Foreword

Because of its many diverse applications, fixed point theory has been a flourishing area of mathematical research for decades. S. Banach's formulation of the contraction mapping principle in the early twentieth century signaled the advent of an intense interest in "metric" related aspects of the theory. The metric theory, often cast in a Banach space framework, has usually involved an intertwining of geometric and topological conditions. Fixed point theory in modular function spaces is closely related to the metric theory, in that it provides modular equivalents of norm and metric concepts. Modular spaces are extensions of the classical Lebesgue and Orlicz spaces, and in many instances conditions cast in this framework are more natural and more easily verified than their metric analogs.

This book is devoted to a comprehensive treatment of what is currently known about fixed point theory in modular function spaces. A unified treatment of this subject was initiated in 1988 with the appearance of W. Kozlowski's Marcel Dekker monograph *Modular Function Spaces*. Since the appearance of Kozlowski's monograph, there have been numerous further developments, both in the theory of modular function spaces and in metric fixed point theory. This book takes full account of these developments. It covers the foundations of the theory, taking as a point of departure a review of the central themes of metric fixed point theory and the basic structures of modular function spaces. Anyone interested in these topics would want to have this book readily at hand. The book is essentially self-contained and should be easily accessible to students. Also it provides a valuable resource for those already involved in this avenue of research.

Iowa City, May, 2014 *W. A. Kirk*

Preface

The single valued fixed point problem, which forms the basis of the fixed point theory, may be stated as :

Let X be a set, A and B two nonempty subsets of X such that $A \cap B \neq \emptyset$, and $f : A \to B$ be a map. When does a point $x \in A$ such that $f(x) = x$, also called a fixed point of f, exist? And if yes, how many such points exist and how to find them?

From the perspective of different settings, methods and applications, the fixed point theory is typically divided into the three major areas:

- Topological fixed point theory
- Metric fixed point theory
- Discrete fixed point theory

Historically, the boundary lines between the three areas were defined by the discovery of three major theorems:

- Brouwer's fixed point theorem [31]
- Banach's fixed point theorem [14]
- Tarski's fixed point theorem [198]

In this book, we will focus mainly on the second area, although from time to time we may touch the other areas as well. Conceptually, the metric fixed point theory deals with situations where the set X is equipped with some kind of a method allowing to assign to every two elements of X a numeric value that measures the level of difference between them; in other words—a distance between them. We need to keep in mind that this idea of a "distance" can take many forms and does have not to be a distance in a popular meaning of the word, or a metric in the strict mathematical sense. Such a situation is very common when dealing with any quantifiable events in science and technology. Many problems like solving equations or finding stationary points of a time dependent system can be actually reduced to finding fixed points of a suitably defined mapping acting within a suitably selected set equipped with a suitably selected "distance."

The authors would like to invite the reader to join them in a journey taking them from a well-known base of classical fixed point theory in Banach and metric spaces (and yes, with Banach's fixed point theorem as the starting point) to the world of the theory of fixed points of mappings defined in a class of spaces of measurable functions, known in the literature as modular function spaces.

The results and methods of fixed point theory, applied to spaces of measurable functions, have been used extensively in the field of integral and differential equations. Since the 1930s many prominent mathematicians like Orlicz and Birnbaum recognized that using the methods of L^p-norms alone created many complications and in some cases did not allow to solve some nonpower type integral equations. They considered spaces of functions with some growth properties different from the power type growth control provided by the L^p-norms. The possibility of introducing the structure of a linear metric as well as the interesting properties of these spaces, later named Orlicz spaces, and many applications to differential and integral equations with kernels of nonpower types were among the reasons for the development of the theory of Orlicz spaces, their applications and generalizations. Using the apparatus of the modular function spaces we can go much further: the operator itself is used for the construction of a function modular and hence of a space in which this operator has required properties. These techniques together with relevant modular function space fixed point theorems can be efficiently applied to solving many mathematical problems. The aim of this book is to familiarize the readers with the main concepts and results of the fixed point theory in modular function spaces, as well as to encourage them to use these results in the course of their research activities.

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