Chapter 2
Spaces of Test Functions

The spaces of test functions we are going to use are vector spaces of smooth (i.e., sufficiently often continuously differentiable) functions on open nonempty subsets $\Omega \subseteq \mathbb{R}^n$ equipped with a "natural" topology. Accordingly we start with a general method to equip a vector space V with a topology such that the vector space operations of addition and scalar multiplication become continuous, i.e., such that

$$\begin{aligned} A : V \times V \to V, &\quad A(x, y) = x + y, \quad x, y \in V, \\ M : \mathbb{K} \times V \to V, &\quad M(\lambda, x) = \lambda x, \qquad \lambda \in \mathbb{K},\ x \in V \end{aligned}$$

become continuous functions for this topology. This can be done in several different but equivalent ways. The way we describe has the advantage of being the most natural one for the spaces of test functions we want to construct. A vector space V which is equipped with a topology $\mathcal{T}$ such that the functions A and M are continuous is called a *topological vector space*, usually abbreviated as *TVS*. The test function spaces used in distribution theory are concrete examples of topological vector spaces where, however, the topology has the additional property that every point has a neighborhood basis consisting of (absolutely) convex sets. These are called *locally convex topological vector spaces*, abbreviated as *LCVTVS*.

2.1 Hausdorff Locally Convex Topological Vector Spaces

To begin we recall the concept of a topology. To define a *topology* on a set X means to define a system $\mathcal{T}$ of subsets of X which has the following properties:

T_1 $X, \emptyset \in \mathcal{T}$ ($\emptyset$ denotes the empty set);
T_2 $W_i \in \mathcal{T},\ i \in I \Rightarrow \bigcup_{i \in I} W_i \in \mathcal{T}$ (I any index set);
T_3 $W_1, \dots, W_N \in \mathcal{T}, N \in \mathbb{N} \Rightarrow \bigcap_{j=1}^{N} W_j \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called *open* and their complements *closed* sets of the *topological space* $(X, \mathcal{T})$.

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Example 2.1

1. Define $\mathcal{T}_t = \{\emptyset, X\}$. $\mathcal{T}_t$ is called the *trivial topology* on X.
2. Define $\mathcal{T}_d$ to be the system of all subsets of X including X and $\emptyset$. $\mathcal{T}_d$ is called the *discrete topology* on X.
3. The usual topology on the real line $\mathbb{R}$ has as open sets all unions of open intervals $]a, b[\ = \{x \in \mathbb{R} : a < x < b\}$.

Note that according to T_3 only finite intersections are allowed. If one would take here the intersection of infinitely many sets, the resulting concept of a topology would not be very useful. For instance, every point $a \in \mathbb{R}$ is the intersection of infinitely many open intervals $I_n =]a - \frac{1}{n}, a + \frac{1}{n}[$, $a = \cap_{n \in \mathbb{N}}$. Hence, if in T_3 infinite intersections were allowed, all points would be open, thus every subset would be open (see discrete topology), a property which in most cases is not very useful.

If we put any topology on a vector space, it is not assured that the basic vector space operations of addition and scalar multiplication will be continuous. A fairly concrete method to define a topology $\mathcal{T}$ on a vector space V so that the resulting topological space $(V, \mathcal{T})$ is actually a topological vector space, is described in the following paragraphs. The starting point is the concept of a *seminorm* on a vector space as a real valued, subadditive, positive homogeneous and symmetric function.

Definition 2.1 Let V be a vector space over $\mathbb{K}$. Any function $q : V \to \mathbb{R}$ with the properties

(i) $\forall x, y \in V$: $q(x + y) \leq q(x) + q(y)$ (subadditive),
(ii) $\forall \lambda \in \mathbb{K}, \forall x \in V$ $q(\lambda x) = |\lambda| q(x)$, (symmetric and positive homogeneous),

is called a **seminorm** on V. If a seminorm q has the additional property

(iii) $q(x) = 0 \Rightarrow x = 0$,

then it is called a **norm**.

There are some immediate consequences which are used very often:

Lemma 2.1 *For every seminorm q on a vector space V one has*

1. $q(0) = 0$;
2. $\forall x, y \in V$: $|q(x) - q(y)| \leq q(x - y)$;
3. $\forall x \in V$: $0 \leq q(x)$.

Proof The second condition in the definition of a seminorm gives for $\lambda = 0$ that $q(0x) = 0$. But for any $x \in V$ one has $0x = 0 \equiv$ the neutral element 0 in V and the first part follows. Apply subadditivity of q to $x = y + (x - y)$ to get $q(x) = q(y + (x - y)) \leq q(y) + q(x - y)$. Similarly one gets for $y = x + (y - x)$ that $q(y) \leq q(x) + q(y - x)$. The symmetry condition ii) of a seminorm says in particular $q(-x) = q(x)$, hence $q(x - y) = q(y - x)$, and thus the above two estimates together say $\pm(q(x) - q(y)) \leq q(x - y)$ and this proves the second part. For $y = 0$ the second part says $|q(x) - q(0)| \leq q(x)$, hence by observing $q(0) = 0$

we get $|q(x)| \leq q(x)$ and therefore a seminorm takes only nonnegative values and we conclude. □

Example 2.2

1. It is easy to show that the functions $q_i : \mathbb{R}^n \to \mathbb{R}$ defined by $q_i(x) = |x_i|$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ are seminorms on the real vector space $\mathbb{R}^n$ but not norms if $n > 1$. And it is well known that the system $\mathcal{P} = \{q_1, \ldots, q_n\}$ can be used to define the usual Euclidean topology on $\mathbb{R}^n$.
2. More generally, consider any vector space V over the field $\mathbb{K}$ and its **algebraic dual space** $V^* = L(V; \mathbb{K})$ defined as the set of all linear functions $T : V \to \mathbb{K}$, i.e. those functions which satisfy

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \forall x, y \in V, \quad \forall \alpha, \beta \in \mathbb{K}.$$

Each such $T \in V^*$ defines a seminorm q_T on V by

$$q_T(x) = |T(x)| \qquad \forall x \in V.$$

3. For an open nonempty set $\Omega \subset \mathbb{R}^n$, the set $\mathcal{C}^k(\Omega)$ of all functions $f : \Omega \to \mathbb{K}$ which have continuous derivatives up to order k is actually a vector space over $\mathbb{K}$ and on it the following functions $p_{K,m}$ and $q_{K,m}$ are indeed seminorms. Here $K \subset \Omega$ is any compact subset and $k \in \mathbb{N}$ is any nonnegative integer. For $0 \leq m \leq k$ and $\phi \in \mathcal{C}^k(\Omega)$ define

$$p_{K,m}(\phi) = \sup_{x \in K, |\alpha| \leq m} |D^\alpha \phi(x)|, \tag{2.1}$$

$$q_{K,m}(\phi) = \left(\sum_{|\alpha| \leq m} \int_K |D^\alpha \phi(x)|^2 \mathrm{d}x \right)^{1/2}. \tag{2.2}$$

The notation is as follows. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we denote by $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}}$ the *derivative monomial* of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$, i.e., $D^\alpha \phi(x) = \frac{\partial^{|\alpha|} \phi}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}}(x), x = (x_1, \ldots, x_n)$. Thus, for example for $f \in \mathcal{C}^3(\mathbb{R}^3)$, one has in this notation: If $\alpha = (1, 0, 0)$, then $|\alpha| = 1$ and $D^\alpha f = \frac{\partial f}{\partial x_1}$; if $\alpha = (1, 1, 0)$, then $|\alpha| = 2$ and $D^\alpha f = \frac{\partial^2 f}{\partial x_1 \partial x_2}$; if $\alpha = (0, 0, 2)$, then $|\alpha| = 2$ and $D^\alpha f = \frac{\partial^2 f}{\partial^2 x_3}$; if $\alpha = (1, 1, 1)$ then $|\alpha| = 3$ and $D^\alpha f = \frac{\partial^3 f}{\partial x_1 \partial x_2 \partial x_3}$.

A few comments on these examples are in order. The seminorms given in the second example play an important role in general functional analysis, those of the third will be used later in the definition of the topology on the test function spaces used in distribution theory.

Recall that in a Euclidean space $\mathbb{R}^n$ the open ball $B_r(x)$ with radius $r > 0$ and centre x is defined by

$$B_r(x) = \left\{y \in \mathbb{R}^n : |y - x| < r\right\}$$

where $|y - x| = \sqrt{\sum_{i=1}^n (y_i - x_i)^2}$ is the Euclidean distance between the points $y = (y_1, \dots, y_n)$ and $x = (x_1, \dots, x_n)$. Similarly one proceeds in a vector space V on which a seminorm p is given: The *open p-ball* in V with centre x and radius $r > 0$ is defined by

$$B_{p,r}(x) = \{y \in V : p(y - x) < r\}.$$

In this definition the Euclidean distance is replaced by the semidistance $d_p(y, x) = p(y - x)$ between the points $y, x \in V$. Note: If p is not a norm, then one can have $d_p(y, x) = 0$ for $y \neq x$. In this case the open p-ball $B_{p,r}(0)$ contains the nontrivial subspace $N(p) = \{y \in V : p(y) = 0\}$. Nevertheless these p-balls share all essential properties with balls in Euclidean space.

1. $B_{p,r}(x) = x + B_{p,r}$, i.e., every point $y \in B_{p,r}(x)$ has the unique representation $y = x + z$ with $z \in B_{p,r} \equiv B_{p,r}(0)$;
2. $B_{p,r}$ is circular, i.e., $y \in B_{p,r}$, $\alpha \in \mathbb{K}$, $|\alpha| \leq 1$ implies $\alpha x \in B_{p,r}$;
3. $B_{p,r}$ is convex, i.e., $x, y \in B_{p,r}$ and $0 \leq \lambda \leq 1$ implies $\lambda x + (1 - \lambda)y \in B_{p,r}$;
4. $B_{p,r}$ absorbs the points of V, i.e., for every $x \in V$ there is a $\lambda > 0$ such that $\lambda x \in B_{p,r}$;
5. The nonempty intersection $B_{p_1,r_1}(x_1) \cap B_{p_2,r_2}(x_2)$ of two open p-balls contains an open p-ball: $B_{p,r}(x) \subset B_{p_1,r_1}(x_1) \cap B_{p_2,r_2}(x_2)$.

For the proof of these statements see the Exercises.

In a finite dimensional vector space all norms are equivalent, i.e., they define the same topology. However, this statement does not hold in an infinite dimensional vector space (see Exercises). As the above examples indicate, in an infinite dimensional vector space there are many different seminorms. This raises naturally two questions: How do we compare seminorms? When do two systems of seminorms define the same topology? A natural way to compare two seminorms is to compare their values in all points. Accordingly one has:

Definition 2.2 For two seminorms p and q on a vector space V one says

a) p is **smaller than** q, in symbols $p \leq q$ if, and only if, $p(x) \leq q(x)\ \forall x \in V$;
b) p and q are **comparable** if, and only if, either $p \leq q$ or $q \leq p$.

The seminorms q_i in our first example above are not comparable. Among the seminorms $q_{K,m}$ and $p_{K,m}$ from the third example there are many which are comparable. Suppose two compact subsets K_1 and K_2 satisfy $K_1 \subset K_2$ and the nonnegative integers m_1 is smaller than or equal to the nonnegative integer m_2, then obviously

$$p_{K_1,m_1} \leq p_{K_2,m_2} \qquad \text{and} \qquad q_{K_1,m_1} \leq q_{K_2,m_2}.$$

In the Exercises we show the following simple facts about seminorms: If p is a seminorm on a vector space V and r a positive real number, then rp defined by

$(rp)(x) = rp(x)$ for all $x \in V$ is again a seminorm on V. The maximum $p = \max\{p_1, \dots, p_n\}$ of finitely many seminorms $p_1, \dots, p_n$ on V, which is defined by $p(x) = \max\{p_1(x), \dots, p_n(x)\}$ for all $x \in V$, is a seminorm on V such that $p_i \le p$ for $i = 1, \dots, n$. This prepares us for a discussion of systems of seminorms on a vector space.

Definition 2.3 A system $\mathcal{P}$ of seminorms on a vector space V is called **filtering** if, and only if, for any two seminorms $p_1,\ p_2 \in \mathcal{P}$ there is a seminorm $q \in \mathcal{P}$ and there are positive numbers $r_1,\ r_2 \in \mathbb{R}^+$ such that $r_1 p_1 \le q$ and $r_2 p_2 \le q$ hold.

Certainly, not all systems of seminorms are filtering (see our first finite-dimensional example). However it is straightforward to construct a filtering system which contains a given system: Given a system $\mathcal{P}_0$ on a vector space V one defines the system $\mathcal{P} = \mathcal{P}(\mathcal{P}_0)$ generated by $\mathcal{P}_0$ as follows:

$$q \in \mathcal{P} \Leftrightarrow \exists\, p_1, \dots, p_n \in \mathcal{P}_0\ \exists\, r_1, \dots, r_n \in \mathbb{R}^+ :\ q = \max\{r_1 p_1, \dots, r_n p_n\}\,.$$

One can show that $\mathcal{P}(\mathcal{P}_0)$ is the minimal filtering system of seminorms on V that contains $\mathcal{P}_0$. In our third example above we considered the following two systems of seminorms on $V = \mathcal{C}^k(\Omega)$:

$$\begin{aligned} \mathcal{P}_k(\Omega) &= \left\{p_{K,m} :\ K \subset \Omega,\ K \text{ compact},\ 0 \le m \le k\right\}, \\ \mathcal{Q}_k(\Omega) &= \left\{q_{K,m} :\ K \subset \Omega,\ K \text{ compact},\ 0 \le m \le k\right\}. \end{aligned}$$

In the Exercises it is shown that both are filtering.

Our first use of the open p-balls is to define a topology.

Theorem 2.1 *Suppose that $\mathcal{P}$ is a filtering system of seminorms on a vector space V. Define a system $\mathcal{T}_{\mathcal{P}}$ of subsets of V as follows: A subset $U \subset V$ belongs to $\mathcal{T}_{\mathcal{P}}$, if and only if, either $U = \emptyset$ or*

$$\forall x \in U\ \exists\, p \in \mathcal{P},\ \exists\, r > 0 :\ B_{p,r}(x) \subset U.$$

Then $\mathcal{T}_{\mathcal{P}}$ is a topology on V in which every point $x \in V$ has a neighborhood basis V_x consisting of open p-balls, $V_x = \left\{B_{p,r}(x) :\ p \in \mathcal{P},\ r > 0\right\}$.

Proof Suppose we are given $U_i \in \mathcal{T}_{\mathcal{P}},\ i \in I$. We are going to show that $U = \cup_{i \in I} U_i \in \mathcal{T}_{\mathcal{P}}$. Take any $x \in U$, then $x \in U_i$ for some $i \in I$. Thus $U_i \in \mathcal{T}_{\mathcal{P}}$ implies: There are $p \in \mathcal{P}$ and $r > 0$ such that $B_{p,r}(x) \subset U_i$. It follows that $B_{p,r}(x) \subset U$, hence $U \in \mathcal{T}_{\mathcal{P}}$. Next assume that $U_1, \dots, U_n \in \mathcal{T}_{\mathcal{P}}$ are given. Denote $U = \cap_{i=1}^n U_i$ and consider $x \in U \subset U_i,\ i = 1, \dots, n$. Therefore, for $i = 1, \dots, n$, there are $p_i \in \mathcal{P}$ and $r_i > 0$ such that $B_{p_i,r_i}(x) \subset U_i$. Since the system $\mathcal{P}$ is filtering, there is a $p \in \mathcal{P}$ and there are $\rho_i > 0$ such that $\rho_i p_i \le p$ for $i = 1, \dots, n$. Define $r = \min\{\rho_1 r_1, \dots, \rho_n r_n\}$. It follows that $B_{p,r}(x) \subset B_{p_i,r_i}(x)$ for $i = 1, \dots, n$ and therefore $B_{p,r}(x) \subset \cap_{i=1}^n U_i = U$. Hence the system $\mathcal{T}_{\mathcal{P}}$ satisfies the three axioms of a topology. By definition $\mathcal{T}_{\mathcal{P}}$ is the topology defined by the system V_x of open p-balls as a neighborhood basis of a point $x \in V$. □

This result shows that there is a unique way to construct a topology on a vector space as soon as one is given a filtering system of seminorms. Suppose now that two filtering systems $\mathcal{P}$ and $\mathcal{Q}$ of seminorms are given on a vector space V. Then we get two topologies $\mathcal{T}_{\mathcal{P}}$ and $\mathcal{T}_{\mathcal{Q}}$ on V and naturally one would like to know how these topologies compare, in particular when they are equal. This question is answered in the following proposition.

Proposition 2.1 *Given two filtering systems $\mathcal{P}$ and $\mathcal{Q}$ on a vector space V, construct the topologies $\mathcal{T}_{\mathcal{P}}$ and $\mathcal{T}_{\mathcal{Q}}$ on V according to Theorem (2.1). Then the following two statements are equivalent:*

(i) $\mathcal{T}_{\mathcal{P}} = \mathcal{T}_{\mathcal{Q}}$.
(ii) $\forall p \in \mathcal{P} \exists\, q \in \mathcal{Q}\, \exists\, \lambda > 0 : p \leq \lambda q$ *and* $\forall q \in \mathcal{Q}\, \exists\, p \in \mathcal{P}\, \exists\, \lambda > 0 : q \leq \lambda p$.

Two systems $\mathcal{P}$ and $\mathcal{Q}$ of seminorms on a vector space V are called **equivalent***, if, and only if, any of these equivalent conditions holds.*

The main technical element of the proof of this proposition is the following elementary but widely used lemma about the relation of open p-balls and their defining seminorms. Its proof is left as an exercise.

Lemma 2.2 *Suppose that p and q are two seminorms on a vector space V. Then, for any $r > 0$ and $R > 0$, the following holds:*

$$p \leq \frac{r}{R} q \quad \Leftrightarrow \quad \text{for any } x \in V : \; B_{q,R}(x) \subseteq B_{p,r}(x). \tag{2.3}$$

Proof (Proof of 2.1) Assume condition (i). Then every open p-ball $B_{p,r}(x)$ is open for the topology $\mathcal{T}_{\mathcal{Q}}$, hence there is an open q-ball $B_{q,R}(x) \subset B_{p,r}(x)$. By the lemma we conclude that $p \leq \frac{r}{R} q$. Condition (i) also implies that every open q-ball is open for the topology $\mathcal{T}_{\mathcal{P}}$, hence we deduce $p \leq \lambda q$ for some $0 < \lambda$. Therefore condition (ii) holds.

Conversely, suppose that condition (ii) holds. Then, using again the lemma one deduces: For every open p-ball $B_{p,r}(x)$ there is an open q-ball $B_{q,R}(x) \subset B_{p,r}(x)$ and for every open q-ball $B_{q,R}(x)$ there is an open p-ball $B_{p,r}(x) \subset B_{q,R}(x)$. This then implies that the two topologies $\mathcal{T}_{\mathcal{P}}$ and $\mathcal{T}_{\mathcal{Q}}$ coincide. □

Recall that a topological space is called *Hausdorff* if any two distinct points can be separated by disjoint neighborhoods. There is a convenient way to decide when the topology $\mathcal{T}_{\mathcal{P}}$ defined by a filtering system of seminorms is Hausdorff.

Proposition 2.2 *Suppose $\mathcal{P}$ is a filtering system of seminorms on a vector space V. Then the topology $\mathcal{T}_{\mathcal{P}}$ is Hausdorff if, and only if, for every $x \in V$, $x \neq 0$, there is a seminorm $p \in \mathcal{P}$ such that $p(x) > 0$.*

Proof Suppose that the topological space $(V, \mathcal{T}_{\mathcal{P}})$ is Hausdorff and $x \in V$ is given, $x \neq 0$. Then there are two open balls $B_{p,r}(0)$ and $B_{q,R}(x)$ which do not intersect. By definition of these balls it follows that $p(x) \geq r > 0$ and the condition of the proposition holds. Conversely assume that the condition holds and two points $x, y \in V$, $x - y \neq 0$ are given. There is a $p \in \mathcal{P}$ such that $0 < 2r = p(x - y)$. Then the open balls $B_{p,r}(x)$ and $B_{p,r}(y)$ do not intersect. (If $z \in V$ were a point belonging to both balls, then we would have $p(z - x) < r$ and $p(z - y) < r$ and therefore $2r = p(x-y) = p(x-z+z-y) \le p(x-z) + p(z-y) < r + r = 2r$, a contradiction). Hence the topology $\mathcal{T}_{\mathcal{P}}$ is Hausdorff. □

Finally, we discuss the continuity of the basic vector space operations of addition and scalar multiplication with respect to the topology $\mathcal{T}_{\mathcal{P}}$ defined by a filtering system $\mathcal{P}$ of seminorms on a vector space V. Recall that a function $f : E \to F$ from a topological space E into a topological space F is continuous at a point $x \in E$ if, and only if, the following condition is satisfied: For every neighborhood U of the point $y = f(x)$ in F there is a neighborhood V of x in E such that $f(V) \subset U$, and it is enough to consider instead of general neighborhoods U and V only elements of a neighborhood basis of $f(x)$, respectively x.

Proposition 2.3 *Let $\mathcal{P}$ be a filtering system of seminorms on a vector space V. Then addition (A) and scalar multiplication (M) of the vector space V are continuous with respect to the topology $\mathcal{T}_{\mathcal{P}}$, hence $(V, \mathcal{T}_{\mathcal{P}})$ is a topological vector space. This topological vector space is usually denoted by*

$$(V, \mathcal{P}) \qquad \text{or} \qquad V[\mathcal{P}].$$

Proof We show that the addition $A : V \times V \to V$ is continuous at any point $(x, y) \in V \times V$. Naturally, the product space $V \times V$ is equipped with the product topology of $\mathcal{T}_{\mathcal{P}}$. Given any open p-ball $B_{p,2r}(x+y)$ for some $r > 0$, then $A(B_{p,r}(x) \times B_{p,r}(y)) \subset B_{p,2r}(x+y)$ since for all $(x', y') \in B_{p,r}(x) \times B_{p,r}(y)$ we have $p(A(x', y') - A(x, y)) = p((x'+y') - (x+y)) = p(x'-x+y-y') \le p(x'-x) + p(y'-y) < r + r = 2r$. Continuity of scalar multiplication M is proved in a similar way. □

We summarize our results in the following theorem.

Theorem 2.2 *Let $\mathcal{P}$ be a filtering system of seminorms on a vector space V. Equip V with the induced topology $\mathcal{T}_{\mathcal{P}}$. Then $(V, \mathcal{T}_{\mathcal{P}}) = V[\mathcal{T}_{\mathcal{P}}]$ is a locally convex topological vector space. It is Hausdorff or a* **HLCVTVS** *if, and only if, for every $x \in V$, $x \neq 0$, there is a $p \in \mathcal{P}$ such that $p(x) > 0$.*

Proof By Theorem 2.1 every point $x \in V$ has a neighbourhood basis V_x consisting of open p-balls. These balls are absolutely convex (i.e. $y, z \in B_{p,r}(x)$, $\alpha, \beta \in \mathbb{K}$, $\alpha + \beta = 1$, $|\alpha| + |\beta| \le 1$ implies $\alpha y + \beta z \in B_{p,r}(x)$) by the properties of p-balls listed earlier. Hence by Proposition 2.3 $V[\mathcal{T}_{\mathcal{P}}]$ is a LCTVS. Finally by Proposition 2.2 we conclude. □

2.1.1 Examples of HLCTVS

The examples of HLCTVS which we are going to discuss serve a dual purpose. Naturally they are considered in order to illustrate the concepts and results introduced above. Then later they will be used as building blocks of the test function spaces used in distribution theory.

1. Recall the filtering systems of seminorms $\mathcal{P}_k(\Omega)$ and $\mathcal{Q}_k(\Omega)$ introduced earlier on the vector space $\mathcal{C}^k(\Omega)$ of k times continuously differentiable functions on

an open nonempty subset $\Omega \subseteq \mathbb{R}^n$. With the help of Theorem 2.2 it is easy to show that both $(\mathcal{C}^k(\Omega), \mathcal{P}_k(\Omega))$ and $(\mathcal{C}^k(\Omega), \mathcal{Q}_k(\Omega))$ are Hausdorff locally convex topological vector spaces.

2. Fix a compact subset K of some open nonempty set $\Omega \subseteq \mathbb{R}^n$ and consider the space $\mathcal{C}_K^\infty(\Omega)$ of all functions $\phi : \Omega \to \mathbb{K}$ which are infinitely often differentiable on Ω and which have their support in K, i.e., supp $f \subseteq K$. On $\mathcal{C}_K^\infty(\Omega)$ consider the systems of semi-norms

$$\mathcal{P}_K(\Omega) = \left\{p_{K,m} : m = 0, 1, 2, \dots\right\} \quad \mathcal{Q}_K(\Omega) = \left\{q_{K,m} : m = 0, 1, 2, \dots\right\}$$

introduced in Eq. (2.1), respectively in Eq. (2.2). Both systems are obviously filtering, and both $p_{K,m}$ and $q_{K,m}$ are norms on $\mathcal{C}_K^\infty(\Omega)$. In the Exercises it is shown that both systems are equivalent and thus we get that

$$\mathcal{D}_K(\Omega) = (\mathcal{C}_K^\infty(\Omega), \mathcal{P}_K(\Omega)) = (\mathcal{C}_K^\infty(\Omega), \mathcal{Q}_K(\Omega)) \tag{2.4}$$

is a Hausdorff locally convex topological vector space.

3. Now let $\Omega \subseteq \mathbb{R}^n$ be an open nonempty subset which may be unbounded. Consider the vector space $\mathcal{C}^k(\Omega)$ of functions $\phi : \Omega \to \mathbb{K}$ which have continuous derivatives up to order k. Introduce two families of symmetric and subadditive functions $\mathcal{C}^k(\Omega) \to [0, +\infty]$ by defining, for $l = 0, 1, 2, \dots, k$ and $m = 0, 1, 2, \dots$,

$$\begin{aligned} p_{m,l}(\phi) &= \sup_{x\in\Omega,\,|\alpha|\le l} (1 + x^2)^{m/2} |D^\alpha \phi(x)|, \\ q_{m,l}(\phi) &= \left(\textstyle\sum_{|\alpha|\le l} \int_\Omega (1 + x^2)^{m/2} |D^\alpha \phi(x)|^2 \mathrm{d}x\right)^{1/2}. \end{aligned}$$

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we use the notation $x^2 = x_1^2 + \dots + x_n^2$ and $|x| = \sqrt{x^2}$. Define the following subspace of $\mathcal{C}^k(\Omega)$:

$$\mathcal{C}_m^k(\Omega) = \left\{\phi \in \mathcal{C}^k(\Omega) : p_{m,l}(\phi) < \infty,\ l = 0, 1, \dots, k\right\}.$$

Then the system of norms $\left\{p_{m,l} : 0 \le l \le k\right\}$ is filtering on this subspace and thus $\left(\mathcal{C}_m^k(\Omega), \left\{p_{m,l} : 0 \le l \le k\right\}\right)$ is a HLCTVS. $\mathcal{C}_m^k(\Omega)$ is the space of continuously differentiable functions which decay at infinity (if Ω is unbounded), with all derivatives of order $\le k$, at least as $|x|^{-m}$. Similarly one can build a HLCTVS space by using the system of norms $q_{m,l}$, $0 \le l \le k$.

4. In this example we use some basic facts from Lebesgue integration theory [1]. Let $\Omega \subset \mathbb{R}^n$ be a nonempty measurable set. On the vector space $L_{loc}^1(\Omega)$ of all measurable functions $f : \Omega \to \mathbb{K}$ which are *locally integrable*, i.e., for which

$$\|f\|_K = \int_K |f(x)| \mathrm{d}x$$

is finite for every compact subset $K \subset \Omega$, consider the system of seminorms $\mathcal{P} = \{\|\cdot\|_K : K \subset \Omega,\ K \text{ compact}\}$. Since the finite union of compact sets is compact, it follows easily that this system is filtering. If $f \in L_{loc}^1(\Omega)$ is given and if $f \neq 0$, then there is a compact set K such that $\|f\|_K > 0$, since $f \neq 0$ means that f is different from zero on a set of positive Lebesgue measure. Therefore, by Theorem 2.2, the space

$$(L^1_{loc}(\Omega), \{\|\cdot\|_K \,:\, K \subset \Omega,\ K \text{ compact}\})$$

is a HLCTVS.

2.1.2 Continuity and Convergence in a HLCVTVS

Since the topology of a LCTVS $V[\mathcal{P}]$ is defined in terms of a filtering system $\mathcal{P}$ of seminorms it is, in most cases, much more convenient to have a characterization of the basic concepts of convergence, of a Cauchy sequence, and of continuity in terms of the seminorms directly instead of having to rely on the general topological definitions. Such characterizations will be given in this subsection.

Recall: A sequence $(x^i)_{i\in\mathbb{N}}$ of points $x^i = (x^i_1, \dots, x^i_n) \in \mathbb{R}^n$ is said to converge if, and only if, there is a point $x \in \mathbb{R}^n$ such that for every open Euclidean ball $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ only a finite number of elements of the sequence are not contained in this ball, i.e., there is an index i_0, depending on $r > 0$, such that $x^i \in B_r(x)$ for all $i \geq i_0$, or expressed directly in terms of the Euclidean norm, $|x^i - x| < r$ for all $i \geq i_0$.

Similarly one proceeds in a general HLCTVS $V[\mathcal{P}]$ where now however instead of the Euclidean norm $|\cdot|$ all the seminorms $p \in \mathcal{P}$ have to be taken into account.

Definition 2.4 Let $V[\mathcal{P}]$ be a HLCTVS and $(x_i)_{i\in\mathbb{N}}$ a sequence in $V[\mathcal{P}]$. Then one says:

1. The sequence $(x_i)_{i\in\mathbb{N}}$ **converges** (in $V[\mathcal{P}]$) if, and only if, there is an $x \in V$ (called a **limit point** of the sequence) such that for every $p \in \mathcal{P}$ and for every $r > 0$ there is an index $i_0 = i_0(p, r)$ depending on p and r such that $p(x - x_i) < r$ for all $i \geq i_0$.
2. The sequence $(x_i)_{i\in\mathbb{N}}$ is a **Cauchy sequence** if, and only if, for every $p \in \mathcal{P}$ and every $r > 0$ there is an index $i_0 = i_0(p, r)$ such that $p(x_i - x_j) < r$ for all $i, j \geq i_0$.

The following immediate results are well known in $\mathbb{R}^n$.

Theorem 2.3

(a) Every convergent sequence in a LCTVS $V[\mathcal{P}]$ is a Cauchy sequence.
(b) In a HLCTVS $V[\mathcal{P}]$ the limit point of a convergent sequence is unique.

Proof Suppose a sequence $(x_i)_{i\in\mathbb{N}}$ converges in $V[\mathcal{P}]$ to $x \in V$. Then, for any $p \in \mathcal{P}$ and any $r > 0$, there is an $i_0 \in \mathbb{N}$ such that $p(x - x_i) < r/2$ for all $i \geq i_0$. Therefore, for all $i, j \geq i_0$, one has $p(x_i - x_j) = p((x - x_j) + (x_i - x)) \leq p(x - x_j) + p(x_i - x) < \frac{r}{2} + \frac{r}{2} = r$, hence $(x_i)_{i\in\mathbb{N}}$ is a Cauchy sequence and part (a) follows.

Suppose $V[\mathcal{P}]$ is a HLCTVS and $(x_i)_{i\in\mathbb{N}}$ is a convergent sequence in $V[\mathcal{P}]$. Assume that for $x, y \in V$ the condition in the definition of convergence holds, i.e., for every $p \in \mathcal{P}$ and every $r > 0$ there is an i_1 such that $p(x - x_i) < r$ for all $i \geq i_1$ and there is an i_2 such that $p(y - x_i) < r$ for all $i \geq i_2$. Then, for all $i \geq \max\{i_1, i_2\}$, $p(x - y) = p(x - x_i + x_i - y) \leq p(x - x_i) + p(x_i - y) < r + r = 2r$, and since $r > 0$ is arbitrary, it follows that $p(x - y) = 0$. Since this holds for every $p \in \mathcal{P}$ and $V[\mathcal{P}]$ is Hausdorff, we conclude (see Proposition 2.2) that $x = y$ and thus part (b) follows. □

Part (a) of Theorem 2.3 raises naturally the question whether the converse holds too, i.e. whether every Cauchy sequence converges. In general, this is not the case. Spaces in which this statement holds are distinguished according to the following definition.

Definition 2.5 A HLCTVS in which every Cauchy sequence converges is called **sequentially complete**.

Example 2.3

1. Per construction, the field $\mathbb{R}$ of real numbers equipped with the absolute value $|\cdot|$ as a norm is a sequentially complete HLCTVS.
2. The Euclidean spaces $(\mathbb{R}^n, |\cdot|)$, n=1,2, ... are HLCTVS. Here $|\cdot|$ denotes the Euclidean norm.
3. For any $\Omega \subset \mathbb{R}^n$, Ω open and nonempty, and k=0,1,2, ... , the space

$$\mathcal{C}^k(\Omega)[\mathcal{P}_k(\Omega)]$$

   is a sequentially complete HLCTVS. This is shown in the Exercises. Recall the definition

$$\mathcal{P}_k(\Omega) = \left\{ p_{K,m} : K \subset \Omega,\ K \text{ compact},\ 0 \leq m \leq k \right\}.$$

   Note that $\mathcal{C}^k(\Omega)[\mathcal{P}_k(\Omega)]$ is equipped with the **topology of uniform convergence of all derivatives of order $\leq k$ on all compact subsets of Ω**.

Compared to a general topological vector space one has a fairly explicit description of the topology in a locally convex topological vector space. Here, as we have learned, each point has a neighborhood basis consisting of open balls, and thus formulating the definition of continuity one can completely rely on these open balls. This then has an immediate translation into conditions involving only the systems of seminorms which define the topology. Suppose that $X[\mathcal{P}]$ and $Y[\mathcal{Q}]$ are two LCTVS. Then a function $f : X \to Y$ is said to be continuous at $x_0 \in X$ if, and only if, for every open q-ball $B_{q,R}(f(x_0))$ in $Y[\mathcal{Q}]$ there is an open p-ball $B_{p,r}(x)$ in $X[\mathcal{P}]$ which is mapped by f into $B_{q,R}(f(x_0))$. This can also be expressed as follows:

Definition 2.6 Assume that $X[\mathcal{P}]$ and $Y[\mathcal{Q}]$ are two LCTVS. A function $f : X \to Y$ is said to be **continuous at** $x_0 \in X$ if, and only if, for every seminorm $q \in \mathcal{Q}$ and every $R > 0$ there are $p \in \mathcal{P}$ and $r > 0$ such that for all $x \in X$ the condition $p(x - x_0) < r$ implies $q(f(x) - f(x_0)) < R$. f is called **continuous on** X if, and only if, f is continuous at every point $x_0 \in X$.

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