

Contents

Part I Distributions

1	Introduction	3
	Reference	6
2	Spaces of Test Functions	7
2.1	Hausdorff Locally Convex Topological Vector Spaces	7
2.1.1	Examples of HLCTVS	13
2.1.2	Continuity and Convergence in a HLCVTVS	15
2.2	Basic Test Function Spaces of Distribution Theory	18
2.2.1	The Test Function Space $\mathcal{D}(\Omega)$ of \mathcal{C}^∞ Functions of Compact Support	18
2.2.2	The Test Function Space $\mathcal{S}(\Omega)$ of Strongly Decreasing \mathcal{C}^∞ -Functions on Ω	20
2.2.3	The Test Function Space $\mathcal{E}(\Omega)$ of All \mathcal{C}^∞ -Functions on Ω ..	21
2.2.4	Relation Between the Test Function Spaces $\mathcal{D}(\Omega)$, $\mathcal{S}(\Omega)$, and $\mathcal{E}(\Omega)$	21
2.3	Exercises	22
	Reference	24
3	Schwartz Distributions	25
3.1	The Topological Dual of an HLCTVS	25
3.2	Definition of Distributions	27
3.2.1	The Regular Distributions	29
3.2.2	Some Standard Examples of Distributions	31
3.3	Convergence of Sequences and Series of Distributions	33
3.4	Localization of Distributions	38
3.5	Tempered Distributions and Distributions with Compact Support ...	40
3.6	Exercises	42

4	Calculus for Distributions	45
4.1	Differentiation	46
4.2	Multiplication	49
4.3	Transformation of Variables	52
4.4	Some Applications	55
4.4.1	Distributions with Support in a Point	55
4.4.2	Renormalization of $\left(\frac{1}{x}\right)_+ = \frac{\theta(x)}{x}$	57
4.5	Exercises	59
	References	60
5	Distributions as Derivatives of Functions	63
5.1	Weak Derivatives	63
5.2	Structure Theorem for Distributions	65
5.3	Radon Measures	67
5.4	The Case of Tempered and Compactly Supported Distributions	69
5.5	Exercises	71
	References	71
6	Tensor Products	73
6.1	Tensor Product for Test Function Spaces	73
6.2	Tensor Product for Distributions	77
6.3	Exercises	84
	Reference	84
7	Convolution Products	85
7.1	Convolution of Functions	85
7.2	Regularization of Distributions	89
7.3	Convolution of Distributions	93
7.4	Exercises	100
	References	100
8	Applications of Convolution	101
8.1	Symbolic Calculus—Ordinary Linear Differential Equations	102
8.2	Integral Equation of Volterra	106
8.3	Linear Partial Differential Equations with Constant Coefficients	107
8.4	Elementary Solutions of Partial Differential Operators	110
8.4.1	The Laplace Operator $\Delta_n = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ in \mathbb{R}^n	111
8.4.2	The PDE Operator $\frac{\partial}{\partial t} - \Delta_n$ of the Heat Equation in \mathbb{R}^{n+1}	112
8.4.3	The Wave Operator $\square_4 = \partial_0^2 - \Delta_3$ in \mathbb{R}^4	114
8.5	Exercises	117
	References	117

9	Holomorphic Functions	119
9.1	Hypoellipticity of $\bar{\partial}$	119
9.2	Cauchy Theory	122
9.3	Some Properties of Holomorphic Functions	125
9.4	Exercises	131
	References	131
10	Fourier Transformation	133
10.1	Fourier Transformation for Integrable Functions	134
10.2	Fourier Transformation on $\mathcal{S}(\mathbb{R}^n)$	141
10.3	Fourier Transformation for Tempered Distributions	144
10.4	Some Applications	153
10.4.1	Examples of Tempered Elementary Solutions	155
10.4.2	Summary of Properties of the Fourier Transformation	159
10.5	Exercises	160
	References	162
11	Distributions as Boundary Values of Analytic Functions	163
11.1	Exercises	167
	References	168
12	Other Spaces of Generalized Functions	169
12.1	Generalized Functions of Gelfand Type \mathcal{S}	170
12.2	Hyperfunctions and Fourier Hyperfunctions	173
12.3	Ultradistributions	177
	References	178
13	Sobolev Spaces	181
13.1	Motivation	181
13.2	Basic Definitions	181
13.3	The Basic Estimates	184
13.3.1	Morrey's Inequality	184
13.3.2	Gagliardo-Nirenberg-Sobolev Inequality	188
13.4	Embeddings of Sobolev Spaces	193
13.4.1	Continuous Embeddings	193
13.4.2	Compact Embeddings	195
13.5	Exercises	198
	References	198
Part II Hilbert Space Operators		
14	Hilbert Spaces: A Brief Historical Introduction	201
14.1	Survey: Hilbert Spaces	201
14.2	Some Historical Remarks	208
14.3	Hilbert Spaces and Physics	210
	References	211

15	Inner Product Spaces and Hilbert Spaces	213
15.1	Inner Product Spaces	213
15.1.1	Basic Definitions and Results	214
15.1.2	Basic Topological Concepts	218
15.1.3	On the Relation Between Normed Spaces and Inner Product spaces	219
15.1.4	Examples of Hilbert Spaces	221
15.2	Exercises	224
	References	225
16	Geometry of Hilbert Spaces	227
16.1	Orthogonal Complements and Projections	227
16.2	Gram Determinants	231
16.3	The Dual of a Hilbert Space	233
16.4	Exercises	237
17	Separable Hilbert Spaces	239
17.1	Basic Facts	239
17.2	Weight Functions and Orthogonal Polynomials	245
17.3	Examples of Complete Orthonormal Systems for $L^2(I, \rho dx)$	249
17.4	Exercises	253
	References	254
18	Direct Sums and Tensor Products	255
18.1	Direct Sums of Hilbert Spaces	255
18.2	Tensor Products	258
18.3	Some Applications of Tensor Products and Direct Sums	261
18.3.1	State Space of Particles with Spin	261
18.3.2	State Space of Multi Particle Quantum Systems	261
18.4	Exercises	262
	References	263
19	Topological Aspects	265
19.1	Compactness	265
19.2	The Weak Topology	267
19.3	Exercises	275
	Reference	276
20	Linear Operators	277
20.1	Basic Facts	277
20.2	Adjoints, Closed and Closable Operators	280
20.3	Symmetric and Self-Adjoint Operators	286
20.4	Examples	289
20.4.1	Operator of Multiplication	289
20.4.2	Momentum Operator	290
20.4.3	Free Hamilton Operator	291
20.5	Exercises	292

21 Quadratic Forms	295
21.1 Basic Concepts, Examples	295
21.2 Representation of Quadratic Forms	298
21.3 Some Applications	302
21.4 Exercises	304
22 Bounded Linear Operators	307
22.1 Preliminaries	307
22.2 Examples	309
22.3 The Space $\mathcal{B}(\mathcal{H}, \mathcal{K})$ of Bounded Linear Operators	313
22.4 The C^* -Algebra $\mathcal{B}(\mathcal{H})$	315
22.5 Calculus in the C^* -Algebra $\mathcal{B}(\mathcal{H})$	318
22.5.1 Preliminaries	318
22.5.2 Polar Decomposition of Operators	320
22.6 Exercises	321
Reference	323
23 Special Classes of Linear Operators	325
23.1 Projection Operators	325
23.2 Unitary Operators	329
23.2.1 Isometries	329
23.2.2 Unitary Operators and Groups of Unitary Operators	330
23.2.3 Examples of Unitary Operators	333
23.3 Some Applications of Unitary Operators in Ergodic Theory	333
23.3.1 Poincaré Recurrence Results	334
23.3.2 The Mean Ergodic Theorem of von Neumann	335
23.4 Self-Adjoint Hamilton Operators	337
23.4.1 Kato Perturbations	337
23.4.2 Kato Perturbations of the Free Hamiltonian	339
23.5 Exercises	341
References	342
24 Elements of Spectral Theory	343
24.1 Basic Concepts and Results	344
24.2 The Spectrum of Special Operators	348
24.3 Comments on Spectral Properties of Linear Operators	350
24.4 Exercises	352
Reference	353
25 Compact Operators	355
25.1 Basic Theory	355
25.2 Spectral Theory	359
25.2.1 The Results of Riesz and Schauder	359
25.2.2 The Fredholm Alternative	361
25.3 Exercises	363
Reference	363

26 Hilbert–Schmidt and Trace Class Operators	365
26.1 Basic Theory	365
26.2 Dual Spaces of the Spaces of Compact and of Trace Class Operators	373
26.3 Related Locally Convex Topologies on $\mathcal{B}(\mathcal{H})$	377
26.4 Partial Trace and Schmidt Decomposition in Separable Hilbert Spaces	382
26.4.1 Partial Trace	382
26.4.2 Schmidt Decomposition	386
26.5 Some Applications in Quantum Mechanics	387
26.6 Exercises	390
References	391
27 The Spectral Theorem	393
27.1 Geometric Characterization of Self-Adjointness	394
27.1.1 Preliminaries	394
27.1.2 Subspaces of Controlled Growth	395
27.2 Spectral Families and Their Integrals	402
27.2.1 Spectral Families	402
27.2.2 Integration with Respect to a Spectral Family	404
27.3 The Spectral Theorem	410
27.4 Some Applications	414
27.5 Exercises	416
References	417
28 Some Applications of the Spectral Representation	419
28.1 Functional Calculus	419
28.2 Decomposition of the Spectrum—Spectral Subspaces	421
28.3 Interpretation of the Spectrum of a Self-Adjoint Hamiltonian	429
28.4 Probabilistic Description of Commuting Observables	435
28.5 Exercises	435
References	436
29 Spectral Analysis in Rigged Hilbert Spaces	439
29.1 Rigged Hilbert Spaces	439
29.1.1 Motivation for the Use of Generalized Eigenfunctions	439
29.1.2 Rigged Hilbert Spaces	440
29.1.3 Examples of Nuclear Spaces	442
29.1.4 Structure of the Natural Embedding in a Gelfand Triple	443
29.2 Spectral Analysis of Self-adjoint Operators and Generalized Eigenfunctions	445
29.2.1 Direct Integral of Hilbert Spaces	445
29.2.2 Classical Versions of Spectral Representation	447
29.2.3 Generalized Eigenfunctions	449
29.2.4 Completeness of Generalized Eigenfunctions	450
29.3 Exercises	453
References	453

30	Operator Algebras and Positive Mappings	455
30.1	Representations of C^* -Algebras	455
30.1.1	Representations of $\mathcal{B}(\mathcal{H})$	456
30.2	On Positive Elements and Positive Functionals	460
30.2.1	The GNS-Construction	462
30.3	Normal States	465
30.4	Completely Positive Maps	470
30.4.1	Positive Elements in $M_k(\mathcal{A})$	470
30.4.2	Some Basic Properties of Positive Linear Mappings	472
30.4.3	Completely Positive Maps Between C^* -Algebras	473
30.4.4	Stinespring Factorization Theorem for Completely Positive Maps	475
30.4.5	Completely Positive Mappings on $\mathcal{B}(\mathcal{H})$	479
30.5	Exercises	482
	References	482
31	Positive Mappings in Quantum Physics	483
31.1	Gleason's Theorem	483
31.2	Kraus Form of Quantum Operations	486
31.2.1	Operations and Effects	487
31.2.2	The Representation Theorem for Operations	490
31.3	Choi's Results for Finite Dimensional Completely Positive Maps	493
31.4	Open Quantum Systems, Reduced Dynamics and Decoherence	496
31.5	Exercises	498
	References	499
Part III Variational Methods		
32	Introduction	503
32.1	Roads to Calculus of Variations	504
32.2	Classical Approach Versus Direct Methods	505
32.3	The Objectives of the Following Chapters	508
	References	508
33	Direct Methods in the Calculus of Variations	511
33.1	General Existence Results	511
33.2	Minimization in Banach Spaces	513
33.3	Minimization of Special Classes of Functionals	515
33.4	Exercises	516
	References	517
34	Differential Calculus on Banach Spaces and Extrema of Functions	519
34.1	The Fréchet Derivative	520
34.2	Extrema of Differentiable Functions	526
34.3	Convexity and Monotonicity	528

34.4	Gâteaux Derivatives and Variations	530
34.5	Exercises	534
	Reference	535
35	Constrained Minimization Problems (Method of Lagrange Multipliers)	537
35.1	Geometrical Interpretation of Constrained Minimization	538
35.2	Tangent Spaces of Level Surfaces	539
35.3	Existence of Lagrange Multipliers	541
35.3.1	Comments on Dido's Problem	543
35.4	Exercises	545
	References	546
36	Boundary and Eigenvalue Problems	547
36.1	Minimization in Hilbert Spaces	547
36.2	The Dirichlet–Laplace Operator and Other Elliptic Differential Operators	551
36.3	Nonlinear Convex Problems	554
36.4	Exercises	560
	References	562
37	Density Functional Theory of Atoms and Molecules	563
37.1	Introduction	563
37.2	Semiclassical Theories of Density Functionals	565
37.3	Hohenberg–Kohn Theory	566
37.3.1	Hohenberg–Kohn Variational Principle	570
37.3.2	The Kohn–Sham Equations	571
37.4	Exercises	572
	References	573
	Appendix A Completion of Metric Spaces	575
	Appendix B Metrizable Locally Convex Topological Vector Spaces	579
	Appendix C The Theorem of Baire	581
	Appendix D Bilinear Functionals	589
	Index	591

Mathematical Methods in Physics

Distributions, Hilbert Space Operators, Variational
Methods, and Applications in Quantum Physics

Blanchard, P.; Brüning, E.

2015, XXVII, 598 p. 4 illus., Hardcover

ISBN: 978-3-319-14044-5

A product of Birkhäuser Basel