Chapter 2
Algebras of Fuzzy Sets

2.1 Introduction

From now on it will be only considered the case in which $(L, \leqslant) = ([0, 1], \leqslant)$, that is, of Zadeh's fuzzy sets, with predicates P in X known through a degree $\mu_P : X \to [0, 1]$, and without knowing, necessarily, its primary use $\leqslant_P$. The set of all fuzzy sets in X, $[0, 1]^X$, will be also denoted by $F(X)$. In this case, the preorder $\leqslant_{\mu_P}$ is linear, or total, since for all x, y in X it is either $\mu_P(x) \leqslant \mu_P(y)$, or $\mu_P(y) \leqslant \mu_P(x)$, that is, it is either $x \leqslant_{\mu_P} y$ or $y \leqslant_{\mu_P} x$ for all x, y in X. Hence, $\leqslant_{\mu_P}$ rarely will perfectly reflect the primary use of P in X, since $\leqslant_P$ is usually not linear.

In the case in which X is finite, $X = \{x_1, \dots, x_n\}$, the fuzzy sets $\mu \in [0, 1]^X$, will be represented by

$$\mu = \mu(x_1)/x_1 + \mu(x_2)/x_2 + \cdots + \mu(x_n)/x_n,$$

with the convention that if some term $\mu(x_j)/x_j$ does not appear, is that it is $\mu(x_j) = 0$. For example, with $X = \{1, 2, 3, 4\}$, the expression

$$\mu = 0.5/x_1 + 0.7/x_2 + 1/x_4,$$

refers to the fuzzy set in X given by $\mu(x_1) = 0.5, \mu(x_2) = 0.7, \mu(x_3) = 0, \mu(x_4) = 1$. Analogously, the fuzzy set $\mu' = N_0 \circ \mu$ ($N_0 = 1 - \text{id}$) is

$$\mu' = 0.5/x_1 + 0.3/x_2 + 1/x_3,$$

2.1.1 Cartesian Product

If A, B are crisp subsets in X and Y, respectively, that is, $A \in \mathbb{P}(X)$ and $B \in \mathbb{P}(X)$, its cartesian product $A \times B = \{(a, b); a \in A, b \in B\} \subset X \times Y$, is with the

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membership function $\mu_{A\times B} : X \times Y \to \{0, 1\}$, given by

$$\mu_{A\times B}(x, y) = \min(\mu_A(x), \mu_B(y)$$

for all $x \in X$, $y \in Y$. It is $\mu_{A\times B}(x, y) = 1 \Leftrightarrow \mu_A(x) = \mu_B(y) = 1$.

In the same vein, if $\mu \in F(X)$, and $\sigma \in F(Y)$, the cartesian product $\mu \times \sigma$ is defined by directly generalizing the classical case:

$$\mu \times \sigma = \min \circ (\mu, \sigma).$$

For example, with $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$, and

$$\mu = 1/x_1 + 0.8/x_2,\ \ \sigma = 0.9/y_1 + 0.7/y_2,$$

it is

$$\mu \times \sigma = 0.9/(x_1, y_1) + 0.7/(x_1, y_2) + 0.8/(x_2, y_1) + 0.7/(x_2, y_2),$$

with $(\mu \times \sigma)(x_3, y_1) = (\mu \times \sigma)(x_3, y_2) = 0$.

With $\mu_{big}(x) = \frac{x}{5}$ if $x \in [0, 5]$ and $\mu_{small}(y) = 1 - \frac{y}{7}$ if $y \in [0, 7]$, it is

$$(\mu_{big} \times \mu_{small})(x, y) = \min(\frac{x}{5}, 1 - \frac{y}{7}),$$

the representation of the cartesian product as a surface contained in the cube $[0, 5] \times [0, 7] \times [0, 1]$.

Of course, if $\mu = \mu_A \in \{0, 1\}^X$, $\sigma = \mu_B \in \{0, 1\}^X$, it is not only $\mu \times \sigma \in \{0, 1\}^X$ but $\mu \times \sigma = \mu_{A\times B}$.

2.1.2 Extension Principle

If $f : X \to Y$ is a mapping and A is a crisp subset of X, $A \subset X$, it is $f(A) = \{y \in Y; f(a) = y, a \in A\}$ the f-image of A in Y. Notice that

$$\mu_{f(A)}(y) = \sup\{\mu_A(x); f(x) = y\} = \begin{cases} 1, & \text{if it exists } x \in A \text{ such that } f(x) = y, \\ 0, & \text{otherwise.} \end{cases}$$

With these f-image, the mapping $f : X \to Y$ is extended to the respective power sets by

$$\widehat{f} : \mathbb{P}(X) \to \mathbb{P}(Y), \quad A \mapsto f(A).$$

In the same vein, given a mapping $f : X \to Y$, it can be extended to the fuzzy power sets $F(X)$, $F(Y)$ by

$$\widehat{f} : F(X) \to F(Y)$$

$$\widehat{f}(\mu)(y) = \sup\{\mu(x); f(x) = y\}, \quad \text{for all } y \in Y,$$

and $\widehat{f}$ is known as the 'extension' of f to the fuzzy parts, and the definition as the Zadeh's *Extension Principle*.

For example, if $f : [0, 10] \to [0, 1]$, is given by $f(x) = 1 - \frac{x}{10}$, the fuzzy set $\mu(x) = \frac{x}{10}$ in $[0, 1]^{[0,10]}$ extends to the fuzzy set in $[0, 1]$, $\widehat{f}(\mu)(y) = \sup\{\mu(x); f(x) = y\} = \sup\{\frac{x}{10}; 1 - \frac{x}{10} = y\} = 1 - y$, for all $y \in [0, 1]$.

If $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c\}$, the mapping $f : X \to Y$ such that

$$f(1) = f(2) = a, \;\; f(3) = f(4) = b,$$

extends the fuzzy set $\mu = 1/1 + 0.4/2 + 1/3 + 0.7/4$ in $F(X)$, to the fuzzy set $\widehat{f}(\mu)$ in $F(Y)$ with values,

$$\begin{aligned}
&\widehat{f}(\mu)(a) = \max\{\mu(x); x \in f^{-1}(a)\} = \max\{\mu(1), \mu(2)\} = \max(1, 0.4) = 1\\
&\widehat{f}(\mu)(b) = \max\{\mu(3), \mu(4)\} = 1\\
&\widehat{f}(\mu)(c) = 0, \text{ since } f^{-1}(c) = \varnothing.
\end{aligned}$$

Hence,

$$\widehat{f}(\mu) = 1/a + 1/b,$$

that corresponds to the crisp subset $\{a, b\}$ of Y.

Notice that if $\mu = \mu_A \in \{0, 1\}^X$, it is $\widehat{f}(\mu_A) = \mu_{f(A)}$, that is not only a crisp subset of Y, but coincides with the classical extension $f(A)$ of A. Nevertheless, as it is shown by the above example, it can happen that $\widehat{f}(\mu) \in \mathbb{P}(Y)$ with $\mu \in F(X) - \mathbb{P}(X)$.

2.1.3 Preservation of the Classical Case

Like with the cartesian product and with the extension principle, all operations with fuzzy sets must reproduce, when the data are crisp, the corresponding result obtained in the crisp theory. This is the *principle of preservation* of the classical case, that is forced by the will, and the necessity, of including all 'the classical' as a particular case of the algebras of fuzzy sets.

To illustrate this preservation's principle, let us show a negative example. With $X = [0, 1]$, and all $\mu \in [0, 1]^{[0,1]}$, the function

$$\mu^*(x) = 1 - \mu(1 - x),$$

verifies:

- $\mu_0^*(x) = 1 - \mu_0(1-x) = 1$: $\mu_0^* = \mu_1$
- $\mu_1^*(x) = 1 - \mu_1(1-x) = 1 - 1 = 0$: $\mu_1^* = \mu_0$
- $\mu \leqslant \sigma \Rightarrow 1 - \sigma(1-x) \leqslant 1 - \mu(1-x) \Rightarrow \sigma^* \leqslant \mu^*$
- $\mu^{**}(x) = 1 - \mu^*(1-x) = 1 - [1 - \mu(x)] = \mu(x)$: $\mu^{**} = \mu$.

Hence, *it could seem* that the function $\mu \mapsto \mu^*$ can be taken as a "strong negation" for the fuzzy sets in [0, 1], but it is not the case. Notice that if $\mu \in \{0, 1\}^{[0,1]}$, then it should be also $\mu^* \in \{0, 1\}^{[0,1]}$, that is, if μ represents a classical subset of [0, 1], also μ^* should represent not only a classical subset but precisely the complement of μ. But with $A = [0, \frac{1}{2}] \subset X$,

$$\mu_A(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 0.5, \\ 0, & 0.5 < x \leqslant 1, \end{cases}$$

follows,

$$\mu_A^*(x) = 1 - \mu_A(1-x) = 1 - \begin{cases} 1, & 0.5 < x \leqslant 1, \\ 0, & 0 \leqslant x \leqslant 0.5, \end{cases} = \begin{cases} 0, & 0.5 < x \leqslant 1, \\ 1, & 0 \leqslant x \leqslant 0.5 \end{cases}$$

that represents the subset $[0, 0.5]$, but not $A^c = (0.5, 1]$. *The unary operation* $*$ *violates the preservation principle*, and hence it cannot be taken into account to negate fuzzy sets.

2.1.4 Resolution

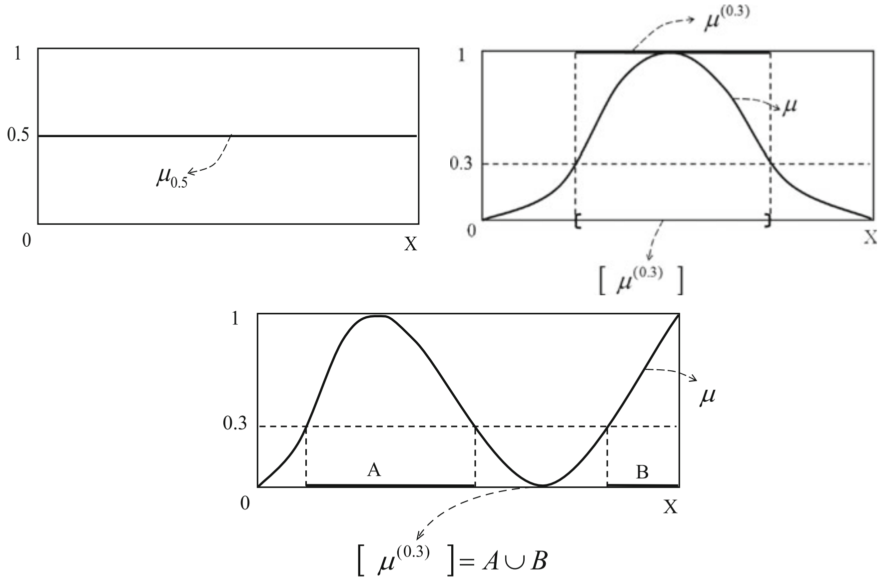
Let us denote by μ_r the constant fuzzy sets in $[0, 1]^X$, $\mu_r(x) = r$, for $r \in [0, 1]$ and all $x \in X$. Notice that in $\{0, 1\}^X$ there are only the "constants" μ_0 and μ_1, that correspond to the sets $\varnothing$ and X, respectively.

Given $\mu \in [0, 1]^X$, let us denote by $\mu^{(r)}$ the fuzzy (crisp) set

$$\mu^{(r)}(x) = \begin{cases} 1, & \text{if } r \leqslant \mu(x), \\ 0, & \text{otherwise}, \end{cases}$$

for all $r \in [0, 1]$, and by $[\mu^{(r)}]$ the corresponding classical subset $\{x \in X; r \leqslant \mu(x)\}$. These sets are called the *r-cuts* of μ and it is always $[\mu^{(0)}] = X$.

For example, in the following figures are shown, respectively, the constant fuzzy set $\mu_{0.5}$, and the 0.3-cut of two different fuzzy sets.



Notice that when $\mu_A \in \{0, 1\}^X$, it is

- $[\mu_A^{(0)}] = X$, since for all $x \in X$ it is $0 \leqslant \mu(x)$
- If $r > 0$, $[\mu_A^{(r)}] = A$, since for all $x \in A$ it is $0 < r \leqslant 1 = \mu_A(x)$,

then, *the only* r-cuts of a crisp subset A of X are X and A.

If $r \leqslant s$, since $s \leqslant \mu(x)$ implies $r \leqslant \mu(x)$, it results $[\mu^{(s)}] \subset [\mu^{(r)}]$: r-cuts are decreasing when their indices increase.

Let us show an example with $X = \{1, 2, 3, 4, 5\}$ and $\mu = 0.8/1 + 0.6/2 + 0.7/3 + 1/4 + 1/5$, where the only significative values for the r-cuts are 0.6, 0.7, 0.8, and 1:

- $[\mu^{(0.6)}] = \{1, 2, 3, 4, 5\} = X$
- $[\mu^{(0.7)}] = \{1, 3, 4, 5\}$
- $[\mu^{(0.8)}] = \{1, 4, 5\}$
- $[\mu^{(1)}] = \{4, 5\}$.

It is clear that $0.6 \leqslant 0.7 \leqslant 0.8 \leqslant 1$, and $[\mu^{(1)}] \subset [\mu^{(0.8)}] \subset [\mu^{(0.7)}] \subset [\mu^{(0.6)}]$.

Theorem 2.1.1 (Theorem of resolution) *For all* $\mu \in [0, 1]^X$, *is* $\mu(x) = \max\{r \in [0, 1]; \min(\mu_r(x), \mu^{(r)}(x)\}$, *for all* $x \in X$.

Proof

$$\max_{0 \leqslant r \leqslant 1} \min(\mu_r(x), \mu^{(r)}(x)) = \max_{0 \leqslant r \leqslant 1} \min(r, \mu^{(r)}(x))$$

$$= \max_{0 \leqslant r \leqslant 1} \min\left(r, \begin{cases} 1, & \text{if } r \leqslant \mu(x), \\ 0, & \text{otherwise}, \end{cases}\right) = \max_{0 \leqslant r \leqslant 1} \begin{cases} r, & \text{if } r \leqslant \mu(x), \\ 0, & \text{otherwise}, \end{cases} = \mu(x).$$

$\square$

Example 2.1.2 With X and μ in the last example, it is

$$\mu(x) = \max(\min(0.6, \mu^{(0.6)}(x)), \min(0.7, \mu^{(0.7)}(x)), \min(0.8, \mu^{(0.8)}(x)), \min(1, \mu^{(1)}(x))),$$

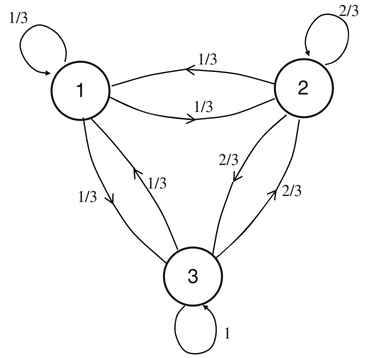
and, for instance,

$\mu(1) = \max(\min(0.6, 1), \min(0.7, 1), \min(0.8, 1), \min(1, 0)) = 0.8$
$\mu(4) = \max(\min(0.6, 1), \min(0.7, 1), \min(0.8, 1), \min(1, 1)) = 1$
etc.

Example 2.1.3 With $X = \{1, 2, 3\}$, take $\mu : X \times X \to [0, 1]$, given by $\mu(i, j) = \frac{\min(i,j)}{3}$. This fuzzy set in $X \times X$ can be represented either by the matrix

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & 1 \end{pmatrix},$$

or by the graph



Since the matrices of $\mu^{(1)}$, $\mu^{(\frac{2}{3})}$, and $\mu^{(\frac{1}{3})}$ are respectively

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

it results

$$\max\left(\min\left(1, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right), \min\left(\frac{2}{3}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\right), \min\left(\frac{1}{3}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right)\right)$$
$$= \max\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & 1 \end{pmatrix},$$

accordingly with the theorem of resolution.

2.2 The Concept of an 'Algebra of Fuzzy Sets'

2.2.1 Introduction

Functions

$$\mu \in F(X) = [0, 1]^X,$$

will be labeled only when it is some predicate P in X such that $\mu_P = \mu$, and it is obvious that it could be the fact of having $\mu_P = \mu_Q = \mu_R \cdots = \mu$, in which case the predicates $P, Q, R, \ldots$ are exact synonyms in X. Notwithstanding there are much more functions in $[0, 1]^X$ than predicates in X, and given a not previously labeled $\mu \in [0, 1]^X$, it can be 'artificially' introduced the predicate M $(= \mu)$ such that,

$$\text{Degree up to which } x \text{ is } M = \mu(x), \quad \text{for all } x \text{ in } X.$$

- Notice that $F(X)$ will be taken as 'ordered' (partially) by means of the binary pointwise relation

$$\mu \leqslant \sigma \Leftrightarrow \mu(x) \leqslant \sigma(x), \quad \text{for all } x \in X,$$

that induces the pointwise identity

$$\mu = \sigma \Leftrightarrow \mu \leqslant \sigma \text{ and } \sigma \leqslant \mu \Leftrightarrow \mu(x) = \sigma(x), \quad \text{for all } x \in X.$$

The pointwise relation $\leqslant$ is also called the 'inclusion', and $\mu \leqslant \sigma$ denoted by 'μ is included in σ'. It enjoys the laws reflexive, antisymmetric and transitive.

It will be always considered that $F(X)$ denotes, at least, the structure $([0, 1]^X; \leqslant; =)$. Observe that if $\mu_A, \mu_B \in \{0, 1\}^X$, that is, A and B are in $\mathbb{P}(X)$, then it follows

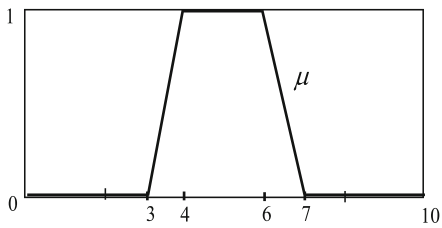
$$\mu_A \leqslant \mu_B \Leftrightarrow A \subset B; \quad \mu_A = \mu_B \Leftrightarrow A = B,$$

and

$$x \in A \Leftrightarrow \mu_A(x) = 1; \quad x \notin A \Leftrightarrow \mu_A(x) = 0.$$

The classical symbol $\in$ is the fuzzy symbol $\underset{1}{\in}$, and $\notin$ is $\underset{0}{\in}$.

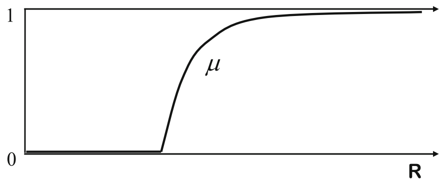
For example, with the fuzzy set $\underset{\sim}{P}$ given by function μ in the next figure it is $x \notin \underset{\sim}{P}$ if $0 \leqslant x \leqslant 3$, and $7 \leqslant x \leqslant 10$, but $x \in \underset{\sim}{P}$ if $4 \leqslant x \leqslant 6$, and $x \underset{\mu(x)}{\in} \underset{\sim}{P}$, if $x \in (3, 4) \cup (6, 7)$ with $0 < \mu(x) < 1$. If $x = 3.5$, since it is $\mu(x) = x - 3$, when $x \in (3, 4)$, it is $3.5 \underset{0.5}{\in} \underset{\sim}{P}$.



- The height of $\mu \in F(X)$ is $H(\mu) = \underset{x \in X}{Sup}\, \mu(x) = \sup \mu$. In the last example, it is $H(\mu) = 1$. In the finite example

$$\mu = 0.7/x_1 + 0.9/x_2 + 0.7/x_3,$$

in $X = \{x_1, x_2, x_3, x_4\}$, it is $H(\mu) = 0.9$. In a case like



it is $H(\mu) = 1$, although there is not any $x \in \mathbb{R}$ such that $\mu(x) = 1$. If there is some $x \in X$ such that $\mu(x) = 1$, it is said that μ is a *normalized* fuzzy set.
- In the case X is finite, $X = \{x_1, \ldots, x_n\}$, the number

$$|\mu| = \sum_{i=1}^{n} \mu(x_i)$$

is the *crisp-cardinality*, or sigma-count, of μ, a name coming from the fact that if $A \subseteq X$ has p elements it is $\sum_{i=1}^{n} \mu(x_i) = p$. Obviously, $\mu_\varnothing = \mu_0$, gives $|\mu_0| = 0$, $\mu_X = \mu_1$, gives $|\mu_1| = n$, and, $\mu \leqslant \sigma$ implies $|\mu| \leqslant |\sigma|$.

Remark 2.2.1 The pointwise definition of fuzzy sets inclusion implies that, for example, the fuzzy sets

$$\mu = 0.7/x_1 + 0.8/x_2 + 1/x_3 + 0.7/x_4$$
$$\sigma = 0.70001/x_1 + 0.7/x_2 + 1/x_3 + 0.6/x_4$$

in X, do not verify $\mu \leqslant \sigma$ although it is $\sigma(x_2) < \mu(x_2)$, $\sigma(x_3) = \mu(x_3)$, $\sigma(x_4) < \mu(x_4)$, but $\sigma(x_1) > \mu(x_1)$, with $\sigma(x_1) - \mu(x_1) = 0.00001$. Pointwise 'inclusion' is strongly affected by very small variations of the membership values. Actually, it is not a flexible, or fuzzy, concept, but a crisp one.

Because of this, it could be preferable to take the inclusion of fuzzy sets as an gradable concept $\leqslant_r$ $(r \in [0, 1])$, and a used definition of which is

$$\mu \leqslant_r \sigma \Leftrightarrow \frac{|\min(\mu, \sigma)|}{|\mu|} \leqslant r,$$

with $|\min(\mu, \sigma)| = \sum_{i=1}^{n} \min(\mu(x_i), \sigma(x_i))$.

In last example, it is $|\min(\mu, \sigma)| = 0.7 + 0.7 + 1 + 0.6 = 3$, $|\mu| = 0.7 + 0.8 + 1 + 0.7 = 3.2$, and $r = 3/3.2 = 0.9375 \approx 0.94$. That is, $\mu \underset{0.9375}{\leqslant} \sigma : \mu$ 'is almost included in' σ.

Since $|\sigma| = 0.70001 + 0.7 + 1 + 0.6 = 3.00001$, it is $\frac{|\min(\mu,\sigma)|}{|\sigma|} = 0.9999$, or $\sigma \leqslant_{0.9999} \mu$. That is, σ is more included in μ, than μ is included in σ!

Remark 2.2.2 Of course, if $\mu \leqslant \sigma$, it is $\min(\mu, \sigma) = \mu$, and $r = 1$, that is,

$$\mu \leqslant \sigma \Rightarrow \mu \underset{1}{\leqslant} \sigma.$$

Nevertheless, since it is only

$$\sum \min(\mu(x_i), \sigma(x_i)) \leqslant \min(\sum \mu(x_i), \sum \sigma(x_i)),$$

from $\mu \underset{1}{\leqslant} \sigma$ (or $|\min(\mu, \sigma)| \leqslant |\mu|$) it does not necessarily follow $\mu \leqslant \sigma$.

Let us show an example with crisp subsets. If $X = \{1, 2, 3, 4, 5, 6, 7\}$, and $A = \{1, 3, 5, 7\}$, $B = \{1, 3, 5, 6\}$, it is

$$\sum_{i=1}^{7} \min(\mu_A(i), \mu_B(i)) = 3, \sum_{i=1}^{7} \mu_B(i) = 4, \sum_{i=1}^{7} \mu_A(i) = 4.$$

hence

$$\mu_A \leqslant_{\frac{3}{4}} \mu_B, \text{ or } A \subset_{\frac{3}{4}} B$$

$$\mu_B \leqslant_{\frac{3}{4}} \mu_A, \text{ or } B \subset_{\frac{3}{4}} A$$

2.2.2 Algebras of Fuzzy Sets

Once $F(X) = ([0, 1]^X; \leqslant; =)$ is taken, a *general algebra of fuzzy sets* comes from endowing $F(X)$ with three operations:

1. $\prime : [0, 1]^X \to [0, 1]^X$,
2. $\cdot : [0, 1]^X \times [0, 1]^X \to [0, 1]^X$,
3. $+ : [0, 1]^X \times [0, 1]^X \to [0, 1]^X$,

respectively called the *complement* μ' of μ, the *intersection* $\mu \cdot \sigma$ of 'μ and σ', and the *union* $\mu + \sigma$ of 'μ or σ'. Then $([0, 1]^X; \leqslant; =; \cdot; +; ')$, is called an *algebra of fuzzy sets*, provided the following laws do hold:

(a) If $\mu \leqslant \sigma$, then $\gamma \cdot \mu \leqslant \gamma \cdot \sigma$, and $\mu \cdot \gamma \leqslant \sigma \cdot \gamma$, for all $\gamma \in [0, 1]^X$
(b) If $\mu \leqslant \sigma$, then $\mu + \gamma \leqslant \sigma + \gamma$, and $\gamma + \mu \leqslant \gamma + \sigma$ for all $\gamma \in [0, 1]^X$
(c) If $\mu \leqslant \sigma$, then $\sigma' \leqslant \mu'$
(d) For any $\mu \in [0, 1]^X$, $\mu \cdot \mu_1 = \mu_1 \cdot \mu = \mu$, $\mu + \mu_0 = \mu_0 + \mu = \mu$
(e) For all $\mu_A, \mu_B \in \{0, 1\}^X$, $\mu'_A = \mu_{A^c}$, $\mu_A \cdot \mu_B = \mu_{A\cap B}$, $\mu_A + \mu_B = \mu_{A\cup B}$ (preservation of the classical case).

Remark 2.2.3 It is not difficult to prove that no general algebra of fuzzy sets is a Boolean algebra. The proof comes from the fact that to be a Boolean algebra would imply $\mu \cdot \mu' = \mu_0$ and $\mu + \mu' = \mu_1$ for all $\mu \in [0, 1]^X$, and consists in finding some μ for which these equalities are not satisfied.

Remark 2.2.4 It is immediate to prove that $\mu \cdot \mu_0 = \mu_0 \cdot \mu = \mu_0$, $\mu + \mu_1 = \mu_1 + \mu = \mu_1$ for all $\mu \in [0, 1]^X$

Remark 2.2.5 Notice that the laws $\mu \cdot \sigma = \sigma \cdot \mu$ (commutativity of the intersection), $\mu + \sigma = \sigma + \mu$ (commutativity of the union), and $\mu'' = \mu$ (involution of the complement) are not supposed to be always verified. Nor it is supposed that the algebras $([0, 1]^X \cdot, +, ')$ *are dual ones*, that is, the so-called De Morgan laws,

$$(\mu + \sigma)' = \mu' \cdot \sigma', (\mu \cdot \sigma)' = \mu' + \sigma',$$

are not supposed to hold in general. It is neither supported $\mu \cdot \mu = \mu$, and $\sigma + \sigma = \sigma$, nor $\mu \cdot (\sigma \cdot \lambda) = (\mu \cdot \sigma) \cdot \lambda$ and $\mu + (\sigma + \lambda) = (\mu + \sigma) + \lambda$, nor that $\cdot$ and $+$ are associative, $\mu \cdot (\sigma \cdot \lambda) = (\mu \cdot \sigma) \cdot \lambda$, and $\mu + (\sigma + \lambda) = (\mu + \sigma) + \lambda$.

Theorem 2.2.6 *For any* $\mu, \sigma \in [0, 1]^X : \mu \cdot \sigma \leqslant \min(\mu, \sigma) \leqslant \max(\mu, \sigma) \leqslant \mu + \sigma$.

Proof From $\mu \leqslant \mu_1$ ($\mu(x) \leqslant 1$, for all $x \in X$), follows $\mu \cdot \sigma \leqslant \mu_1 \cdot \sigma = \sigma$. From $\sigma \leqslant \mu_1$, follows $\mu \cdot \sigma \leqslant \mu_1 \cdot \mu = \mu$. Thus,

$$\begin{aligned}&(\mu \cdot \sigma)(x) \leqslant \sigma(x), (\mu \cdot \sigma)(x) \leqslant \mu(x)\\ &\quad \Rightarrow (\mu \cdot \sigma)(x) \leqslant \min(\mu(x), \sigma(x)) = \min(\mu, \sigma)(x),\end{aligned}$$

or $\mu \cdot \sigma \leqslant \min(\mu, \sigma)$. Hence, the operation min is the greatest possible intersection of fuzzy sets. Analogously, $\mu_0 \leqslant \mu, \mu_0 \leqslant \sigma \Rightarrow \mu_0 + \sigma = \sigma \leqslant \mu + \sigma, \mu + \mu_0 = \mu \leqslant \mu + \sigma$, and $\max(\mu, \sigma) \leqslant \mu + \sigma$: The operation *max* is the smallest possible union of fuzzy sets. □

Obviously, for all $\mu, \sigma \in [0, 1]^X$: $\mu \cdot \sigma \leqslant \mu \leqslant \mu + \sigma$, $\mu \cdot \sigma \leqslant \sigma \leqslant \mu + \sigma$.

Theorem 2.2.7 *An operation* $* : [0, 1]^X \times [0, 1]^X \to [0, 1]^X$ *is called idempotent if and only if* $\mu * \mu = \mu$*, for all* $\mu \in [0, 1]^X$

- *The intersection* $\cdot$ *is idempotent if and only if* $\cdot = \min$
- *The union* $+$ *is idempotent if and only if* $+ = \max$

Proof The operations min, and max are obviously idempotent. Let us show that if $\cdot$ is idempotent, it must be $\cdot = \min$. By Theorem 2.2.6, it is always $\mu \cdot \sigma \leqslant \min(\mu, \sigma)$, and the idempotency of $\cdot$ implies

$$\min(\mu, \sigma) \cdot \min(\mu, \sigma) = \min(\mu, \sigma).$$

But $\min(\mu, \sigma) \leqslant \mu$, $\min(\mu, \sigma) \leqslant \sigma$, imply $\min(\mu, \sigma) \cdot \min(\mu, \sigma) \leqslant \mu \cdot \sigma$, that is

$$\min(\mu, \sigma) \leqslant \mu \cdot \sigma,$$

and, by Theorem 2.2.6, $\min(\mu, \sigma) = \mu \cdot \sigma$. A similar proof applies to $+$ and *max*. □

Theorem 2.2.8 (Absortion laws)

- $\mu \cdot (\mu + \sigma) = \mu$ *holds for all* $\mu, \sigma \in [0, 1]^X \Leftrightarrow \cdot = \min$
- $\mu + (\mu \cdot \sigma) = \mu$ *holds for all* $\mu, \sigma \in [0, 1]^X \Leftrightarrow + = \max$

Proof If $\cdot = \min$, the formula $\min(\mu, \mu + \sigma) = \mu$ does hold, since $\mu \leqslant \mu + \sigma$. Provided it is always $\mu \cdot (\mu + \sigma) = \mu$, taking $\sigma = \mu_0$ follows $\mu \cdot (\mu + \mu_0) = \mu \cdot \mu = \mu$, that holds if and only if $\cdot = \min$. If $+ = \max$, the formula $\max(\mu, \mu \cdot \sigma) = \mu$ does hold since $\mu \cdot \sigma \leqslant \mu$. Provided it is always $\mu + (\mu \cdot \sigma) = \mu$, taking $\sigma = \mu_1$ follows $\mu + (\mu \cdot \mu_1) = \mu + \mu = \mu$, that holds if and only if $+ = \max$. □

Theorem 2.2.9 (Duality, or De Morgan laws) *Provided the* complement $'$ is involutive ($(\mu')' = \mu'' = \mu$, *for all* $\mu \in [0, 1]^X$), *the algebra of fuzzy sets* ($[0, 1]^X$, min, max, $'$) *is a dual algebra.*

Proof If $'$ is involutive, from $\mu' + \sigma' = (\mu \cdot \sigma)'$ it follows $\mu' \cdot \sigma' = (\mu + \sigma)'$ since $\mu + \sigma = \mu'' + \sigma'' = (\mu' \cdot \sigma')'$, hence $(\mu + \sigma)' = \mu' \cdot \sigma'$. The converse is proven in the same way. And the two De Morgan laws

$$(\mu \cdot \sigma)' = \mu' + \sigma', \quad (\mu + \sigma)' = \mu' \cdot \sigma'$$

result equivalent. Then, it is enough to prove $\max(\mu, \sigma) = (\min(\mu', \sigma'))'$, for all μ, σ in $[0, 1]^X$. Since

$$\min(\mu', \sigma') \leqslant \mu', \quad \min(\mu', \sigma') \leqslant \sigma',$$

it follows

$$\mu = \mu'' \leqslant (\min(\mu', \sigma'))', \quad \sigma = \sigma'' \leqslant (\min(\mu', \sigma'))',$$

and

$$\max(\mu, \sigma) \leqslant (\min(\mu', \sigma'))' \tag{2.1}$$

On the other hand,

$$\mu \leqslant \max(\mu, \sigma) \Rightarrow (\max(\mu, \sigma))' \leqslant \mu'$$

$$\sigma \leqslant \max(\mu, \sigma) \Rightarrow (\max(\mu, \sigma))' \leqslant \sigma'$$

imply $(\max(\mu, \sigma))' \leqslant \min(\mu', \sigma')$, or

$$(\min(\mu', \sigma'))' \leqslant \max(\mu, \sigma). \tag{2.2}$$

Now, from (2.2) and (2.1), follows the result. □

Theorem 2.2.10 (Kleene's Law) *In all general algebra of fuzzy sets it holds the law*

$$\mu \cdot \mu' \leqslant \sigma + \sigma',$$

for all μ, σ *in* $[0, 1]^X$.

Proof We have to prove that, for any $x \in X$, it is $(\mu \cdot \mu')(x) \leqslant (\sigma + \sigma')(x)$. But it only can be either $\mu(x) \leqslant \sigma(x)$, or $\sigma(x) \leqslant \mu(x)$ for each $x \in X$. In the firs case, it is $(\mu \cdot \mu')(x) \leqslant \min(\mu(x), \mu'(x)) \leqslant \mu(x) \leqslant \sigma(x) \leqslant \max(\sigma(x), \sigma'(x)) = (\max(\sigma, \sigma')(x) \leqslant (\sigma + \sigma')(x)$. In the second case, it is $\mu'(x) \leqslant \sigma'(x)$, and $(\mu \cdot \mu')(x) \leqslant \min(\mu(x), \mu'(x)) \leqslant \mu'(x) \leqslant \sigma'(x) \leqslant \max(\sigma(x), \sigma'(x)) = (\max(\sigma, \sigma')(x) \leqslant (\sigma + \sigma')(x)$. Notice that provided μ and σ were crisp sets, the Kleene's law is reduced to $\mu_0 \leqslant \mu_1$. □

Remark 2.2.11 Concerning duality, Theorem 2.2.9 only states that the algebra given by the triplets (min, max, $'$), with $'$ involutive, are dual algebras. But they are not the only dual algebras. For example, with $\cdot =$ product,

$$(\mu \cdot \sigma)(x) = \mu(x) \cdot \sigma(x), \quad \forall x \in X,$$

it is easy to proof that taking $\mu'(x) = 1 - \mu(x)$, and

$$(\mu + \sigma)(x) = 1 - (1 - \mu(x))(1 - \sigma(x)) = \mu(x) + \sigma(x) - \mu(x) \cdot \sigma(x),$$

it is $([0, 1]^X, \cdot, +, ')$ an algebra of fuzzy sets that since it is

$$\mu + \sigma = (\mu' \cdot \sigma')',$$

is a dual algebra. Nevertheless, with $\mu'(x) = 1 - \mu(x)$, $(\mu \cdot \sigma)(x) = \mu(x) \cdot \sigma(x)$, and $(\mu + \sigma)(x) = \max(\mu(x), \sigma(x))$, we get an algebra that is not dual since

$$(\mu' \cdot \sigma')'(x) = \mu(x) + \sigma(x) - \mu(x) \cdot \sigma(x)$$

does not coincides with $\max(\mu(x), \sigma(x))$, as it is easy to see.

Remark 2.2.12 It is easy to prove that, for each algebra of fuzzy sets $([0, 1]^X, \cdot, +, ')$, the operation

$$\mu +' \sigma = (\mu' \cdot \sigma')',$$

gives the new algebra $([0, 1]^X, \cdot, +', ')$. If the complement $'$ is involutive ($\mu'' = \mu$), then $(\mu +' \sigma)' = \mu' \cdot \sigma'$.

Analogously, with the operation $\mu \cdot' \sigma = (\mu +' \sigma')'$, one has the new algebra $([0, 1]^X, \cdot', +', ')$ and, if $'$ is involutive, $(\mu \cdot' \sigma)' = \mu +' \sigma'$.

2.2.3 Non-contradiction and Excluded-Middle

An statement is self-contradictory whenever entails its negation. For example, the only classical set that is self-contradictory is the empty one:

$$A \subseteq A^c \Rightarrow A \cap A \subseteq A \cap A^c \Rightarrow A \subseteq \varnothing \Rightarrow A = \varnothing.$$

Perhaps, this is the reason of the difficulty children do have on accepting that $\varnothing$ is a set!

Within an algebra of fuzzy sets there are many self-contradictory fuzzy sets. For example, with $N = 1 - \text{id}$ it is

$$\mu \leqslant \mu' \Leftrightarrow \mu(x) \leqslant 1 - \mu(x) \Leftrightarrow \mu(x) \leqslant \frac{1}{2}, \quad \forall x \in X,$$

hence: μ is self-contradictory if and only if $\mu \leqslant \mu_{\frac{1}{2}}$. Analogously, with the strong negation $N(x) = \frac{1-x}{1+x}$, it is

$$\begin{aligned}\mu \leqslant \mu' \Leftrightarrow \mu(x) \leqslant \frac{1-\mu(x)}{1+\mu(x)} &\Leftrightarrow \mu(x)^2 + 2\mu(x) - 1 \leqslant 0\\ \Leftrightarrow \mu(x) \leqslant \sqrt{2} - 1, &\quad \forall x \in X,\end{aligned}$$

that is, μ is self-contradictory if and only id $\mu \leqslant \mu_{\sqrt{2}-1}$.

Notice that 1/2 is the fixed-point of the strong negation $N = 1 - \text{id}$ ($1 - n = n \Leftrightarrow n = 1/2$), and that $\sqrt{2} - 1$ is the fixed-point of the strong negation $N = \frac{1-id}{1+id}$ ($\frac{1-n}{1+n} = n \Leftrightarrow n = \sqrt{2} - 1$).

Notice that if $\mu_P \leqslant \mu_{aP}$, since it is always supposed that $\mu_{aP} \leqslant \mu_{not\,P}$, it follows $\mu_P \leqslant \mu_{not\,P}$, and μ_P is self-contradictory.

The classical principle of non-contradiction, "it is impossible to have both an statement and its negation", could be interpreted as "P and not P is impossible", or "P and not P is self-contradictory". All general algebras of fuzzy sets do verify the principle of non-contradiction once stated in this form.

Theorem 2.2.13 *If $([0, 1]^X, \cdot, +, ')$ is an algebra of fuzzy sets, it holds the principle of non-contradiction stated by: $\mu \cdot \mu' \leqslant (\mu \cdot \mu')'$ for all $\mu \in [0, 1]^X$. That is, for all $\mu \in [0, 1]^X$, $\mu \cdot \mu'$ is self-contradictory.*

Proof It is $\mu \cdot \mu' \leqslant \min(\mu, \mu') \leqslant \mu$, and $\mu \cdot \mu' \leqslant \min(\mu, \mu') \leqslant \mu'$; from the first inequality it follows $\mu' \leqslant (\mu \cdot \mu')'$, and then with the second follows $\mu \cdot \mu' \leqslant (\mu \cdot \mu')'$. □

Notice that no additional hypotheses on the connective $\cdot$, and the complement $'$, are needed for the proof of this theorem. In the algebras of fuzzy sets the non-contradiction principle is a theorem: the algebra's axioms imply the principle. It is not true, as it is sometimes stated, that fuzzy sets do not verify the principle of non-contradiction in which science is based.

The classical principle of Excluded-Middle, "It is always P or not P", can be interpreted as "Not (P or Not P) is a self-contradiction"and it is verified by all algebra of fuzzy sets.

Theorem 2.2.14 *If* $([0, 1]^X, \cdot, +, ')$ *is an algebra of fuzzy sets, it holds the principle of Excluded-Middle stated by*: $(\mu + \mu')' \leqslant ((\mu + \mu')')'$ *for all* $\mu \in [0, 1]^X$. *That is, for all* $\mu \in [0, 1]^X$, $(\mu + \mu')'$ *is self-contradictory.*

Proof It is,

- $\mu \leqslant \max(\mu, \mu') \leqslant \mu + \mu' \Rightarrow (\mu + \mu')' \leqslant \mu' \Rightarrow (\mu')' \leqslant ((\mu + \mu')')'$
- $\mu' \leqslant \max(\mu, \mu') \leqslant \mu + \mu' \Rightarrow (\mu + \mu')' \leqslant (\mu')'$

then $(\mu + \mu')' \leqslant ((\mu + \mu')')'$. □

Notice that no additional hypotheses on the connective $+$, and the complement $'$, are needed for the proof of this theorem: In the algebra of fuzzy sets the excluded-middle principle is a theorem. In conclusion,

In all algebra of fuzzy sets $([0, 1]^X, \cdot, +, ')$,

- *The logic principles of non-contradiction and excluded-middle are theorems, once stated through the concept of self-contradiction.*

A very different situation appears if these two principles are stated as it is currently done within logic and classical set theory, that is, by stating

- "P and not P" is false
- "P or not P" is true,

or,

- There is no x in X such that "x is P and x is not P"
- For all x in X it is "x is P or x is not P"

translated into

- $(\mu_P \cdot \mu_{not\ P})(x) = 0$, for all x in X
- $(\mu_P + \mu_{not\ P})(x) = 1$, for all x in X

that corresponds to "solve" the equations with fuzzy sets,

$$\mu \cdot \mu' = \mu_0, \ \mu + \mu' = \mu_1,$$

that is, to find for which intersections $\cdot$ and which unions $+$, these equations do hold.

Of course, they do not hold in all cases, for example, with $N = 1 - \text{id}$,

- If $\cdot = \min$, it is not always $\min(\mu(x), 1 - \mu(x)) = 0$,
- If $+ = \max$, it is not always $\max(\mu(x), 1 - \mu(x)) = 1$,
- If $\cdot = W$, $(W(a, b) = \max(0, a + b - 1))$ it is $W(a, 1 - a) = \max(0, a + 1 - a - 1)) = 0$, and $W(\mu(x), 1 - \mu(x))) = 0$ for all x in X
- If $+ = W^*$, $(W^*(a, b) = \min(1, a + b))$, it is $W^*(\mu(x), 1 - \mu(x))) = 1$ for all x in X.

That is, *there are algebras* of fuzzy sets *where this forms* of non-contradiction or excluded-middle *hold*, and *algebras where* this principles *do not jointly hold*. In the algebras with the triplet (min, max, $1 - id$) do not hold both principles, in the algebras with $(W, \max, 1 - id)$ it holds the principle of non-contradiction but not that of excluded-middle, in the algebras with $(\min, W^*, 1 - id)$ it holds the excluded-middle but not the principle of non-contradiction, and in the algebras with $(W, W^*, 1 - id)$ both principles hold.

Remark 2.2.15 Let us show that with $\mu \in \{0, 1\}^X$ it is always $\mu \cdot \mu' = \mu_0$ and $\mu + \mu' = \mu_1$. If $\mu \in \{0, 1\}^X$, denote $A = \{x \in X; \mu(x) = 1\}$. Obviously, $\mu = \mu_A$ and $\mu' = \mu_{A^c}$, hence

- $(\mu \cdot \mu') = \mu_A \cdot \mu_{A^c} = \mu_{A \cap A^c} = \mu_\varnothing = \mu_0$
- $(\mu + \mu') = \mu_A + \mu_{A^c} = \mu_{A \cup A^c} = \mu_X = \mu_1$.

Remark 2.2.16 Results in Theorems 2.2.13 and 2.2.14 challenge the usual statement that in fuzzy sets the basic principles of Non-contradiction and Excluded-middle fail. A statement that could conduct to believe that fuzzy set algebras are not properly grounded in a solid ground.

The fact is, notwithstanding, that these two principles were established before the current ways of considering the problems of logic and, of course, before the *nomenclature* of set theory. In set theory (or Boolean algebras), $A \cap A^c = \varnothing$ and $A \cap A^c \subset (A \cap A^c)^c$ are equivalent formulas since, as it was said, it is

$$B = \varnothing \Leftrightarrow B \subset B^c,$$

an equivalence only verified in the setting of ortholattices (of which Boolean algebras are a particular case), but that does not hold on weaker algebraic structures like it is, for example, the case of the above defined algebras of fuzzy sets. Let us call 'restricted' the principles stated by $\mu \cdot \mu' = \mu_0$ and $\mu + \mu' = \mu_1$.

2.2.4 Decomposable Algebras

Definition 2.2.17 An operation with fuzzy sets $* : [0, 1]^X \times [0, 1]^X \to [0, 1]^X$ is *decomposable, or functionally expressible*, if it exists a numerical operation $\widehat{*} : [0, 1] \times [0, 1] \to [0, 1]$, such that

$$(\mu * \sigma)(x) = \widehat{*}(\mu(x), \sigma(x)),$$

for all μ, σ in $[0, 1]^X$ and all x in X. Of course, by this formula, a numerical operation $\widehat{*}$ allows to define an operation $*$ for fuzzy sets.

For example, the operation min in $[0, 1]^X$ is decomposable since, by definition, it is

$$(\min(\mu, \sigma))(x) = \min(\mu(x), \sigma(x))$$

for all μ, σ in $[0, 1]^X$ and all x in X.

Definition 2.2.18 A function $f : [0, 1]^X \to [0, 1]^X$ is *decomposable, or functionally expressible*, if it exists a numerical function $\widehat{f} : [0, 1] \to [0, 1]$, such that

$$(f(\mu))(x) = \widehat{f}(\mu(x)),$$

for all μ in $[0, 1]^X$ and all x in X.

For example, the function $'$ defined by

$$\mu'(x) = 1 - \mu(x),$$

is decomposable because of $N_0 = 1 - \text{id}$ gives $\mu'(x) = N_0(\mu(x))$. With $X = [0, 1]$, the function defined by $\mu^*(x) = 1 - \mu(1 - x)$ is not decomposable, since if it were such, that is, if there is $N : [0, 1] \to [0, 1]$ such that $1 - \mu(1 - x) = N(\mu(x))$, it suffices to take $\mu, \sigma \in [0, 1]^X$ such that

- $\mu(0) = 0, \mu(1) = 1 \Rightarrow N(0) = 1 - \mu(1 - 0) = 1 - \mu(1) = 0, N(1) = 1 - \mu(1 - 1) = 1 - \mu(0) = 1$
- $\sigma(0) = 1, \sigma(1) = 0 \Rightarrow N(0) = 1 - \sigma(1 - 0) = 1 - \sigma(1) = 1, N(1) = 1 - \sigma(1 - 1) = 1 - \sigma(0) = 0$

that is absurd.

The algebras of fuzzy sets $([0, 1]^X, \cdot, +, ')$, can be

Decomposable	if the three operation $\cdot, +, '$ are decomposable
Partially decomposable	if at least one of the three operations $\cdot, +, '$ is decomposable
Non decomposable	if no one of the three operations is decomposable

In what follows we will only deal with decomposable algebras, that is, such that:

There are three functions $F : [0, 1] \times [0, 1] \to [0, 1]$, $G : [0, 1] \times [0, 1] \to [0, 1]$, and $N : [0, 1] \to [0, 1]$ with which

$$(\mu \cdot \sigma)(x) = F(\mu(x), \sigma(x)), (\mu + \sigma)(x) = G(\mu(x), \sigma(x)), \mu'(x) = N(\mu(x)),$$

for all $\mu, \sigma \in [0, 1]^X$, and all $x \in X$. For short, $\mu \cdot \sigma = F \circ (\mu \times \sigma)$, $\mu + \sigma = G \circ (\mu \times \sigma)$, $\mu' = N \circ \mu$.

In these cases, instead of $([0, 1]^X, \cdot, +, ')$ it is written $([0, 1]^X, F, G, N)$.

The laws verified by the triplet $(\cdot, +, ')$ force analogous laws for the triplet (F, G, N). For example, linked to the axioms, it is

(a) If $a \leqslant b$, then $F(a, c) \leqslant F(b, c)$, $F(c, a) \leqslant F(c, b)$, for all $c \in [0, 1]$
(b) If $a \leqslant b$, then $G(a, c) \leqslant G(b, c)$, $G(c, a) \leqslant G(c, b)$, for all $c \in [0, 1]$
(c) If $a \leqslant b$, then $N(b) \leqslant N(a)$
(d) $F(1, a) = F(a, 1) = a$, $G(0, a) = G(a, 0) = a$
(e) $N(0) = 1$, $N(1) = 0$, $F(0, a) = F(a, 0) = 0$, $G(1, a) = G(a, 1) = 1$

and linked to the Theorems 2.2.6–2.2.13 it is

- $F(a, b) \leqslant \min(a, b) \leqslant \max(a, b) \leqslant G(a, b)$ for all a, b in $[0, 1]$. In particular, it is $F \leqslant G$.
- F is idempotent ($F(a, a) = a$, for all a in $[0, 1]$), if and only if $F = \min$.
- G is idempotent ($G(a, a) = a$, for all a in $[0, 1]$), if and only if $G = \max$.
- It is $F(a, G(a, b)) = a$, for all a, b in $[0, 1]$, if and only if $F = \min$.
- It is $G(a, F(a, b)) = a$, for all a, b in $[0, 1]$, if and only if $G = \max$.
- A triplet (F, G, N) is called dual, or De Morgan triplet, if $F = N \circ G \circ (N \times N)$, or, $G = N \circ F \circ (N \times N)$, that is, $F(a, b) = N(G(N(a), N(b))$, or $G(a, b) = N(F(N(a), N(b))$, for all a, b in $[0, 1]$. Notice that, in this case, it is enough to know N and F to have G, or N and G to have F.
- If N is involutive ($N(N(a)) = a$, for all $a \in [0, 1]$ or $N^2 = N$), the triplet $(\min, \max, N)$ is a dual one.
- All triplet (F, G, N) verifies $F(a, N(a)) \leqslant G(b, N(b))$, for all a, b in $[0, 1]$,
- It is $F(a, N(a)) \leqslant N(F(a, N(a)))$, for all a in $[0, 1]$
- It is $N(F(a, N(a))) \leqslant N(N(F(a, N(a))))$, for all a in $[0, 1]$
- Given N involutive and F, and denoting by G_N the dual of F respect to N, $G_N(a, b) = N(F(N(a), N(b)))$, it follows $G_N(N(a), a) \leqslant N(G_N(N(a), a))$, that with
$$\mu + \sigma = G_N \circ (\mu \times \sigma),$$
gives $\mu' + \mu \leqslant (\mu' + \mu)'$.
- It is always $F(a, N(a)) \leqslant G(b, N(b))$, for all a, b in $[0, 1]$.

Remark 2.2.19 With classical sets, from $A \cap B \subseteq A \cup B$ it results $(A \cup B) \cup (A \cap B) = A \cup B$. In a decomposable theory, to have the analogous law $(\mu + \sigma) + (\mu \cdot \sigma) = \mu + \sigma$ it should be $G(G(a, b), F(a, b)) = G(a, b)$, for all a, b in $[0, 1]$, that is verified if, for example, $G = \max$, $F = \min$, or $G = W^*$, $F = W$

Remark 2.2.20 Let us show an example of a 'union' that is not-decomposable. Define

$$(\mu + \sigma)(x) = \begin{cases} \max(\mu(x), \sigma(x)), & \text{if } \mu \text{ or } \sigma \text{ are in} \{0, 1\}^X \text{ (crisp)} \\ \max(H(\mu), H(\sigma)), & \text{otherwise.} \end{cases}$$

It is easy to show that this operation verifies the laws b, d and e in Sect. 2.1.1. Hence, it is a union for fuzzy sets that, in addition, is commutative. It is not idempotent, since if $\mu \in [0, 1]^X - \{0, 1\}^X$, it is $(\mu + \mu)(x) = H(\mu) \neq \mu(x)$. It does not exist a function $G : [0, 1] \times [0, 1] \to [0, 1]$ such that

$$(\mu + \sigma)(x) = G(\mu(x), \sigma(x))$$

for all $x \in X$ and all $\mu, \sigma[0, 1]^X$. Indeed, let us suppose that such a G does exist, and take $\mu = \mu_{0.5}$. Then

- With $\sigma = \mu_0$, it is $(\mu + \sigma)(x) = \max(\frac{1}{2}, 0) = \frac{1}{2}$. Hence $G(\frac{1}{2}, 0) = \frac{1}{2}$.
- With $\sigma(x) = x$, is $(\mu + \sigma)(x) = \max(H(\mu), H(\sigma)) = \max(\frac{1}{2}, 1) = 1$, and $(\mu + \sigma)(0) = 1 = G(\frac{1}{2}, 0)$, that is absurd.
  To have a not-decomposable 'intersection', it is enough to define, with $\mu' = 1 - \mu$, the dual operation,

$$\begin{aligned} (\mu \cdot \sigma)(x) &= [(\mu' + \sigma')]'(x) = 1 - (\mu' + \sigma')(x) \\ &= \begin{cases} \min(\mu(x), \sigma(x)), & \text{if } \mu \text{ or } \sigma \text{ are in} \{0, 1\}^X \\ \max(H(\mu'), H(\sigma')), & \text{otherwise.} \end{cases} \end{aligned}$$

2.2.5 *Standard Algebras of Fuzzy Sets*

An standard algebra of fuzzy sets is a decomposable algebra of fuzzy sets such that:

1. $\mu \cdot \sigma = \sigma \cdot \mu$, for all μ, σ in $[0, 1]^X$ ($\cdot$ is commutative)
2. $\mu + \sigma = \sigma + \mu$, for all μ, σ in $[0, 1]^X$ ($+$ is commutative)
3. $\mu \cdot (\sigma \cdot \lambda) = (\mu \cdot \sigma) \cdot \lambda$, for all μ, σ, λ in $[0, 1]^X$ ($\cdot$ is associative)
4. $\mu + (\sigma + \lambda) = (\mu + \sigma) + \lambda$, for all μ, σ, λ in $[0, 1]^X$ ($+$ is associative)
5. $\mu'' = \mu$, for all μ in $[0, 1]^X$ ($'$ is involutive).

Hence, writing

$$\mu \cdot \sigma = T \circ (\mu \times \sigma), \;\; \mu + \sigma = S \circ (\mu \times \sigma), \;\; \mu' = N \circ \mu,$$

functions $T, S : [0, 1] \times [0, 1] \to [0, 1]$ and $N : [0, 1] \to [0, 1]$, in addition to the corresponding general properties stated before, must verify

- T is commutative, S is commutative
- T is associative, S is associative
- N is involutive,

that is:

- $T(a, b) = T(b, a)$, $S(a, b) = S(b, a)$, for all a, b in $[0, 1]$
- $T(a, T(b, c)) = T(T(a, b), c)$, $S(a, S(b, c)) = S(S(a, b), c)$, for all a, b, c in $[0, 1]$
- $N(N(a)) = a$, for all a in $[0, 1]$, or $N \circ N = \mathrm{id}$, or $N = N^{-1}$.

Functions T and S are called t-norms and t-conorms, respectively. Functions N are strong negations. Hence, $([0, 1], T, \leqslant)$ is an ordered semigroup with neutral 1, and absorbent 0, and $([0, 1], S, \leqslant)$ is also an ordered semigroup but with neutral 0 and absorbent 1. Since $N(1) = 0$, it seems that this two kind of ordered semigroups should show some character of duality. This duality goes in the way:

- If T is a t-norm, $T_N = N \circ S \circ (N \times N)$ is a t-conorm
- If S is a t-conorm, $S_N = N \circ S \circ (N \times N)$ is a t-norm

that are easy to prove. Of course, from Sect. 2.1.4,

- If T is a t-norm, $T \leqslant \min$, and min is a t-norm
- If S is a t-conorm, $\max \leqslant S$, and *max* is a t-conorm

Hence, for all t-norm T and all t-conorm S:

$$T \leqslant \min \leqslant \max \leqslant S,$$

in particular, $T \leqslant S$.[1] Even more, the function

$$Z(a, b) = \begin{cases} b, & \text{if } a = 1 \\ a, & \text{if } b = 1 \\ 0, & \text{otherwise,} \end{cases} = \begin{cases} \min(a, b), & \text{if } a = 1 \text{ or } b = 1 \\ 0, & \text{otherwise,} \end{cases}$$

is obviously a t-norm such that $Z \leqslant T$ for all t-norm T. Consequently,

$$Z^*(a, b) = 1 - Z(1 - a, 1 - b) = \begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } b = 0 \\ 1, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \max(a, b), & \text{if } a = 0 \text{ or } b = 0 \\ 1, & \text{otherwise,} \end{cases}$$

is a t-conorm such that $S \leqslant Z^*$ for all t-conorm S. Hence, for all t-norm T and all t-conorm S,

$$Z \leqslant T \leqslant \min \leqslant \max \leqslant S \leqslant Z^*.$$

[1] Notice that $T \leqslant S$ mean $T(a, b) \leqslant S(a, b)$, for all $(a, b) \in [0, 1] \times [0, 1]$.

Notice that min(max) is a continuous t-norm (t-conorm), but $Z(Z^*)$ is a discontinuous t-norm (t-conorm). The operations in [0, 1] given by

- $T_{prod}(a, b) = \ prod\ (a, b) = a \cdot b$
- $T_W(a, b) = W(a, b) = \max(0, a + b - 1) = (\max(0, Sum - 1))(a, b)$,

are also continuous t-norms. Then, the dual operations,

- $T^*_{prod}(a, b) = 1 - T_{prod}(1 - a, 1 - b) = 1 - (1 - a) \cdot (1 - b) = a + b - a \cdot b = (Sum - prod)(a, b)$
- $W^*(a, b) = 1 - W(1 - a, 1 - b) = \min(1, a + b) = \min(1,\ Sum(a, b))$

are continuous t-conorms. Since it is easy to prove that

$$W \leqslant T_{prod} \leqslant \min,$$

it follows $\max \leqslant T^*_{prod} \leqslant W^*$, and

$$Z \leqslant W \leqslant T_{prod} \leqslant \min \leqslant \max \leqslant T^*_{prod} \leqslant W^*.$$

Remark 2.2.21 Since $Z(0.5, 0.5) = 0$, t-norm Z has zero-divisors. Analogously, from

$$W(a, b) = 0 \Leftrightarrow \max(0, a + b - 1) = 0 \Leftrightarrow a + b \leqslant 1,$$

it follows that t-norm W has zero-divisors, for example, $W(0.5, 0.4) = 0$; t-norms min and T_{prod} do not have zero-divisors:

- $T_{prod}(a, b) = 0 \Leftrightarrow a = 0$, or $b = 0$
- $\min(a, b) = 0 \Leftrightarrow a = 0$, or $b = 0$

Proposition 2.2.22 *The only idempotent t-norm, i.e.* $T(a, a) = a$*, for all* $a \in [0, 1]$*, is* $T = \min$.

Proof If T is idempotent, $\min(a, b) = T(\min(a, b), \min(a, b)) \leqslant T(a, b)$ since $\min(a, b) \leqslant a$, and $\min(a, b) \leqslant b$. Hence, $\min(a, b) \leqslant T(a, b) \leqslant \min(a, b)$ implies $T = \min$. □

Proposition 2.2.23 *The only idempotent t-conorm, i.e.* $S(a, a) = a$*, for all* $a \in [0, 1]$*, is* $S = \max$.

Proof If S is idempotent, $\max(a, b) = S(\max(a, b), \max(a, b)) \geq S(a, b)$, since $\max(a, b) \geq a$, and $\max(a, b) \geq b$. Hence, $\max(a, b) \geq S(a, b) \geq \max(a, b)$ implies $S = \max$. □

Remark 2.2.24 t-norms can be continuous, like min, T_{prod}, and W, or discontinuous, like Z. They can have zero-divisors, like W and Z, or not like min and T_{prod}. They can have all elements in [0, 1] idempotent (only $T = \min$), only have the idempotents

0 and 1 (like T_{prod} and W), or have some idempotents different from 0 and 1. In any case, since it is always $T(0, 0) = 0$ and $T(1, 1) = 1$, 0 and 1 are idempotent elements for all t-norms.

Remark 2.2.25 Analogous considerations can be made for t-conorms. There are discontinuous t-conorms like Z^*, and continuous ones like T^*_{prod} and W^*. The only for which all elements in [0, 1] are idempotent is $S = \max$. Since $S(0, 0) = 0$ and $S(1, 1) = 1$, 0 and 1 are always idempotent, and there are t-conorms that only have these two idempotents (like T^*_{prod} and W^*), as well as those that have some idempotents different from 0,1. There are t-conorms without one-divisors, like max and T^*_{prod}, and t-conorms with one-divisors like W^*, for example, $W^*(0.5, 0.5) = \min(1, 1) = 1$.

Remark 2.2.26 There is not a characterization theorem for all t-norms (t-conorms), *but it is a characterization of the continuous t-norms* (t-conorms) that will be presented by means of the following, and easy to prove, results:

- If $\varphi : [0, 1] \to [0, 1]$ verifies, (1) If $x \leqslant y$, then $\varphi(x) \leqslant \varphi(y)$, (2) φ is bijective, (3) $\varphi(0) = 0, \varphi(1) = 1$ (φ is an order-automorphism of the ordered interval $([0, 1], \leqslant)$), and T is a t-norm, then $T_\varphi = \varphi^{-1} \circ T \circ (\varphi \times \varphi)$ is also a t-norm. Given T, the set $\{T_\varphi; \varphi$ an order-automorphism$\}$ is called the *family* of T.
- T is a continuous t-norm if and only if all t-norms T_φ are continuous.
- If S is a t-conorm, then $S_\varphi = \varphi^{-1} \circ S \circ (\varphi \times \varphi)$ is also a t-conorm, and S is continuous if and only if all t-conorms S_φ are continuous, the set $\{S_\varphi; \varphi$ an order-automorphism$\}$ is called the *family* of S.

In particular,

- The family of $T = \min$, is reduced to the only t-norm min, since $\varphi^{-1}(\min(\varphi(a), \varphi(b)) = \min(\varphi^{-1}(\varphi(a)), \varphi^{-1}(\varphi(b))) = \min(a, b)$
- The family of T_{prod} contains all continuous t-norms of the form $\text{prod}_\varphi(a, b) = \varphi^{-1}(\varphi(a).(\varphi(b))$.
- The family of W contains all t-norms of the form $W_\varphi(a, b) = \varphi^{-1}(W(\varphi(a), \varphi(b)) = \varphi^{-1}(\max(0, \varphi(a) + \varphi(b) - 1))$, and all of them are continuous t-norms. Notice that no t-norm in the family $\{prod_\varphi\}$ has zero-divisors, since $prod_\varphi(a, b) = 0 \Leftrightarrow \varphi(a) \cdot \varphi(b) = 0 \Leftrightarrow \varphi(a) = 0$ or $\varphi(b) = 0 \Leftrightarrow a = 0$, or $b = 0$. Instead all t-norms W_φ have zero-divisors, since $W_\varphi(a, b) = 0 \Leftrightarrow \max(0, \varphi(a) + \varphi(b) - 1) \Leftrightarrow \varphi(a) + \varphi(b) \leqslant 1$. Of course, neither t-norms $prod_\varphi$, nor W_φ, have more idempotents than 0 and 1:
- $a = W_\varphi(a, a) = \varphi^{-1}(\max(0, 2\varphi(a) - 1)) \Leftrightarrow \varphi(a) = \max((0, 2\varphi(a) - 1) \Leftrightarrow \varphi(a) = 0$ or $\varphi(a) = 1$ or $a = 0$ or $a = 1$.
- $a = prod_\varphi(a, a) = \varphi^{-1}(\varphi(a).(\varphi(a)) \Leftrightarrow \varphi(a) = \varphi(a).(\varphi(a) \Leftrightarrow \varphi(a) = 0$ or $\varphi(a) = 1$ or $a = 0$ or $a = 1$.

Analogously,

- The family of $S = \max$, only contains this t-conorm.
- The family of T^*_{prod} contains all t-conorms of the form

$$prod^*_\varphi(a, a) = \varphi^{-1}(prod^*(\varphi(a), \varphi(b))) = \varphi^{-1}(\varphi(a) + \varphi(b) - \varphi(a) \cdot \varphi(b))$$

- The family of W^* contains all t-conorms of the form

$$W^*(a, a) = \varphi^{-1}(W^*(\varphi(a), \varphi(b)) = \varphi^{-1}(\min(1, \varphi(a) + \varphi(b)))$$

Remark 2.2.27 The order-automorphism φ plays the role of a functional parameter. By taking, $\varphi(x) = x^r$, it follows, for example,

$$W_\varphi(a, b) = \sqrt[r]{\max((0, a^r + b^r - 1)}, \quad W^*_\varphi(a, b) = \sqrt[r]{\min((1, a^r + b^r)}$$

giving a family of t-norms (t-conorms) depending on the numerical parameter $r > 0$. Notice that with $\varphi(x) = x^r$,

$$Prod_\varphi(a, b) = \sqrt[r]{a^r \cdot b^r} = a \cdot b = Prod(a, b),$$

but

$$Prod^*_\varphi(a, b) = \varphi^{-1}(\varphi(a) + \varphi(b) - \varphi(a) \cdot \varphi(b)) = \sqrt[r]{a^r + b^r - a^r \cdot b^r}.$$

2.2.6 Strong Negations

As it was said before, an strong negation is a function $N : [0, 1] \to [0, 1]$ such that

- $N(0) = 1$
- If $a \leqslant b$, then $N(b) \leqslant N(a)$
- $N(N(a)) = a$, for all $a \in [0, 1]$, or $N^2 = \mathrm{id}$.

Notice that $N^2 = \mathrm{id}$ is equivalent to $N = N^{-1}$, that shows N is a continuous function: It is $N(1) = N(N(0)) = 0$, and if $a < b$ it should be $N(b) < N(a)$ since $N(b) = N(a)$ would imply $N(N(b)) = N(N(a))$, or $a = b$. Hence, N is strictly decreasing.

Since N is continuous, the equation $N(x) = x$ has solutions, but there is only one. Suppose $N(x_1) = x_1$ and $N(x_2) = x_2$. Either $x_1 \leqslant x_2$, or $x_2 < x_1$. In the first case, it follows $N(x_2) \leqslant N(x_1)$, or $x_2 \leqslant x_1$, and $x_1 = x_2$. In the second case, $N(x_1) < N(x_2)$, or $x_1 < x_2$, that is absurd. Then, each strong negation has a single fixed point $N(n) = n$, in the open interval $(0, 1)$, since $N(0) = 1$, $N(1) = 0$ show that 0 and 1 are not fixed points.

Remark 2.2.28 In the classical case (a Boolean algebra L, or a power set $\mathbb{P}(X)$), the transformation

$$F : \mathbb{P}(X) \to \mathbb{P}(X),\ F(A) = A^c,$$

has no fixed points, since $A = A^c$ implies $A \cap A = A \cap A^c$, or $A = \varnothing$, and $\varnothing^c = X$. Nevertheless, with fuzzy sets

$$F : [0,1]^X \to [0,1]^X,\ F(\mu) = \mu' = N \circ \mu,$$

the equation $\mu = \mu'$, $N(\mu(x)) = \mu(x)$ for all x in X, has the only solution $\mu(x) = n$ for all x in X, that is $\mu = \mu_n$, with $n = N(n)$ the fix point of N.

In the fuzzy case, that mapping F shows a kind of symmetry that is not in the crisp case.

An order-automorphism of the ordered unit interval $([0,1], \leqslant)$ $\varphi : [0,1] \to [0,1]$, verifies by definition,

- $\varphi(0) = 0, \varphi(1) = 1$
- If $a < b$, then $\varphi(a) < \varphi(b)$.

Hence, φ is continuous, and its inverse function φ^{-1} verifies,

- $\varphi^{-1}(0) = 0, \varphi^{-1}(1) = 1$
- If $a < b$, then $\varphi^{-1}(a) < \varphi^{-1}(b)$.

Let us denote by N_φ the function $N_\varphi : [0,1] \to [0,1]$ defined by

$$N_\varphi(a) = \varphi^{-1}(1 - \varphi(a)), \quad \text{for all } a \in [0,1]$$

Proposition 2.2.29 *N_φ is a strong negation.*

Proof It is $N_\varphi(0) = \varphi^{-1}(1 - \varphi(0)) = \varphi^{-1}(1) = 1$. In addition, if $a \leqslant b$, it follows $1 - \varphi(b) \leqslant 1 - \varphi(a)$, and $\varphi^{-1}(1 - \varphi(b)) \leqslant \varphi^{-1}(1 - \varphi(a))$, or $N_\varphi(b) \leqslant N_\varphi(a)$. Finally,

$$\begin{aligned} N_\varphi(N_\varphi(a)) &= N_\varphi(\varphi^{-1}(1 - \varphi(a))) = \varphi^{-1}(1 - \varphi(\varphi^{-1}(1 - \varphi(a)))) \\ &= \varphi^{-1}(1 - 1 + \varphi(a)) = \varphi^{-1}(\varphi(a)) = a \end{aligned}$$

□

Theorem 2.2.30 *If N is a strong negation, there exist order-automorphisms φ such that $N = N_\varphi$.*

Proof Let it be $n = N(n) \in (0,1)$ the fixed point of N, and consider an strictly non-decreasing function $h : [0,n] \to [0, \frac{1}{2}]$ such that $h(0) = 0$ and $h(n) = \frac{1}{2}$. With h define the function $\varphi : [0,1] \to [0,1]$ by

$$\varphi(x) = \begin{cases} h(x), & \text{if } x \in [0,n] \\ 1 - h(N(x)), & \text{if } x \in (n,1]. \end{cases}$$

This function φ is, obviously, continuous, strictly increasing,[2] and verifies $\varphi(0) = h(0) = 0$, $\varphi(1) = 1 - h(N(1)) = 1 - h(0) = 1$. Then

- If $x \in [0, n)$, or $N(x) \in (n, 1]$, $\varphi(N(x)) = 1 - h(x) = 1 - \varphi(x)$, and $N(x) = \varphi^{-1}(1 - \varphi(x))$.
- If $x = n$, $N(n) = n = h^{-1}(\frac{1}{2}) = \varphi^{-1}(\frac{1}{2})$, or $N(n) = \varphi^{-1}(1 - \varphi(x)$
- If $x \in (n, 1]$, or $N(x) \in [0, n)$, $\varphi(N(x)) + \varphi(x) = h(N(x)) + 1 - h(N(x)) = 1$

In conclusion, $N(x) = \varphi^{-1}(1 - \varphi(x))$, for all x in $[0, 1]$, or $N = N_\varphi$. □

Notice that the proof of last theorem shows clearly that the order-automorphism φ such that $N = N_\varphi$ is not unique. Notice also that with $\varphi = \mathrm{id}_{[0,1]}$ it follows $N(x) = 1 - x$, the fundamental strong negation, with which it results $N = N_\varphi = \varphi^{-1} \circ (1 - \mathrm{id}_{[0,1]}) \circ \varphi = \varphi^{-1} \circ N \circ \varphi$, that is, all strong negations belong to the family of $N_0(x) = 1 - x$. Nevertheless, in all cases it is $n = \varphi^{-1}(\frac{1}{2})$ the fixed point of N_φ.

If $\varphi(x) = x^2$, it results $N_\varphi(x) = \sqrt{1 - x^2}$, called the circular negation. If $\varphi(x) = \frac{2x}{1+x}$, or $\varphi^{-1}(x) = \frac{x}{2-x}$, it follows $N_\varphi(x) = \varphi^{-1}(1 - \frac{2x}{1+x}) = \varphi^{-1}(\frac{1-x}{1+x}) = \frac{1-x}{1+3x}$, that is the strong negation N_3 of the before mentioned Sugeno's negations.

With $\varphi(x) = \frac{1}{\lambda}\ln(1 + \lambda x^\alpha)$, $\lambda > -1$, $\alpha > 0$, it follows the bi-parametric family $N_\varphi(x) = (\frac{1-x^\alpha}{1+\lambda x^\alpha})^{\frac{1}{\alpha}}$, where with $\alpha = 1$ it is obtained the Sugeno's family of strong negations $N_\lambda = \frac{1-x}{1+\lambda x} (\lambda > -1)$ that only depends on one single parameter.

Remark 2.2.31 The only linear strong negation N is $N = N_0$, since from $N(a) = \alpha a + \beta$, with $N(0) = 1 = \beta$ and $N(1) = 0 = \alpha + 1$, follows $\alpha = -1$ and $N(a) = 1 - a$.

Remark 2.2.32 The only "rational" strong negations N of the form $N(x) = \frac{ax+b}{cx+d}$, a, b different of 0, are those $N_\lambda (\lambda > -1)$ in the Sugeno's family. It follows from:

- $N(0) = 1 = \frac{b}{d}$, or $d = b$
- $N(1) = 0 = \frac{a+b}{c+d}$, or $a = -b$

that gives

$$N(x) = \frac{-bx + b}{cx + b} = \frac{b(1 - x)}{cx + b} = \frac{1 - x}{1 + \frac{c}{b}x}.$$

To have $0 \leqslant N(x) \leqslant 1$, it should be $1 - x \leqslant 1 + \frac{c}{b}$. But $-1 = \frac{c}{b}$ implies $N(x) = 1$, that is not an strong negation. Hence it is $-1 < \frac{c}{b}$, and with $\lambda = \frac{c}{b}$, it follows $N(x) = \frac{1-x}{1+\lambda x} = N_\lambda(x)$, with $-1 < \lambda$.

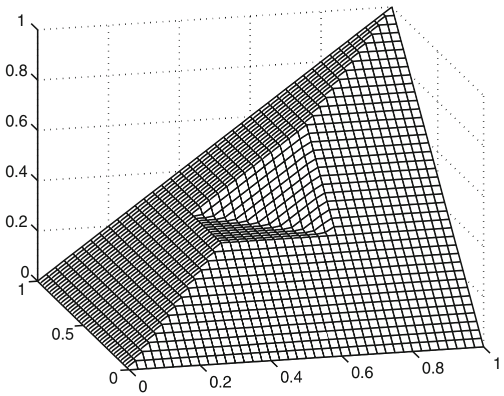
---

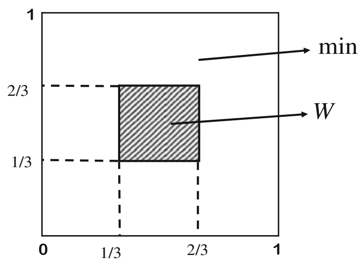
[2] For $(x < y)$ is evident that $\varphi(x) < \varphi(y)$ if either $x, y \in [0, n]$, or $x, y \in (n, 1]$. If $x \in [0, n]$, $y \in (n, 1]$ and $x < y$, since $h(x) + h(N(x)) < 1$, it is $h(x) < 1 - h(N(x))$, or $\varphi(x) < \varphi(y)$.

2.2.7 Continuous T-Norms and T-Conorms

As it was said, the only t-norm that is idempotent for all a in [0, 1], is $T = \min$, and the t-norms in $\{prod\} \cup \{W\}$ only have the idempotents 0 and 1. As it was also said, there are t-norms with several (but not all) idempotent elements. For example, the function

$$T(x, y) = \begin{cases} \frac{1}{3} + \frac{1}{3}W(3x-1, 3y-1), & \text{if } (x, y) \in [\frac{1}{3}, \frac{2}{3}]^2 \\ \min(x, y), & \text{otherwise ,} \end{cases}$$





that as it is easy to prove is a t-norm, verifies

- $T(x, x) = \min(x, x) = x$, if $x \notin [\frac{1}{3}, \frac{2}{3}]$
- $T(\frac{1}{3}, \frac{1}{3}) = \frac{1}{3} + \frac{1}{3}W(0, 0) = \frac{1}{3} + \frac{1}{3}.0 = \frac{1}{3}$
- $T(\frac{2}{3}, \frac{2}{3}) = \frac{1}{3} + \frac{1}{3}W(1, 1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$
- $T(\frac{1}{2}, \frac{1}{2}) = \frac{1}{3} + \frac{1}{3}W(\frac{2}{3} - 1, \frac{2}{3} - 1) = \frac{1}{3} + \frac{1}{3}W(\frac{1}{2}, \frac{1}{2}) = \frac{1}{3} + \frac{1}{3}.0 = \frac{1}{3} \neq \frac{1}{2}$
- etc.

that is, all elements in $[0, 1] - [\frac{1}{3}, \frac{2}{3}]$, as well $\frac{1}{3}$ and $\frac{2}{3}$ are idempotent for T, and the elements in $(\frac{1}{3}, \frac{2}{3})$ are not idempotent.

Look that an analogous result is obtained when changing W by prod in the above expression of T. Without proof it follows the theorem that completely characterizes all continuous t-norms.

Theorem 2.2.33 *T is a continuous t-norm if and only if,*

1. $T = \min$, *T is in the family of min*
2. $T = prod_\varphi$, *T is in the family of prod*
3. $T = W_\varphi$, *T is in the family of W*
4. *There exist an index set (finite or countable infinite), a family of pairwise disjoint open intervals in* [0, 1], $\{(a_1, b_1); i \in I\}$, *and a family of t-norms* $T_i \in \{prod\} \cup \{W\}(i \in I)$, *such that*

$$T(x, y) = \begin{cases} a_i + (b_i - a_i)T_i(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}), & \textit{if } (x, y) \in [a_i, b_i]^2 \\ \min(x, y), & \textit{otherwise,} \end{cases}$$

for any x, y *in* [0, 1.

The continuous t-norm of the fourth type are called *ordinal-sums* of the continuous t-norms $T_i \in \{prod\} \cup \{W\}$.

Remark 2.2.34 Why the names t-norm and t-conorm? The "t" comes from "triangular", because these functions were introduced to formalize the triangular property of probabilistic distances, i.e. distances whose values are something like the probability that the numerical distance between two points is less than a given number. They were introduced by Karl Menger with the name triangular-norms without considering associativity.

The name of t-conorm refers to the duality with a t-norm, since S is a t-conorm if and only if $1 - S(1 - x, 1 - y)$ is a t-norm. In general, it should be pointed out that, for each strong negation N, S is a t-conorm if and only if $N \circ S \circ (N \times N)$ is a t-norm.

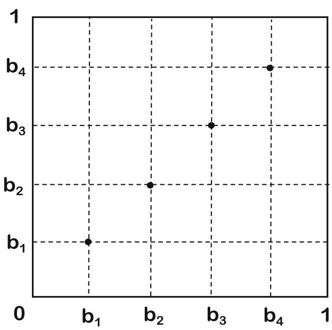
Theorem 2.2.35 S *is a continuous t-conorm if and only if,*

1. $S = \max$, *S is in the family of max*
2. $S = prod^*_\varphi$, *S is in the family of prod**
3. $S = W^*_\varphi$, *S is in the family of* W^*
4. *There exist an index set (finite or countable infinite), a family of pairwise disjoint open intervals of* $[0, 1], \{(a_1, b_1); i \in I\}$, *and a family of t-conorms* $S_i \in \{prod^*\} \cup \{W^*\} (i \in I)$, *such that*

$$S(x, y) = \begin{cases} a_i + (b_i - a_i)S_i(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}), & \textit{if } (x, y) \in [a_i, b_i]^2 \\ \max(x, y), & \textit{otherwise,} \end{cases}$$

for any x, y *in* [0, 1].

Remark 2.2.36 Notice that with both ordinal-sums of t-norms and of t-conorms, provided it is $[0, 1] = [0, b_1] \cup [b_1, b_2] \cdots \cup [b_{n-1}, b_n] \cup [b_n, 1]$, a finite partition of the unit interval [0, 1], like, for example



the only idempotent elements are b_1, b_2, b_3, b_4, etc., as well as 0 and 1, that is, the points giving the partition of [0, 1].

Remark 2.2.37 Although currently only continuous t-norms in $\{\min\} \cup \{prod\} \cup \{W\}$ are taken into account in both theoretic fuzzy logic and its applications, it should be pointed out that provided there are, at least, two statements 'x is P' and 'x is Q' such that $\mu_P(x), \mu_Q(x) \notin \{0, 1\}$ and $\mu_{P\ and\ P} = \mu_P$, $\mu_{Q\ and\ Q} = \mu_Q$, the only possibility for representing $\mu \cdot \sigma = T \circ (\mu \times \sigma)$, is by taking as continuous t-norm T an ordinal-sum with the single interval $(\min(\mu_P(x), \mu_Q(x)), \max(\mu_P(x), \mu_Q(x)))$.

Remark 2.2.38 Which t-norms are strictly non-decreasing in the sense that if $0 < a < b < 1$, then $T(a, c) < T(b, c)$ for all $c \in [0, 1]$?

- If $T = \min$, the answer is negative. For example, $0.3 < 0.5$, but $\min(0.2, 0.3) = \min(0.2, 0.5) = 0.2$
- If $T = W_\varphi$, the answer is also negative. For example, $0.3 < 0.5$, but $W(0.2, 0.3) = W(0.2, 0.5) = 0$
- If $T = prod_\varphi$, the answer is positive, since: $a < b \Rightarrow \varphi(a) < \varphi(b) \Rightarrow \varphi(a) \cdot \varphi(c) < \varphi(b) \cdot \varphi(c) \Rightarrow \varphi^{-1}(\varphi(a) \cdot \varphi(c))) < \varphi^{-1}(\varphi(b) \cdot \varphi(c)))$, or $prod_\varphi(a, c) < prod_\varphi(b, c)$, because $\varphi(c) \in (0, 1]$.
- If T is an ordinal-sum, it can't be strictly non-decreasing because of the values it takes with min.

  Analogously, the only t-conorms that are strictly non-decreasing are those in $\{prod^*_\varphi\}$.

2.2.8 Laws of Fuzzy Sets

As it was said, in all standard algebras $([0, 1]^X, T, S, N)$ of fuzzy sets the triplet (T, S, N) share the following common properties:

1. T and S are commutative and associative
2. 1 is neutral for T, and 0 is neutral for S
3. 0 is absorbent for T, and 1 is absorbent for S
4. For all T and S, it is $T \leqslant \min < \max \leqslant S$
5. Each T (S) is non decreasing in the two variables
6. N is involutive, strictly decreasing and such that $N(0) = 1$,

a list of properties that gives some basic laws for fuzzy sets in the standard algebras, like

- $\mu \cdot \sigma = \sigma \cdot \mu$, $\mu + \sigma = \sigma + \mu$,
- $\mu + (\sigma + \lambda) = (\mu + \sigma) + \lambda = (\sigma + \mu) + \lambda = \lambda + (\sigma + \mu)$
- $\mu \cdot \mu_1 = \mu$, $\mu + \mu_0 = \mu$, $\mu + \mu_1 = \mu_1$, $\mu \cdot \mu_0 = \mu_0$
- If $\mu \leqslant \sigma$, then $\mu \cdot \lambda \leqslant \sigma \cdot \lambda$, and $\lambda + \mu \leqslant \sigma + \lambda$
- etc.

Anyway, *a lot of laws typical of classical sets are not always valid* in all standard algebras of fuzzy sets. For example, $(\mathbb{P}(X), \cap, \cup, ^c)$ is a Boolean algebra and no one $([0,1]^X, T, S, N)$ is a Boolean algebra. In particular, $(\mathbb{P}(X), \cap, \cup)$ is a lattice and the only standard algebra that is a lattice is that with $T = \min$ and $S = \max$. Let us study in which standard algebras some laws of crisp sets do hold.

2.2.8.1 Distributive Laws

With classical sets it always do hold the two distributive laws

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

and the question is for which triplets (T, S, N) do hold the corresponding laws with fuzzy sets

1. $\mu \cdot (\sigma + \lambda) = \mu \cdot \sigma + \mu \cdot \lambda$,
2. $\mu + (\sigma \cdot \lambda) = (\mu + \sigma) \cdot (\mu + \lambda)$.

This questions correspond to solve the functional equations in the unknowns T and S:

$$T(a, S(b, c)) = S(T(a, b), T(a, c)) \tag{2.3}$$

$$S(a, T(b, c)) = T(S(a, b), S(a, c)) \tag{2.4}$$

for all a, b, c in $[0, 1]$.

Lemma 2.2.39 *Equation* (2.3) *does hold if and only if* $S = \max$.

Proof With $b = c = 1$, is $T(a, S(1, 1)) = S(T(a, 1), T(a, 1))$ or $a = S(a, a)$. That is $S = \max$.

Provided $S = \max$, the equation is $T(a, \max(b, c)) = \max(T(a, b), T(a, c))$. If either $b \leqslant c$ or $c \leqslant b$, it is immediate to check its validity for all t-norm T. □

Lemma 2.2.40 *Equation* (2.4) *does hold if and only if* $T = \min$.

Proof With $b = c = 0$, is $S(a, 0) = a = T(a, a)$. That is $T = \min$.

Provided $T = \min$, the equation is $S(a, \min(b, c)) = \min(S(a, b), S(a, c))$. If either $b \leqslant c$ or $c \leqslant b$, it is immediate to check its validity for all t-conorm T. □

Hence,

- In all standard algebras with $(T, \max)$, it holds $\mu \cdot (\sigma + \lambda) = \mu \cdot \sigma + \mu \cdot \lambda$,
- In all standard algebras with $(\min, S)$, it holds $\mu + (\sigma \cdot \lambda) = (\mu + \sigma) \cdot (\mu + \lambda)$
- The two distributive laws (2.3) and (2.4) do jointly hold if and only if $T = \min$ and $S = \max$.

2.2.8.2 De Morgan Laws

With classical sets it always do hold the De Morgan, or duality, laws

1. $A \cup B = (A^c \cap B^c)^c$
2. $A \cap B = (A^c \cup B^c)^c$

showing that one of the two operators $\cap, \cup$ can be defined by the other and the complementation. With fuzzy sets in standard algebras, these laws are

1. $\mu \cdot \sigma = (\mu' + \sigma')'$, or $(\mu \cdot \sigma)' = \mu' + \sigma'$
2. $\mu + \sigma = (\mu' \cdot \sigma')'$, or $(\mu + \sigma)' = \mu' \cdot \sigma'$,

that correspond to the functional equations in the unknowns T, S and N

- $T(a, b) = N(S(N(a), N(b)))$
- $S(a, b) = N(T(N(a), T(b)))$

for all a, b in $[0, 1]$.

Obviously, law (1) does hold if and only if $T = N \circ S \circ (N \times N)$ and law (2) does hold if and only if $S = N \circ T \circ (N \times N)$, two formulas that are equivalent since, for example, from the first it follows (with $N^2 = \mathrm{id}$) $N \circ T = S \circ (N \times N)$ or $N \circ S = T \circ (N \times N)$, that is, the second formula.

Hence, *the two* De Morgan *laws hold in an algebra given by the triplet* (T, S, N) *if and only if* $T = N \circ S \circ (N \times N)$ *, that is,* T *and* S *are* N*-dual.*

2.2.8.3 Restricted Non-contradiction Principle $\mu \cdot \mu' = \mu_0$

With classical sets it always holds $A \cap A^c = \emptyset$. With fuzzy sets, when is it

$$\mu \cdot \mu' = \mu_0 \ ?$$

The equation to be solved is

$$T(a, N(a)) = 0, \quad \text{for all a } \in [0, 1]$$

in the unknowns T and N.

Theorem 2.2.41 *If* T *is a continuous t-norm, and* N *is a strong negation, it is* $T(a, N(a)) = 0$ *for all* $a \in [0, 1]$*, if and only if* $T = W_\varphi$ *and* $N \leqslant N_\varphi$*.*

Proof With the fixed point $n \in (0, 1)$ of N, it follows $T(n, n) = 0$, that is, T has zero-divisors. Hence, $T = W_\varphi$, and $W_\varphi(a, N(a)) = \varphi^{-1}(\max(0, \varphi(a)+\varphi(N(a))-1)) = 0$, or $\max(0, \varphi(a) + \varphi(N(a)) - 1) = 0$, or $\varphi(a) + \varphi(N(a)) - 1 \leqslant 0$, that implies $\varphi(N(a)) \leqslant 1 - \varphi(a)$, or $N(a) \leqslant \varphi^{-1}(1 - \varphi(a)) = N_\varphi(a)$, for all $a \in [0, 1]$. Hence $N \leqslant N_\varphi$. The reciprocal is a simple calculation. $\square$

Then, the (restricted) non-contradiction principle $\mu \cdot \mu' = \mu_0$ holds if and only if $T = W_\varphi$ and $N \leqslant N_\varphi$, for any order-automorphism φ and any t-conorm S. For example, it holds (with $\varphi = \text{id}$) if $T = W$, $S = \max$, $N = N_0$, and it does not hold provided $T = \min$, or $T = prod_\varphi$.

2.2.8.4 Restricted Excluded-Middle Principle $\mu + \mu' = \mu_1$

With classical sets it always holds $A \cup A^c = X$. When is it $\mu + \mu' = \mu_1$ for fuzzy sets? When it does hold the equation $S(a, N(a)) = 1$ for all $a \in [0, 1]$?

Theorem 2.2.42 *If S is a continuous t-conorm, and N is a strong negation, it is $S(a, N(a)) = 1$ for all $a \in [0, 1]$, if and only if $S = W^*_\psi$ and $N_\psi \leqslant N$.*

Proof With $N(n) = n \in (0, 1)$, it follows $S(n, n) = 1$. That is $S = W^*_\psi$, and $1 = W^*_\psi(a, n(a)) = \psi^{-1}(\min(1, \psi(a) + \psi(N(a))))$, or $1 = \min(1, \psi(a) + \psi(N(a)))$. Hence, $1 \leqslant \psi(a)+\psi(N(a))$, or $N_\psi(a) = \psi^{-1}(1-\psi(a) \leqslant N(a))$. That is, $N_\psi \leqslant N$. The reciprocal is a simple calculation. □

Then, the (restricted) excluded-middle principle $\mu + \mu' = \mu_1$ holds if and only if $S = W^*_\psi$ and $N_\psi \leqslant N$, for any order automorphism ψ and any t-norm T. For example, it holds (with $\psi = \text{id}$) if $S = W^*$, $T = \min$, $N = N_0$, but it does not hold provided $S = \max$ or $S = prod^*$.

2.2.8.5 Both Restricted Principles of Non-contradiction and Excluded-Middle

From last theorems it immediately follows that,

Theorem 2.2.43 *In a standard algebra of fuzzy sets with a triplet (T, S, N), it holds $\mu \cdot \mu' = \mu_0$ and $\mu + \mu' = \mu_1$ if and only if $T = W_\varphi$, $S = W^*_\psi$, and $N_\psi \leqslant N \leqslant N_\varphi$.*

In particular, they hold if $T = W$, $S = W^*$, and $N = N_0$, or with $\varphi(x) = x^2$ and $\psi(x) = x$, they hold with the triplet:

$$T(x, y) = \sqrt{\max(0, x^2 + y^2 - 1)}, \ S(x, y) = \min(1, x + y),$$

$$1 - x \leqslant N(x) \leqslant \sqrt{1 - x^2},$$

for all x, y in $[0, 1]$.

Of course, with $\varphi = \psi$, the principles hold with W_φ, W^*_φ, and $N = N_\varphi$.

2.2.8.6 Laws of Absorption

With classical sets, the absorption laws $A \cap (A \cup B) = A$, and $A \cup (A \cap B) = A$, always hold. With fuzzy sets in standard algebras, the formulas and respective equations

- $\mu \cdot (\mu + \sigma) = \mu,\ T(a, S(a, b)) = a$
- $\mu + (\mu \cdot \sigma) = \mu,\ S(a, T(a, b)) = a$

must be studied to find for which algebras these laws do hold.

Lemma 2.2.44 *If T and S are, respectively, a t-norm and a t-conorm, it is $T(a, S(a, b)) = a$ for all a, b in* $[0, 1]$ *if and only if* $T = \min$.

Proof If $T = \min$, since $a \leqslant S(a, b)$, it follows $\min(a, S(a, b)) = a$. With $b = 0$, the equation gives $T(a, a) = a$, and $= \min$. □

Lemma 2.2.45 *If T and S are, respectively, a t-norm and a continuous t-conorm, it is $S(a, T(a, b)) = a$ for all a, b in* $[0, 1]$ *if and only if* $S = \max$.

Proof Since $T(a, b) \leqslant a$, it follows $\max(a, T(a, b)) = a$. With $b = 1$, the equation gives $S(a, a) = a$, and $= \max$. □

Hence,

- The law $\mu \cdot (\mu + \sigma) = \mu$, holds for all S and $T = \min$
- The law $\mu + (\mu \cdot \sigma) = \mu$, holds for all T and $S = \max$
- The two laws hold jointly if and only if $T = \min$ and $S = \max$.

2.2.8.7 The Law of von Neumann

With classical sets it always holds the law of von Neumann, or law of the perfect repartition,

$$A = (A \cap B) \cup (A \cap B^c),$$

that follows from $A = A \cap X = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c)$, and generalizes that of the excluded-middle since $A = X$ implies $X = (B \cap X) \cup (X \cap B^c) = B \cup B^c$.

From that law, by duality it follows $A^c = (A \cap B)^c \cap (A \cap B^c)^c = (A^c \cup B^c) \cap (A^c \cup B)$, that is

$$A = (A \cup B) \cap (A \cup B^c),$$

a law that generalizes that of non-contradiction since $A = \varnothing$ implies $\varnothing = B \cap B^c$.

With fuzzy sets, the question is the validity of the laws

$$\mu = \mu \cdot \sigma + \mu \cdot \sigma', \quad \mu = (\mu + \sigma) \cdot (\mu + \sigma'),$$

or of the functional equations

$$a = S(T(a, b), T(a, N(b))), \quad a = T(S(a, b), S(a, N(b)))$$

Lemma 2.2.46 *The equation* $a = S(T(a, b), T(a, N(b)))$ *holds if and only if* $T = prod_\varphi$, $S = W^*_\varphi$, $N = N_\varphi$.

Proof If $T = prod_\varphi, S = W^*_\varphi, N = N_\varphi$, it is $S(T(a, b), T(a, N(b))) = W^*_\varphi(prod_\varphi(a, b), prod_\varphi(a, N_\varphi(b))) = \varphi^{-1}(W^*(\varphi(prod_\varphi(a, b)), \varphi(prod_\varphi(a, N_\varphi(b))))) = \varphi^{-1}(W^*(\varphi(a)\cdot\varphi(b), \varphi(a)\cdot\varphi(N_\varphi(b)))) = \varphi^{-1}(\min(1, \varphi(a)\cdot\varphi(b)+\varphi(a)\cdot\varphi(N_\varphi(b)))) = \varphi^{-1}(\min(1, \varphi(a) \cdot \varphi(b) + \varphi(a) \cdot (1 - \varphi(b)) = \varphi^{-1}(\min(1, \varphi(a) \cdot \varphi(b) + \varphi(a) - \varphi(a) \cdot \varphi(b))) = \varphi^{-1}(\min(1, \varphi(a)) = a$.

The proof of the reciprocal will be avoided since it is technically complex. Let us say only that $a = S(T(a, b), T(a, N(b)))$ gives, with $a = 1$, $1 = S(b, N(b))$, that implies $S = W^*$ and $N_\varphi \leqslant N$. □

It can be also proven that $a = T(S(a, b), S(a, N(b)))$ if and only if $T = W_\varphi$, $S = prod^*_\varphi$, $N = N_\varphi$. Notice only that $a = 0$ gives $T(b, N(b)) = 0$, or $T = W_\varphi$ and $N = N_\varphi$.

Notice that the verification of von Neumann's law require, in the case of fuzzy sets, non-dual theories, like those given by the triplets $(prod_\varphi, W^*_\varphi, N_\varphi)$, and $(W_\varphi, prod^*_\varphi, N_\varphi)$.

2.2.8.8 Which Standard Algebra Is Closer to a Boolean Algebra?

The results in last section can be summarized in the following Table 2.1.

Hence, the algebras with the triplets (min, max, N) are the ones that preserve more structural Boolean properties. Indeed, these algebras preserve all the basic Boolean laws except those of non-contradiction and excluded-middle. They are distributive pseudo-complemented lattices, that is, De Morgan algebras that, in addition and like all algebras of fuzzy sets, verify the law of Kleene,

$$T(a, N(a)) \leqslant S(b, N(b)),$$

for all a, b in $[0, 1]$. The algebras given by the triplets (min, max, N) are De Morgan-Kleene algebras.

2.2.8.9 Last Comments

It can be considered, in addition to the structural Boolean laws, the cases that can be derived from them, for example,

$$(A \cap B^c)^c = B \cup (A^c \cap B^c),$$

Table 2.1 Basic boolean properties

	T	**S**	**N**
Lattice	Min	Max	All
Identity			
$T(a, 1) = a, T(a, 0) = 0$	All	–	–
$S(a, 0) = a, S(a, 1) = 1$	–	All	–
Commutativity			
$T(a, b) = T(b, a)$	All	–	–
$S(a, b) = S(b, a)$	–	All	–
Associativity			
$T(a, T(b, c)) = T(T(a, b), c)$	All	–	–
$S(a, S(b, c)) = S(S(a, b), c)$	–	All	–
Involution			
$N(N(a)) = a$	–	–	All
B1. Idempotency			
$T(a, a) = a$	Min	–	–
$S(a, a) = a$	–	Max	–
Distributivity			
$T(a, S(b, c)) = S(T(a, b), T(a, c))$	All	Max	–
$S(a, T(b, c)) = T(S(a, b), S(a, c))$	Min	All	–
Absorption			
$T(a, S(a, b)) = a$	Min	All	–
$S(a, T(a, b)) = a$	All	Max	–
Non-contradiction			
$T(a, N(a)) = 0$	W_φ	–	$N \leqslant N_\varphi$
Excluded-middle			
$S(a, N(a)) = 1$	–	W^*_φ	$N \geq N_\varphi$
De Morgan's laws			
$N(T(a, b)) = S(N(a), N(b))$	$T = N \circ S \circ N \times N$		
$N(S(a, b)) = T(N(a), N(b))$	$T = N \circ S \circ N \times N$		

that follows from $B \cup (A^c \cap B^c) = (B \cup A^c) \cap (B \cup B^c) = (B \cup A^c) \cap X = B \cup A^c = (A \cap B^c)^c$. This law, in fuzzy set theory, is translated by

$$(\mu \cdot \sigma')' = \sigma + \mu' \cdot \sigma'$$

or

$$N(T(a, N(b))) = S(b, T(N(a), N(b))),$$

that holds with $T = prod_\varphi$, $S = W^*_\varphi$, $N = N_\varphi$.

Nevertheless, not all derived law has solution within the standard algebras of fuzzy sets, as it is the case with $(A \cup A) \cap (A \cap A^c) = \varnothing$ (or $A \cap \varnothing = \varnothing$), since for

$$\mu \cdot \mu + \mu \cdot \mu' = \mu_0, \quad [\star]$$

there are no triplets (T, S, N) for which it can hold $S(T(a, a), T(a, N(a))) = 0$ for all a in $[0, 1]$.

In the same vein, there are some laws that have solutions when different t-norms, t-conorms and strong negations are considered. For example, $(\mu + \mu) \cdot (\mu \cdot \mu') = \mu_0$, that comes from $(A \cup A) \cap (A \cap A^c) = \varnothing$, translated in the form $T_1(S(a, a), T_2(a, N(a))) = 0$, has infinite solutions like, for example, with an strong negation N, such that $N \leqslant N_0$, $T_1 = \min$, $T_2 = W$ and any t-conorm S, since $\min(S(a, a), T_2(a, N(a)) = T_2(a, N(a)) = W(a, N(a)) = \max(0, a + N(a) - 1) = 0$, because of $T_2(a, N(a)) \leqslant a \leqslant S(a, a)$, and $N(a) \leqslant 1 - a$, or $a + N(a) - 1 \leqslant 0$.

Another case is given by the classical (derived) laws

$$A \cap (A^c \cup B) = A \cap B, \quad A \cup (A^c \cap B) = A \cup B,$$

and the corresponding 'possible' fuzzy laws

$$\mu \cdot (\mu' + \sigma) = \mu \cdot \sigma, \ \mu + (\mu \cdot \sigma) = \mu + \sigma,$$

which functional equations

$$T_1(a, S(N(a), b)) = T_2(a, b), \ S_1(a, T(N(a), b)) = S_2(a, b),$$

do not have solutions with $T_1 = T_2$ and $S_1 = S_2$, respectively, but that with $N = N_0$, W and W^* do verify

- $W(a, W^*(1 - a, b)) = \max(0, \min(a, b)) = \min(a, b)$
- $W^*(a, W(1 - a, b)) = \min(1, \max(a, b)) = \max(a, b)$

that is, they have the solutions $(T_1 = W, S = W^*, T_2 = \min)$ and $(S_1 = W^*, T = W, S_2 = \max)$, respectively. Thus, it is possible to consider more complex algebras of fuzzy sets by means of n-tuples of the type $(T_1, \ldots, T_m; S_1, \ldots, S_r; N_1, \ldots, N_p)$.

Notwithstanding, there are more derived laws than $[\star]$ that have no solutions neither in standard algebras, nor with different t-norms, t-conorms, or different strong negations. The fact that no standard algebra of fuzzy sets is a Boolean algebra, makes impossible to simultaneously deal in such algebras with all formulas that are valid with classical sets.

2.2.9 Examples

Example 2.2.47 In a scale between 10 and 50 °C, the label 'cold' referred to temperature, is graduated by

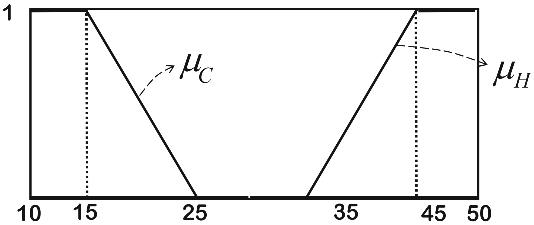
$$\mu_C(x) = \begin{cases} 1, & \text{if } 10 \leqslant x \leqslant 15 \\ \frac{25-x}{10}, & \text{if } 15 \leqslant x \leqslant 25 \\ 0, & \text{if } 25 \leqslant x \leqslant 50. \end{cases}$$

Which one of the following linguistic labels: *cold, hot, warm, more or less cold, more or less warm*, is the more adequate for the temperatures of 20, 21 and 22 °C?

Solution. With C = cold, it is $\mu_C(20) = \frac{5}{10} = 0.5$ and $\mu_{more\ or\ less\ C}(20) = \sqrt{0.5} = 0.71$. To obtain $H = hot$, we can compute μ_H as the opposite of μ_C:

$$\mu_H(x) = \mu_C(50 + 10 - x) = \mu_C(60 - x) = \begin{cases} 1, & \text{if } 45 \leqslant x \leqslant 50 \\ \frac{x-35}{10}, & \text{if } 35 \leqslant x \leqslant 45 \\ 0, & \text{if } 10 \leqslant x \leqslant 35 \end{cases}$$

graphically



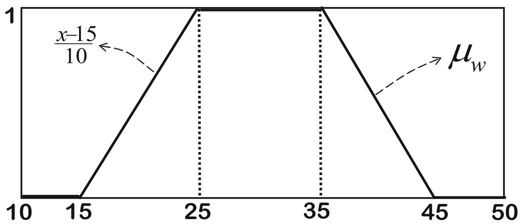
By defining, as it is usual, warm = not cold and not hot, that is

$$\mu_w = \mu'_{cold} \cdot \mu'_{hot}$$

with $\cdot$ represented by min, and $'$ by N_0,

$$\mu_w(x) = \min(1 - \mu_{cold}(x), 1 - \mu_{hot}(x)),$$

for all $x \in [10, 50]$, it results



Then:

- $x = 20$, gives $\mu_c(20) = \frac{5}{10} = 0.5$, $\mu_H(20) = 0$, $\mu_W(20) = \frac{5}{10} = 0.5$,
- $x = 21$, gives $\mu_c(21) = \frac{4}{10} = 0.4$, $\mu_H(21) = 0$, $\mu_W(21) = \frac{6}{10} = 0.6$,
- $x = 22$, gives $\mu_c(22) = \frac{3}{10} = 0.3$, $\mu_H(22) = 0$, $\mu_W(22) = \frac{7}{10} = 0.7$,

and

- $\mu_{more\ or\ less\ cold}(20) = \sqrt{\mu_c(20)} = 0.71$,
  $\mu_{more\ or\ less\ warm}(20) = \sqrt{\mu_W(20)} = 0.71$
- $\mu_{more\ or\ less\ ;cold}(21) = 0.63$, $\mu_{more\ or\ less\ ;warm}(21) = 0.77$,
- $\mu_{more\ or\ less\ ;cold}(22) = 0.55$, $\mu_{more\ or\ less\ ;warm}(22) = 0.84$.

Hence

- The more adequate linguistic label for $x = 20$, cannot be decided but it could be either 'not cold', or 'not warm'. Since, it is not hot at all, we can take 'not cold'.
- For $x = 21$, is 'mol warm' (mol = more or less)
- For $x = 22$, is 'mol warm'

Example 2.2.48 On the age of a person p, it is known that

$$37 \leqslant Age(p) \leqslant 41,$$

and neither $Age(p) \leqslant 32$, nor $43 \leqslant Age(p)$. What can be said on the degree up to which it could be $Age(p) = 35$, and $Age(p) = 42$?

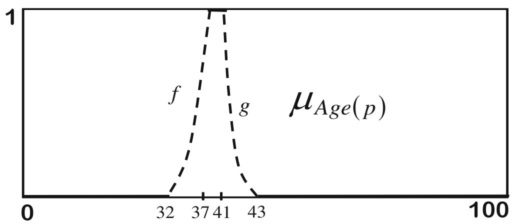
Solution. What is unknown is the variation of $Age(p)$ between 32 and 37, as well as between 41 and 43. Since Age varies continuously, we can suppose there are two functions

$$f : [32, 37] \to [0, 1], g : [41, 43] \to [0, 1]$$

such that $f(32) = 0, f(37) = 1, g(41) = 1, g(43) = 0$, with f strictly non-decreasing, and g strictly decreasing. Then, we can define $\mu_{Age(p)} : [0, 100] \to [0, 1]$, by

$$\mu_{Age(p)}(x) = \begin{cases} 0, & \text{if } x \in [0, 32] \cup [43, 100] \\ 1, & \text{if } x \in [37, 41] \\ f(x), & \text{if } x \in [32, 37] \\ g(x), & \text{if } x \in [41, 43] \end{cases}$$

and $\mu_{Age(p)}(35) = f(35)$, $\mu_{Age(p)}(42) = g(42)$. Graphically



To determine f and g more information is needed, but in the absence of it, we can decide to take the linear models $f(x) = \frac{x-32}{5}$, and $g(x) = \frac{43-x}{2}$, with which

$$\mu_{Age(p)}(35) = \frac{3}{5}, \quad \mu_{Age(p)}(42) = \frac{1}{2}.$$

As it will be seen later on, 0.6 is the *possibility* that $Age(p) = 35$, and 0.5 that of $Age(p) = 42$. Hence, it seems a little bit more possible that it be '$Age(p) = 42$' than '$Age(p) = 35$'.

Example 2.2.49 Knowing that Height(John) = 175 cm, and Height(Peter) = 180 cm, consider the two statements:

$p =$ It is false that John is not very tall or is more or less short
$q =$ It is false that Peter is not very tall or is more or less short.

which is more true?

Solution. Both statements can be written by

Is false that x is P,
with $P =$ '(not very tall) or (more or less short)'.

Hence

$$\mu_P(x) = S(\mu'_{very\ tall}(x), \mu_{not\ short}(x)) = S(N(\mu_{tall}(x)^2), \sqrt{\mu_{tall}(A(x))}\ ),$$

with a continuous t-conorm S, a strong negation N, and a symmetry A on X, provided x varies in a scale of heights.

What should be compared are the two values $N(\mu_P(175))$ and $N(\mu_P(180))$, and for that it is needed to know μ_{tall}. Let us take

$$\mu_{tall}(x) = \begin{cases} 0, & \text{if } x \in [0, 150] \\ \text{strictly non decreasing}, & \text{if } x \in [150, 190] \\ 1, & \text{if } x \in [190, 210] \end{cases}$$

with, perhaps, $\mu_{tall}(x) = 0.025x - 3.75$, $x \in [150, 190]$, if we need to have numbers.

Hence, with $A(x) = 210 - x$, it is $A(175) = 210 - 175 = 35$, and $\mu_{tall}(35) = 0$, as well as $A(180) = 220 - 180 = 30$, and $\mu_{tall}(30) = 0$, because of that

$$\mu_P(175) = S(N(\mu_{tall}(175)), 0) = N(\mu_{tall}(175)^2)$$
$$\mu_P(180) = S(N(\mu_{tall}(180)), 0) = N(\mu_{tall}(180)^2).$$

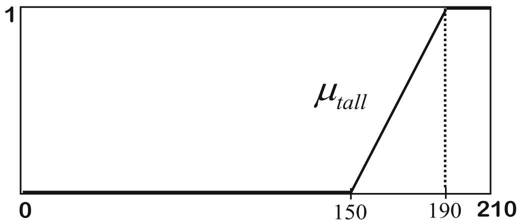
Since μ_{tall} is strictly non-decreasing between 150 and 190, it is $\mu_{tall}(175) < \mu_{tall}(180)$, and $N(\mu_{tall}(180)^2) < N(\mu_{tall}(175)^2)$. Finally,

$$N(N(\mu_{tall}(175)^2)) < N(N(\mu_{tall}(180)^2)), \text{ or } \mu_{tall}(175)^2 < \mu_{tall}(180)^2,$$

and q is strictly more true than p.

Notice that it is not needed to fix S and N, but only a form for μ_{tall}, as well as to accept that $\mu_{very\ P}(x) = \mu_P(x)^2$, $\mu_{mol\ P}(x) = \sqrt{\mu_P(x)}$, and $A(x) = 210 - x$. This last hypotheses is perfectly reasonable since μ_{tall} is non-decreasing, and then the order $\leqslant_{\mu_{tall}}$ is just the order of [0, 210].

Provided we need to know up to which numerical degree p and q do hold, we can use the linear function in the figure,



and fix $N = N_0$. It results

degree up to which p is true $= 1 - (1 - \mu^2_{tall}(175)) = \mu^2_{tall}(175) = 0.391$
degree up to which q is true $= \mu^2_{tall}(180) = 0.563$,

that shows how is q more true than p.

Example 2.2.50 It is known that the algebra with which fuzzy sets must be combined should satisfy the laws

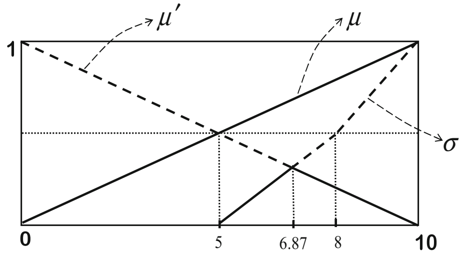
$$\mu + \mu \cdot \sigma = \mu, \quad \mu \cdot (\mu + \sigma) = \mu,$$

as well as that the negation is linear, Determine the triplet (T, S, N) and, with $X =$ [0, 10], and

$$\mu(x) = \frac{x}{10}, \ \sigma(x) = \begin{cases} 0, & \text{if } 0 \leqslant x \leqslant 5 \\ \frac{x-5}{6}, & \text{if } 5 \leqslant x \leqslant 8 \\ \frac{x-6}{4}, & \text{if } 8 \leqslant x \leqslant 10, \end{cases}$$

compute $\mu \cdot \sigma$, $\mu + \sigma$, and $\mu' + \sigma$.

Solution. The first law of absorption $\mu + \mu \cdot \sigma = \mu$, implies $S = \max$ for any T. The second law of absorption $\mu \cdot (\mu + \sigma) = \mu$, implies $T = \min$ for any S. Hence $(T, S) = (\min, \max)$, and the only linear N is $N = N_0$. Hence, $(T, S, N) = (\min, \max, 1 - id)$. With the graphics of μ and σ in the figure,



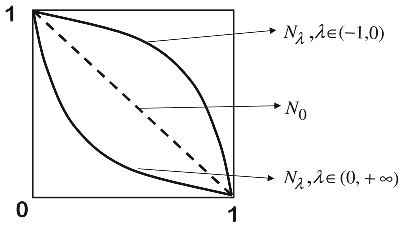
it follows $\mu \cdot \sigma = \min(\mu, \sigma) = \sigma$, $\mu + \sigma = \max(\mu, \sigma) = \mu$, and $\mu' + \sigma = (1 - \mu) + \sigma$ is the pointed curve. That is,

$$(\mu' + \sigma)(x) = \begin{cases} \mu'(x), & \text{if } x \in [0, 6.87] \\ \sigma(x), & \text{if } x \in [6.87, 10]. \end{cases}$$

Notice that $x = 6.87$ comes from the equation $1 - \frac{x}{10} = \frac{x-5}{6}$.

Example 2.2.51 Consider the Sugeno's family of strong negations $N_\lambda(x) = \frac{1-x}{1+\lambda x}(\lambda > -1)$. If $-1 < \lambda_1 < \lambda_2$, it follows $1 + \lambda_1 x < 1 + \lambda_2 x$, or $\frac{1}{1+\lambda_2 x} < \frac{1}{1+\lambda_1 x}$, that is, $N_{\lambda_2}(x) < N_{\lambda_1}(x)$. Hence,

If $\lambda \leqslant 0 : N_0 \leqslant N_\lambda$
If $0 \leqslant \lambda : N_\lambda \leqslant N_0$



Compare the graphics of $\mu' = N_0 \circ \mu$ and $\mu' = N_1 \circ \mu$, in a figure, if

$$\mu(x) = \begin{cases} 0, & \text{if } x \in [0, 3] \cup [7, 10] \\ 1, & \text{if } x \in [4, 6] \\ x - 3, & \text{if } x \in [3, 4] \\ 7 - x, & \text{if } x \in [6, 7] \end{cases}$$

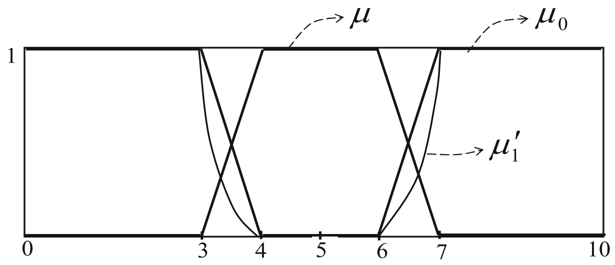
Solution. It is

$$\mu'(x) = \begin{cases} 1, & \text{if } x \in [0, 3] \cup [0, 4] \\ 0, & \text{if } x \in [4, 6] \\ \frac{4-x}{x}, & \text{if } x \in [3, 4] \\ \frac{x-6}{x}, & \text{if } x \in [6, 7], \end{cases}$$

and $\mu_1'(x) = N_1(\mu(x)) = \frac{1-\mu(x)}{1+\mu(x)}$, or

$$\mu_1'(x) = \begin{cases} 0, & \text{if } x \in [0, 3] \cup [7, 10] \\ 1, & \text{if } x \in [4, 5] \\ \frac{4-x}{x-2}, & \text{if } x \in [3, 4] \\ \frac{x-6}{8-x}, & \text{if } x \in [6, 7] \end{cases}$$

Hence,



Look that $N_1(3.2) = 0.\widehat{6}$, $N_1(3.5) = 0.\widehat{3}$, $N_1(3.4) = 0.43$, $N_1(3.6) = 0.25$, $N_1(3.8) = 0.\widehat{1}$, but $N_0(3.2) = 0.8$, $N_0(3.5) = 0.5$, $N_0(3.6) = 0.4$, $N_0(3.4) = 0.6$, , $N_0(3.6) = 0.4$, $N_0(3.8) = 0.2$.

Example 2.2.52 The negation is linear, and the standard algebra must verify the law $\mu = \mu \cdot \sigma + \mu \cdot \sigma'$. Determine the triplet (T, S, N), and with $\mu(x) = \frac{x}{10}$ (in $X = [0, 10]$) and

$$\sigma(x) = \begin{cases} 1, & \text{if } x \in [0, 5] \\ \frac{7-x}{2}, & \text{if } x \in [5, 7] \\ 0, & \text{if } x \in [7, 10] \end{cases}$$

check that $\mu \cdot \sigma + \mu \cdot \sigma' = \mu$, $\mu \cdot \sigma + \mu' \cdot \sigma = \sigma$, and $(\mu \cdot \sigma')' = \sigma + \mu' \cdot \sigma'$.

Solution. From N linear, $N = N_0$, it follows that we can take $\varphi = \text{id}$, and from $\mu \cdot \sigma + \mu \cdot \sigma' = \mu$ it follows $T = \text{prod}_\varphi$, $S = W^*_\varphi$, and $N = N_\varphi$. Hence $(T, S, N) = (\text{prod}, W^*, N)$.

Since $\mu'(x) = 1 - \frac{x}{10}$, and $\sigma'(x) = \begin{cases} 0 \\ \frac{x-5}{2} \\ 1 \end{cases}$, it follows

$$\mu\sigma(x) = \begin{cases} \frac{x}{10} \\ \frac{x(7-x)}{20} \\ 0 \end{cases}, \quad \mu\sigma'(x) = \begin{cases} 0 \\ \frac{x(x-5)}{20} \\ \frac{x}{10} \end{cases}, \quad \mu'\sigma(x) = \begin{cases} 1 - \frac{x}{10} \\ \frac{7-x}{2}(1 - \frac{x}{10}) \\ 0 \end{cases}.$$

Hence,

$$(\mu \cdot \sigma + \mu \cdot \sigma')(x) = \left\{ \begin{array}{l} W^*(\frac{x}{10}, 0) = \frac{x}{10} \\ W^*(\frac{x(7-x)}{20}, \frac{x(x-5)}{20}) = \frac{x}{10} \\ W^*(0, \frac{x}{10}) = \frac{x}{10} \end{array} \right\} = \frac{x}{10} = \mu(x),$$

$$(\sigma \cdot \mu + \sigma \cdot \mu')(x) = \left\{ \begin{array}{l} W^*(\frac{x}{10}, 1 - \frac{x}{10}) = 1 \\ W^*(\frac{x(7-x)}{20}, \frac{7-x}{2}(1 - \frac{x}{10}) = \frac{7-x}{2} \\ W^*(0, 0) = 0 \end{array} \right\} = \sigma(x).$$

Finally, since,

$$(\mu \cdot \sigma')'(x) = 1 - (\mu \cdot \sigma')(x) = \begin{cases} 1 \\ 1 - \frac{x(x-5)}{20} \\ 1 - \frac{x}{10} \end{cases}, \text{ and } (\mu' \cdot \sigma')(x) = \begin{cases} 0 \\ (1 - \frac{x}{10})\frac{x-5}{2} \\ 1 - \frac{x}{10} \end{cases},$$

it results

$$(\sigma + (\mu' \cdot \sigma'))(x) = \begin{cases} W^*(1, 0) = 1 \\ W^*(\frac{7-x}{2}, \frac{x-5}{2}(1 - \frac{x}{10})) = 1 - \frac{x(x-5)}{20} = (\mu \cdot \sigma')'(x) \\ W^*(0, 1 - \frac{x}{10}) = 1 - \frac{x}{10} \end{cases}.$$

Example 2.2.53 Predicate F = *high fever* refers to the interval $[37, 42]$ in a clinical thermometer, in which the values $\{37, 37.5, 38, \ldots, 41.5, 42\}$ are significative. Asking an expert one obtains the following fuzzy set

$$\mu_F = 0.3/38.5 + 0.5/39 + 0.7/39.5 + 0.8/40$$
$$+ 0.9/40.5 + 1/41 + 1/41.5 + 1/42$$

where it is clear that $0/37 + 0/37.5 + 0/38$ is avoided since this values of the body's temperature are not significative for F. With all that, give the membership function of P = *very high fever*, Q = *more or less high fever*, R = *low fever, S* = *not high fever*.

Solution. With the usual definition $\mu_{very\ F} = \mu_F^2$, $\mu_{mol\ F} = +\sqrt{\mu_F}$, $\mu_{low\ F} = \mu_F(37 + 42 - x) = \mu_F(79 - x)$, $\mu_{not\ F} = 1 - \mu_F$, it results:

- $\mu_P = 0.09/38.5 + 0.25/39 + 0.49/39.5 + 0.64/40 + 0.81/40.5 + 1/41 + 1/41.5 + 1/42$.
- $\mu_Q = 0.55/38.5 + 0.7/39 + 0.84/39.5 + 0.89/40 + 0.95/40.5 + 1/41 + 1/41.5 + 1/42$.
- $\mu_R = 1/37 + 1/37.5 + 1/38 + 0.9/38.5 + 0.8/39.5 + 0.7/39.5 + 0.5/40.5 + 0.3/40.5$.
- $\mu_S = 1/37 + 1/37.5 + 1/38 + 0.7/38.5 + 0.5/39.5 + 0.3/39.5 + 0.2/40.5 + 0.1/40.5$.

Notice the incoherence produced by $\mu_S = \mu_{not\ F} \leqslant \mu_{low\ F} = \mu_R$. An incoherence showing that it cannot be taken the representation $\mu_{not\ F} = 1 - \mu_F$, but some $\mu_{not\ F} = N \circ \mu_F$ with $N \geq N_0$.

For example, if $N(x) = \frac{1-x}{1-0.9x}$, it is

$\mu_S = 1/37 + 1/37.5 + 1/38 + +0.96/38.5 + 0.91/39.5 + 0.81/39.5 + 0.71/40 + 0.53/40.5 + 1/41 + 1/41.5 + 1/42$, showing $\mu_{low\ F} \leqslant \mu_{not\ F}$.

Look that

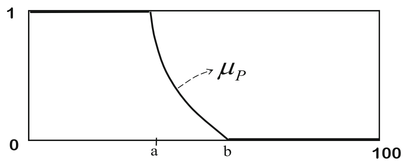
$\mu_{F\ \&\ low\ F} = 0.3/38.5 + 0.5/39.5 + 0.7/39.5 + 0.5/40 + 0.3/40.5$

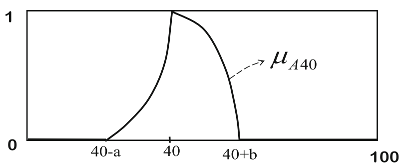
provided $\mu_{F\ \&\ low\ F} = \min(\mu_F, \mu_{low\ F})$.

Example 2.2.54 Describe in fuzzy terms, the statements

$p =$ John is young and around forty,
$q =$ John is old or around forty.

Solution. The solution will come after representing the predicates $P =$ *young, aP = Old, A40 = around forty*, in a scale of 0–100 years. The general forms of μ_P, μ_{aP}, and μ_{A40}, are





with $\mu_{aP}(x) = \mu_P(100 - x)$ since μ_P is non-decreasing. Once these functions were established accordingly with the current use of P and $A40$, it will be:

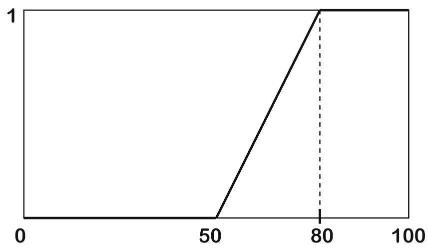
$$\text{Degree}(p) = T(\mu_P(x), \mu_{A40}(x)), \quad \text{Degree}(q) = S(\mu_{aP}(x), \mu_{A40}(x)),$$

with convenient continuous t-norm T and t-conorm S. This formulas are the description of p and q in fuzzy terms.

For example, if $a = 20, b = 50, \mu_P = \frac{50-x}{30}$, if $20 \leqslant x \leqslant 50$, and $40 - a = 30, 40 + b = 50$, with μ_{A40} piece-wise linear, $T = \min$, $S = \max$, with

$$\mu_{aP}(x) = \mu_P(100 - x) = \begin{cases} 0, & \text{if } 0 \leqslant x \leqslant 50 \\ \frac{x-50}{30}, & \text{if } 50 \leqslant x \leqslant 80 \\ 1, & \text{if } 80 \leqslant x \leqslant 100, \end{cases}$$

the graphics is



The slashed function describes p, and the continuous one describes q. Of course, $\text{Degree}(p) \leqslant \min(\mu_P(x), \mu_{A40}(x)) \leqslant \mu_P(35) = 0.5$.

Remark 2.2.55 To select T and S, the following points could be taken into account,

- It could be perfectly the case that 'John is young and not young' with a positive degree. Hence, the laws of Non-contradiction can be avoided, and $T \notin \{W\}$.

- Since it is reasonable to accept that ‘John is old’ *and* ‘John is old’ does coincide with ‘John is old’, and ‘John is young’ *or* ‘John is young’ does coincide with ‘John is young’ , we can decide to take either $T = \min$, $S = \max$, or T and S as ordinal-sums.
- Provided idempotency is avoided i.e., ‘John is young’ *and* ‘John is young’ does coincide with ‘John is very young’, instead of $T = \min$, we can take $T = \text{prod}$, that is more interactively than min. In this case, because it does not seem that duality should be avoided, we could take $S = \text{Prod}^*$, and then $\mu_{very\ young\ or\ mol\ old}(45) = \text{Prod}^*(\frac{1}{36}, \frac{\sqrt{6}}{6}) = \frac{1}{6}(1 + \frac{35\sqrt{6}}{36}) = 0.5636$, that is greater than the value $\frac{\sqrt{6}}{6} = 0.408$ obtained with $= \max$.

Example 2.2.56 In $X = [0, 10]$ the predicate $P = \text{big}$ is represented by $\mu(x) = \frac{x}{10}$. In which points in $[0, 10]$ is the degree of ‘big’ less than that of ‘not big’?

Solution. Given μ, the problem is to find for which $x \in X$ it is $\mu(x) \leqslant \mu'(x) = N_\varphi(\mu(x)) = \varphi^{-1}(1 - \varphi(\mu(x)))$, that is, $\mu(x) \leqslant \varphi^{-1}(\frac{1}{2})$. Then,

- If $N = N_0$, $\frac{x}{10} \leqslant 1 - \frac{x}{10}$, or $x \leqslant 5$.
- If $N = N_1$, $\frac{x}{10} \leqslant \frac{1-\frac{x}{10}}{1+\frac{x}{10}}$, or $x^2 + 20x - 100 \leqslant 0$, that means $x \leqslant 10(\sqrt{2} - 1) = 4.142$
- If $N = N_2$, $\frac{x}{10} \leqslant \frac{1-\frac{x}{10}}{1+\frac{2x}{10}}$, or $x^2 + 10x - 50 \leqslant 0$, that means $x \leqslant \sqrt{75} - 5 = 3.66$

Hence,

- If $N = N_0$, the set is $[0, 5]$, and the threshold (of selfcontradiction) of *big* is 5.
- If $N = N_1$, the set is $[0, 4.142]$, and the threshold is 4.142
- If $N = N_2$, the set is $[0, 3.66]$, and the threshold is 3.66

Notice that changing *big* by *not big*, the thresholds do remain but the sets are, respectively, $[5, 10]$, $[4.142, 10]$, and $[3.66, 10]$.

2.3 On Aggregating Imprecise Information

The kind of problems this section will deal with are like the following. An exam is corrected by three referees R_1, R_2, R_3, each one with a different weight of strongness $W(R_i) \in [0, 1]$, $1 \leqslant i \leqslant 3$, such that $\sum_{i=1}^{3} W(R_i) = 1$. Each referee assigns a numerical qualification $p_i \in [0, 10]$ to the exam delivered by a given student. How these qualification can be “aggregated” to obtain final qualification for the student’s exam? A recognized usual way of doing it is by the *weighted mean*:

$$\frac{1}{10}Q = \frac{p_1}{10} \cdot W(R_1) + \frac{p_2}{10} \cdot W(R_2) + \frac{p_3}{10} \cdot W(R_3),$$

with $\frac{p_i}{10} \in [0, 10]$. For example, if $W = (0.5, 0.3, 0.2)$ and $P = (7, 6, 5)$, it follows

$$\frac{1}{10}Q = 0.7 \times 0.5 + 0.6 \times 0.3 + 0.2 \times 0.5 = 0.63$$

that implies $Q = 6.3$. Provided the three referees have the same weight, it is $W = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and then, $Q = p_1 \cdot \frac{1}{3} + p_2 \cdot \frac{1}{3} + p_3 \cdot \frac{1}{3} = \frac{p_1+p_2+p_3}{3} = \frac{7+6+5}{3} = 6$, is just the arithmetic *mean* of the three qualifications.

Another way of obtaining the final qualification, this time by ignoring the referee's character, is by the *geometric mean*

$$Q = \sqrt[3]{p_1 \cdot p_2 \cdot p_3} = \sqrt[3]{7 \times 6 \times 5} = 5.94,$$

showing that in a problem with $p_1 = p_2 = 10, p_3 = 0$, it results $Q = \sqrt[3]{10 \times 10 \times 0} = 0$, when the arithmetic mean is $\frac{20}{3} = 6.67$.

2.3.1 Aggregation Functions

Most of these problems are "represented" by the so-called *Aggregation Functions*, that is, functions

$$A : [0, 1]^n \to [0, 1],$$

such that

1. A is continuous in all variables
2. $A(0, \ldots, 0) = 0$, and $A(1, \ldots, 1) = 1$
3. If $x_1 \leqslant y_1, \ldots, x_n \leqslant y_n$, then $A(x_1, \ldots, x_n) \leqslant A(y_1, \ldots, y_n)$.

Sometimes it is said that A is an *n-dimensional aggregation function*. Continuous t-norms and continuous t-conorms are 2-dimensional aggregation functions.

Of the many types of aggregation functions, a particular and important type are the *quasi-linear means*,

$$M(x_1, \ldots, x_n) = f^{-1}\left(\sum_{i=1}^{n} p_i . f(x_i)\right)$$

with $(p_1, \ldots, p_n)$ in $[0, 1]$, verifying $\sum_{i=1}^{n} p_i = 1$, and $f : [0, 1] \to \mathbb{R}$, continuous, one-to-one, and monotonic. Function f is called the *generator* of M.

Notice that if f is the identity $f(x) = x$, we get the *weighted means*:

$$M(x_1, \ldots, x_n) = \sum_{i=1}^{n} p_i \cdot x_i,$$

that with $p_1 = \frac{1}{n}$ $(1 \leqslant i \leqslant n)$ is the arithmetic mean

$$M(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

and with $f(x) = -\log x$, and $p_1 = \frac{1}{n}$ $(1 \leqslant i \leqslant n)$ is the geometric mean

$$M(x_1, \ldots, x_n) = \sqrt[n]{p_1 \cdot p_2 \cdots p_n}.$$

With $f(x) = x^\alpha$ $(\alpha > 0)$, is $f^{-1}(x) = x^{\frac{1}{\alpha}}$, and with $p_1 = \frac{1}{n}$, it is obtained the family of quasi-linear means,

$$M_\alpha(x_1, \ldots, x_n) = \left(\frac{x_1^\alpha + \cdots + x_n^\alpha}{n}\right)^{\frac{1}{\alpha}}.$$

In particular, with $\alpha = 1$, is M_1 the arithmetic mean, and with $\alpha = -1$, it follows

$$M_{-1}(x_1, \ldots, x_n) = \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}} \text{ (provided } x_1, \ldots, x_n \neq 0),$$

called *Harmonic Mean*. As it is easy to prove,

$$\lim_{\alpha \to 0} M_\alpha(x_1, \ldots, x_n) = \sqrt[n]{x_1 \cdots x_n}$$

$$\lim_{\alpha \to \infty} M_\alpha(x_1, \ldots, x_n) = \max(x_1, \ldots, x_n)$$

$$\lim_{\alpha \to -\infty} M_\alpha(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n).$$

2.3.2 Ordered Weighted Means

It is said that $M : [0, 1]^n \to [0, 1]$ is a *mean*, when M is continuous, monotonic, and verifies:

$$\min(x_1, \ldots, x_n) \leqslant M(x_1, \ldots, x_n) \leqslant \max(x_1, \ldots, x_n).$$

Since, $\min(0, \ldots, 0) \leqslant M(0, \ldots, 0) \leqslant \max(0, \ldots, 0) = 0$, it results $M(0, \ldots, 0) = 0$. Since, $\min(1, \ldots, 1) \leqslant M(1, \ldots, 1) \leqslant \max(1, \ldots, 1) = 1$, it results $M(1, \ldots, 1) = 1$. Hence, of course, quasi-linear means are means, but there are more of such means. An important and useful example are the *Ordered Weighted Means* (OWA). Its definition is the following:

$O : [0, 1]^n \to [0, 1]$ is an OWA, if $O(x_1, \ldots, x_n)$ is obtained under the process,

- Select weights $p_1, \ldots, p_n$ in [0, 1], such that $\sum_{i=1}^{n} p_i = 1$.
- Permute the n-pla $(x_1, \ldots, x_n)$, to the n-pla $(x_1^*, \ldots, x_n^*)$ such that $x_1^* \leqslant \cdots \leqslant x_n^*$)
- $O(x_1, \ldots, x_n) = \sum_{i=1}^{n} p_i \cdot x_i^*$.

For example, if $n = 2$,

$$O(x_1, x_2) = p_1 \cdot \min(x_1, x_2) + p_2 \cdot \max(x_1, x_2), \text{ with } p_1 + p_2 = 1.$$

If $n = 4$, and the weights are (0.2, 0.4, 0.3, 0.1), it is

$$\begin{aligned} O(0.2, 0.5, 0.7, 0.3) &= O(0.2, 0.3, 0.5, 0.7) \\ &= 0.2 \times 0.2 + 0.4 \times 0.3 + 0.3 \times 0.5 + 0.1 \times 0.7 = 0.74. \end{aligned}$$

2.3.3 More on Aggregations

Because they are associative, continuous t-norms and continuous t-conorms can be extended to n-dimensional aggregation functions. For example, with $n = 3$,

$$T(x_1, x_2, x_3) = T(x_1, T(x_2, x_3)) = T(T(x_1, x_2), x_3)) = \cdots$$

$$S(x_1, x_2, x_3) = S(x_1, S(x_2, x_3)) = S(S(x_1, x_2), x_3)) = \cdots$$

Nevertheless, not all aggregation functions are associative. For example, if M is the arithmetic mean, $M(x_1, M(x_2, x_3)) = \frac{2x_1+x_2+x_3}{4}$, but $M(M(x_1, x_2), x_3)) = \frac{x_1+x_2+2x_3}{4}$. Concerning means, the only associative are min, and max.

In general, Aggregation Functions are not commutative. For example, a 2-dimensional quasi-linear mean

$$M(x_1, x_2) = f^{-1}(p_1 f(x_1) + p_2 f(x_2)), \quad p_1 + p_2 = 1,$$

is commutative if and only if $p_1 = p_2 = \frac{1}{2}$. Arithmetic and geometric means are commutative, but weighted means in general are not.

If T is a continuous t-norm, and S a continuous t-conorm, the function

$$A(x_1, x_2) = p_1 T(x_1, x_2) + p_2 S(x_1, x_2), \quad p_1 + p_2 = 1$$

is an aggregation function that, since $T \leqslant \min \leqslant \max \leqslant S$, in general is not a mean. The only exception is with $T = \min$, and $S = \max$, as it was said before. For example,

- $A(x_1, x_2) = 0.7x_1.x_2 + 0.3W^*(x_1, x_2)$
- $A(x_1, x_2) = 0.6\min(x_1, x_2) + 0.4(x_1 + x_2 - x_1.x_2)$
- $A(x_1, x_2) = 0.6W(x_1, x_2) + 0.4\max(x_1, x_2)$,

are aggregation functions.

2.3.4 Examples

The pointwise aggregation of classical sets is not, in general, a classical set, but a fuzzy one. For example, the arithmetic mean verifies

$$M(0, 0) = 0, \quad M(0, 1) = M(1, 0) = \frac{1}{2}, \quad M(1, 1) = 1$$

and, if A, B are crisp subsets, $M(A, B)$ is not a crisp subset if given by $M(\mu_A, \mu_B)(x) = M(\mu_A(x), \mu_B(x))$. On the contrary, with the geometric mean G, it is

$$G(0, 0) = G(0, 1) = G(1, 0) = 0, \quad G(1, 1) = 1,$$

and $G(A, B)$ is a crisp set.

In all cases, if $\mu \in [0, 1]^X$, $\sigma \in [0, 1]^Y$, and A is an aggregation function, then

$$A(\mu, \sigma)(x, y) = A(\mu(x), \sigma(y)),$$

for all $x \in X$, $y \in Y$, is a fuzzy set $A(\mu, \sigma) \in [0, 1]^{X\times Y}$ called the aggregation of μ and σ. When $X = Y$ it could be defined the fuzzy set $A(\mu, \sigma) \in [0, 1]^X$,

$$A(\mu, \sigma)(x) = A(\mu(x), \sigma(x)), \quad \text{for all } x \in X.$$

Example 2.3.1 If $X = \{1, 2, 3, 4, 5\}$, and $\mu = 0.6/1 + 0.7/2 + 0.5/3 + 1/4$, $\sigma = 0.9/1 + 0.5/3 + 0.7/4 + 0.8/5$, compute $M(\mu, \sigma)$, $G(\mu, \sigma)$, and $O(\mu, \sigma)$ with O the OWA with weights $p_1 = 0.4$, $p_2 = 0.6$.

Solution.

$M(\mu, \sigma) = 0.75/1 + 0.35/2 + 0.5/3 + 0.85/4 + 0.4/5$
$G(\mu, \sigma) = 0.735/1 + 0/2 + 0.5/3 + 0.837/4 + 0/5$
$O(\mu, \sigma) = (0.4 \times 0.6 + 0.6 \times 0.9)/1 + (0.4 \times 0 + 0.6 \times 0.7)/2 + (0.4 \times 0.5 + 0.6 \times 0.5)/3 + (0.4 \times 0.7 + 0.6 \times 1)/4 + (0.4 \times 0 + 0.6 \times 0.8)/5 = 0.72/1 + 0.42/2 + 0.5/3 + 0.88/4 + 0.48/5.$

Notice that $G(\mu, \sigma) \leqslant M(\mu, \sigma)$, but that neither $G(\mu, \sigma)$ and $O(\mu, \sigma)$, nor $M(\mu, \sigma)$ and $O(\mu, \sigma)$, are order-comparable.

Example 2.3.2 A linguistic variable has the fuzzy values $H = high$, $S = short$ and $M = medium$, with H represented by

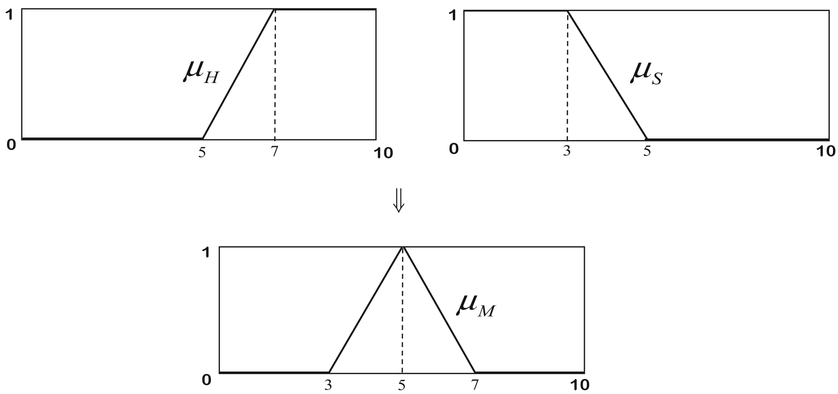
$$\mu_H(x) = \begin{cases} 0, & \text{if } 0 \leqslant x \leqslant 5 \\ 1, & \text{if } 7 \leqslant x \leqslant 10 \\ \frac{x-5}{2}, & \text{if } 5 \leqslant x \leqslant 7. \end{cases}$$

In the two suppositions, $M = H' \cdot S'$, and that M is the aggregation of H and S under the weighted mean $A(x_1, x_2) = 0.3x_1 + 0.7x_2$, compute μ_M.

Solution. Since S is an antonym of H, and μ_H is monotonic, $\mu_S(x) = \mu_H(10 - x)$, that is

$$\mu_S(x) = \begin{cases} 1, & \text{if } 0 \leqslant x \leqslant 3 \\ 0, & \text{if } 5 \leqslant x \leqslant 10 \\ \frac{5-x}{2}, & \text{if } 3 \leqslant x \leqslant 5. \end{cases}$$

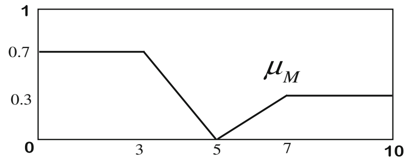
The solution for the first supposition appears in the following sequence of figures (with $\cdot = \min, ' = 1 - \mathrm{id}$).



The solution for the second supposition is

$$\mu_H(x) = \mu_{A(H,S)}(x) = \begin{cases} 0.7, & \text{if } 0 \leqslant x \leqslant 3 \\ \frac{0.7(5-x)}{2}, & \text{if } 3 \leqslant x \leqslant 5 \\ \frac{0.3(x-5)}{2}, & \text{if } 5 \leqslant x \leqslant 7 \\ 0.3, & \text{if } 7 \leqslant x \leqslant 10, \end{cases}$$

graphically,



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