmatrix} \mathcal{U}A_{d1j} & 0 \\ \mathcal{V}A_{d1j} & 0 \end{bmatrix}, \\ \Pi_{66j} \triangleq \begin{bmatrix} -\mathcal{Q}_{11j} & -\mathcal{Q}_{12j} \\ * & -\mathcal{Q}_{13j} \end{bmatrix}, \\ \Pi_{17j} \triangleq \begin{bmatrix} \mathcal{U}A_{2j} + \mathcal{B}_{2f}C_j & \mathcal{A}_{2f} \\ \mathcal{V}A_{2j} + \mathcal{B}_{2f}C_j & \mathcal{A}_{2f} \end{bmatrix}, \\ \Pi_{77j} \triangleq \begin{bmatrix} \mathcal{Q}_{21j} - \mathcal{Q}_{1j} & \mathcal{Q}_{22j} - \mathcal{Q}_{2j} \\ * & \mathcal{Q}_{23j} - \mathcal{Q}_{3j} \end{bmatrix}, \\ \Pi_{18j} \triangleq \begin{bmatrix} \mathcal{U}B_{2j} + \mathcal{B}_{2f}D_j \\ \mathcal{V}B_{2j} + \mathcal{B}_{2f}D_j \end{bmatrix}, \\ \Pi_{19j} \triangleq \begin{bmatrix} \mathcal{U}A_{d2j} & 0 \\ \mathcal{V}A_{d2j} & 0 \end{bmatrix}, \\ \Pi_{99j} \triangleq \begin{bmatrix} -\mathcal{Q}_{21j} & -\mathcal{Q}_{22j} \\ * & -\mathcal{Q}_{23j} \end{bmatrix}. \end{array} \right.$$

2.4.2 Norm-Bounded Uncertain Case

An alternative way of dealing with uncertain systems is to assume that the deviation of the system parameters from their nominal values is norm-bounded, which has also been widely used in the robust filtering problem.

Assumption 2.3 The matrices $A_1, A_2, A_{d1}, A_{d2}, B_1, B_2, C, D$ and E of the system (2.1a–2.1c) are assumed to have the following form

$$\begin{cases} A_1 \triangleq \hat{A}_1 + \Delta A_1, & B_1 \triangleq \hat{B}_1 + \Delta B_1, & A_{d1} \triangleq \hat{A}_{d1} + \Delta A_{d1}, \\ A_2 \triangleq \hat{A}_2 + \Delta A_2, & B_2 \triangleq \hat{B}_2 + \Delta B_2, & A_{d2} \triangleq \hat{A}_{d2} + \Delta A_{d2}, \\ C \triangleq \hat{C} + \Delta C, & D \triangleq \hat{D} + \Delta D, & E \triangleq \hat{E} + \Delta E, \end{cases} \quad (2.28)$$

where $\hat{A}_1, \hat{A}_2, \hat{A}_{d1}, \hat{A}_{d2}, \hat{B}_1, \hat{B}_2, \hat{C}, \hat{D}$ and $\hat{E}$ are known constant matrices with appropriate dimensions; and $\Delta A_1, \Delta A_2, \Delta A_{d1}, \Delta A_{d2}, \Delta B_1, \Delta B_2, \Delta C, \Delta D$ and ΔE are real-valued time-varying matrix functions representing norm-bounded parameter uncertainties satisfying

$$\begin{bmatrix} \Delta A_1 & \Delta B_1 & \Delta A_{d1} \\ \Delta A_2 & \Delta B_2 & \Delta A_{d2} \\ \Delta C & \Delta D & \Delta E \end{bmatrix} \triangleq \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} \Delta \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix},$$

where Δ is a real uncertain matrix function with Lebesgue measurable elements satisfying $\Delta^T \Delta \leq I$, and M_1, M_2, M_3, N_1, N_2 and N_3 are known real constant matrices of appropriate dimensions.

Before proceeding further, we give the following lemma which will be used in the proof of this section.

Lemma 2.11 ([212]) *Given appropriately dimensioned matrices Σ, Σ_1 and Σ_2 with $\Sigma^T = \Sigma$. Then*

$$\Sigma + \Sigma_1 \Delta \Sigma_2 + \Sigma_2^T \Delta^T \Sigma_1^T < 0,$$

holds for all Δ satisfying $\Delta^T \Delta \leq I$ if and only if for some $\varepsilon > 0$,

$$\Sigma + \varepsilon^{-1} \Sigma_1 \Sigma_1^T + \varepsilon \Sigma_2^T \Sigma_2 < 0.$$

We now present the robust $\mathcal{H}_\infty$ filtering result for system (2.1a–2.1c) with norm-bounded uncertainties in the following theorem.

Theorem 2.12 *Consider the uncertain 2-D state-delayed system in (2.1a–2.1c) with Assumption 2.3, and let $\gamma > 0$ be a prescribed constant scalar. There exists a robust filter in the form of (2.4a, 2.4b) such that the filtering error system in (2.5a, 2.5b) is robustly asymptotically stable with an $\mathcal{H}_\infty$ performance if there exist matrices $\mathcal{U} > 0, \mathcal{V} > 0, \mathcal{Q}_1 > 0, \mathcal{Q}_3 > 0, \mathcal{Q}_{11} > 0, \mathcal{Q}_{13} > 0, \mathcal{Q}_{21} > 0, \mathcal{Q}_{23} > 0, \mathcal{Q}_2, \mathcal{Q}_{12}, \mathcal{Q}_{22}, \mathcal{A}_{1f}, \mathcal{A}_{2f}, \mathcal{B}_{1f}, \mathcal{B}_{2f}$ and $\mathcal{C}_f$, scalars $\varepsilon_j > 0$ ($j = 1, 2, \dots, 6$) such that (2.17b–2.17d) and the following LMI holds:*

$$\begin{bmatrix}
\Pi_{11} & 0 & 0 & \hat{\Pi}_{14} & \hat{\Pi}_{15} & \hat{\Pi}_{16} & \hat{\Pi}_{17} & \hat{\Pi}_{18} & \hat{\Pi}_{19} & \hat{\Pi}_{110} \\
* & -I & 0 & \hat{\Pi}_{24} & 0 & 0 & 0 & 0 & 0 & \hat{\Pi}_{210} \\
* & * & -I & 0 & 0 & 0 & \Pi_{24} & 0 & 0 & \hat{\Pi}_{310} \\
* & * & * & \hat{\Pi}_{44} & \hat{\Pi}_{45} & \hat{\Pi}_{46} & 0 & 0 & 0 & 0 \\
* & * & * & * & -\gamma^2 I & \hat{\Pi}_{56} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \hat{\Pi}_{66} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & \hat{\Pi}_{77} & \hat{\Pi}_{78} & \hat{\Pi}_{79} & 0 \\
* & * & * & * & * & * & * & -\gamma^2 I & \hat{\Pi}_{89} & 0 \\
* & * & * & * & * & * & * & * & \hat{\Pi}_{99} & 0 \\
* & * & * & * & * & * & * & * & * & \hat{\Pi}_{1010}
\end{bmatrix} < 0. \quad (2.29)$$

Moreover, a robust $\mathcal{H}_\infty$ filter is given in (2.4a, 2.4b) with parameters can be computed from (2.18). The notions in (2.29) are given as follows.

$$\begin{aligned}
\hat{\Pi}_{14} &\triangleq \begin{bmatrix} \mathcal{U}\hat{A}_1 + \mathcal{B}_{1f}\hat{C} & \mathcal{A}_{1f} \\ \mathcal{V}\hat{A}_1 + \mathcal{B}_{1f}\hat{C} & \mathcal{A}_{1f} \end{bmatrix}, \\
\hat{\Pi}_{24} &\triangleq \begin{bmatrix} \hat{E} & -\mathcal{C}_f \end{bmatrix}, \\
\hat{\Pi}_{15} &\triangleq \begin{bmatrix} \mathcal{U}\hat{B}_1 + \mathcal{B}_{1f}\hat{D} \\ \mathcal{V}\hat{B}_1 + \mathcal{B}_{1f}\hat{D} \end{bmatrix}, \\
\hat{\Pi}_{45} &\triangleq \begin{bmatrix} (\varepsilon_1 + \varepsilon_3) N_1^T N_2 \\ 0 \end{bmatrix}, \\
\hat{\Pi}_{16} &\triangleq \begin{bmatrix} \mathcal{U}\hat{A}_{d1} & 0 \\ \mathcal{V}\hat{A}_{d1} & 0 \end{bmatrix}, \\
\hat{\Pi}_{46} &\triangleq \begin{bmatrix} \varepsilon_1 N_1^T N_3 & 0 \\ 0 & 0 \end{bmatrix}, \\
\hat{\Pi}_{56} &\triangleq \begin{bmatrix} \varepsilon_1 N_2^T N_3 & 0 \end{bmatrix}, \\
\hat{\Pi}_{66} &\triangleq \begin{bmatrix} -\mathcal{Q}_{11} + \varepsilon_1 N_3^T N_3 & -\mathcal{Q}_{12} \\ * & -\mathcal{Q}_{13} \end{bmatrix}, \\
\hat{\Pi}_{17} &\triangleq \begin{bmatrix} \mathcal{U}\hat{A}_2 + \mathcal{B}_{2f}\hat{C} & \mathcal{A}_{2f} \\ \mathcal{V}\hat{A}_2 + \mathcal{B}_{2f}\hat{C} & \mathcal{A}_{2f} \end{bmatrix}, \\
\hat{\Pi}_{18} &\triangleq \begin{bmatrix} \mathcal{U}\hat{B}_2 + \mathcal{B}_{2f}\hat{D} \\ \mathcal{V}\hat{B}_2 + \mathcal{B}_{2f}\hat{D} \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\hat{\Pi}_{78} &\triangleq \begin{bmatrix} (\varepsilon_2 + \varepsilon_4) N_1^T N_2 \\ 0 \end{bmatrix}, \\
\hat{\Pi}_{19} &\triangleq \begin{bmatrix} \mathcal{U} \hat{A}_{d2} & 0 \\ \mathcal{V} \hat{A}_{d2} & 0 \end{bmatrix}, \\
\hat{\Pi}_{79} &\triangleq \begin{bmatrix} \varepsilon_2 N_1^T N_3 & 0 \\ 0 & 0 \end{bmatrix}, \\
\hat{\Pi}_{89} &\triangleq [\varepsilon_2 N_2^T N_3 \quad 0], \\
\hat{\Pi}_{99} &\triangleq \begin{bmatrix} -\mathcal{Q}_{21} + \varepsilon_2 N_3^T N_3 & -\mathcal{Q}_{22} \\ * & -\mathcal{Q}_{23} \end{bmatrix}, \\
\hat{\Pi}_{110} &\triangleq \begin{bmatrix} \mathcal{U} M_1 & \mathcal{U} M_2 & \mathcal{B}_{1f} M_3 & \mathcal{B}_{2f} M_3 & 0 & 0 \\ \mathcal{V} M_1 & \mathcal{V} M_2 & \mathcal{B}_{1f} M_3 & \mathcal{B}_{2f} M_3 & 0 & 0 \end{bmatrix}, \\
\hat{\Pi}_{210} &\triangleq [0 \ 0 \ 0 \ 0 \ M_3 \ 0], \\
\hat{\Pi}_{310} &\triangleq [0 \ 0 \ 0 \ 0 \ 0 \ M_3], \\
\hat{\Pi}_{1010} &\triangleq \text{diag} \{ -\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_3 I, -\varepsilon_4 I, -\varepsilon_5 I, -\varepsilon_6 I \}, \\
\hat{\Pi}_{77} &\triangleq \begin{bmatrix} \mathcal{Q}_{21} - \mathcal{Q}_1 + (\varepsilon_2 + \varepsilon_4) N_1^T N_1 + \varepsilon_6 N_3^T N_3 & \mathcal{Q}_{22} - \mathcal{Q}_2 \\ * & \mathcal{Q}_{23} - \mathcal{Q}_3 \end{bmatrix}, \\
\hat{\Pi}_{44} &\triangleq \begin{bmatrix} \mathcal{Q}_1 + \mathcal{Q}_{11} - \mathcal{U} + (\varepsilon_1 + \varepsilon_3) N_1^T N_1 + \varepsilon_5 N_3^T N_3 & \mathcal{Q}_2 + \mathcal{Q}_{12} - \mathcal{V} \\ * & \mathcal{Q}_3 + \mathcal{Q}_{13} - \mathcal{V} \end{bmatrix}.
\end{aligned}$$

Proof The desired result can be obtained by Lemma 2.11 based on the proof of Theorem 2.8. $\square$

2.5 Illustrative Example

Example 2.13 In a real world, some dynamical processes in gas absorption, water stream heating and air drying can be described by the Darboux equation with time-delays:

$$\begin{aligned}
\frac{\partial^2 s(x, t)}{\partial x \partial t} &= a_{11} \frac{\partial s(x, t)}{\partial t} + a_{12} \frac{\partial s(x, t - \tau_1)}{\partial t} + a_{21} \frac{\partial s(x, t)}{\partial x} + a_{22} \frac{\partial s(x, t - \tau_1)}{\partial x} \\
&\quad + a_0 s(x, t) + b f(x, t), \tag{2.30a}
\end{aligned}$$

$$y(x, t) = c_1 s(x, t) + c_2 \left[\frac{\partial s(x, t)}{\partial t} - a_{21} s(x, t) \right], \tag{2.30b}$$

where $s(x, t)$ is an unknown function at $x(\text{space}) \in [0, x_f]$ and $t(\text{time}) \in [0, \infty)$, τ_1 is the time delay, $a_0, a_{11}, a_{12}, a_{21}, a_{22}, b, c_1$ and c_2 are real coefficients, $f(x, t)$ is the input function, and $y(x, t)$ is the measured output.

Note that (2.30a, 2.30b) is a partial differential equation (PDE) and, in practice, it is often desired to predict the unknown state function $s(x, t)$ through the available measurement $y(x, t)$, which renders the filtering problem. Similar to the technique used in [55], we define

$$\begin{aligned} r(x, t) &:= \frac{\partial s(x, t)}{\partial t} - a_{21}s(x, t) + \frac{\partial s(x, t - \tau_1)}{\partial t} - a_{22}s(x, t - \tau_1) \\ x_{i,j}^1 &:= r(i, j) := r(i\Delta x, j\Delta t), \quad x_{i,j}^2 := s(i, j) := s(i\Delta x, j\Delta t), \end{aligned}$$

and then the PDE model (2.30a, 2.30b) can be converted into the form of a 2-D state-delayed system in (2.1a).

In fact, the discrepancy between the PDE model and its 2-D difference approximation depends on the step sizes Δx and Δt which may be treated as uncertainty in the difference model. Obviously, the smaller the step sizes Δx and Δt , the closer between the PDE model and the difference model.

Now, subject to the selection of the parameters $a_0, a_{11}, a_{12}, a_{21}, a_{22}, b, c_1$ and c_2 , we let the system matrices be given as follows:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.3 & 0 \\ 0.2 & 0.1 + 0.02\delta \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.3 \\ 0.5 + 0.01\delta \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 + 0.02\delta \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0.1 & 0 \\ 0.2 & 0.2 + 0.02\delta \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.2 \\ 0.4 + 0.01\delta \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0 & 0.1 \\ 0 & 0.2 + 0.02\delta \end{bmatrix}, \\ C &= \begin{bmatrix} 1.0 & 0 \\ 1.0 & 0.6 + 0.02\delta \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0.3 + 0.01\delta \end{bmatrix}, \quad E = \begin{bmatrix} -1.0 & 1.0 \\ 0 & -0.8 + 0.02\delta \end{bmatrix}. \end{aligned}$$

First, we assume that the system matrices are perfectly known, that is, $\delta = 0$. Solving the LMIs condition obtained in Theorem 2.8 by applying the well-developed LMI-Toolbox in the Matlab environment directly, we obtain that the minimum γ is $\gamma^* = 3.8207$ and

$$\begin{aligned} A_{1f} &= \begin{bmatrix} -0.0117 & 0.0086 \\ 0.0086 & -0.0063 \end{bmatrix}, \quad B_{1f} = \begin{bmatrix} -2.1209 & 1.0000 \\ 1.5539 & -0.7343 \end{bmatrix}, \\ A_{2f} &= \begin{bmatrix} -0.0101 & 0.0074 \\ 0.0072 & -0.0053 \end{bmatrix}, \quad B_{2f} = \begin{bmatrix} -1.5607 & 1.9339 \\ 1.1429 & -1.4179 \end{bmatrix}, \\ C_f &= \begin{bmatrix} 1.3918 & -1.0222 \\ -1.1895 & 0.8761 \end{bmatrix}. \end{aligned}$$

Now, we assume $|\delta| \leq 1$, that is, the system considered has parameter uncertainties. As mentioned in the previous section, there are two types of parameter uncertainties, namely, polytopic and norm-bounded uncertainties. In the following, firstly, we consider the polytopic uncertainties case. In this case, according to Assumption 2.2,

the parameter uncertainties can be represented by a two-vertex polytope. Using Theorem 2.10, the minimum γ obtained is $\gamma^* = 5.4379$, and the obtained filter parameter matrices are given as follows:

$$\begin{aligned} A_{1f} &= \begin{bmatrix} -0.0661 & 0.0378 \\ 0.0375 & -0.0215 \end{bmatrix}, \quad B_{1f} = \begin{bmatrix} -6.3037 & 2.3277 \\ 3.6136 & -1.3374 \end{bmatrix}, \\ A_{2f} &= \begin{bmatrix} -0.0428 & 0.0245 \\ 0.0237 & -0.0136 \end{bmatrix}, \quad B_{2f} = \begin{bmatrix} -4.6924 & 4.1119 \\ 2.6911 & -2.3608 \end{bmatrix}, \\ C_f &= \begin{bmatrix} 1.8108 & -1.0445 \\ -1.5350 & 0.8889 \end{bmatrix}. \end{aligned}$$

Finally, we consider the norm-bounded uncertainties case, and the uncertainties are characterized as follows according to Assumption 2.3:

$$\begin{aligned} \hat{A}_1 &= \begin{bmatrix} 0.3 & 0 \\ 0.2 & 0.1 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 0.1 & 0 \\ 0.2 & 0.2 \end{bmatrix}, \quad \hat{A}_{d1} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad \hat{A}_{d2} = \begin{bmatrix} 0 & 0.1 \\ 0 & 0.2 \end{bmatrix}, \\ \hat{B}_1 &= \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 1.0 & 0 \\ 1.0 & 0.6 \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix}, \quad N_2 = 0.02, \\ \hat{E} &= \begin{bmatrix} -1.0 & 1.0 \\ 0 & -0.8 \end{bmatrix}, \quad M_1 = M_2 = M_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad N_1 = N_3 = \begin{bmatrix} 0 & 0.02 \end{bmatrix}. \end{aligned}$$

Using Theorem 2.12, the minimum γ is obtained as $\gamma^* = 5.2074$, and the obtained filter parameter matrices are given as follows:

$$\begin{aligned} A_{1f} &= \begin{bmatrix} 0.6830 & -0.4439 \\ -0.4496 & 0.2917 \end{bmatrix}, \quad B_{1f} = \begin{bmatrix} -2.2071 & 0.4063 \\ 1.4350 & -0.2651 \end{bmatrix}, \\ A_{2f} &= \begin{bmatrix} 1.8433 & -1.2020 \\ -1.2065 & 0.7864 \end{bmatrix}, \quad B_{2f} = \begin{bmatrix} 4.4037 & 0.6876 \\ -2.8895 & -0.4487 \end{bmatrix}, \\ C_f &= \begin{bmatrix} 1.6334 & -1.0734 \\ -1.4913 & 0.9842 \end{bmatrix}. \end{aligned} \tag{2.31}$$

In the following, we shall show the usefulness of the designed $\mathcal{H}_\infty$ filters by presenting simulation results. Here, we consider only the norm-bounded uncertainty case. Let the initial and boundary conditions be

$$x_{0,i} = x_{i,0} = \begin{cases} \begin{bmatrix} 1 & 1.5 \end{bmatrix}^T, & 0 \leq i \leq 15, \\ \begin{bmatrix} 0 & 0 \end{bmatrix}^T, & i > 15. \end{cases}$$

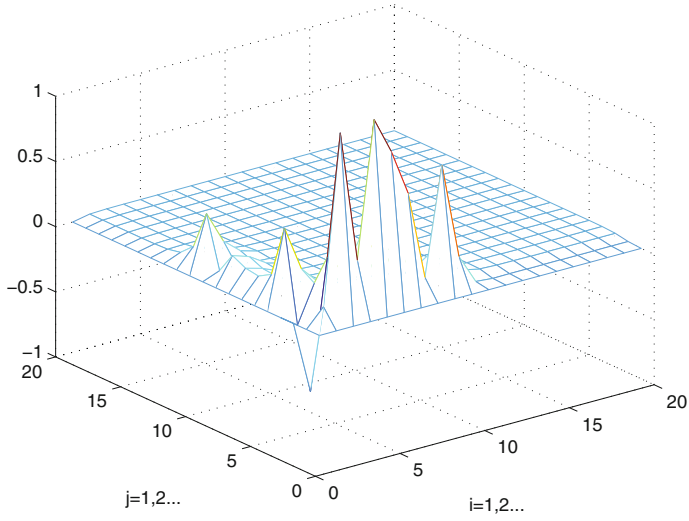


Fig. 2.1 State of the filter $\hat{x}_{i,j}$: the 1st component

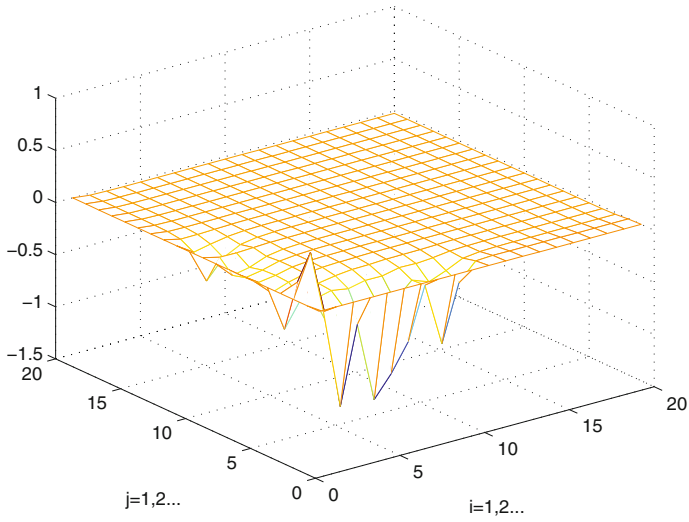


Fig. 2.2 State of the filter $\hat{x}_{i,j}$: the 2nd component

and let the disturbance input $\omega_{i,j}$ be

$$\omega_{i,j} = \begin{cases} 0.05, & 3 \leq i, j \leq 19, \\ 0, & \text{otherwise.} \end{cases}$$

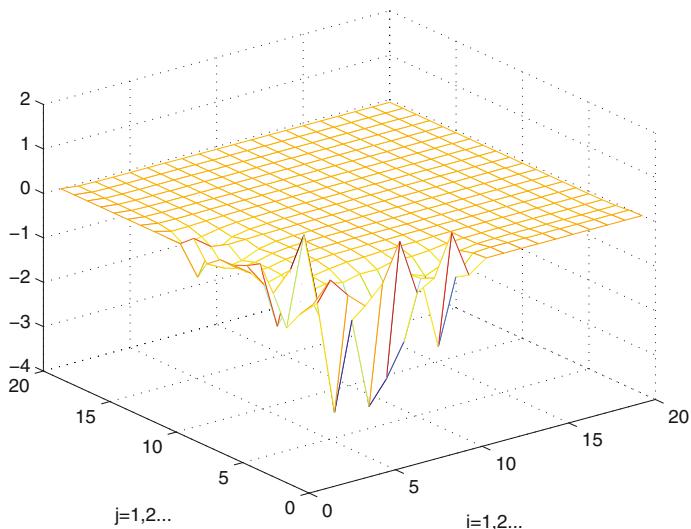


Fig. 2.3 Filtering error $e_{i,j}$: the 1st component

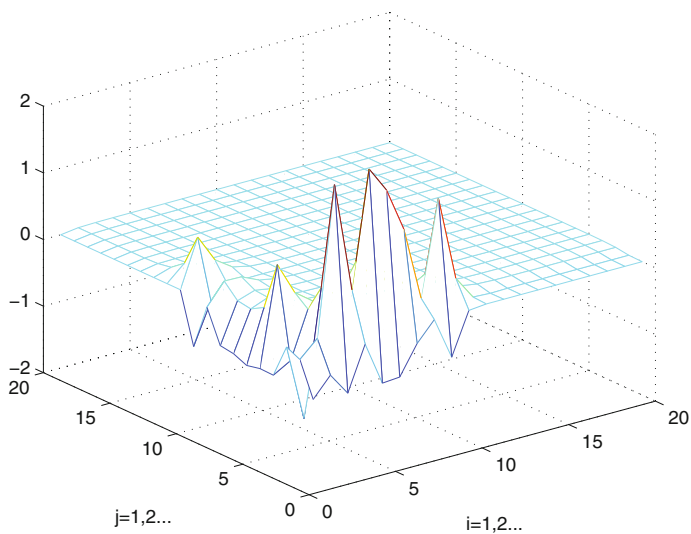


Fig. 2.4 Filtering error: $e_{i,j}$: the 2nd component

The states of the designed $\mathcal{H}_\infty$ filter with (2.31) are given in Figs. 2.1, 2.2, 2.3 and 2.4 are the error response for $e_{i,j}$.

2.6 Conclusion

In this chapter, the problem of $\mathcal{H}_\infty$ filtering for 2-D state-delayed systems has been investigated. Some sufficient conditions have been proposed for the existence of $\mathcal{H}_\infty$ filter in terms of LMI. The designed robust $\mathcal{H}_\infty$ filter guarantees robust asymptotic stability with a prescribed $\mathcal{H}_\infty$ performance of the filtering error system, and the desired filter can be found by solving a convex optimization problem. In addition, the obtained results have been further extended to more general cases where the system matrices also contain uncertain parameters. The most frequently used methods of dealing with parameter uncertainties, including polytopic and norm-bounded characterizations, have been taken into consideration.

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