Chapter 2
Fixed Point Theory in Metric Spaces: An Introduction

This chapter is primarily intended to serve as an introduction to metric fixed point theory. It will set the foundation for the coming chapters. In terms of content this chapter overlaps in places with the following popular books on fixed point theory by A. Aksoy and M. A. Khamsi [5], by K. Goebel and W. A. Kirk [81], by J. Dugundji and A. Granas, [67], by M. A. Khamsi and W. A. Kirk [111], by I.A. Rus, A. Petrusel, and G. Petrusel [186], and by E. Zeidler [203]. Material on the general theory of Banach space geometry is drawn from many sources but the following books by B. Beauzamy [16], and by J. Diestel [59] are worth a special mention.

2.1 Banach Contraction Principle

In 1922, Banach [14] published his fixed point theorem, also known as *the Banach Contraction Principle*, which uses the concept of Lipschitz mappings .

Definition 2.1. Let (M,d) be a metric space. The map $T : M \to M$ is said to be *Lipschitzian* if there exists a constant $k > 0$ (called *Lipschitz constant*) such that

$$d\Big(T(x),T(y)\Big) \leq k\, d(x,y)$$

for all $x,y \in M$. A Lipschitzian mapping with a Lipschitz constant $k < 1$, is called *contraction* . Moreover if the Lipschitz constant $k = 1$, the Lipschitzian mapping is called *nonexpansive* . Observe that all contractions and nonexpansive mappings are uniformly continuous.

Theorem 2.1 (Banach Contraction Principle). *Let (M,d) be a complete metric space and let $T : M \to M$ be a contraction mapping, with Lipschitz constant $k < 1$. Then T has a unique fixed point ω in M, and for each $x \in M$, we have*

$$\lim_{n\to\infty} T^n(x) = \omega.$$

M. A. Khamsi, W. Kozlowski, *Fixed Point Theory in Modular Function Spaces*,
DOI 10.1007/978-3-319-14051-3_2

Moreover, for each $x \in M$, we have

$$d\left(T^n(x), \omega\right) \leq \frac{k^n}{1-k} d\left(T(x), x\right).$$

Proof. Since T is a contraction mapping we know that for each $x \in M$ and $n \geq 1$,

$$d(T^n(x), T^{n+1}(x)) \leq k\, d(T^{n-1}(x), T^n(x)) \leq k^n\, d(x, T(x)).$$

Let $n, m \in \mathbb{N}$, with $n \geq 1$, we get

$$d(T^n(x), T^{n+m}(x)) \leq \sum_{i=n}^{n+m} d(T^i(x), T^{i+1}(x)) \leq \sum_{i=n}^{n+m} k^i\, d(x, T(x)),$$

for any $x \in M$. Since $k < 1$, we get

$$\sum_{i=n}^{n+m} k^i = k^n \frac{1-k^{m+1}}{1-k} \leq \frac{k^n}{1-k}.$$

Hence

$$d(T^n(x), T^{n+m}(x)) \leq \frac{k^n}{1-k} d(x, T(x)). \tag{2.1}$$

This forces $\{T^n(x)\}$ to be a Cauchy sequence. Since M is complete, then $\lim_{n\to\infty} T^n(x) = \omega_x$ exists in M. Note that a priori the limit point depends on the starting point x. Since T is continuous

$$\omega_x = \lim_{n\to\infty} T^n(x) = \lim_{n\to\infty} T^{n+1}(x) = T(\omega_x).$$

Thus ω_x is a fixed point of T, for any $x \in M$. Next we show that ω_x is independent of the starting point x. Let $y \in M$ and ω_y be the fixed point of T associated to y. Then

$$d(\omega_x, \omega_y) = d(T(\omega_x), T(\omega_y)) \leq k\, d(\omega_x, \omega_y).$$

Since $k < 1$, we get $d(\omega_x, \omega_y) = 0$, i.e., $\omega_x = \omega_y$. This shows that T has one fixed point ω and for any $x \in M$, we have $\lim_{n\to\infty} T^n(x) = \omega$. If we let $m \to \infty$ in the inequality (2.1), we get

$$d(T^n(x), \omega) \leq \frac{k^n}{1-k} d(x, T(x)),$$

for any $x \in M$ and $n \in \mathbb{N}$. This completes the proof of Theorem 2.1.
□

An easy implication of the Banach Contraction Principle is the following theorem.

Theorem 2.2. *Let (M, d) be a complete metric space. Suppose $T : M \to M$ is a mapping for which T^N is a contraction mapping, for some positive integer $N \geq 1$. Then T has a unique fixed point.*

Proof. Since T^N is a contraction, Theorem 2.1 implies the existence of a unique fixed point x_0 of T^N. Since

$$T^N(T(x_0)) = T^{N+1}(x_0) = T(T^N(x_0)) = T(x_0),$$

then $T(x_0)$ is also a fixed point of T^N. Since T^N has only one fixed point, we conclude that $T(x_0) = x_0$.
□

It is not clear in general whether T has a fixed point whenever T^N has a fixed point. Note that fixed points of T^N are also known as periodic points of T.

Remark 2.1. We all have learned that the origins of the metric contraction principle and, ergo, metric fixed point theory itself, rest in the method of successive approximations for proving existence and uniqueness of solutions of differential equations. This method is associated with the names of such celebrated nineteenth century mathematicians as Cauchy, Liouville, Lipschitz, Peano, and specially Picard. In fact the iterative process used in the proof of the Banach Contraction Theorem bears the name of Picard iterates. It is quite interesting to know that in 1429, Al-Kashani [23] published a book entitled: "The Calculator's Key", where he used Picard iterates. In fact, Al-Kashani set the stage for the so-called numerical techniques some 600 years ago. He was keen to develop ideas with practical matters, like the approximate values of $\sin(1^o)$ which enabled scientists after him to come up with very good approximations to the circumference of the Earth.

2.2 Pointwise Lipschitzian Mappings

The notion of pointwise Lipschitzian mappings was introduced in [125, 127, 128]. The main motivation behind this generalization is to find a larger class of mappings for which the conclusion of Banach's celebrated contraction mapping theorem is still valid. Pointwise Lipschitzian mappings are defined as follows.

Definition 2.2. Let (M,d) be a metric space. A mapping $T : M \to M$ is called a pointwise Lipschitzian mapping if there exists a mapping $\alpha : M \to [0,+\infty)$ such that

$$d(T(x),T(y)) \leq \alpha(x)\, d(x,y),$$

for any $x,y \in M$. The map T is called pointwise contraction if $\alpha(x) < 1$, for any $x \in M$.

Note that if T is a pointwise contraction, then it is continuous although not necessarily uniformly continuous. Moreover, if $\alpha(x) = 0$ for some $x \in M$, then T is a constant map. It is easy to prove that a pointwise contraction $T : M \to M$ has at most one fixed point, and if x_0 is its fixed point, then the orbit $\{T^n(x)\}$ converges to x_0, for each $x \in M$. Indeed, we have

$$d(x_0,T^n(x)) \leq \alpha(x_0)^n d(x_0,x)$$

for any $x \in M$. The above conclusion follows because $\alpha(x_0) < 1$. It is not clear how to prove the existence of the fixed point from the convergence of the orbits which is the case in the classical proof given to the Banach Contraction Principle, Theorem 2.1.

In order to extend the main conclusion of [128] to metric spaces, the authors in [96] needed the following definition.

Definition 2.3. Let (M,d) be a metric space. A subset C of M is *admissible* if it is a nonempty intersection of closed balls. The family of all admissible subsets of M is denoted by $\mathscr{A}(M)$. We will say that $\mathscr{A}(M)$ is compact if any family $(A_\alpha)_{\alpha\in\Gamma}$ of elements of $\mathscr{A}(M)$, has a nonempty intersection provided $\bigcap\limits_{\alpha\in F} A_\alpha \neq \emptyset$ for any finite subset $F \subset \Gamma$. Moreover for any nonempty subset A of M, the cover of A is defined by

$$\mathrm{cov}(A) = \bigcap \{B : B \text{ is a closed ball and } A \subset B\}.$$

The admissible subsets were introduced in metric fixed point theory because of their similarities with convex subsets in linear vector spaces.

Banach Contraction Principle for pointwise contraction mappings may be stated as:

Theorem 2.3. *Let M be a bounded metric space. Assume that the convexity structure $\mathscr{A}(M)$ is compact. Let $T : M \to M$ be a pointwise contraction. Then T has a unique fixed point x_0. Moreover the orbit $\{T^n(x)\}$ converges to x_0, for each $x \in M$.*

Proof. Since $\mathscr{A}(M)$ is compact, there exists a minimal nonempty $K \in \mathscr{A}(M)$ such that $T(K) \subset K$. It is easy to check that $\mathrm{cov}(T(K)) = K$. Let $a \in K$, then we have $K \subset B(a, r_a(K))$, where

$$r_a(K) = \sup\{d(a,x) : x \in K\}.$$

Since T is a pointwise contraction, there exists a mapping $\alpha : M \to [0,1)$ such that

$$d(T(x),T(y)) \leq \alpha(x)d(x,y) \ \text{ for any } x,y \in M.$$

In particular, we have then $T(K) \subset B(T(a), \alpha(a)r_a(K))$, which implies

$$K = \mathrm{cov}(T(K)) \subset B(T(a), \alpha(a)r_a(K)).$$

So $r_{T(a)}(K) \leq \alpha(a)r_a(K)$. This will force $\mathrm{diam}(K) = 0$. Indeed let $a \in K$ and define

$$K_a = \{x \in K : r_x(K) \leq r_a(K)\}.$$

Clearly K_a is not empty. Moreover, we have

$$K_a = \bigcap_{x\in K} B(x, r_a(K)) \cap K \in \mathscr{A}(M).$$

And since $r_{T(x)}(K) \leq \alpha(x) r_x(K)$, for any $x \in K$, we get $T(K_a) \subset K_a$. The minimality behavior of K implies $K_a = K$. In particular we have $r_x(K) = r_a(K)$ for any $x \in K$. Hence $\mathrm{diam}(K) = r_a(K)$, for any $a \in K$, i.e., a is a diametral point of K. Hence $\mathrm{diam}(K) \leq \alpha(a)\mathrm{diam}(K)$. And since $\alpha(a) < 1$, we get $\mathrm{diam}(K) = 0$, i.e., K is reduced to one point which is fixed by T. The remaining conclusion of the theorem follows from the general properties of pointwise contractions.
□

As we did in the case of Banach's Contraction Principle, we extend Theorem 2.3 to the case when iterates are pointwise contractions. First we need the following definition.

Definition 2.4. Let (M,d) be a metric space. A mapping $T : M \to M$ is called an *asymptotic pointwise mapping* if there exists a sequence of mappings $\alpha_n : M \to [0,\infty)$ such that

$$d(T^n(x), T^n(y)) \leq \alpha_n(x) d(x,y), \text{ for any } y \in M.$$

(i) If $\{\alpha_n\}$ converges pointwise to $\alpha : M \to [0,1)$, then T is called an *asymptotic pointwise contraction* .
(ii) If $\limsup\limits_{n\to\infty} \alpha_n(x) \leq 1$, then T is called *asymptotic pointwise nonexpansive* .
(ii) If $\limsup\limits_{n\to\infty} \alpha_n(x) \leq k$, with $0 < k < 1$, then T is called a *strongly asymptotic pointwise contraction* .

Let (M,d) be a metric space. We will say that a function $\Phi : M \to [0,\infty)$ is *convex* if $\{x : \Phi(x) \leq r\} \in \mathscr{A}(M)$, for any $r \geq 0$. Also we define a *type* to be a function $\Phi : M \to [0,\infty)$ defined as

$$\Phi(u) = \limsup_{n\to\infty} d(x_n, u)$$

where $\{x_n\}$ is a bounded sequence in M. Types are very useful in the study of the geometry of Banach spaces and the existence of fixed point of mappings. We will say that $\mathscr{A}(M)$ is *T-stable* if types are convex. We have the following lemma.

Lemma 2.1. *Let M be a metric space. Assume that $\mathscr{A}(M)$ is compact and T-stable. Then for any type Φ, there exists $x_0 \in M$ such that*

$$\Phi(x_0) = \inf\{\Phi(x) : x \in M\}.$$

Proof. Let Φ be a type. Then there exists a bounded sequence $\{x_n\}$ in M such that $\Phi(x) = \limsup\limits_{n\to\infty} d(x_n, x)$, for any $x \in M$. Set $\Phi_0 = \inf\limits_{x\in M} \Phi(x)$. Then the set

$$M_n = \left\{ x \in M : \Phi(x) \leq \Phi_0 + \frac{1}{n} \right\},$$

is a nonempty admissible subset of M because $\mathscr{A}(M)$ is convex. Since $\{M_n\}$ is a decreasing sequence and $\mathscr{A}(M)$ is compact, then $M_\infty = \bigcap_{n\geq 1} M_n$ is not empty. Note that for any $x \in M_\infty$ we have $\Phi(x) = \Phi_0$.
□

We have the following fixed point theorem for strongly asymptotic pointwise contractions in metric spaces.

Theorem 2.4. *Let M be a fixed bounded metric space. Assume that $\mathscr{A}(M)$ is compact. Let $T : M \to M$ be a strongly asymptotic pointwise contraction. Then T has a unique fixed point x_0. Moreover any orbit $\{T^n(x)\}$ converges to x_0.*

Proof. First note that T has, at most, one fixed point. Indeed, let $a,b \in M$ be two fixed points of T. Then we have

$$d(a,b) = d(T^n(a), T^n(b)) \leq \alpha_n(a)\, d(a,b).$$

If we let n go to infinity, we get $d(a,b) \leq k\, d(a,b)$ for some $k \in (0,1)$. This will force $d(a,b) = 0$. Let us fix $x \in M$ and define the type

$$\Phi(u) = \limsup_{n\to\infty}\, d(T^n(x), u), \text{ for } u \in M.$$

Since $\mathscr{A}(M)$ is compact, then

$$\Omega(x) = \bigcap_{n\geq 1} \mathrm{cov}\Big(\{T^k(x) : k \geq n\}\Big) \neq \emptyset.$$

Let $\omega \in \Omega(x)$. Then for every $n,m \in \mathbb{N}$ we have

$$d(T^{m+n+h}(x), T^{m+h}(x)) \leq \alpha_h(T^m(x)) d(T^n(x), T^m(x)).$$

If we let n go to infinity, we get

$$\Phi(T^{m+h}(x)) \leq \alpha_h(T^m(x)) \Phi(T^m(x)).$$

Next we let h go to infinity to get

$$\limsup_{n\to\infty}\, \Phi(T^n(x)) \leq k\, \Phi(T^n(x))$$

for some $k \in (0,1)$, which easily implies that $\limsup_{n\to\infty}\, \Phi(T^n(x)) = 0$. Notice that

$$\Phi(\omega) \leq \limsup_{n\to\infty}\, \Phi(T^n(x)) = 0.$$

Indeed let $u \in M$, then for any $\varepsilon > 0$, then there exists $n_0 \geq 1$ such that for any $n \geq n_0$

$$d(T^n(x), u) \leq \Phi(u) + \varepsilon.$$

In particular we have $T^n(x) \in B(u, \Phi(u)+\varepsilon)$, for any $n \geq n_0$. So

$$\Omega(x) \subset \mathrm{cov}\Big(\{T^n(x) : n \geq n_0\}\Big) \subset B(u, \Phi(u)+\varepsilon),$$

which implies $\omega \in B(u, \Phi(u)+\varepsilon)$. This is true for any $\varepsilon > 0$. Hence for any $u \in M$ we have $d(\omega, u) \leq \Phi(u)$, which implies that

$$\Phi(\omega) = \limsup_{n\to\infty}\ d(T^n(x), \omega) \leq \limsup_{n\to\infty}\ \Phi(T^n(x)).$$

Therefore we have $\Phi(\omega) = 0$ which implies that $\{T^n(x)\}$ converges to ω. Since T is continuous, then ω is a fixed point of T. Since T has at most one fixed point, then the limit of the orbit $\{T^n(x)\}$ is independent of $x \in M$. Therefore T has a unique fixed point x_0 and any orbit converges to x_0.
□

Let us relax now the strong behavior of T but assume that types are convex to obtain the following result.

Theorem 2.5. *Let M be a bounded metric space. Assume that $\mathscr{A}(M)$ is compact and T-stable. Let $T : M \to M$ be an asymptotic pointwise contraction. Then T has a unique fixed point x_0. Moreover the orbit $\{T^n(x)\}$ converges to x_0, for each $x \in M$.*

Proof. As we did in the beginning of the proof of Theorem 2.4, we can easily show that T has at most one fixed point. Let $x \in M$ and define the type

$$\Phi(u) = \limsup_{n\to\infty}\ d(T^n(x), u), \ \text{ for each } u \in M.$$

Since $\mathscr{A}(M)$ is compact and T-stable, then by Lemma 2.1 there exists $x_0 \in M$ such that

$$\Phi(x_0) = \inf\,\{\Phi(u) : u \in M\}.$$

Let us show that $\Phi(x_0) = 0$. Indeed we have

$$d(T^{n+m}(x), T^m(x_0)) \leq \alpha_m(x_0)\ d(T^n(x), x_0),$$

for any $n, m \geq 1$. If we let n go to infinity, we get

$$\Phi(T^m(x_0)) \leq \alpha_m(x_0)\ \Phi(x_0),$$

which implies

$$\Phi(x_0) = \inf\,\{\Phi(u) : u \in M\} \leq \Phi(T^m(x_0)) \leq \alpha_m(x_0)\ \Phi(x_0).$$

If we let m go to infinity, we get $\Phi(x_0) \leq \alpha(x_0)\Phi(x_0)$. Since $\alpha(x_0) < 1$, we get $\Phi(x_0) = 0$, which implies that $\{T^n(x)\}$ converges to x_0. This will force x_0 to be a fixed point of T. Since we already noticed that T has at most one fixed point, then T has a fixed point x_0 and any orbit converges to x_0.
□

2.3 Caristi-Ekeland Extension

This is one of the most interesting extensions of the Banach Contraction Principle. In order to understand its context, let us first go over the proof of Theorem 2.1 given by Caristi [40, 41] . Let $T : M \to M$ be a contraction with Lipschitz constant $k < 1$. Then we have

$$d(T(x), T^2(x)) \leq k\, d(x, T(x)),$$

for any $x \in M$. Adding $d(x, T(x))$ to both sides of the above inequality yields

$$d(x, T(x)) + d(T(x), T^2(x)) \leq d(x, T(x)) + k\, d(x, T(x))$$

which can be rewritten

$$d(x, T(x)) - k\, d(x, T(x)) \leq d(x, T(x)) - d(T(x), T^2(x)).$$

This in turn is equivalent to

$$d(x, T(x)) \leq \frac{1}{1-k}\Big[d(x, T(x)) - d(T(x), T^2(x))\Big],$$

for any $x \in M$. Now define the function $\varphi : M \to \mathbb{R}^+$ by setting

$$\varphi(x) = \frac{1}{1-k} d(x, T(x)).$$

This gives us the basic inequality

$$d(x, T(x)) \leq \varphi(x) - \varphi(T(x)),$$

for any $x \in M$. As a generalization to contraction mappings, Caristi [41] and Ekeland [73] considered mappings $T : M \to M$ which satisfy the following property

$$d\,(x, T(x)) \leq \varphi(x) - \varphi(T(x)), \quad x \in M,$$

where $\varphi : M \to \mathbb{R}^+ = [0, +\infty)$. Both Caristi and Ekeland investigated this new class of mappings to find out when a fixed point exists. Recall that the function φ is said to be *lower semicontinuous(l.s.c.)*, if for any sequence $\{x_n\} \subset M$, if $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} \varphi(x_n) = r$, then $\varphi(x) \leq r$.

Theorem 2.6 (Ekeland Variational Principle, 1974). *Let (M, d) be a complete metric space and $\varphi : M \to \mathbb{R}^+$ l.s.c. Define the Brønsted partial order :*

$$x \preceq y \Leftrightarrow d(x, y) \leq \varphi(x) - \varphi(y),\ x, y \in M.$$

Then $(M, \preceq)$ has a maximal element.

Theorem 2.7 (Caristi Fixed Point Theorem, 1975). *Let (M,d) be a complete metric space and $\varphi : M \to \mathbb{R}^+$ l.s.c. Suppose $T : M \to M$ satisfies:*

$$d(x,T(x)) \leq \varphi(x) - \varphi(T(x)),\ x \in M.$$

Then T has a fixed point.

Although both theorems have different settings, in fact they are equivalent, [171]. The proof of Caristi-Ekeland's theorems is based on the discrete technique: Zorn's lemma and Axiom of Choice. There are some attempts to find a pure metric proof of Caristi's fixed point theorem, without success so far.

2.4 Some Applications

There are many known examples of the Banach Contraction Principle. Here, we will discuss only two of them.

2.4.1 ODE and Integral Equations

Consider the integral equation

$$f(x) = \lambda \int_a^x K(x,t) f(t) dt + \phi(x)$$

for a fixed real number λ, where $K(x,t)$ is continuous on $[a,b] \times [a,b]$. Consider the metric space $\mathscr{C}[a,b]$ of continuous real-valued functions defined on $[a,b]$. Consider the map $T : \mathscr{C}[a,b] \to \mathscr{C}[a,b]$ defined by

$$T(f)(x) = \lambda \int_a^x K(x,t) f(t) dt + \phi(x).$$

For $f_1, f_2 \in \mathscr{C}[a,b]$, we have

$$d\left(T^n(f_1), T^n(f_2)\right) \leq |\lambda|^n M^n \frac{(b-a)^n}{n!} d(f_1,f_2)$$

where

$$d(f_1,f_2) = \max\{|f_1(x) - f_2(x)| :\ x \in [a,b]\},$$

and

$$M = \max\left\{|K(x,t)| :\ (x,t) \in [a,b] \times [a,b]\right\}.$$

Clearly there exists $n \geq 1$ such that T^n is a contraction, which implies that the above integral equation has a unique solution $f(x)$. In general, the map T may not be a contraction on $[a,b]$. Bielecki [26] discovered another way to find a remedy to this problem. Indeed, for $\lambda > 0$, set

$$\|f\|_\lambda = \max_{a \le x \le b} \left\{ e^{-\lambda\,(x-a)} |f(x)| \right\},$$

it is now possible to prove that for any $f_1, f_2 \in \mathscr{C}[a,b]$, we have

$$d_\lambda\left(T(f_1), T(f_2)\right) = \|T(f_1) - T(f_2)\|_\lambda \le \frac{M}{\lambda}\, \|f_1 - f_2\|_\lambda = \frac{M}{\lambda}\, d_\lambda(f_1, f_2),$$

where $M = \max_{a \le x,y \le b} |K(x,y)|$ is as before. It is then clear that for λ sufficiently large T is a contraction for the new distance d_λ.

2.4.2 Cantor and Fractal sets

Let (M,d) be a complete metric space and let $\mathscr{M}$ denote the family of all nonempty bounded closed subsets of M, and let $\mathscr{C}$ denote the subfamily of $\mathscr{M}$ consisting of all compact sets. For $A \in \mathscr{M}$ and $\varepsilon > 0$ define the ε-neighborhood of A to be the set

$$N_\varepsilon(A) = \{x \in M : dist(x,A) < \varepsilon\}$$

where $dist(x,A) = \inf_{y \in A} d(x,y)$. Now for $A, B \in \mathscr{M}$, set

$$H(A,B) = \inf\{\varepsilon > 0 : A \subseteq N_\varepsilon(B) \text{ and } B \subseteq N_\varepsilon(A)\}.$$

Then $(\mathscr{M}, H)$ is a metric space, and H is called the *Hausdorff metric* on $\mathscr{M}$. Notice that if (M,d) is complete, then $(\mathscr{M}, H)$ is also complete. Let $T_i : M \to M$, $i = 1, \ldots, n$, be a family of contractions. Define the map $T^* : \mathscr{C} \to \mathscr{C}$ by

$$T^*(X) = \bigcup_{i=1}^{n} T_i(X).$$

Then T^* is a contraction and its Lipschitz constant is smaller than the maximum of all Lipschitz constants of the mappings T_i, $i = 1, \ldots, n$. Then Banach Contraction Principle implies the existence of a unique nonempty compact subset X of M such that

$$X = \bigcup_{i=1}^{n} T_i(X).$$

As an application of this, consider the real interval $[0,1]$ and the two contractions

$$T_1(x) = \frac{1}{3}x \text{ and } T_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

http://www.springer.com/978-3-319-14050-6

Fixed Point Theory in Modular Function Spaces
Khamsi, M.A.; Kozlowski, W.M.
2015, X, 245 p., Hardcover
ISBN: 978-3-319-14050-6
A product of Birkhäuser Basel