Chapter 2
Stochastic Models of Risk Management Concepts

Abstract The formulation and investigation of stochastic models for the fundamental quantitative concepts of risk management constitute the purpose of this chapter. The concepts of the main quantitative components of risk and the concepts of the main quantitative components of risk control and risk financing operations constitute the fundamental quantitative concepts of risk management. This chapter consists of two parts. The first part concentrates on the formulation and investigation of stochastic models for risk severity, risk duration, risk frequency, and total risk severity which are the main quantitative components of risk. The second part concentrates on the formulation and investigation of stochastic models for the time required for treating a risk occurrence, the time of the first occurrence of a major risk, the minimum time of a random number of risk occurrences, the number of ongoing risk occurrences, the multiplicative risk severity, and the riskiness which are the main quantitative components of risk control and risk financing operations.

2.1 Introduction

The purpose of the present chapter is the formulation and investigation of stochastic models for the fundamental quantitative concepts of the discipline of risk management. These concepts include the concepts of the main quantitative components of risk and the concepts of the main quantitative components of risk control and risk financing operations. Risk severity, risk duration, risk frequency, and total risk severity are the main quantitative components of risk. The concepts of the main quantitative components of risk and the stochastic models of these concepts constitute the first part of the present chapter. The time required for treating a risk occurrence, the time of the first occurrence of a major risk, the minimum time of a random number of risk occurrences, the number of ongoing risk occurrences, the multiplicative risk severity, and the riskiness are the main quantitative components of risk control and risk financing operations. The concepts of the main quantitative components of risk control and risk financing operations and the stochastic models of these concepts constitute the second part of the present

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chapter. The stochastic models of the fundamental quantitative concepts of risk management are generally recognized as very strong analytical tools for investigating problems and making decisions in various areas of this discipline. These models provide risk experts with valuable information for the implementation of risk management principles. The implementation of risk management principles is realized with the contribution of the stochastic models of the main quantitative components of risk as structural elements in stochastic modeling activities of the main quantitative components of risk control and risk financing operations. In conclusion, the importance of the stochastic models of the main quantitative components of risk substantially supports the importance of the stochastic models of the main quantitative components of risk control and risk financing operations.

The investigation of the stochastic models of the fundamental concepts of risk management uses the results of the theory of mixed probability distributions. In particular, the very strong results of the theory of characteristic functions corresponding to mixed probability distributions are very useful for investigating properties of such stochastic models. The establishment of unimodality, infinite divisibility, selfdecomposability, and other properties for the probability distributions of the stochastic models describing the fundamental concepts of risk management is facilitated by the use of the corresponding characteristic functions.

2.2 Model of Risk Severity

The economic loss due to the occurrence of a risk is defined as risk severity. The suitable stochastic model for the description and analysis of risk severity is a continuous random variable X with values in the interval $(0, \infty)$.

The distribution function $F_X(x)$, the probability density function $f_X(x)$, and the characteristic function $\varphi_X(u),\ u \in \mathbf{R}$ of the random variable X are the concepts of probability theory which constitute the basic analytical tools for investigating the behaviour of risk severity. The establishment of infinite divisibility, selfdecomposability, unimodality and other properties of the probability distribution of risk severity substantially supports the activities of developing, investigating, and applying of various risk control operations. These operations are the structural elements of proactive risk management which is generally recognized as the modern perspective of this discipline.

2.3 Model of Risk Duration

The length of the time interval in which the cause of a risk creates economic loss is defined as risk duration. The suitable stochastic model for the description and analysis of risk duration is a continuous random variable S with values in the interval $(0, \infty)$.

The distribution function $F_S(s)$, the probability density function $f_S(s)$, and the characteristic function $\varphi_S(u)$ of the random variable S are the concepts of probability theory which constitutes the basic analytical tools for investigating the behavior of risk duration. The establishment of theoretical properties for the probability distribution of risk duration substantially supports the activities of developing, investigating, and applying of various risk control operations. Risk duration can be used in the formulation of stochastic multiplicative models for the description and analysis of risk severity.

2.4 Model of Risk Frequency

The number of the occurrences of a risk in a given time interval is defined as risk frequency. The suitable stochastic model for the description and analysis of risk frequency is a discrete random variable N with values in the set $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$.

The probability function $P(N = n) = p_n, n = 0, 1, 2, \ldots$ and the probability generating function $P_N(z), |z| \leq 1$ of the random variable N are the concepts of probability theory which constitute the basic analytical tools for investigating the behavior of risk frequency. The establishment of theoretical properties for the probability distribution of risk frequency supports the activities of developing, investigating, and applying of various risk control operations. Risk frequency is particularly useful in formulating stochastic models for the description and analysis of total risk severity. These models are structural elements of risk control and risk financing operations.

The consideration of risk frequency in a time interval of the form $[0, t]$ is of significant practical interest. In this interval, risk frequency is represented by the random variable $N(t)$ with probability generating function

$$P_{N(t)}(z, t). \tag{2.4.1}$$

In this case $\{N(t), t \geq 0\}$ is a counting stochastic process.

The consideration of risk frequency in a time interval of the form $[0, T]$, where T is a continuous random variable with probability density function

$$f_T(t), \tag{2.4.2}$$

is particularly useful in a wide variety of practical disciplines. The risk frequency in the time interval $[0, T]$ is represented by the random variable K.

The probability generating function of this random variable has the form

$$P_K(z) = E\big(E\big(z^K | T\big)\big). \tag{2.4.3}$$

From (2.4.2) and (2.4.3) it follows that

$$P_K(z) = \int_0^\infty E\left(z^K | T = t\right) f_T(t) dt. \tag{2.4.4}$$

Since the random variable $K|T = t$ is equally distributed with the random variable $N(t)$ then we get that

$$E\left(z^K | T = t\right) = E\left(z^{N(t)}\right). \tag{2.4.5}$$

From (2.4.1) and (2.4.5) we get that

$$E\left(z^K | T = t\right) = P_{N(t)}(z, t). \tag{2.4.6}$$

If we use (2.4.6) in (2.4.4) then the probability generating function of the random variable K has the form

$$P_K(z) = \int_0^\infty P_{N(t)}(z, t) f_T(t) dt. \tag{2.4.7}$$

If the counting stochastic process $\{N(t), t \geq 0\}$ is a homogeneous Poisson process with probability generating function $P_{N(t)}(z, t) = e^{\lambda t(z-1)}, \lambda > 0,$ then (2.4.7) has the form

$$P_K(z) = \int_0^\infty e^{\lambda t(z-1)} f_T(t) dt. \tag{2.4.8}$$

Since the homogeneous Poisson process is the most important counting process, from a theoretical and a practical point of view, and the discrete distributions with probability generating functions of the form (2.4.8) are strong tools of probability theory then the evaluation of special cases of (2.4.8) is very interesting.

If the random variable T follows the uniform distribution with probability density function

$$f_T(t) = 1, \quad 0 < t < 1, \tag{2.4.9}$$

then from (2.4.8) and (2.4.9) it follows that the discrete random variable K, which denotes the risk frequency in the time interval $[0, T]$, has probability generating function $P_K(z) = \int_0^1 e^{\lambda t(z-1)} dt$ or equivalently

$$P_K(z) = \frac{1 - e^{\lambda(z-1)}}{\lambda(1-z)}. \tag{2.4.10}$$

The probability generating function (2.4.10) belongs to the renewal distribution which corresponds to the Poisson distribution.

If the random variable T follows the exponential distribution with probability density function

$$f_T(t) = \mu e^{-\mu t},\ t > 0,\ \mu > 0, \tag{2.4.11}$$

then from (2.4.8) and (2.4.11) it follows that the random variable K, which denotes the risk frequency in the time interval $[0, T]$, has probability generating function

$$P_K(z) = \int_0^\infty e^{\lambda t(z-1)} \mu e^{-\mu t} dt$$

or equivalently

$$P_K(z) = \frac{\mu}{\mu + \lambda(1-z)}. \tag{2.4.12}$$

If we set $p = \frac{\mu}{\mu+\lambda}$ and $q = \frac{\lambda}{\lambda+\mu}$, then the probability generating function (2.4.12) has the form

$$P_K(z) = \frac{p}{1-qz}. \tag{2.4.13}$$

The probability generating function (2.4.13) belongs to the geometric type I distribution.

We suppose that the Laplace transform of the random variable T is

$$\omega(\rho) = \int_0^\infty e^{-\rho t} f_T(t) dt,\ \ \rho \geq 0. \tag{2.4.14}$$

Since the probability generating function (2.4.8) has the form

$$P_K(z) = \int_0^\infty e^{-\lambda t(1-z)} f_T(t) dt, \tag{2.4.15}$$

then from (2.4.14) and (2.4.15) it follows that

$$P_K(z) = \omega(\lambda(1-z)). \tag{2.4.16}$$

A particular case of (2.4.16) is the following. We suppose that the distribution of the random variable T belongs to the class of continuous stable distributions with Laplace transform

$$\omega(\rho) = e^{-\rho^{\gamma}},\ 0 < \gamma \leq 1. \tag{2.4.17}$$

From (2.4.16) and (2.4.17) it follows that the random variable K, which denotes the frequency of risk in the time interval $[0, T]$, has probability generating function

$$P_K(z) = e^{-\lambda^{\gamma}(1-z)^{\gamma}}. \tag{2.4.18}$$

The probability generating function (2.4.18) belongs in a distribution which is a member of the class of discrete stable distributions. The class of discrete stable distributions is very important, in theory and practice, for five reasons. The first reason is the unimodality of the discrete stable distributions. The infinite divisibility of the discrete stable distributions is the second reason. The fact that the class of discrete stable distributions includes distributions of significant theoretical and practical interest constitutes the third reason. An example of such distribution is the Poisson distribution with probability generating function $P_K(z) = e^{\lambda(z-1)}$ being of the form (2.4.18) with $\gamma = 1$.

The fourth reason is the representation of a discrete random variable L, with values in the set $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ and distribution belonging to the class of discrete stable distributions, as a Poisson random sum of random variables following the Sibuya distribution. The representation is implemented in the following way. Since the distribution of the random variable L belongs to the class of discrete stable distributions then the probability generating function of the random variable L has the form

$$P_L(z) = e^{-c(1-z)^{\gamma}},\ c > 0,\ 0 < \gamma \leq 1. \tag{2.4.19}$$

The probability generating function (2.4.18) is of the form (2.4.19) with $c = \lambda^{\gamma}$.

We consider the discrete random variable E which follows the Poisson distribution with probability generating function $P_E(z) = e^{c(z-1)}$ and the sequence of independent random variables $\{X_{\varepsilon}, \varepsilon = 1, 2, \ldots\}$.

The random variable E is independent of the sequence $\{X_{\varepsilon}, \varepsilon = 1, 2, \ldots\}$ of random variables distributed as the random variable X which follows the Sibuya distribution with probability generating function $P_X(z) = 1 - (1-z)^{\gamma}$.

Since the probability generating function $P_L(z) = e^{-c(1-z)^{\gamma}}$ has the form $P_L(z) = e^{c[1-(1-z)^{\gamma}-1]}$ then the random variable L has the form of a Poisson random sum of random variables following the Sibuya distribution, or equivalently the random variable L has the form $L = X_1 + X_2 + \cdots + X_E$.

The fifth reason is the construction of important mixed distributions with the use of the class of discrete stable distributions.

2.5 Total Risk Severity

The discrete random variable N with values in the set $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ and probability generating function $P_N(z)$ is independent of the sequence of continuous, positive, independent, and identically distributed random variables $\{X_n, n = 1, 2, \ldots\}$.

The random variables of the above sequence are equally distributed with the random variable X which has characteristic function $\varphi_X(u)$.

If the random variable N represents the frequency of a risk and the random variable X_n represents the severity of the nth risk occurrence then the random sum $L = X_1 + X_2 + \cdots + X_N$ represents the total risk severity. The study of the total risk severity is based on the characteristic function $\varphi_L(u)$ since it is not possible the evaluation of the distribution function $F_L(\ell)$ and the evaluation of the probability density function $f_L(\ell)$.

The characteristic function $\varphi_L(u)$ is evaluated in the following way. We have that $\varphi_L(u) = E(e^{iuL})$ or equivalently

$$\varphi_L(u) = E\big(E\big(e^{iuL}|N\big)\big). \tag{2.5.1}$$

From (2.5.1) it follows that

$$\varphi_L(u) = \sum_{n=0}^{\infty} E\big(e^{iuL}|N = n\big)\ P(N = n)$$

or equivalently

$$\varphi_L(u) = \sum_{n=0}^{\infty} E\big(e^{iuX_1 + \cdots + iuX_n}|N = n\big)P(N = n). \tag{2.5.2}$$

From (2.5.2) it follows that

$$\varphi_L(u) = \sum_{n=0}^{\infty} E\big(e^{iuX_1} \cdots e^{iuX_n}|N = n\big)\ P(N = n). \tag{2.5.3}$$

The independence of the random variable N from the sequence of random variables $\{X_n, n = 1, 2, \ldots\}$ implies the independence of the random variables $N, X_1, \ldots, X_n$.

Hence the random variables $N, e^{iuX_1}, \ldots, e^{iuX_n}$ are independent and the random variables $e^{iuX_1}, \ldots, e^{iuX_n}$ are also independent. It is easily seen that (2.5.3) has the form

$$\varphi_L(u) = \sum_{n=0}^{\infty} E\left(e^{iuX_1}\right) \ldots E\left(e^{iuX_n}\right) P(N = n). \tag{2.5.4}$$

Since

$$\varphi_X(u) = E\left(e^{iuX_n}\right),\ n = 1, 2, \ldots$$

then (2.5.4) has the form

$$\varphi_L(u) = \sum_{n=0}^{\infty} \varphi_X^n(u) P(N = n). \tag{2.5.5}$$

From (2.5.5) it follows that the characteristic function of the total risk severity has the form

$$\phi_L(u) = P_N(\phi_X(u)). \tag{2.5.6}$$

The characteristic function (2.5.6), the theorem of inversion of characteristic functions, and the Fast Fourier Transform algorithm make possible the study of the probabilistic behavior of the total risk severity. This behavior is important for making decisions in the areas of risk control and risk financing operations.

The mean value of the total risk severity $E(L)$ can be used in risk classification operations. The mean value of the total risk severity can be evaluated in the following way. From (2.5.6) we get that

$$\varphi_L'(u) = \varphi_X'(u) P_N'(\varphi_X(u)). \tag{2.5.7}$$

Hence (2.5.7) implies that

$$\varphi_L'(0) = \varphi_X'(0) P_N'(\varphi_X(0)). \tag{2.5.8}$$

From (2.5.8) it follows that

$$E(L) = E(X)E(N). \tag{2.5.9}$$

A special case of the total risk severity, with significant practical interest, is the following.

We suppose that the frequency of a risk follows the Bernoulli distribution with probability function $P(N = n) = p^n q^{1-n}, n = 0, 1$.

The probability generating function of the random variable N is $P_N(z) = q + pz$.

The mean value of the random variable N is $E(N) = p$.

We suppose that the severity of a risk is a continuous and positive random variable X with characteristic function $\varphi_X(u)$ and mean value $E(X)$.

The total severity of risk is the random variable

$$L = \begin{cases} 0, & N = 0 \\ X, & N = 1. \end{cases}$$

The characteristic function of the total risk severity L is $\varphi_L(u) = q + p\varphi_X(u)$.

From (2.5.9) we get that the mean value of the total severity L is $E(L) = pE(X)$.

An extension of the random sum $L = X_1 + X_2 + \cdots + X_N$ as a model of total risk severity is the following. Let N be a discrete random variable with values in the set $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ and probability generating function $P_N(z)$.

Let $\{V_n, n = 1, 2, \ldots\}$ be a sequence of discrete and independent random variables, distributed as the random variable V with values in the set $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ and probability generating function $P_V(z)$.

We suppose that the random variable N is independent of the sequence of random variables $\{V_n, n = 1, 2, \ldots\}$ and we set $K = V_1 + V_2 + \cdots + V_N$.

Let $\{X_\kappa, \kappa = 1, 2, \ldots\}$ be a sequence of continuous, positive and independent random variables, distributed as the random variable X with characteristic function $\varphi_X(u)$, and we set $L = X_1 + X_2 + \cdots + X_K$.

An interpretation of the random variable $L = X_1 + X_2 + \cdots + X_K$ in the area of stochastic models of total risk frequency is the following.

We suppose that the random variable N denotes the frequency of a risk and the random variable V_n denotes the number of different damages due to the nth risk occurrence. The random variable $K = V_1 + V_2 + \cdots + V_N$ denotes the number of different damages due to the N risk occurrences. The random variable X_κ denotes the size of the κth damage. Hence the random variable $L = X_1 + X_2 + \cdots + X_K$ denotes the total severity of risk.

The following result establishes sufficient conditions for the evaluation of the characteristic function $\varphi_L(u)$ of the random variable $L = X_1 + X_2 + \cdots + X_K$.

Theorem 2.5.1 *Let N be a discrete random variable with values in the set $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ and probability generating function $P_N(z)$.*

Let $\{V_n, n = 1, 2, \ldots\}$ be a sequence of discrete and independent random variables, distributed as the random variable V with values in the set $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$, and probability generating function $P_V(z)$.

We set

$$K = V_1 + V_2 + \cdots + V_N.$$

Let $\{X_\kappa, \kappa = 1, 2, \ldots\}$ be a sequence of continuous, positive, and independent random variables distributed as the random variable X with characteristic function $\varphi_X(u)$.

We set

$$L = X_1 + X_2 + \cdots + X_K.$$

If N, $\{V_n, n = 1, 2, \ldots\}$ *and* $\{X_\kappa, \kappa = 1, 2, \ldots\}$ *are independent then the characteristic function of the random variable* $L = X_1 + X_2 + \cdots + X_K$ *is* $\varphi_L(u) = P_N(P_V(\varphi_X(u)))$.

Proof We have $\varphi_L(u) = E(e^{iuL})$ or equivalently

$$\varphi_L(u) = E\big(E\big(e^{iuL}|K\big)\big). \tag{2.5.10}$$

From (2.5.10) we get that

$$\varphi_L(u) = \sum_{\kappa=0}^{\infty} E\big(e^{iuL}|K = \kappa\big)P(K = \kappa)$$

or equivalently we get that

$$\varphi_L(u) = \sum_{\kappa=0}^{\infty} E\Big(e^{iu(X_1+\cdots+X_K)}|K = \kappa\Big)\ P(K = \kappa). \tag{2.5.11}$$

From (2.5.11) it follows that

$$\varphi_L(u) = \sum_{\kappa=0}^{\infty} E\Big(e^{iu(X_1+\cdots+X_\kappa)}|K = \kappa\Big)\ P(K = \kappa). \tag{2.5.12}$$

Since $K = V_1 + V_2 + \cdots + V_N$ then (2.5.12) has the form

$$\varphi_L(u) = \sum_{\kappa=0}^{\infty} E\Big(e^{iu(X_1+\cdots+X_\kappa)}|V_1 + V_2 + \cdots + V_N = \kappa\Big)P(V_1 + V_2 + \cdots + V_N = \kappa). \tag{2.5.13}$$

We shall prove that the random variables $\Lambda = X_1 + X_2 + \cdots + X_\kappa$ and $K = V_1 + V_2 + \cdots + V_N$ are independent. If $\varphi_\Lambda(u)$ is the characteristic function of the random variable $\Lambda = X_1 + X_2 + \cdots + X_\kappa$ then we get that $\varphi_\Lambda(u) = E\big(e^{iu\Lambda}\big)$ or equivalently we get that

$$\varphi_\Lambda(u) = E\Big(e^{iu(X_1+\cdots+X_\kappa)}\Big). \tag{2.5.14}$$

From (2.5.14) it follows that

$$\varphi_{\Lambda}(u) = E\left(e^{iuX_1+\cdots+iuX_\kappa}\right). \tag{2.5.15}$$

Since the random variables of the sequence $\{X_\kappa, \kappa = 1, 2, \ldots\}$ are independent then the random variables $X_1, \ldots, X_\kappa$ are independent. The independence of the above random variables implies the independence of the random variables $e^{iuX_1}, \ldots, e^{iuX_\kappa}$.

Hence (2.5.15) has the form

$$\varphi_{\Lambda}(u) = E\left(e^{iuX_1}\right)\ldots E\left(e^{iuX_\kappa}\right). \tag{2.5.16}$$

Since the random variables of the sequence $\{X_\kappa, \kappa = 1, 2, \ldots\}$ are equally distributed with the random variable X having characteristic function $\varphi_X(u)$ then (2.5.16) has the form $\varphi_{\Lambda}(u) = \varphi_X^{\kappa}(u)$.

Let $\varphi_{\mathrm{K}}(\xi)$ be the characteristic function of the random variable $\mathrm{K} = V_1 + V_2 + \cdots + V_N$.

The independence of $\{V_n, n = 1, 2, \ldots\}$, N and $\{X_\kappa, \kappa = 1, 2, \ldots\}$ implies the independence of $\{V_n, n = 1, 2, \ldots\}$ and N.

Since the random variables of the sequence $\{V_n, n = 1, 2, \ldots\}$ are independent and equally distributed with the random variable V and since the random variable N has probability generating function $P_N(z)$ then from (2.5.6) it follows that the characteristic function of the random variable $K = V_1 + V_2 + \cdots + V_N$ is $\varphi_K(\xi) = P_N(\varphi_V(\xi))$ where $\varphi_V(\xi)$ is the characteristic function of the random variable V.

The proof of the independence of the random variables $\Lambda = X_1 + X_2 + \cdots + X_\kappa$ and $K = V_1 + V_2 + \cdots + V_N$ requires the establishment of the relationship $\varphi_{\Lambda,K}(u, \xi) = \varphi_{\Lambda}(u)\varphi_K(\xi)$ where $\varphi_{\Lambda,K}(u, \xi)$ is the characteristic function of the vector (Λ, K).

We have $\varphi_{\Lambda,K}(u, \xi) = E\left(e^{iu\Lambda+i\xi K}\right)$ or equivalently we have $\varphi_{\Lambda,K}(u, \xi) = E\left(E\left(e^{iu\Lambda+i\xi K}|N\right)\right)$.

Hence

$$\varphi_{\Lambda,K}(u, \xi) = \sum_{n=0}^{\infty} E\left(e^{iu(X_1+X_2+\cdots+X_\kappa)+i\xi K}|N = n\right)P(N = n). \tag{2.5.17}$$

From (2.5.17) it follows that

$$\varphi_{\Lambda,K}(u, \xi) = \sum_{n=0}^{\infty} E\left(e^{iu(X_1+X_2+\cdots+X_\kappa)+i\xi(V_1+V_2+\cdots+V_N)}|N = n\right)P(N = n)$$

or equivalently it follows that

$$\varphi_{\Lambda,K}(u,\xi)=\sum_{n=0}^{\infty}E\Big(e^{iu(X_1+X_2+\cdots+X_\kappa)+i\xi(V_1+V_2+\cdots+V_n)}|N=n\Big)P(N=n). \quad (2.5.18)$$

From (2.5.18) it follows that

$$\varphi_{\Lambda,K}(u,\xi)=\sum_{n=0}^{\infty}E\big(e^{iuX_1}\ldots e^{iuX_\kappa}e^{i\xi V_1}\ldots e^{i\xi V_n}|N=n\big)P(N=n). \quad (2.5.19)$$

The independence of $\{V_n, n=1,2,\ldots\}$, N and $\{X_\kappa, \kappa=1,2,\ldots\}$ implies the independence of the random variables $V_1,\ \ldots,\ V_n,\ X_1,\ \ldots,\ X_\kappa,\ N$.

The independence of the above random variables implies the independence of the random variables $e^{i\xi V_1},\ldots,\ e^{i\xi V_n},\ e^{iuX_1},\ldots,\ e^{iuX_\kappa},\ N$.

Hence (2.5.19) has the form

$$\varphi_{\Lambda,K}(u,\xi)=\sum_{n=0}^{\infty}E\big(e^{iuX_1}\ldots e^{iuX_\kappa}e^{i\xi V_1}\ldots e^{i\xi V_n}\big)P(N=n). \quad (2.5.20)$$

The independence of the random variables $e^{iuX_1},\ldots,\ e^{iuX_\kappa},\ e^{i\xi V_1},\ldots,\ e^{i\xi V_n},\ N$ implies the independence of the random variables $e^{iuX_1},\ldots,e^{iuX_\kappa},\ e^{i\xi V_1},\ldots,\ e^{i\xi V_n}$.

Hence (2.5.20) has the form

$$\varphi_{\Lambda}(u)=\sum_{n=0}^{\infty}E\big(e^{iuX_1}\big)\ldots E\big(e^{iuX_\kappa}\big)E\big(e^{i\xi V_1}\big)\ldots E\big(e^{i\xi V_n}\big)P(N=n). \quad (2.5.21)$$

Since the random variables of the sequence $\{X_\kappa, \kappa=1,2,\ldots\}$ are equally distributed with the random variable X having characteristic function $\varphi_X(u)$ and the random variables of the sequence $\{V_n, n=1,2,\ldots\}$ are equally distributed with the random variable V with characteristic function $\varphi_V(u)$ then (2.5.21) has the form

$$\varphi_{\Lambda,K}(u,\xi)=\varphi_X^{\kappa}(u)\sum_{n=0}^{\infty}\varphi_V^{n}(\xi)P(N=n)$$

or equivalently the form

$$\varphi_{\Lambda,K}(u,\xi)=\varphi_X^{\kappa}(u)P_N(\varphi_V(\xi)). \quad (2.5.22)$$

Hence (2.5.22) has the form $\varphi_{\Lambda,K}(u,\xi)=\varphi_{\Lambda}(u)\varphi_K(\xi)$, which means that the random variables $\Lambda=X_1+X_2+\cdots+X_\kappa$ and $K=V_1+V_2+\cdots+V_N$ are independent. The independence of the above random variables implies that (2.5.13) has the form

$$\varphi_L(u) = \sum_{\kappa=0}^{\infty} E\Big(e^{iu(X_1+\cdots+X_\kappa)}\Big) P(V_1 + V_2 + \cdots + V_N = \kappa)$$

or equivalently the form

$$\varphi_L(u) = \sum_{\kappa=0}^{\infty} \varphi_X^{\kappa}(u) P(V_1 + V_2 + \cdots + V_N = \kappa). \tag{2.5.23}$$

Since the random variables of the sequence $\{V_n, n = 1, 2, \ldots\}$ are equally distributed with the random variable V having probability generating function $P_V(z)$ and $P_N(z)$ is the probability generating function of the random variable N.

Then from the form (2.5.6), for probability generating functions, it follows that the probability generating function of the random sum $K = V_1 + V_2 + \cdots + V_N$ is $P_K(z) = P_N(P_V(z))$.

Since

$$P_K(z) = \sum_{\kappa=0}^{\infty} z^{\kappa} P(K = \kappa)$$

or equivalently

$$P_N(P_V(z)) = \sum_{\kappa=0}^{\infty} z^{\kappa} P(V_1 + V_2 + \cdots + V_N = \kappa)$$

then (2.5.23) implies that the characteristic function of the random sum $L = X_1 + X_2 + \cdots + X_K$ with $K = V_1 + V_2 + \cdots + V_N$ is

$$\varphi_L(u) = P_N(P_V(\varphi_X(u))). \tag{2.5.24}$$

The consideration of special cases of the probability generating function (2.5.24) when the random variable N follows the Poisson distribution is very important because the Poisson distribution is the most usual distribution of risk frequency.

We suppose that the random variable N follows the Poisson distribution with probability generating function

$$P_N(z) = e^{\lambda(z-1)} \tag{2.5.25}$$

and the random variable V follows the Poisson distribution with probability generating

$$P_V(z) = e^{\theta(z-1)} \tag{2.5.26}$$

then from (2.5.24), (2.5.25) and (2.5.26) it follows that the probability generating function $P_K(z) = P_N(P_V(z))$ of the random sum $K = V_1 + V_2 + \cdots + V_N$ has the form

$$P_K(z) = \exp\left[\lambda\left(e^{\theta(z-1)} - 1\right)\right]. \tag{2.5.27}$$

Hence the random sum $K = V_1 + V_2 + \cdots + V_N$ follows the Neyman type A distribution. We suppose that the random variable X follows the exponential distribution with characteristic function

$$\varphi_X(u) = \frac{\mu}{\mu - iu}. \tag{2.5.28}$$

From (2.5.24), (2.5.27) and (2.5.28) it follows that the characteristic function of the random sum $L = X_1 + X_2 + \cdots + X_K$ has the form

$$\varphi_L(u) = \exp\left\{\lambda\left[e^{\theta\left(\frac{\mu}{\mu-iu}-1\right)} - 1\right]\right\}.$$

If the random variable N follows the Poisson distribution with probability generating function

$$P_N(z) = e^{\lambda(z-1)} \tag{2.5.29}$$

and the random variable V follows the binomial distribution with probability generating function

$$P_V(z) = (pz + q)^m \tag{2.5.30}$$

then from (2.5.24), (2.5.29) and (2.5.30) it follows that the probability generating function $P_K(z) = P_N(P_V(z))$ of the random sum $K = V_1 + V_2 + \cdots + V_N$ has the form

$$P_K(z) = \exp\{\lambda[(pz + q)^m - 1]\}. \tag{2.5.31}$$

Hence the random sum $K = V_1 + V_2 + \cdots + V_N$ follows the Poisson – binomial distribution. We suppose that the random variable X follows the uniform distribution with characteristic function

$$\varphi_X(u) = \frac{e^{iu} - 1}{iu}. \tag{2.5.32}$$

From (2.5.24), (2.5.31) and (2.5.32) it follows that the characteristic function of the random sum $L = X_1 + X_2 + \cdots + X_K$ has the form

$$\varphi_L(u) = \exp\left\{\lambda\left[\left(p\frac{e^{iu} - 1}{iu} + q\right)^m - 1\right]\right\}.$$

If the random variable N follows the Poisson distribution with probability generating function

$$P_N(z) = e^{\lambda(z-1)} \tag{2.5.33}$$

and the random variable V follows the geometric type II distribution with probability generating function

$$P_V(z) = \frac{pz}{1 - qz} \tag{2.5.34}$$

then from (2.5.33) and (2.5.34) it follows that the probability generating function $P_K(z) = P_N(P_V(z))$ of the random sum $K = V_1 + V_2 + \cdots + V_N$ has the form

$$P_K(z) = \exp\left\{\lambda\left[\frac{pz}{1 - qz} - 1\right]\right\}. \tag{2.5.35}$$

Hence the random sum $K = V_1 + V_2 + \cdots + V_N$ follows the Polya-Aeppli distribution. We suppose that the random variable X follows the exponential distribution with characteristic function

$$\varphi_X(u) = \frac{\mu}{\mu - iu}. \tag{2.5.36}$$

From (2.5.24), (2.5.35) and (2.5.36) it follows that the characteristic function of the random sum $L = X_1 + X_2 + \cdots + X_K$ has the form

$$\varphi_L(u) = \exp\left\{\lambda\left[\frac{p\mu}{p\mu - iu}\right] - 1\right\}.$$

We suppose that the random variable N follows the Poisson distribution with probability generating function

$$P_N(z) = e^{\lambda(z-1)} \tag{2.5.37}$$

and the random variable V has the form $V = \Pi + 1$, where Π is a random variable following the Poisson distribution with probability generating function $P_\Pi(z) = e^{\theta(z-1)}$.

The probability generating function of the random variable V is

$$P_V(z) = ze^{\theta(z-1)}. \tag{2.5.38}$$

From (2.5.24), (2.5.37) and (2.5.38) it follows that the probability generating function $P_K(z) = P_N(P_V(z))$ of the random sum $K = V_1 + V_2 + \cdots + V_N$ has the form

$$P_K(z) = \exp\left\{\lambda\left[ze^{\theta(z-1)} - 1\right]\right\}. \tag{2.5.39}$$

Hence the random sum $K = V_1 + V_2 + \cdots + V_N$ follows the Thomas distribution. We suppose that the random variable X follows the gamma distribution with characteristic function

$$\varphi_X(u) = \left(\frac{\mu}{\mu - iu}\right)^a. \tag{2.5.40}$$

From (2.5.24), (2.5.39) and (2.5.40) it follows that the characteristic function of the random sum $L = X_1 + X_2 + \cdots + X_K$ has the form

$$\varphi_L(u) = \exp\left\{\lambda\left[\left(\frac{\mu}{\mu - iu}\right)^a e^{\theta\left[\left(\frac{\mu}{\mu - iu}\right)^a - 1\right]} - 1\right]\right\}.$$

We suppose that the random variable N follows the Poisson distribution with probability generating function

$$P_N(z) = e^{\lambda(z-1)} \tag{2.5.41}$$

and the random variable V follows the renewal distribution corresponding to the distribution of the random variable D, which follows the Poisson distribution with probability generating function $P_D(z) = e^{\theta(z-1)}$. The probability generating function of the random variable V is

$$P_V(z) = \frac{1 - e^{\theta(z-1)}}{\theta(1 - z)}. \tag{2.5.42}$$

From (2.5.41) and (2.5.42) it follows that the probability generating function $P_K(z) = P_N(P_V(z))$ of the random sum $K = V_1 + V_2 + \cdots + V_N$ has the form

$$P_K(z) = \exp\left\{\lambda\left[\frac{1 - e^{\theta(z-1)}}{\theta(1 - z)} - 1\right]\right\}. \tag{2.5.43}$$

Hence the random sum $K = V_1 + V_2 + \cdots + V_N$ follows the Neyman type B distribution. We suppose that the random X follows the uniform distribution with characteristic function

$$\varphi_X(u) = \frac{e^{iu} - 1}{iu}. \tag{2.5.44}$$

From (2.5.24), (2.5.43) and (2.5.44) it follows that the characteristic function of the random sum $L = X_1 + X_1 + \cdots + X_K$ has the form

$$\varphi_L(u) = \exp\left\{ \lambda \left[\frac{1 - e^{\theta\left(\frac{e^{iu}-1}{iu}-1\right)}}{\theta\left(1 - \frac{e^{iu}-1}{iu}\right)} - 1 \right] \right\}.$$

□

2.6 Recovery Time of a Partially Damaged System

We consider the occurrence time of a risk as the time point 0. The occurrence of the risk interrupts N operations of a system, where N is a discrete random variable with values in the set $\mathbf{N} = \{1, 2, \ldots\}$ and probability generating function $P_N(z)$.

We suppose that $\{X_n, n = 1, 2, \ldots\}$ is a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence are distributed as the random variable X with distribution function $F_X(x)$.

The sequence of random variables $\{X_n, n = 1, 2, \ldots\}$ is independent of the random variable N.

If the random variable X_n denotes the time required for the recovery of the nth interrupted operation of the system then the random variable $T = \max(X_1, X_2, \ldots, X_N)$ denotes the time required for the recovery of the system.

The evaluation of the distribution function $F_T(t)$ of the random variable $T = \max(X_1, X_2, \ldots, X_N)$ is particularly important for the study of the behavior of the system after the occurrence of the risk. Since $F_T(t) = P(T \leq t)$ or equivalently

$$F_T(t) = P[\max(X_1, X_2, \ldots, X_N) \leq t] \tag{2.6.1}$$

then (2.6.1) implies that

$$F_T(t) = \sum_{n=1}^{\infty} P[\max(X_1, X_2, \ldots, X_N) \leq t | N = n] P(N = n)$$

or equivalently

$$F_T(t) = \sum_{n=1}^{\infty} P[\max(X_1, X_2, \ldots, X_n) \le t | N = n] P(N = n). \tag{2.6.2}$$

Since the event $[\max(X_1, X_2, \ldots, X_n) \le t | N = n]$ implies the event $(X_1 \le t, X_2 \le t, \ldots, X_n \le t | N = n)$ then (2.6.2) has the form

$$F_T(t) = \sum_{n=1}^{\infty} P(X_1 \le t, X_2 \le t, \ldots, X_n \le t | N = n) P(N = n). \tag{2.6.3}$$

The independence of the random variable N from the sequence of continuous, positive, independent, and identically distributed random variables $\{X_n, n = 1, 2, \ldots\}$ means the independence of the random variables $N, X_1, X_2, \ldots, X_n$.

Hence (2.6.3) has the form

$$F_T(t) = \sum_{n=1}^{\infty} P(X_1 \le t, X_2 \le t, \ldots, X_n \le t) P(N = n). \tag{2.6.4}$$

Since the random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ are independent then (2.6.4) has the form

$$F_T(t) = \sum_{n=1}^{\infty} P(X_1 \le t) P(X_2 \le t) \ldots P(X_n \le t) P(N = n). \tag{2.6.5}$$

Moreover, the assumption that the random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ are distributed as the random variable X having distribution function $F_X(x)$ then (2.6.5) has the form

$$F_T(t) = \sum_{n=1}^{\infty} F_X^n(t) P(N = n). \tag{2.6.6}$$

From (2.6.6) it follows that the distribution function of the random variable $T = \max(X_1, X_2, \ldots, X_N)$ is

$$F_T(t) = P_N(F_X(t)), \ t > 0. \tag{2.6.7}$$

An interesting special case of (2.6.7) arises if the random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ follow the exponential distribution with distribution function

$$F_X(x) = 1 - e^{-\mu x}, \ x > 0, \ \mu > 0 \tag{2.6.8}$$

and the random variable N follows the Sibuya distribution with probability generating function

$$P_N(z) = 1 - (1-z)^\gamma,\ 0 < \gamma \le 1. \tag{2.6.9}$$

From (2.6.7), (2.6.8) and (2.6.9) it follows that the distribution function of the random variable $T = \max(X_1, X_2, \ldots, X_N)$ has the form $F_T(t) = 1 - [1 - (1 - e^{-\mu t})]^\gamma,\ t > 0$ or equivalently the form $F_T(t) = 1 - e^{-\mu\gamma t},\ t > 0.$

Hence, in this special case, the random variable $T = \max(X_1, X_2, \ldots, X_N)$ follows the exponential distribution with parameter $\mu\gamma$.

Another special case of (2.6.7) arises if the random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ follow the uniform distribution with distribution function

$$F_X(x) = x\ ,\ 0 < x < 1, \tag{2.6.10}$$

and the random variable N follows the geometric type II distribution with probability generating function

$$P_N(z) = \frac{pz}{1 - qz}. \tag{2.6.11}$$

From (2.6.7), (2.6.10) and (2.6.11) it follows that the distribution function of the random variable $T = \max(X_1, X_2, \ldots, X_N)$ has the form $F_T(t) = \frac{pt}{1-qt},\ 0 < t < 1.$

The present section is based on the assumption that the risk occurrence, interrupting N operations of a system, is realized at given time point called time point 0. This assumption does not agree with the random character of risk management. In practice, the time point 0 is a sufficiently small and closed time interval where a risk occurs with probability 1. Consequently, any point of such an interval can be considered as the occurrence time point of a risk and the random variable $T = \max(X_1, X_2, \ldots, X_N)$ can the be interpreted as a stochastic model for the recovery time of a partially damaged system.

The distribution function $F_T(t) = P_N(F_X(t))$ of the random variable $T = \max(X_1, X_2, \ldots, X_N)$ provides probabilistic information which makes the time interval $[0, T]$ particularly important for the implementation of the risk management principles and operations.

2.7 Time of First Damage of a System Threatened by a Random Number of Risks

The discrete random variable N with values in the set $\mathbf{N} = \{1, 2, \ldots\}$ and probability generating function $P_N(z)$ is independent of the sequence of continuous, positive, independent, and identically distributed random variables $\{X_n, n = 1, 2, \ldots\}$.

The random variables of the above sequence are distributed as the random variable X having distribution function $F_X(x)$.

If the random variable N denotes the number of risks threatening a system at the time point 0 and the random variable X_n denotes the occurrence time of the nth risk then the random variable $T = \min(X_1, X_2, \ldots, X_N)$ denotes the time of the first risk occurrence. The consideration of a system under a random number N of independent competing risks means the use of the random variable $T = \min(X_1, X_2, \ldots, X_N)$ as a fundamental stochastic model for investigating the evolution of this system.

The evaluation of the distribution function $F_T(t)$ of the random variable $T = \min(X_1, X_2, \ldots, X_N)$ is very important for the consideration of a system under a random number of independent competing risks. We have $F_T(t) = P(T \le t)$ or equivalently

$$F_T(t) = P[\min(X_1, X_2, \ldots, X_N) \le t]. \tag{2.7.1}$$

Since the event $[\min(X_1, X_2, \ldots, X_N) > t]$ is the complement of the event $[\min(X_1, X_2, \ldots, X_N) \le t]$ then (2.7.1) implies that $F_T(t) = 1 - P[\min(X_1, X_2, \ldots, X_N) > t]$ or equivalently

$$F_T(t) = 1 - \sum_{n=1}^{\infty} P(\min(X_1, X_2, \ldots, X_N) > t | N = n) P(N = n). \tag{2.7.2}$$

From (2.7.2) it follows that

$$F_T(t) = 1 - \sum_{n=1}^{\infty} P(\min(X_1, X_2, \ldots, X_n) > t | N = n) P(N = n). \tag{2.7.3}$$

Since the event $[\min(X_1, X_2, \ldots, X_n) > t | N = n]$ implies the event $(X_1 > t, X_2 > t, \ldots, X_n > t | N = n)$ then (2.7.3) has the form

$$F_T(t) = 1 - \sum_{n=1}^{\infty} P(X_1 > t, X_2 > t, \ldots, X_n > t | N = n) P(N = n). \tag{2.7.4}$$

The independence of the random variable N and the sequence of continuous, positive, independent, and identically distributed random variables $\{X_n, n = 1, 2, \ldots\}$ means the independence of the random variables $N, X_1, X_2, \ldots, X_n$.

Hence (2.7.4) has the form

$$F_T(t) = 1 - \sum_{n=1}^{\infty} P(X_1 > t, X_2 > t, \ldots, X_n > t) P(N = n). \tag{2.7.5}$$

Since the random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ are independent then (2.7.5) has the form $F_T(t) = 1 - \sum_{n=1}^{\infty} P(X_1 > t)P(X_2 > t)\ldots P(X_n > t) P(N = n)$ or equivalently the form

$$F_T(t) = 1 - \sum_{n=1}^{\infty} [1 - P(X_1 \leq t)]\,[1 - P(X_2 \leq t)]\ldots[1 - P(X_n \leq t) \leq t]P(N = n). \tag{2.7.6}$$

Moreover, the assumption that the random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ are distributed as the random variable X having distribution function $F_X(x)$ implies that (2.7.6) has the form

$$F_T(t) = 1 - \sum_{n=1}^{\infty} [1 - F_X(t)]^n P(N = n). \tag{2.7.7}$$

From (2.7.7) it follows that the distribution function of the random variable $T = \min(X_1, X_2, \ldots, X_N)$ has the form

$$F_T(t) = 1 - P_N(1 - F_X(t)),\ t > 0. \tag{2.7.8}$$

An interesting special case of (2.7.8) arises if the random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ follow the beta distribution with parameters $\alpha, 1$ or equivalently the distribution function of the random variables of the above sequence has the form

$$F_X(x) = x^{\alpha},\ 0 < x < 1 \tag{2.7.9}$$

and the random variable N follows the Sibuya distribution with probability generating function

$$P_N(z) = 1 - (1 - z)^{\gamma},\ 0 < \gamma \leq 1. \tag{2.7.10}$$

From (2.7.8), (2.7.9) and (2.7.10) it follows that the distribution function of the random variable $T = \min(X_1, X_2, \ldots, X_N)$ has the form $F_T(t) = 1 - \{1 - [1 - (1 - t^{a})]^{\gamma}\}$ or equivalently the form $F_T(t) = t^{\alpha\gamma}$, $0 < t < 1$.

Hence, in this case, the random variable $T = \min(X_1, X_2, \ldots, X_N)$ follows the beta distribution with parameters $\alpha\gamma, 1$.

The role of the random variable $T = \min(X_1, X_2, \ldots, X_N)$ in the consideration of a system under a random number N of independent and competing risks becomes very important if the occurrence of one of these risks implies the destruction of the system. In this case the random variable $T = \min(X_1, X_2, \ldots, X_N)$ denotes the life time of the system.

2.8 Time of First Major Damage

We consider the sequence of continuous, positive, independent, and identically distributed random variables $\{C_n, n = 1, 2, \ldots\}$. The random variables of the sequence are distributed as the random variable C having characteristic function

$$\varphi_C(u). \tag{2.8.1}$$

The random variable $C_n, n = 1, 2, \ldots$ denotes the time between the $(n-1)$th and the nth occurrence of a risk

We consider the sequence of continuous, positive, independent, and identically distributed random variables $\{X_n, n = 1, 2, \ldots\}$.

The random variables of the sequence are distributed as the random variable X having distribution function

$$F_X(x). \tag{2.8.2}$$

The random variable X_n denotes the size of the damage from the nth occurrence of the risk.

Let θ be a positive real number. If $X_n > \theta$ then the damage due to the nth occurrence of the risk is considered as major one. If p is the probability of the event that the damage due to the nth occurrence of the risk is a major one then $p = P(X_n > \theta)$ or equivalently $p = 1 - P(X_n \le \theta)$.

Hence (2.8.2) implies that $p = 1 - F_X(\theta)$.

Let N be a random variable denoting the number of risk occurrences required to get the first major damage. The random variable N follows the geometric type II distribution with probability function $P(N = n) = pq^{n-1},\ q = 1 - p,\ n = 1, 2, \ldots$ and probability generating function

$$P_N(z) = \frac{pz}{1 - qz}. \tag{2.8.3}$$

Moreover, the random variable N is independent of the sequence of continuous, positive, independent, and identically distributed random variables $\{C_n, n = 1, 2, \ldots\}$.

The random sum $Y = C_1 + C_2 + \cdots + C_N$ denotes the occurrence time of the first major damage. From Sect. (2.5), (2.8.1), and (2.8.3) it follows that the characteristic function of the random sum $Y = C_1 + C_2 + \cdots + C_N$ is

$$\varphi_Y(u) = \frac{p\varphi_C(u)}{1 - q\varphi_C(u)}. \tag{2.8.4}$$

A special case of the characteristic function $\varphi_Y(u)$ arises if the random variables of the sequence $\{C_n, n = 1, 2, \ldots\}$ follow the exponential distribution with characteristic function

$$\varphi_C(u) = \frac{\mu}{\mu - iu}, \ \mu > 0. \tag{2.8.5}$$

From (2.8.4) and (2.8.5) it follows that $\varphi_Y(u) = \frac{v}{v - iu}$ where $v = p\mu$.

Hence, in this case, the random sum $Y = C_1 + C_2 + \cdots + C_N$ follows the exponential distribution with parameter $v = p\mu$.

The time $Y = C_1 + C_2 + \cdots + C_N$, of the first major damage from the occurrence of a risk, is particularly significant in practice if the realization of the first major damage implies the destruction of the organization threatened by the risk. In that case the random sum $Y = C_1 + C_2 + \cdots + C_N$ denotes the life time of the organization. A direct consequence of that case is the recognition of the importance of the random sum $Y = C_1 + C_2 + \cdots + C_N$ and the corresponding characteristic function $\varphi_Y(u) = \frac{p\varphi_C(u)}{1 - q\varphi_C(u)}$ in the formulation and investigation of stochastic models describing the behavior and evolution of an organization. The form of the characteristic function $\varphi_C(u)$ reflects the difficulty of investigating of such stochastic models. The presence of the time value of money in stochastic models, having as a constituent element the random sum $Y = C_1 + C_2 + \cdots + C_N$, and describing the evolution of a system, is of significant practical importance.

2.9 Number of Ongoing Risk Occurrences

A risk occurrence is considered as an ongoing one, at a given time point, if the risk cause is active at that time point. The present section concentrates on the establishment of an application of a result of service systems theory for evaluating the distribution of the random variable denoting the ongoing occurrences of a risk. The presentation of that application is based on the concept of ordered sample of continuous, independent, and identically distributed random variables, and a property of the homogeneous Poisson process.

Let $C_1, C_2, \ldots, C_n$ be random variables. The random variables $L_1, L_2, \ldots, L_n$ is an ordered random sample corresponding to the random variables $C_1, C_2, \ldots, C_n$ if the random variable $L_\kappa, \kappa = 1, 2, \ldots, n$ denotes the κth smallest value among $C_1, C_2, \ldots, C_n$.

We suppose that the random variables $C_1, C_2, \ldots, C_n$ are continuous, independent, and identically distributed. Moreover, we suppose that the random variables $C_1, C_2, \ldots, C_n$ are equally distributed with the random variable C having probability generating function $f_C(c)$.

In this case the joint probability density function of the ordered random sample $L_1, L_2, \ldots, L_n$ has the form

$$f_{L_1,L_2,\ldots,L_n}(l_1, l_2, \ldots, l_n) = n!\, f_C(l_1) f_C(l_2) \ldots f_C(l_n). \tag{2.9.1}$$

We consider the homogeneous Poisson process $\{N(t), t \geq 0\}$ with $E(N(t)) = \lambda t$.

The following theorem establishes the property of homogeneous Poisson process $\{N(t), t \geq 0\}$ which constitutes the structural element of the present section.

Theorem 2.9.1 *Let* $\{N(t), t \geq 0\}$ *be a homogeneous Poisson process with* $E(N(t)) = \lambda t$ *and* $W_\kappa, \kappa = 1, 2, \ldots$ *a random variable denoting the waiting time for the occurrence of the* κth *event of the homogeneous Poisson process* $\{N(t), t \geq 0\}$.

If $g(w_1, w_2, \ldots, w_n | N(t) = n)$ *is the joint probability density function of the random variable* $W_1, W_2, \ldots, W_n$ *when* $N(t) = n$ *then* $g(w_1, w_2, \ldots, w_n | N(t) = n) = \frac{n!}{t^n}$, $0 < w_1 < w_2 < \ldots < w_n < t$.

Proof We consider the time points $t_1, t_2, \ldots, t_n, t_{n+1}$ satisfying $t_1 < t_2 < \ldots < t_n < t_{n+1}$ and $t = t_{n+1}$.

Moreover, we consider the positive real numbers $h_1, h_2, \ldots, h_n$ satisfying $t_1 + h_1 < t_2, \ldots, t_n + h_n < t_{n+1}$.

We have

$$\begin{gathered} P\{[t_1 \leq W_1 \leq t_1 + h_1, \ldots, t_n \leq W_n \leq t_n + h_n] | (N(t) = n)\} = \\ \frac{P\{[t_1 \leq W_1 \leq t_1 + h_1, \ldots, t_n \leq W_n \leq t_n + h_n], (N(t) = n)\}}{P(N(t) = n)}. \end{gathered} \tag{2.9.2}$$

Since the event $\{[t_1 \leq W_1 \leq t_1 + h_1, \ldots, t_n \leq W_n \leq t_n + h_n], (N(t) = n)\}$ is equivalent to the event $\{[N(t_1 + h_1) - N(t_1) = 1, \ldots, N(t_n + h_n) - N(t_n) = 1], (N(t) = n)\}$ then (2.9.2) implies that

$$\begin{gathered} P\{[t_1 \leq W_1 \leq t_1 + h_1, \ldots, t_n \leq W_n \leq t_n + h_n] | (N(t) = n)\} = \\ \frac{P\{[N(t_1 + h_1) - N(t_1) = 1, \ldots, N(t_n + h_n) - N(t_n) = 1], (N(t) = n)\}}{P(N(t) = n)}. \end{gathered} \tag{2.9.3}$$

Since the event $\{[N(t_1 + h_1) - N(t_1) = 1, \ldots, N(t_n + h_n) - N(t_n) = 1], (N(t) = n)\}$ is equivalent to the event $\{[N(t_1 + h_1) - N(t_1) = 1, \ldots, N(t_n + h_n) - N(t_n) = 1, 0\ \textit{no events elsewhere in}\ [0, t]]\}$ then (2.9.3) implies that

$$\begin{gathered} P\{[t_1 \leq W_1 \leq t_1 + h_1, \ldots, t_n \leq W_n \leq t_n + h_n] | (N(t) = n)\} = \\ \frac{P\{[N(t_1 + h_1) - N(t_1) = 1, \ldots, N(t_n + h_n) - N(t_n) = 1,\ \textit{no events elsewhere in}\ [0, t]]\}}{P(N(t) = n)}. \end{gathered} \tag{2.9.4}$$

Since the process $\{N(t), t \geq 0\}$ is a homogeneous Poisson process with $E(N(t)) = \lambda t$ then (2.9.4) implies that

$$P\{[t_1 \leq W_1 \leq t_1 + h_1, \ldots, t_n \leq W_n \leq t_n + h_n] | (N(t) = n)\}$$
$$= \frac{\lambda h_1 e^{-\lambda h_1} \ldots e^{-\lambda(t-h_1-\ldots-h_n)}}{e^{-\lambda t} \frac{(\lambda t)^n}{n!}}$$

or equivalently

$$P\{[t_1 \leq W_1 \leq t_1 + h_1, \ldots, t_n \leq W_n \leq t_n + h_n] | (N(t) = n)\} = \frac{n!}{t^n} h_1 \ldots h_n. \quad (2.9.5)$$

From (2.9.5) it follows that

$$\frac{P\{[t_1 \leq W_1 \leq t_1 + h_1, \ldots, t_n \leq W_n \leq t_n + h_n] | (N(t) = n)\}}{h_1 h_2 \ldots h_n} = \frac{n!}{t^n}. \quad (2.9.6)$$

Hence the joint probability generating function of the random variables $W_1, W_2, \ldots, W_n$ given that $N(t) = n$ is

$$g(w_1, w_2, \ldots, w_n | N(t) = n) =$$
$$\lim \frac{P\{[t_1 \leq W_1 \leq t_1 + h_1, \ldots, t_n \leq W_n \leq t_n + h_n] | (N(t) = n)\}}{h_1 \ldots h_n}$$
$$h_1 \to 0, \ldots, h_n \to 0 \quad (2.9.7)$$

From (2.9.6) and (2.9.7) it follows that $g(w_1, w_2, \ldots, w_n | N(t) = n) = \lim\limits_{h_1 \to 0,\, h_2 \to 0, \ldots,\, h_n \to 0} \frac{n!}{t^n}$, or equivalently

$$g(w_1, w_2, \ldots, w_n | N(t) = n) = \frac{n!}{t^n}. \quad (2.9.8)$$

From Theorem 2.9.1 and (2.9.8) we get the following conclusion. If $N(t) = n$, that is n events of the homogeneous Poisson process have occurred in the time interval $[0, t]$ and the continuous, independent, positive, and identically distributed random variables $V_1,\ V_2,\ \ldots,\ V_n$ represent the unordered occurrence times of these events then the random variables $W_1,\ W_2,\ \ldots,\ W_n$ is the ordered sample of the random variables $V_1,\ V_2,\ \ldots,\ V_n$.

From (2.9.1) and Theorem 2.9.1 it follows that the random variables $V_1,\ V_2,\ \ldots,\ V_n$ are equally distributed with the random variable V which follows the uniform distribution with probability density function $f_V(v) = \frac{1}{t},\ 0 < v < t$.

The following result constitutes an application in risk management of a result of service systems theory. □

Theorem 2.9.2 *Let* $\{N(t), t \geq 0\}$ *be a homogeneous Poisson process with* $E(N(t)) = \lambda t$.

We suppose that the random variable $N(t)$ *denotes the frequency of a risk in the time interval* $[0, t]$ *and* $\{Y_n, n = 1, 2, \ldots\}$ *is a sequence of continuous, positive,* *independent, and identically distributed random variables. The random variables of the sequence represent the durations of the risk occurrences. Moreover, these random variables are equally distributed with the random variable Y having distribution function $F_Y(y)$.*

If the random variable $\Pi(t)$ denotes the number of the ongoing risk occurrences at the time point t then the probability generating function of the random variable $\Pi(t)$ is $P_{\Pi(t)}(z) = e^{\lambda pt(z-1)}$ where

$$p = \int_0^t \frac{1 - F_Y(y)}{t} dy.$$

Proof We have $P_{\Pi(t)}(z) = E\left(z^{\Pi(t)}\right)$ or equivalently

$$P_{\Pi(t)}(z) = E\left(E\left(z^{\Pi(t)} | N(t)\right)\right). \tag{2.9.9}$$

From (2.9.9) it follows that

$$P_{\Pi(t)}(z) = \sum_{n=0}^{\infty} E\left(z^{\Pi(t)} | N(t) = n\right) P(N(t) = n). \tag{2.9.10}$$

Since

$$P(N(t) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

then (2.9.10) has the form

$$P_{\Pi(t)}(z) = \sum_{n=0}^{\infty} E\left(z^{\Pi(t)} | N(t) = n\right) e^{-\lambda t} \frac{(\lambda t)^n}{n!}. \tag{2.9.11}$$

We suppose that v is the time point of a risk occurrence where $0 < v < t$.

Since the continuous and positive random variable Y denotes the duration of the risk occurrence arising at the time point v then the probability of the event that this risk occurrence will be an ongoing one at the time point t is $P(Y > t - v)$.

Since $P(Y > t - v) = 1 - P(Y \leq t - v)$ and $F_Y(y)$ is the distribution function of the random variable Y then

$$P(Y > t - v) = 1 - F_Y(t - v). \tag{2.9.12}$$

If $N = n$, that is n risk occurrences arise in the time interval $[0, t]$, then Theorem 2.9.1 implies that the unordered time points of the n risk occurrences in the time interval $[0, t]$ are continuous, independent random variables $V_1, V_2, \ldots, V_n$ equally distributed with the random variable V which follows the uniform distribution with probability density function $f_V(v) = \frac{1}{t},\ 0 < v < t$.

Hence the probability of the event that a risk occurrence arising in the time interval $[0, t]$ is ongoing at the time point t independently of the other risk occurrences in the time interval $[0, t]$, according to (2.9.12), has the form

$$p = \int_0^t \frac{1 - F_Y(t - v)}{t} dv$$

or equivalently the form

$$p = \int_0^t \frac{1 - F_Y(y)}{t} dy.$$

In this case if $N(t) = n$, that is n risk occurrences arise in the time interval $[0, t]$ then the random variable $\Pi(t)|N(t) = n$, which denotes the number of risk occurrences in the time interval $[0, t]$ and which risk occurrences are ongoing at the time point t, follows the binomial distribution with parameters n and p.

Hence the probability generating function of the random variable $\Pi(t)|N(t) = n$ is

$$E\left(z^{\Pi(t)}|N(t) = n\right) = (pz + q)^n \tag{2.9.13}$$

where $q = 1 - p$.

From (2.9.11) and (2.9.13) it follows that the probability generating function $P_{\Pi(t)}(z)$ of the random variable $\Pi(t)$ has the form

$$P_{\Pi(t)}(z) = \sum_{n=0}^{\infty} (pz + q)^n e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

or equivalently the form

$$P_{\Pi(t)}(z) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{[\lambda t(pz + q)]^n}{n!}. \tag{2.9.14}$$

From (2.9.14) it follows that $P_{\Pi(t)}(z) = e^{-\lambda t} e^{\lambda t(pz+q)}$ or equivalently $P_{\Pi(t)}(z) = e^{\lambda pt(z-1)}$.

Hence the random variable $\Pi(t)$ follows the Poisson distribution with parameter λpt.

Since the random variable $\Pi(t)$ denotes the number of the occurrences of a risk in the time interval $[0, t]$ and which occurrences are ongoing at the time point t then this random variable can be used in formulating stochastic models for describing operations suitable for financing damages due to the occurrences of that risk. The constituent elements of the contribution of the present section are the following two. The first constituent element is the introduction of the concept of ongoing risk occurrence. The second element is the application of a significant result of service systems theory in modeling the concept of the number of ongoing occurrences of a risk. That application provides risk managers with the ability to get a holistic consideration of the number of ongoing occurrences of a risk. The fundamental assumption of the proposed application is that the frequency of the risk in the time interval $[0, t]$ is represented by a homogeneous Poisson $\{N(t), t \geq 0\}$.

This assumption does not restrict the significance of the proposed application since the homogeneous Poisson process is considered as a very efficient model of the frequency of a risk in the time interval $[0, t]$.

The conclusion that the number of ongoing occurrences of a risk at a given time point t is represented by the random variable $\Pi(t)$ following the Poisson distribution with parameter λpt can be considered as a very good reason for modeling the number of ongoing risk occurrences at a random time point. □

2.10 Multiplicative Models of Risk Severity

The purpose of the present section is the formulation and investigation of a stochastic multiplicative model for the description and investigation of risk severity. The model is based on the product of two continuous, positive, and independent random variables.

We suppose that the duration of a risk is represented by the continuous, and positive random variable S with distribution function $F_S(s)$, probability density function $f_S(s)$, and characteristic function $\varphi_S(u)$.

We suppose that U is a continuous and positive random variable with distribution function $F_U(\upsilon)$, probability density function $f_U(\upsilon)$, and characteristic function $\varphi_U(u)$.

The random variable U denotes the damage, per unit of time, due to the occurrence of a risk. We suppose that the random variable S is independent of the random variable U.

The random variable $X = SU$ represents the severity of risk. The independence of random variables S, U permits the evaluation of the distribution function $F_X(x)$, the evaluation of the probability density function $f_X(x)$, and the evaluation of the characteristic function $\varphi_X(u)$ of the random variable $X = SU$.

We have $F_X(x) = P(X \leq x)$ or equivalently $F_X(x) = P(SU \leq x)$.

Hence

$$F_X(x) = \int_0^\infty P(SU \leq x | U = \upsilon) f_U(\upsilon) d\upsilon$$

or equivalently

$$F_X(x) = \int_0^\infty P(\upsilon S \le x | U = \upsilon) f_U(\upsilon) d\upsilon. \tag{2.10.1}$$

Since the random variable S is independent of the random variable U then (2.10.1) has the form

$$F_X(x) = \int_0^\infty P(\upsilon S \le x) f_U(\upsilon) d\upsilon$$

or equivalently the form

$$F_X(x) = \int_0^\infty P\left(S \le \frac{x}{\upsilon}\right) f_U(\upsilon) d\upsilon. \tag{2.10.2}$$

Since

$$P\left(S \le \frac{x}{\upsilon}\right) = F_S\left(\frac{x}{\upsilon}\right)$$

then (2.10.2) implies that the distribution function $F_X(x)$ of the random variable $X = SU$ is

$$F_X(x) = \int_0^\infty F_S\left(\frac{x}{\upsilon}\right) f_U(\upsilon) d\upsilon. \tag{2.10.3}$$

It is obvious that the following formula is also valid

$$F_X(x) = \int_0^\infty F_U\left(\frac{x}{s}\right) f_S(s) ds. \tag{2.10.4}$$

From (2.10.3) and (2.10.4) it follows that the probability density function of the random variable $X = SU$ has the form

$$f_X(x) = \int_0^\infty \frac{1}{\upsilon} f_S\left(\frac{x}{\upsilon}\right) f_U(\upsilon) d\upsilon$$

or equivalently the form

$$f_X(x) = \int_0^{\infty} \frac{1}{s} f_U\left(\frac{x}{s}\right) f_S(s)\,ds.$$

If $\varphi_X(u)$ is the characteristic function of the random variable $X = SU$ then $\varphi_X(u) = E(e^{iuX})$ or equivalently $\varphi_X(u) = E(e^{iuSU})$.

Hence

$$\varphi_X(u) = \int_0^{\infty} E\left(e^{iuSU} | U = \upsilon\right) f_U(\upsilon)\,d\upsilon$$

or equivalently

$$\varphi_X(u) = \int_0^{\infty} E\left(e^{iu\upsilon S} | U = \upsilon\right) f_U(\upsilon)\,d\upsilon. \tag{2.10.5}$$

Since the random variable S is independent of the random variable U then (2.10.5) has the form

$$\varphi_X(u) = \int_0^{\infty} E\left(e^{iu\upsilon S}\right) f_U(\upsilon)\,d\upsilon. \tag{2.10.6}$$

Since $E(e^{iu\upsilon S}) = \varphi_S(u\upsilon)$ then (2.10.6) implies that the characteristic function $\varphi_X(u)$ of the random variable $X = SU$ is

$$\varphi_X(u) = \int_0^{\infty} \phi_S(u\upsilon) f_U(\upsilon)\,d\upsilon \tag{2.10.7}$$

It is obvious that the following formula is also valid

$$\varphi_X(u) = \int_0^{\infty} \phi_U(us) f_S(s)\,ds$$

The consideration of special cases of the distribution of the stochastic model $X = SU$, with use of the corresponding characteristic function $\varphi_X(u)$, is very significant for the practical applications of the stochastic model $X = SU$ in the description and analysis of concepts and operations of risk management.

Special cases of the distribution of the stochastic model $X = SU$, having probability distribution functions with unique mode a the point 0, are of particular practical importance.

The probability density function $f_\Pi(\Pi)$ of the continuous random variable Π is said unimodal at the point 0 if this probability density function has a unique maximum at the point 0. The establishment of the property of unimodality at the point 0 for the probability density function $f_X(x)$ of risk severity $X = SU$ is based on a result of Khintchine and a result of Medgyessy. The result of Khintchine states that the probability density function $f_\Pi(\pi)$ is unimodal at the point 0 if the corresponding characteristic function $\varphi_\Pi(u)$ has the form

$$\varphi_\Pi(u) = \int_0^1 \varphi_D(uy)dy \tag{2.10.8}$$

where $\varphi_D(u)$ is the characteristic function of a continuous random variable D.

The result of Medgyessy states that if the continuous random variable Π has probability generating function $f_\Pi(\pi)$ which is unimodal at the point 0 and B is a continuous random variable independent of the random variable Π then the random variable ΠB has a probability density function with a unique mode at the point 0.

If the random variable U follows the uniform distribution with probability density function $f_U(\upsilon) = 1,\ 0 < \upsilon < 1$ then (2.10.7) implies that the characteristic function of risk severity $X = SU$ has the form

$$\varphi_X(u) = \int_0^1 \varphi_S(u\upsilon)d\upsilon. \tag{2.10.9}$$

From (2.10.8) and (2.10.9) it follows that the probability density function

$$f_X(x) = \int_0^1 \frac{1}{\upsilon} f_S\left(\frac{x}{\upsilon}\right) d\upsilon$$

of risk severity $X = SU$ is unimodal at the point 0.

If the continuous and positive random variable S has probability density function $f_S(s)$ with unique mode at the point 0 or the continuous and positive random variable U has probability density function $f_U(\upsilon)$ with unique mode at the point 0 then the result of Medgyessy implies that the random variable $X = SU$ has probability density function

$$f_X(x) = \int_0^\infty \frac{1}{\upsilon} f_S\left(\frac{x}{\upsilon}\right) f_U(\upsilon) d\upsilon,$$

or equivalently probability density function

$$f_X(x) = \int_0^\infty \frac{1}{s} f_U\left(\frac{x}{s}\right) f_S(s) ds$$

which has a unique mode at the point 0. The existence of a unique mode at the point 0 for the probability density function $f_X(x)$ of the random variable $X = SU$ implies that the event the size of the damage, due to an occurrence of the risk, to be in an area right to the point 0 has a significant probability. That means that the organization threatened by the risk can select the retention of the risk instead of the transfer of the risk.

Significant theoretical and practical interest has the special case of the stochastic multiplicative model $X = SU$ if the continuous and positive random variable S follows the exponential distribution with characteristic function

$$\varphi_S(u) = \frac{\mu}{\mu - iu}. \tag{2.10.10}$$

From (2.10.7) and (2.10.10) it follows that the characteristic function of the stochastic multiplicative model $X = SU$ has the form

$$\varphi_X(u) = \int_0^\infty \frac{\mu}{\mu - iu\upsilon} f_U(\upsilon) d\upsilon.$$

Since the probability density function $f_S(s) = \mu e^{-\mu s}$ has a unique mode at the point 0 then the probability density function

$$f_X(x) = \int_0^\infty \frac{\mu}{\upsilon} e^{-\mu\frac{x}{\upsilon}} f_U(\upsilon) d\upsilon$$

corresponding to the characteristic function

$$\varphi_X(u) = \int_0^\infty \frac{\mu}{\mu - iu\upsilon} f_U(\upsilon) d\upsilon$$

has a unique mode at the point 0. Characteristic functions of the form

$$\varphi_X(u) = \int_0^\infty \frac{\mu}{\mu - iu\upsilon} f_U(\upsilon) d\upsilon$$

belong to the class of infinitely divisible characteristic functions having important applications to stochastic processes.

From a theoretical and practical point of view, it is of particular interest to investigate properties of the probability density function $f_U(\upsilon)$ of the random variable U which are transferred to the probability density function

$$f_X(x) = \int_0^\infty \frac{\mu}{\upsilon} e^{-\mu\frac{x}{\upsilon}} f_U(\upsilon)d\upsilon$$

of the random variable $X = SU$.

An interpretation of the stochastic multiplicative model $X = SU$ in the area of fundamental risk control operations is the following.

If the continuous and positive random variable U takes values in the interval $(0, 1)$ then the random variable U can be considered as a coefficient describing the impact of a risk control operation. In this case the presence of the random variable U and the presence of the random variable S in the stochastic multiplicative model $X = SU$ permit the interpretation of the random variable X as the consequence of the application of a risk control operation. This interpretation of the stochastic multiplicative model $X = SU$ in the area of fundamental risk control operations requires the consideration of the continuous and positive random variable S as a positive component of the concept of risk.

Since the random variable S is continuous and positive then this random variable can represent the severity or the duration of a risk. Hence the stochastic multiplicative model $X = SU$ can be used for the description and analysis of risk severity and risk duration reduction operations. Such applications of the stochastic multiplicative model $X = SU$ constitute the main purpose of the third chapter of the present work.

Particular practical interest has the stochastic multiplicative model $X = SU$ with the continuous and positive random variable S having the form of a random sum. In this case the investigation of the stochastic multiplicative model $X = SU$ is based on the corresponding characteristic function $\varphi_X(u)$.

2.11 Riskiness

We consider a risk with frequency denoted by the discrete random variable N taking values in the set $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$, severity denoted by the continuous and positive random variable X, and duration denoted by the continuous and positive random variable S.

The continuous and positive random variable $R = NXS$ is said riskiness of the risk or simply riskiness. Since riskiness $R = NXS$ is proportional to risk frequency N, risk severity X, and risk duration S then riskiness can be used as a model for describing the aversion of a person for a risk with frequency N, severity X, and duration S.

The distribution function $F_R(r)$, the probability density function $f_R(r)$, and the characteristic function $\varphi_R(u)$ of riskiness $R = NXS$ are the analytical tools implementing the theoretical and practical applicability of that concept.

The present section concentrates on the implementation of two purposes. The first purpose is the evaluation of the characteristic function $\varphi_R(u)$ of riskiness $R = NXS$.

The choice for evaluating the characteristic function $\varphi_R(u)$ is based on the important applications of the results of the theory of characteristic functions. The evaluation of the characteristic function $\varphi_R(u)$ of riskiness $R = NXS$ is based on the independence of the random variables N, X, S.

The second purpose is the establishment of the unimodality at the point 0 of the probability density function of riskiness $R = NXS$ by making use of the independence of the random variables N, X, S, the unimodality at the point 0 of the probability density function of the random variable X or equivalently the unimodality at the point 0 of the probability density function of the random variable S, the integral representation of a characteristic function corresponding to a probability density function with unique mode at the point 0, and the probability density function of the product of two independent and continuous random variables one of which has a probability density function with unique mode at the point 0.

The significance of the purposes of the present section is based on the presence of the fundamental quantitative components of risk, that is risk frequency N, risk severity X, and risk duration S in the definition of riskiness $R = NXS$.

The present section makes quite clear that the characteristic function $\varphi_R(u)$ constitutes a strong analytical tool for investigating the probabilistic behavior of riskiness. The establishment of a sufficient condition for evaluating the characteristic function of riskiness is provided by the following theorem.

Theorem 2.11.1 *We suppose that N is a discrete random variable with values in the set $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ and probability function $P(N = n) = p_n,\ n = 0, 1, 2, \ldots$, X is a continuous and positive random variable with probability density function $f_X(x)$ and characteristic function $\varphi_X(u)$, and S is a continuous and positive random variable with probability density function $f_S(s)$ and characteristic function $\varphi_S(u)$. If the random variables N, X, S are independent then the characteristic function of the random variable $R = NXS$ has the form*

$$\varphi_R(u) = \sum_{n=0}^{\infty} p_n \int_0^{\infty} \varphi_X(nus) f_S(s)\,ds$$

or equivalently the form

$$\varphi_R(u) = \sum_{n=0}^{\infty} p_n \int_0^{\infty} \varphi_S(nux) f_X(x)\,dx.$$

Proof The independence of the random variables N, X, S implies the independence of the random variables X, S.

We consider the random variable XS with characteristic function $\varphi_{XS}(u)$. We have $\varphi_{XS}(u) = E(E(e^{iuXS}|S))$ or equivalently

$$\varphi_{XS}(u) = \int_0^\infty E\left(e^{iuXS}|S = s\right) f_S(s) ds. \qquad (2.11.1)$$

From (2.11.1) it follows that

$$\varphi_{XS}(u) = \int_0^\infty E\left(e^{iusX}|S = s\right) f_S(s) ds. \qquad (2.11.2)$$

Since the random variable X is independent of the random variable S then (2.11.2) implies that

$$\varphi_{XS}(u) = \int_0^\infty E\left(e^{iusX}\right) f_S(s) ds. \qquad (2.11.3)$$

Since

$$E\left(e^{iusX}\right) = \varphi_X(us) \qquad (2.11.4)$$

then (2.11.3) and (2.11.4) imply that the characteristic function of the random variable XS is

$$\varphi_{XS}(u) = \int_0^\infty \varphi_X(us) f_S(s) ds. \qquad (2.11.5)$$

It is easily seen that (2.11.5) has the equivalent form

$$\varphi_{XS}(u) = \int_0^\infty \varphi_S(ux) f_X(x) dx. \qquad (2.11.6)$$

The characteristic function $\varphi_{XS}(u)$ in (2.11.5) or (2.11.6) of the random variable XS and the proof of independence of the random variable N and the random variable XS are required for the evaluation of the characteristic function $\varphi_R(u)$ of riskiness $R = NXS$.

Let $\varphi_N(\xi)$ be the characteristic function of the random variable N, $\varphi_{XS}(u)$ be the characteristic function of the random variable XS and $\varphi_{N,XS}(\xi, u)$ be the characteristic function of the vector (N, XS) of random variables N, XS.

The proof of independence of the random variables N, XS requires the proof of the relationship

$$\varphi_{N,XS}(\xi, u) = \varphi_N(\xi)\varphi_{XS}(u). \tag{2.11.7}$$

We have

$$\varphi_{N,XS}(\xi, u) = E\left(e^{i\xi N + iuXS}\right).$$

or equivalently

$$\varphi_{N,XS}(\xi, u) = E\left(E\left(e^{i\xi N + iuXS}|S\right)\right). \tag{2.11.8}$$

From (2.11.8) it follows that

$$\varphi_{N,XS}(\xi, u) = \int_0^\infty E\left(e^{i\xi N + iuXS}|S = s\right)f_S(s)ds$$

or equivalently

$$\varphi_{N,XS}(\xi, u) = \int_0^\infty E\left(e^{i\xi N + iusX}|S = s\right)f_S(s)ds. \tag{2.11.9}$$

From (2.11.9) and the independence of the random variables N, X, S. It follows that

$$\varphi_{N,XS}(\xi, u) = \int_0^\infty E\left(e^{i\xi N + iusX}\right)f_S(s)ds. \tag{2.11.10}$$

Since the independence of the random variables N, X, S implies the independence of the random variables N, X then the random variables $e^{i\xi N}, e^{iusX}$ are also independent. Hence (2.11.10) has the form

$$\varphi_{N,XS}(\xi, u) = \int_0^\infty E\left(e^{i\xi N}\right)E\left(e^{iusX}\right)f_S(s)ds$$

or equivalently the form

$$\varphi_{N,XS}(\xi, u) = E\left(e^{i\xi N}\right)\int_0^\infty E\left(e^{iusX}\right)f_S(s)ds. \tag{2.11.11}$$

Since $\varphi_N(\xi) = E\left(e^{i\xi N}\right)$ and $\varphi_X(us) = E(e^{iusX})$ then (2.11.11) implies that

$$\varphi_{N,XS}(\xi, u) = \varphi_N(\xi) \int_0^\infty \varphi_X(us) f_S(s) ds. \tag{2.11.12}$$

From (2.11.5) it follows that the characteristic function of the random variable XS is

$$\varphi_{XS}(u) = \int_0^\infty \varphi_X(us) f_S(s) ds. \tag{2.11.13}$$

From (2.11.12) and (2.11.13) it follows that (2.11.7) is valid that is $\varphi_{N,XS}(\xi, u) = \varphi_N(\xi)\varphi_{XS}(u)$. Hence the random variables N, XS are independent.

The characteristic function $\varphi_{XS}(u)$ of the random variable XS, the probability function $P(N = n) = p_n,\ n = 0, 1, 2, \ldots$ of the random variable N and the independence of the random variables N, XS permit the evaluation of the characteristic function $\varphi_R(u)$ of the random variable $R = NXS$ in the following way. We have $\varphi_R(u) = E(e^{iuNXS})$ or equivalently

$$\varphi_R(u) = E\left(E\left(e^{iuNXS}|N\right)\right). \tag{2.11.14}$$

From (2.11.14) it follows that

$$\varphi_R(u) = \sum_{n=0}^{\infty} E\left(e^{iuNXS}|N = n\right) P(N = n)$$

or equivalently

$$\varphi_R(u) = \sum_{n=0}^{\infty} E\left(e^{iunXS}|N = n\right) p_n. \tag{2.11.15}$$

Since the random variable N is independent of the random variable XS then (2.11.15) has the form

$$\varphi_R(u) = \sum_{n=0}^{\infty} E\left(e^{iunXS}\right) p_n. \tag{2.11.16}$$

Since $\varphi_{XS}(u) = E(e^{iuXS})$ or equivalently

$$\varphi_{XS}(u) = \int_0^\infty \varphi_X(us) f_S(s) ds$$

then $\varphi_{XS}(nu) = E(e^{inuXS})$ or equivalently

$$\varphi_{XS}(nu) = \int_0^\infty \phi_X(nus) f_S(s) ds. \tag{2.11.17}$$

From (2.11.16) and (2.11.17) it follows that the characteristic function $\varphi_R(u)$ of riskiness $R = NXS$ has the form

$$\varphi_R(u) = \sum_{n=0}^\infty p_n \int_0^\infty \varphi_X(nus) f_S(s) ds.$$

Since (2.11.6) has the form

$$\varphi_{XS}(u) = \int_0^\infty \varphi_S(ux) f_X(x) dx$$

then for the characteristic function $\varphi_R(u)$ of riskiness $R = NXS$ is also valid the formula

$$\varphi_R(u) = \sum_{n=0}^\infty p_n \int_0^\infty \varphi_S(nux) f_X(x) dx.$$

From a theoretical and a practical point of view it is completely understood that the evaluation of the characteristic function $\varphi_R(u)$ of riskiness $R = NXS$, if the random variables N, X, S are independent, is a very important factor for the probabilistic description, investigation, and solution of problems related with operations of analysis, measurement, evaluation, communication, control, and financing of risks.

The riskiness $R = NXS$ has particular practical interest if risk frequency N follows the Bernoulli distribution. In this case the characteristic function of riskiness has the form

$$\varphi_R(u) = q + p \int_0^\infty \varphi_X(us) f_S(s) ds. \tag{2.11.18}$$

The significance of Bernoulli distribution in the probabilistic consideration of risk frequency makes necessary the evaluation of some special cases of (2.11.18).

We suppose that risk severity X follows the uniform distribution with characteristic function

$$\varphi_X(u) = \frac{e^{iu} - 1}{iu}$$

and risk duration S follows the beta distribution with probability density function $f_S(s) = 2s,\ 0 < s < 1.$

In this case (2.11.18) implies that the characteristic function of riskiness has the form

$$\varphi_R(u) = q + 2p \int_0^1 \frac{e^{ius} - 1}{ius} s ds$$

or equivalently the form

$$\varphi_R(u) = q + 2p \frac{1 + iu - e^{iu}}{u^2}.$$

We suppose that risk severity X follows the exponential distribution with characteristic function

$$\varphi_X(u) = \frac{\mu}{\mu - iu},\ \mu > 0$$

and risk duration S follows the uniform distribution with probability density function $f_S(s) = 1,\ 0 < s < 1.$

The characteristic function of riskiness has the form

$$\varphi_R(u) = q + p \int_0^1 \frac{\mu}{\mu - ius} ds$$

or equivalently the form

$$\varphi_R(u) = q + p \frac{\mu}{iu} \log\left(\frac{\mu}{\mu - iu}\right).$$

We suppose that risk severity X follows the gamma distribution with characteristic function

$$\varphi_X(u) = \left(\frac{\mu}{\mu - iu}\right)^2,\ \mu > 0$$

and risk duration follows S follows the uniform distribution with probability density function $f_S(s) = 1,\ 0 < s < 1$

In this case (2.11.18) implies that the characteristic function of riskiness has the form

$$\varphi_R(u) = q + p \int_0^1 \left(\frac{\mu}{\mu - ius}\right)^2 ds.$$

or equivalently the form

$$\varphi_R(u) = q + p\frac{\mu}{\mu - iu}.$$

We suppose that risk severity X follows the renewal distribution corresponding to the gamma distribution with parameters μ and 2. The characteristic function of the random variable X is

$$\varphi_X(u) = \mu\left[\left(\frac{\mu}{\mu - iu}\right)^2 - 1\right] \Big/ 2iu. \qquad (2.11.19)$$

From (2.11.19) it follows that

$$\varphi_X(u) = \frac{1}{2}\left(\frac{\mu}{\mu - iu}\right)^2 + \frac{1}{2}\frac{\mu}{\mu - iu}.$$

Moreover, we suppose that risk duration S follows the uniform distribution with probability density function $f_S(s) = 1,\ 0 < s < 1.$

In this case we have that

$$\varphi_R(u) = q + \frac{p}{2}\int_0^1 \left(\frac{\mu}{\mu - ius}\right)^2 ds + \frac{p}{2}\int_0^1 \frac{\mu}{\mu - ius} ds. \qquad (2.11.20)$$

From (2.11.20) it follows that

$$\varphi_R(u) = q + \frac{p}{2}\left(\frac{\mu}{\mu - iu}\right) + \frac{p}{2}\frac{\mu}{iu}\log\left(\frac{\mu}{\mu - iu}\right).$$

The establishment of a sufficient condition for the unimodality at the point 0 of the probability density function $f_R(r)$ of riskiness $R = NXS$ is based on the integral representation of the characteristic function of a probability density function with unique mode at the point 0 and the unimodality at the point 0 of the probability density function of a continuous random variable which is the product of two continuous independent random variables, one of which has a probability density function with unique mode at the point 0.

The existence of a unique mode at the point 0 for the probability density function $f_R(r)$ of riskiness $R = NXS$ substantially facilitates decision making for operations treating a risk of which the frequency is represented by the discrete random variable N with values in the set $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$, the severity is represented by the continuous and positive random variable X and the duration is represented by the continuous and positive random variable S.

The unimodality at the point 0 of the probability density function $f_R(r)$ of riskiness $R = NXS$ can be used for the study of very complex risks related with the evolution of modern organizations. □

Theorem 2.11.2 *We suppose that N is a discrete random variable with values in the set $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$, and probability function $P(N = n) = p_n,\ n = 0, 1, 2, \ldots$, X is a continuous and positive random variable with probability density function $f_X(x)$ and characteristic function $\varphi_X(u)$, and S is a continuous and positive random variable with probability density function $f_S(s)$ and characteristic function $\varphi_S(u)$.*

If the random variables N, X, S are independent and the probability density function $f_X(x)$ is unimodal at the point 0 or the probability density function $f_S(s)$ is unimodal at the point 0 then the probability density function $f_R(r)$ of the random variable $R = NXS$ is unimodal at the point 0.

Proof We suppose that the continuous and positive random variable X has probability density function $f_X(x)$ which is unimodal at the point 0. Since the independence of the random variables N, X, S implies the independence of the random variables X, S then the random variable XS has probability density function which is unimodal at the point 0. Hence the characteristic function

$$\varphi_{XS}(u) = \int_0^\infty \varphi_X(us) f_S(s)ds \tag{2.11.21}$$

of the random variable XS has the form

$$\varphi_{XS}(u) = \int_0^1 \varphi_H(uy)dy \tag{2.11.22}$$

where $\varphi_H(u)$ is the characteristic function of a continuous and positive random variable H.

We consider the sequence of continuous and positive random variables

$$\{nXS, n = 0, 1, 2, \ldots\}. \tag{2.11.23}$$

From (2.11.21) and (2.11.22) it is obvious that the corresponding sequence of characteristic functions of the sequence (2.11.23) is

$$\left\{\int_0^\infty \varphi_X(nus)f_S(s)ds, n = 0, 1, 2, \ldots\right\} \tag{2.11.24}$$

or equivalently

$$\left\{\int_0^1 \varphi_H(nuy)dy, n = 0, 1, 2, \ldots\right\}. \tag{2.11.25}$$

From the independence of the random variables N, X, S and Theorem 2.11.1 it follows that the characteristic function of the random variable $R = NXS$ is

$$\varphi_R(u) = \sum_{n=0}^\infty p_n \int_0^\infty \varphi_X(nus)f_S(s)ds. \tag{2.11.26}$$

From (2.11.24), (2.11.25) and (2.11.26) it follows that

$$\varphi_R(u) = \sum_{n=0}^\infty p_n \int_0^1 \varphi_H(nuy)dy$$

or equivalently

$$\varphi_R(u) = \int_0^1 \left(\sum_{n=0}^\infty p_n\varphi_H(nuy)\right)dy. \tag{2.11.27}$$

We consider the sequence of continuous and positive random variables

$$\{nH, n = 0, 1, 2, \ldots\}. \tag{2.11.28}$$

It is obvious that the corresponding sequence of characteristic functions of the sequence (2.11.28) is $\{\varphi_H(nu), n = 0, 1, 2, \ldots\}$.

We consider the function

$$\varphi_V(u) = \sum_{n=0}^\infty p_n\varphi_H(nu)$$

which is a discrete mixture of the characteristic functions of the sequence $\{\varphi_H(nu), n = 0, 1, 2, \ldots\}$ with mixing probability function $p_n, n = 0, 1, 2, \ldots$ which belongs to the random variable N.

Hence the function

$$\varphi_V(u) = \sum_{n=0}^{\infty} p_n \varphi_H(nu) \tag{2.11.29}$$

is the characteristic function of a continuous and positive random variable V.

From (2.11.27) and (2.11.29) it follows that

$$\varphi_R(u) = \int_0^1 \varphi_V(uy)dy. \tag{2.11.30}$$

Hence (2.10.8) and (2.11.30) imply that the random variable $R = NXS$ has probability density function $f_R(r)$ which is unimodal at the point 0.

It is obvious that if the assumption of unimodality at the point 0 for the probability density function $f_X(x)$ of the random variable X is replaced by the assumption of unimodality at the point 0 for the probability density function $f_S(s)$ of the random variable S then the random variable $R = NXS$ has probability density function $f_R(r)$ which is also unimodal at the point 0.

The unimodality at the point 0 of the probability density function $f_R(r)$ means that the probability of the event for the riskiness $R = NXS$ to be in an area right to the point 0 is significant.

The independence of the random variables N, X, S implies that the mean value of the riskiness $R = NXS$ is $E(R) = E(N)E(X)E(S)$. In the case of independence of the random variables N, X, S the evaluation of the mean value of riskiness $R = NXS$ is based on the characteristic function

$$\varphi_R(u) = \sum_{n=0}^{\infty} p_n \int_0^{\infty} \varphi_X(nus) f_S(s)ds \tag{2.11.31}$$

of riskiness $R = NXS$.

From (2.11.31) we get that

$$\varphi'_R(u) = \sum_{n=0}^{\infty} np_n \int_0^{\infty} \varphi'_X(nus) s f_S(s)ds. \tag{2.11.32}$$

Moreover, from (2.11.32) it follows that

$$\varphi_R'(0) = \sum_{n=0}^{\infty} np_n \varphi_X'(0) \int_0^{\infty} sf_S(s)ds$$

or equivalently $E(R) = E(N)E(X)E(S)$. If the probabilistic information for the independent random variables N, X, S are provided by the mean values $E(N)$, $E(X)$, $E(S)$ then the mean value of riskiness $E(R) = E(N)E(X)E(S)$ is a very useful analytical tool for the development and application of risk classification operations. □

2.12 Total Risk Severity and Asset Liquidation

Let N be a discrete random variable with values in the set $\mathbf{N}_0$ and probability generating function $P_N(z)$. Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ are equally distributed with the random variable X having characteristic function $\varphi_X(u)$.

We set $T = X_1 + X_2 + \cdots + X_N$.

Let $\{C_n, n = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence $\{C_n, n = 1, 2, \ldots\}$ are equally distributed with the random variable C having characteristic function $\varphi_C(\xi)$.

We set $L = C_1 + C_2 + \cdots + C_N$.

We consider the vector (T, L).

The purpose of the present section is the establishment of properties and applications in risk management of the above vector.

The following result establishes sufficient conditions for the evaluation of the characteristic function $\varphi_T(u, \xi)$ of the vector (T, L).

Theorem 2.12.1 *Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ are equally distributed with the random variable X having characteristic function*

$$\varphi_X(u). \tag{2.12.1}$$

We consider the discrete random variable N with values in the set $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ and probability generating function

$$P_N(z) \tag{2.12.2}$$

and we set $T = X_1 + X_2 + \cdots + X_N$.

Let $\{C_n, n = 1, 2, \ldots\}$ *be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence* $\{C_n, n = 1, 2, \ldots\}$ *are equally distributed with the random variable* C *having characteristic function*

$$\varphi_C(\xi). \tag{2.12.3}$$

We set $L = C_1 + C_2 + \cdots + C_N$. If $\{X_n, n = 1, 2, \ldots\}$, N *and* $\{C_n, n = 1, 2, \ldots\}$ *are independent then the characteristic function of the vector* (T, L) *is* $\varphi_{T,L}(u, \xi) = P_N(\varphi_X(u)\varphi_C(\xi))$.

Proof We have $\varphi_{T,L}(u, \xi) = E\left(e^{iuT+i\xi L}\right)$ or equivalently we have

$$\varphi_{T,L}(u, \xi) = E\left(E\left(e^{iuT+i\xi L}|N\right)\right). \tag{2.12.4}$$

From (2.12.4) it follows that

$$\varphi_{T,L}(u, \xi) = \sum_{n=0}^{\infty} E\left(e^{iuT+i\xi L}|N = n\right)P(N = n). \tag{2.12.5}$$

It is easily seen that (2.12.5) implies that

$$\varphi_{T,L}(u, \xi) = \sum_{n=0}^{\infty} E\left(e^{iu(X_1+\cdots+X_N)+i\xi(C_1+\cdots+C_N)}|N = n\right)P(N = n). \tag{2.12.6}$$

From (2.12.6) we get that

$$\varphi_{T,L}(u, \xi) = \sum_{n=0}^{\infty} E\left(e^{iu(X_1+\cdots+X_n)+i\xi(C_1+\cdots+C_n)}|N = n\right)P(N = n). \tag{2.12.7}$$

Moreover (2.12.7) implies that

$$\varphi_{T,L}(u, \xi) = \sum_{n=0}^{\infty} E\left(e^{iuX_1+\cdots+iuX_n+i\xi C_1+\cdots+i\xi C_n}|N = n\right)P(N = n). \tag{2.12.8}$$

From (2.12.8) it follows that

$$\varphi_{T,L}(u, \xi) = \sum_{n=0}^{\infty} E\left(e^{iuX_1}\ldots e^{iuX_n}e^{iuC_1}\ldots e^{iuC_n}|N = n\right)P(N = n). \tag{2.12.9}$$

From the assumption that $\{X_n, n = 1, 2, \ldots\}$, N and $\{C_n, n = 1, 2, \ldots\}$ are independent it follows the independence of the random variables $X_1, \ldots, X_n$, N, $C_1, \ldots, C_n$.

The independence of the above random variables implies the independence of the random variables $e^{iuX_1},\ldots,e^{iuX_n},\ldots,\ N,\ e^{i\xi C_1},\ldots,\ e^{i\xi C_n}$.

Hence (2.12.9) has the form

$$\varphi_{T,L}(u,\xi)=\sum_{n=0}^{\infty}E\left(e^{iuX_1}\ldots e^{iuX_n}e^{i\xi C_1}\ldots e^{i\xi C_n}\right)P(N=n). \tag{2.12.10}$$

Moreover, the independence of the random variables $e^{iuX_1},\ldots,e^{iuX_n},\ldots,\ N,\ e^{i\xi C_1},\ldots,\ e^{i\xi C_n}$ implies the independence of the random variables $e^{iuX_1},\ldots,e^{iuX_n},\ldots,\ e^{i\xi C_1},\ldots,\ e^{i\xi C_n}$.

Hence (2.12.10) has the form

$$\varphi_{T,L}(u,\xi)=\sum_{n=0}^{\infty}E\left(e^{iuX_1}\right)\ldots E\left(e^{iuX_n}\right)E\left(e^{i\xi C_1}\right)\ldots E\left(e^{i\xi C_n}\right)P(N=n). \tag{2.12.11}$$

Since the random variables of the sequence $\{X_n, n=1,2,\ldots\}$ are equally distributed with the random variable X and the random variables of the sequence $\{C_n, n=1,2,\ldots\}$ are equally distributed with the random variable C then (2.12.1), (2.12.3) and (2.12.11) imply that

$$\varphi_{T,L}(u,\xi)=\sum_{n=0}^{\infty}\varphi_X^n(u)\varphi_C^n(\xi)P(N=n). \tag{2.12.12}$$

From (2.12.2) and (2.12.12) it follows that the characteristic function of the vector $(T,L)$is

$$\varphi_{T,L}(u,\xi)=P_N(\varphi_X(u)\varphi_C(\xi)).$$

An interpretation of the vector (T,L), where $T=X_1+X_2+\cdots+X_N$ and $L=C_1+C_2+\cdots+C_N$, in risk management is the following.

We consider a firm under conditions of risk and asset liquidation in a given time interval. We suppose that the random variable N denotes the frequency of risk in that time interval. The random variable X_n denotes the economic loss due to the nth occurrence of risk. Hence the random variable $T=X_1+X_2+\cdots+X_N$ denotes the risk severity in the given time interval. We suppose that the random variable C_n denotes the income of the firm from asset liquidation at the time of the nth occurrence of the risk. Hence the random variable $L=C_1+C_2+\cdots+C_N$ denotes the total income of the firm from asset liquidation in the given time interval. In this case the vector (T,L) constitutes a strong analytical tool for investigating the evolution of the firm under conditions of risk and asset liquidation in a given time interval. The independence for $\{X_n, n=1,2,\ldots\}$ N and $\{C_n, n=1,2,\ldots\}$ is a sufficient condition for evaluating the characteristic function $\varphi_{T,L}(u,\xi)=P_N(\varphi_X(u)\varphi_C(\xi))$ of the vector (T,L).

In this case the applicability of the above vector is substantially extended. □

2.13 Total Risk Severity and Total Income

Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ are equally distributed with the random variable X having characteristic function $\varphi_X(u)$.

We consider the discrete random variable N with values in the set $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ and probability generating function $P_N(z)$.

We set $T = X_1 + X_2 + \cdots + X_N$.

Let $\{C_s, s = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence $\{C_s, s = 1, 2, \ldots\}$ are equally distributed with the random variable C having characteristic function $\varphi_C(\xi)$.

We consider the discrete random variable S with values in the set $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ and probability generating function $P_S(z)$ and we set $L = C_1 + C_2 + \cdots + C_S$.

We consider the vector (T, L).

The purpose of the present section is the establishment of properties and applications in risk management of the above vector.

The following theorem establishes sufficient conditions for the evaluation of the characteristic function of the vector (T, L).

Theorem 2.13.1 *Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ are equally distributed with the random variable X having characteristic function*

$$\varphi_X(u). \tag{2.13.1}$$

We consider the discrete random variable N with values in the set $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ and probability generating function

$$P_N(z) \tag{2.13.2}$$

and we set $T = X_1 + X_2 + \cdots + X_N$.

Let $\{C_s, s = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence $\{C_s, s = 1, 2, \ldots\}$ are equally distributed with the random variable C having characteristic function

$$\varphi_C(\xi). \tag{2.13.3}$$

We consider the discrete random variable S *with values in the set* $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ *and probability generating function*

$$P_S(z) \tag{2.13.4}$$

and we set $L = C_1 + C_2 + \cdots + C_S$.

If $\{X_n, n = 1, 2, \ldots\}$, N, $\{C_s, s = 1, 2, \ldots\}$ *and* S *are independent then the random variables* $T = X_1 + X_2 + \cdots + X_N$ *and* $L = C_1 + C_2 + \cdots + C_S$ *are independent and* $\varphi_{T,L}(u, \xi) = P_N(\varphi_X(u)P(\varphi_C(\xi)))$ *is the characteristic function of the vector* (T, L).

Proof The independence of $\{X_n, n = 1, 2, \ldots\}$, N, $\{C_s, s = 1, 2, \ldots\}$ and S implies the independence of $\{X_n, n = 1, 2, \ldots\}$ and N, and the independence of $\{C_s, s = 1, 2, \ldots\}$ and S.

Hence (2.13.1) and (2.13.2) imply that the characteristic function of the random variable $T = X_1 + X_2 + \cdots + X_N$ is

$$\varphi_T(u) = P_N(\varphi_X(u)) \tag{2.13.5}$$

and (2.13.3), (2.13.4) imply that the characteristic function of the random variable $L = C_1 + C_2 + \cdots + C_S$ is

$$\varphi_L(\xi) = P_S(\varphi_C(\xi)). \tag{2.13.6}$$

Let $\varphi_{T,L}(u, \xi)$ be the characteristic function of the vector (T, L).

The establishment of the independence of the random variables $T = X_1 + X_2 + \cdots + X_N$ and $L = C_1 + C_2 + \cdots + C_S$ requires the establishment of the relationship $\varphi_{T,L}(u, \xi) = \varphi_T(u)\varphi_L(\xi)$.

We have $\varphi_{T,L}(u, \xi) = E\left(e^{iuT + i\xi L}\right)$ or equivalently

$$\varphi_{T,L}(u, \xi) = E\left(E\left(e^{iuT + i\xi L} | N, S\right)\right). \tag{2.13.7}$$

From (2.13.7) it follows that

$$\varphi_{T,L}(u, \xi) = \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} E\left(e^{iuT + i\xi L} | N = n, S = s\right) P(N = n, S = s)$$

or equivalently

$$\varphi_{T,L}(u, \xi) = \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} E\left(e^{iu(X_1 + \cdots + X_N) + i\xi(C_1 + \cdots + C_S)} | N = n, S = s\right) P(N = n, S = s). \tag{2.13.8}$$

From (2.13.8) it follows that

$$\varphi_{T,L}(u,\xi) = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} E\Big(e^{iu(X_1+\cdots+X_n)+i\xi(C_1+\cdots+C_s)}|N=n,S=s\Big)P(N=n,S=s). \tag{2.13.9}$$

Moreover, (2.13.9) implies that

$$\varphi_{T,L}(u,\xi) = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} E\big(e^{iuX_1}\ldots e^{iuX_n}, e^{i\xi C_1}\ldots e^{i\xi C_s}|N=n,S=s\big)P(N=n,S=s). \tag{2.13.10}$$

The independence of $\{X_n, n=1,2,\ldots\}$, N, $\{C_s, s=1,2,\ldots\}$ and S implies the independence of the random variables $X_1,\ldots,X_n,\ N,\ C_1,\ldots,C_s,\ S.$

The independence of the above random variables implies the independence of the random variables $e^{iuX_1},\ldots,e^{iuX_n},\ N,\ e^{i\xi C_1},\ldots,e^{i\xi C_s},\ S.$

Hence (2.13.10) has the form

$$\varphi_{\mathrm{T},L}(u,\xi) = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} E\big(e^{iuX_1}\ldots e^{iuX_n}e^{i\xi C_1}\ldots e^{i\xi C_S}\big)P(N=n,S=s). \tag{2.13.11}$$

The independence of the random variables $e^{iuX_1},\ldots,e^{iuX_n},\ N,\ e^{i\xi C_1},\ldots,e^{i\xi C_s},\ S$ implies the independence of the random variables $e^{iuX_1},\ldots,e^{iuX_n},\ e^{i\xi C_1},\ldots,e^{i\xi C_s}.$ and the independence of the random variables N, S.

Hence (2.13.11) has the form

$$\varphi_{T,L}(u,\xi) = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} E\big(e^{iuX_1}\big)\ldots E\big(e^{iuX_n}\big)E\big(e^{i\xi C_1}\big)\ldots E\big(e^{iuC_s}\big)P(N=n)P(S=s)$$

or equivalently the form

$$\varphi_{T,L}(u,\xi) = \sum_{n=0}^{\infty} E\big(e^{iuX_1}\big)\ldots E\big(e^{iuX_n}\big)P(N=n)\sum_{s=0}^{\infty} E\big(e^{i\xi C_1}\big)\ldots E\big(e^{i\xi C_s}\big)P(S=s). \tag{2.13.12}$$

Since the random variables of the sequence $\{X_n, n=1,2,\ldots\}$ are equally distributed with the random variable X having characteristic function $\varphi_X(u)$ and the random variables of the sequence $\{C_s, s=1,2,\ldots\}$ are equally distributed with the random variable C having characteristic function $\varphi_C(\xi)$ then (2.13.12) has the form

$$\varphi_{T,L}(u,\xi) = \sum_{n=0}^{\infty} \varphi_X^n(u)P(N=n)\sum_{s=0}^{\infty}\varphi_C^s(\xi)P(S=s). \tag{2.13.13}$$

From (2.13.1), (2.13.2), (2.13.3),(2.13.4) and (2.13.13) it follows that

$$\varphi_{T,L}(u,\xi) = P_N(\varphi_X(u))P_S(\varphi_C(\xi)). \tag{2.13.14}$$

Moreover (2.13.5), (2.13.6) and (2.13.14) imply that $\varphi_{T,L}(u,\xi) = \varphi_T(u)\varphi_L(\xi)$. Hence the random variables $T = X_1 + X_2 + \cdots + X_N$ and $L = C_1 + C_2 + \cdots + C_S$ are independent and the characteristic function of the vector (T,L) is $\varphi_{T,L}(u,\xi) = P_N(\varphi_X(u))P_S(\varphi_C(\xi))$.

The interpretation of the vector (T,L) as the concept of a quantitative component of risk is the following.

We consider a firm under the occurrences of a risk and the creation of incomes in a given time interval. We suppose that the random variable N denotes the frequency of risk in that time interval and the random variable X_n denotes the economic loss due to the nth occurrence of risk. Hence the random variable $T = X_1 + X_2 + \cdots + X_N$ denotes the total risk severity in the given time interval. We suppose that the random variable S denotes the number of incomes created by the firm in the same time interval and the random variable C_s denotes the size of the sth income created by the production activities of the firm. Hence the random variable $L = C_1 + C_2 + \cdots + C_S$ denotes the total income created by the firm in that time interval. In this case, the vector (T,L), where $T = X_1 + X_2 + \cdots + X_N$ and $L = C_1 + C_2 + \cdots + C_S$ constitutes a strong analytical tool for investigating the behavior of the firm under the occurrences of a risk and the creation of incomes in a given time interval. The independence of $\{X_n, n = 1,2,\ldots\}$, N, $\{C_s, s = 1,2,\ldots\}$ and S is a sufficient condition for evaluating the characteristic function $\varphi_{T,L}(u,\xi) = P_N(\varphi_X(u))P_S(\varphi_C(\xi))$ of the vector (T,L).

In this case the applicability of the above vector is substantially extended. □

2.14 Recovery Time of a Partially Damaged System and Release Time of a Backup System

Let N be a discrete random variable with values in the set $\mathbf{N} = \{1,2,\ldots\}$ and probability generating function $P_N(z)$.

We suppose that $\{X_n, n = 1,2,\ldots\}$ is a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence are equally distributed with the random variable X having distribution function $F_X(x)$.

We set $T = \max(X_1, X_2, \ldots, X_N)$.

Let $\{C_n, n = 1,2,\ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence are equally distributed with the random variable C having distribution function $F_C(c)$.

We set $L = \max(C_1, C_2, \ldots, C_N)$.

We consider the vector (T,L).

The purpose of the present section is the establishment of properties and applications in risk management of the vector (T, L).

The following result establishes sufficient conditions for evaluating the distribution function $F_{T,L}(t, \ell)$ of the vector (T, L).

Theorem 2.14.1 *Let N be a discrete random variable with values in the set $\mathbf{N} = \{1, 2, \ldots\}$ and probability generating function*

$$P_N(z). \tag{2.14.1}$$

We suppose that $\{X_n, n = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. Moreover, we suppose that the random variables of the sequence are equally distributed with the random variable X having distribution function

$$F_X(x) \tag{2.14.2}$$

and we set $T = \max(X_1, X_2, \ldots, X_N)$.

Let $\{C_n, n = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. Moreover, we suppose that the random variables of the sequence are equally distributed with the random variable C having distribution function

$$F_C(c) \tag{2.14.3}$$

and we set $L = \max(C_1, C_2, \ldots, C_N)$.

We consider the vector (T, L). If N, $\{X_n, n = 1, 2, \ldots\}$ and $\{C_n, n = 1, 2, \ldots\}$ are independent then the distribution function $F_{T,L}(t, \ell)$ of the vector (T, L) is $F_{T,L}(t, \ell) = P_N(F_X(t) F_C(\ell))$.

Proof We have $F_{T,L}(t, \ell) = P(T \le t, L \le \ell)$ or equivalently

$$F_{T,L}(t, \ell) = \sum_{n=1}^{\infty} P(T \le t, L \le \ell | N = n) P(N = n). \tag{2.14.4}$$

From (2.14.4) it follows that

$$F_{T,L}(t, \ell) = \sum_{n=1}^{\infty} P[\max(X_1, X_2, \ldots, X_N) \le t, \max(C_1, C_2, \ldots, C_N) \le \ell | N = n] P(N = n). \tag{2.14.5}$$

Hence (2.14.5) implies that

$$F_{T,L}(t, \ell) = \sum_{n=1}^{\infty} P[\max(X_1, \ldots, X_n) \le t, \max(C_1, \ldots, C_n) \le \ell | N = n] P(N = n). \tag{2.14.6}$$

From (2.14.6) it follows that

$$F_{T,L}(t,\ell) = \sum_{n=1}^{\infty} P(X_1 \le t, \ldots, X_n \le t, C_1 \le \ell, \ldots, C_n \le \ell | N = n)P(N = n). \tag{2.14.7}$$

The independence of N, $\{X_n, n = 1, 2, \ldots\}$ and $\{C_n, n = 1, 2, \ldots\}$ implies the independence of the random variables $N,\ X_1, \ldots, X_n,\ C_1, \ldots, C_n$.

Hence (2.14.7) has the form

$$F_{T,L}(t,\ell) = \sum_{n=1}^{\infty} P(X_1 \le t, \ldots, X_n \le t, C_1 \le \ell, \ldots, C_n \le \ell)P(N = n). \tag{2.14.8}$$

From the independence of the random variables $N,\ X_1, \ldots, X_n,\ C_1, \ldots, C_n$. It follows the independence of the random variables $X_1, \ldots, X_n,\ C_1, \ldots, C_n$.

The independence of the above random variables implies that (2.14.8) has the form

$$F_{T,L}(t,\ell) = \sum_{n=1}^{\infty} P(X_1 \le t) \ldots P(X_n \le t)P(C_1 \le \ell) \ldots P(C_n \le \ell)P(N = n). \tag{2.14.9}$$

Since the random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ are equally distributed with the random variable X and the random variables of the sequence $\{C_n, n = 1, 2, \ldots\}$ are equally distributed with the random variable C then (2.14.2), (2.14.3) and (2.14.9) imply that

$$F_{T,L}(t,\ell) = \sum_{n=1}^{\infty} F_X^n(t)F_C^n(\ell)P(N = n). \tag{2.14.10}$$

From (2.14.1) and (2.14.10) it follows that the distribution function $F_{T,L}(t,\ell)$ of the vector (T, L) has the form $F_{T,L}(t,\ell) = P_N(F_X(t)F_C(\ell))$.

An interpretation of the vector (T, L), where $T = \max(X_1, X_2, \ldots, X_N)$ and $L = \max(C_1, C_2, \ldots, C_N)$, in risk management is the following.

We suppose that the occurrence of a risk, at the time point 0, interrupts N operations of a system. This system is called system I. The random variable X_n denotes the time required for the recovery of the nth interrupted operation of system I. Hence the random variable $T = \max(X_1, X_2, \ldots, X_N)$ denotes the time required for the recovery of system I. We suppose that the N operations of system I, which are interrupted by the occurrence of risk at the time point 0, are undertaken by another system which is called system II. The random variable C_n denotes the available time of system II for undertaking the nth interrupted operation of system II. Hence the random variable

$L = \max(C_1, C_2, \ldots, C_N)$ denotes the release time of system II from the undertaking of the N interrupted operations of system II. In this case the vector (T, L) is particularly useful for investigating the behavior of system I.

The independence of N, $\{X_n, n = 1, 2, \ldots\}$ and $\{C_n, n = 1, 2, \ldots\}$ permits the evaluation of the distribution function $F_{T,L}(t, \ell) = P_N(F_X(t)F_C(\ell))$ of the vector (T, L).

In this case the the practical applicability of the above vector in risk management is substantially extended.

The interpretation of the random variable $L = \max(C_1, C_2, \ldots, C_N)$ as the release time of system II from the undertaking of the N interrupted operations of system I means that the consideration of system II and the corresponding vector (T, L) facilitates the development and implementation of risk treatment operations. □

2.15 Vector of Recovery Times of Two Partially Damaged Systems

Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ are equally distributed with the random variable X having distribution function $F_X(x)$.

We consider the discrete random variable N with values in the set $\mathbf{N} = \{1, 2, \ldots\}$ and probability generating function $P_N(z)$, and we set $T = \max(X_1, X_2, \ldots, X_N)$.

Let $\{C_s, s = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence $\{C_s, s = 1, 2, \ldots\}$ are equally distributed with the random variable C having distribution function $F_C(c)$.

We consider the discrete random variable S with values in the set $\mathbf{N} = \{1, 2, \ldots\}$ and probability generating function $P_S(z)$, and we set $L = \max(C_1, C_2, \ldots, C_S)$.

We consider the vector (T, L).

The purpose of the present section is the establishment properties and applications in risk management of the above vector.

The following result establishes sufficient conditions for evaluating the distribution function of the vector (T, L)

Theorem 2.15.1 *Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variable. The random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ are equally distributed with the random variable X having distribution function*

$$F_X(x). \tag{2.15.1}$$

We consider the discrete random variable N with values in the set $\mathbf{N} = \{1, 2, \ldots\}$ *and probability generating function*

$$P_N(z) \tag{2.15.2}$$

and we set $T = \max(X_1, X_2, \ldots, X_N)$.

Let $\{C_s, s = 1, 2, \ldots\}$ *be a sequence of continuous, positive, independent, and identically distributed random variables. The sequence of the random variables* $\{C_s, s = 1, 2, \ldots\}$ *are equally distributed with the random variable C having distribution function*

$$F_C(c). \tag{2.15.3}$$

We consider the discrete random variable S with values in the set $\mathbf{N} = \{1, 2, \ldots\}$ *and probability generating function*

$$P_S(z) \tag{2.15.4}$$

and we set $L = \max(C_1, C_2, \ldots, C_S)$.

If $\{X_n, n = 1, 2, \ldots\}$, N, $\{C_s, s = 1, 2, \ldots\}$ *and S are independent then* $T = \max(X_1, X_2, \ldots, X_N)$ *and* $L = \max(C_1, C_2, \ldots, C_S)$ *are independent and* $F_{T,L}(t, \ell) = P_N(F_X(t))P_S(F_C(\ell))$ *is the distribution function of the vector* (T, L).

Proof The assumption that $\{X_n, n = 1, 2, \ldots\}$, N, $\{C_s, s = 1, 2, \ldots\}$ and S are independent implies that $\{X_n, n = 1, 2, \ldots\}$ and N are independent and also that $\{C_s, s = 1, 2, \ldots\}$ and S are independent. Hence (2.15.1) and (2.15.2) imply that the distribution function of the random variable $T = \max(X_1, X_2, \ldots, X_N)$ is

$$F_T(t) = P_N(F_X(t)) \tag{2.15.5}$$

and (2.15.3), (2.15.4) imlply that the distribution function of the random variable $L = \max(C_1, C_2, \ldots, C_S)$ is

$$F_L(\ell) = P_S(F_C(\ell)). \tag{2.15.6}$$

Let $F_{T,L}(t, \ell)$ be the distribution function of the vector (T, L).

The proof of the independence of the random variables $T = \max(X_1, X_2, \ldots, X_N)$ and $L = \max(C_1, C_2, \ldots, C_S)$ requires the establishment of the relationship $F_{T,L}(t, \ell) = F_T(t)F_L(\ell)$. Since

$$F_{T,L}(t, \ell) = P(T \le t, L \le \ell). \tag{2.15.7}$$

We get that

$$F_{T.L}(t, \ell) = \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} P(T \le t, L \le \ell | N = n, S = s) P(N = n, S = s). \tag{2.15.8}$$

From (2.15.8) it follows that

$$F_{T,L}(t,\ell) = \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} P(\max(X_1,\ldots,X_N) \le t, \max(C_1,\ldots,C_S) \le \ell | N=n, S=s) P(N=n, S=s)$$

or equivalently

$$F_{T,L}(t,\ell) = \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} P(\max(X_1,\ldots,X_n) \le t, \max(C_1,\ldots,C_s) \le \ell | N=n, S=s) P(N=n, S=s). \tag{2.15.9}$$

From (2.15.9) it follows that

$$F_{T,L}(t,\ell) = \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} P(X_1 \le t,\ldots,X_n \le t, C_1 \le \ell,\ldots,C_s \le \ell | N=n, S=s) P(N=n, S=s). \tag{2.15.10}$$

From the fact that $\{X_n, n=1,2,\ldots\}$, N, $\{C_s, s=1,2,\ldots\}$ and S are independent it follows the independence of the random variables $X_1,\ldots,X_n,\ N,\ \ C_1,\ldots,C_s,\ S$.

Hence (2.15.10) has the form

$$F_{T,L}(t,\ell) = \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} P(X_1 \le t,\ldots,X_n \le t, C_1 \le \ell,\ldots,C_s \le \ell) P(N=n, S=s). \tag{2.15.11}$$

The independence of the random variables $X_1,\ldots,X_n,\ N,\ \ C_1,\ldots,C_s,\ S$ implies the independence of the random variables $X_1,\ldots,X_n,\ C_1,\ldots,C_s$ and the independence of the random variables $N,\ S$.

Hence (2.15.11) has the form

$$F_{T,L}(t,\ell) = \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} P(X_1 \le t)\ldots P(X_n \le t) P(C_1 \le \ell)\ldots P(C_s \le \ell) P(N=n) P(S=s)\,. \tag{2.15.12}$$

Since the random variables of the sequence $\{X_n, n=1,2,\ldots\}$ are equally distributed with the random variable X having distribution function $F_X(x)$ and the random variables of the sequence $\{C_s, s = 1, 2, \ldots\}$ are equally distributed with the random variable C having distribution function $F_C(c)$ then (2.15.12) has the form

$$F_{T,L}(t,\ell) = \sum_{n=1}^{\infty} P(X_1 \le t) \ldots P(X_n \le t) P(N = n) \sum_{s=1}^{\infty} P(C_1 \le \ell) \ldots P(C_s \le \ell) P(S = s)$$

or equivalently the form

$$P(T \le t, L \le \ell) = \sum_{n=1}^{\infty} F_X^n(t) P(N = n) \sum_{s=1}^{\infty} F_C^s(\ell) P(S = s). \tag{2.15.13}$$

From (2.15.13) it follows that

$$F_{T,L}(t,\ell) = P_N(F_X(t)) P_S(F_C(\ell)). \tag{2.15.14}$$

Moreover (2.15.5), (2.15.6), (2.15.7) and (2.15.14) imply that $F_{T,L}(t,\ell) = F_T(t) F_L(\ell)$.

Hence the random variables $T = \max(X_1, X_2, \ldots, X_N)$ and $L = \max(C_1, C_2, \ldots, C_S)$ are independent and the distribution function of the random vector (T, L) is $F_{T,L}(t,\ell) = P_N(F_X(t)) P_S(F_C(\ell))$.

An interpretation of the vector (T, L) where $T = \max(X_1, X_2, \ldots, X_N)$ and $L = \max(C_1, C_2, \ldots, C_S)$ in risk management is the following.

We consider two systems. The occurrence of a risk, at the time point 0, interrupts N operations of the first system and S operations of the second system. The random variable X_n denotes the time required for the recovery of the nth interrupted operation of the first system. Hence the random variable $T = \max(X_1, X_2, \ldots, X_N)$ denotes the time required for the recovery of the first system. The random variable C_s denotes the time required for the recovery of the sth interrupted operation of the second system. Hence the random variable $L = \max(C_1, C_2, \ldots, C_S)$ denotes the time required for the recovery of the second system. In this case the vector (T, L) is particularly useful for investigating the evolution of the pair of two systems. The independence of $\{X_n, n = 1, 2, \ldots\}$, N, $\{C_s, s = 1, 2, \ldots\}$, and S permits the evaluation of the distribution function $F_{T,L}(t,\ell) = P_N(F_X(t)) P_S(F_C(\ell))$ of the vector (T, L).

In this case the practical applicability in risk management of the above vector is substantially extended.

It is obvious that the results of the present section for the vector (T, L) can be extended for vectors of many dimensions. $\square$

2.16 Recovery Time of a Partially Damaged System Under a Random Number of Competing Risks

Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ are equally distributed with the random variable X having distribution function $F_X(x)$.

We consider the discrete random variable N with values in the set $\mathbf{N} = \{1, 2, \ldots\}$ and probability generating function $P_N(z)$, and set $T = \max(X_1, X_2, \ldots, X_N)$.

Let $\{C_s, s = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence $\{C_s, s = 1, 2, \ldots\}$ are equally distributed with the random variable C having distribution function $F_C(c)$.

We consider the discrete random variable S with values in the set $\mathbf{N} = \{1, 2, \ldots\}$ and probability generating function $P_S(z)$, and set $L = \min(C_1, C_2, \ldots, C_S)$.

We consider the vector (T, L).

The purpose of the present section is the establishment of properties and applications in risk management of the above vector.

The following result establishes sufficient conditions for evaluating the distribution function of the vector (T, L).

Theorem 2.16.1 *Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ are equally distributed with the random variable X having distribution function*

$$F_X(x). \tag{2.16.1}$$

We consider the discrete random variable N with values in the set $\mathbf{N} = \{1, 2, \ldots\}$ and probability generating function

$$P_N(z) \tag{2.16.2}$$

and we set $T = \max(X_1, X_2, \ldots, X_N)$.

Let $\{C_s, s = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence $\{C_s, s = 1, 2, \ldots\}$ are equally distributed with the random variable C having distribution function

$$F_C(c). \tag{2.16.3}$$

We consider the discrete random variable S with values in the set $\mathbf{N} = \{1, 2, \ldots\}$ *and probability generating function*

$$P_S(z) \tag{2.16.4}$$

and set $L = \min(C_1, C_2, \ldots, C_S)$.

If $\{X_n, n = 1, 2, \ldots\}$, N, $\{C_s, s = 1, 2, \ldots\}$, *and S are independent then the random variables* $T = \max(X_1, X_2, \ldots, X_N)$ *and* $L = \min(C_1, C_2, \ldots, C_S)$ *are independentvaı and* $F_{T,L}(t, \ell) = P_N(F_X(t))(1 - P_S(1 - F_C(\ell)))$ *is the distribution function of the vector* (T, L).

Proof The independence of $\{X_n, n = 1, 2, \ldots\}$, N, $\{C_s, s = 1, 2, \ldots\}$ and S implies the independence of $\{X_n, n = 1, 2, \ldots\}$ and N and the independence of $\{C_s, s = 1, 2, \ldots\}$ and S.

Hence (2.16.1) and (2.16.2) imply that the distribution function of the random variable $T = \max(X_1, X_2, \ldots, X_N)$ is

$$F_T(t) = P_N(F_X(t)) \tag{2.16.5}$$

and (2.16.3), (2.16.4) imply that the distribution function of the random variable $L = \min(C_1, C_2, \ldots, C_S)$ is

$$F_L(\ell) = 1 - P_S(1 - F_C(\ell)). \tag{2.16.6}$$

Let $F_{T,L}(t, \ell)$ be the distribution function of the vector (T, L).

The establishment of independence of the random variables $T = \max(X_1, X_2, \ldots, X_N)$ and $L = \min(C_1, C_2, \ldots, C_S)$ requires the establishment of the relationship $F_{T,L}(t, \ell) = F_T(t)F_L(\ell)$.

We have

$$F_{T,L}(t, \ell) = P(T \le t, L \le \ell). \tag{2.16.7}$$

Since $P(T \le t) = P(T \le t, L \le \ell) + P(T \le t, L > \ell)$ then we get

$$P(T \le t, L \le \ell) = P(T \le t) - P(T \le t, L > \ell). \tag{2.16.8}$$

From (2.16.8) it follows that

$$\begin{aligned} P(T \le t, L \le \ell) = {} & P(T \le t) \\ & - \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} P(T \le t, L > \ell | N = n, S = s) P(N = n, S = s) \end{aligned} \tag{2.16.9}$$

Moreover (2.16.5), (2.16.7) and (2.16.9) imply that

$$F_{T,L}(t,\ell) = P_N(F_X(t)) - \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} P[\max(X_1,X_2,\ldots,X_N) \le t, \min(C_1,C_2,\ldots,C_S) > \ell | N=n, S=s] P(N=n,S=s). \tag{2.16.10}$$

From (2.16.10) it follows that

$$F_{T,L}(t,\ell) = P_N(F_X(t)) - \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} P[\max(X_1,X_2,\ldots,X_n) \le t, \min(C_1,C_2,\ldots,C_s) > \ell | N=n, S=s] P(N=n,S=s). \tag{2.16.11}$$

From (2.16.11) it follows that

$$F_{T,L}(t,\ell) = P_N(F_X(t)) - \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} P[X_1 \le t,\ldots,X_n \le t, C_1 > \ell,\ldots,C_s > \ell | N=n, S=s] P(N=n,S=s). \tag{2.16.12}$$

The independence of $\{X_n, n=1,2,\ldots\}$, N, $\{C_s, s=1,2,\ldots\}$ and S implies the independence of the random variables $X_1,\ldots,X_n,\ N,\ \ C_1,\ldots,C_s,\ S$.

Hence (2.16.12) has the form

$$F_{T,L}(t,\ell) = P_N(F_X(t)) - \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} P(X_1 \le t,\ldots X_n \le t, C_1 > \ell,\ldots,C_s > \ell) P(N=n,S=s). \tag{2.16.13}$$

The independence of the random variables $X_1,\ldots,X_n,\ N,\ \ C_1,\ldots,C_s,\ S$ implies the independence of the random variables $X_1,\ldots,X_n,\ C_1,\ldots,C_s$ and the independence of the random variables $N,\ S$.

Hence (2.16.13) has the form

$$F_{T,L}(t,\ell) = P_N(F_X(t)) - \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} P(X_1 \le t)\ldots P(X_n \le t) P(C_1 > \ell)\ldots P(C_s > \ell) P(N=n) P(S=s) \tag{2.16.14}$$

From (2.16.14) it follows that

$$F_{T,L}(t,\ell) = P_N(F_X(t)) - \sum_{n=1}^{\infty} P(X_1 \le t)\ldots P(X_n \le t) P(N=n) \sum_{s=1}^{\infty} P(C_1 > \ell)\ldots P(C_s > \ell) P(S=s). \tag{2.16.15}$$

Since the random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ are equally distributed with the random variable X having distribution function $F_X(x)$ and the random variables of the sequence $\{C_s, s = 1, 2, \ldots\}$ are equally distributed with the random variable C having distribution function $F_C(c)$ then (2.16.15) has the form

$$F_{T,L}(t,\ell) = P_N(F_X(t)) - \sum_{n=1}^{\infty} F_X^n(t)P(N=n)\sum_{s=1}^{\infty}[1 - F_C(\ell)]^s P(S=s)$$

or equivalently the form

$$F_{T,L}(t,\ell) = P_N(F_X(t)) - P_N(F_X(t))P_S(1 - F_C(\ell)). \tag{2.16.16}$$

From (2.16.16) it follows that

$$F_{T,L}(t,\ell) = P_N(F_X(t))[1 - P_S(1 - F_C(\ell))]. \tag{2.16.17}$$

Moreover (2.16.5), (2.16.6) and (2.16.17) imply that $F_{T,L}(t,\ell) = F_T(t)F_L(\ell)$. Hence the random variable $T = \max(X_1, X_2, \ldots, X_N)$ is independent of the random variable $L = \min(C_1, C_2, \ldots, C_S)$ and the distribution function $F_{T,L}(t,\ell)$ of the vector (T, L) is $F_{T,L}(t,\ell) = P_N(F_X(t))[1 - P_S(1 - F_C(\ell))]$..

An interpretation of the vector (T, L), where $T = \max(X_1, X_2, \ldots, X_N)$ and $L = \min(C_1, C_2, \ldots, C_S)$, as a concept of risk management is the following. A risk occurs at the time point 0. The occurrence of risk interrupts N operations of a system. The random variable X_n denotes the time required for the recovery of the nth interrupted operation of the system. Hence the random variable $T = \max(X_1, X_2, \ldots, X_N)$ denotes the time required for the recovery of the system. The random variable $T = \max(X_1, X_2, \ldots, X_N)$ is a fundamental stochastic model for describing and analyzing the recovery process of a system. Moreover S risks threaten the system at the time point 0. The random variable C_s denotes the occurrence time of the sth risk. Hence the random variable $L = \min(C_1, C_2, \ldots, C_S)$ denotes the minimum of the risk occurrence times. The consideration of a system under a random number S of independent competing risks means the use of the random variable $L = \min(C_1, C_2, \ldots, C_S)$ as a fundamental stochastic model for describing and analyzing the evolution of that system. Hence the vector (T, L) can be considered as a strong analytical tool for investigating the behaviour of a system which has experienced the occurrence of a risk and then the system recovers under a random number of independent competing risks. □

2.17 Considering a System Under a Random Number of Competing Risks

Let N be a discrete random variable with values in the set $\mathbf{N} = \{1, 2, \ldots\}$ and probability generating function $P_N(z)$. We suppose that $\{X_n, n = 1, 2, \ldots\}$ is a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence are equally distributed with the random variable X having distribution function $F_X(x)$ and we set $T = \min(X_1, X_2, \ldots, X_N)$.

Let $\{C_n, n = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence are equally distributed with the random variable C having distribution function $F_C(c)$ and we set $L = \min(C_1, C_2, \ldots, C_N)$.We consider the vector (T, L).

The purpose of the present section is the establishment properties and applications in risk management of the vector (T, L).

The following result establishes sufficient conditions for evaluating the distribution function of the vector (T, L).

Theorem 2.17.1 *Let N be a discrete random variable with values in the set* $\mathbf{N} = \{1, 2, \ldots\}$ *and probability generating function*

$$P_N(z). \tag{2.17.1}$$

We suppose that $\{X_n, n = 1, 2, \ldots\}$ is a sequence of continuous, positive, independent, and identically distributed random variables random variables. We also suppose that the random variables of the sequence are equally distributed with the random variable X having distribution function

$$F_X(x) \tag{2.17.2}$$

and we set $T = \min(X_1, X_2, \ldots, X_N)$.

Let $\{C_n, n = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence are equally distributed with the random variable random variables C having distribution function

$$F_C(c) \tag{2.17.3}$$

and we set $L = \min(C_1, C_2, \ldots, C_N)$.

We consider the vector (T, L).

If N, $\{X_n, n = 1, 2, \ldots\}$ and $\{C_n, n = 1, 2, \ldots\}$ are independent then the distribution function $F_{T,L}(t, \ell)$ of the vector (T, L) is

$$F_{T,L}(t, \ell) = 1 - P_N(1 - F_X(t)) - P_N(1 - F_C(\ell)) + P_N[(1 - F_X(t))(1 - F_C(\ell))].$$

Proof The independence of N, $\{X_n, n = 1, 2, \ldots\}$ and $\{C_n, n = 1, 2, \ldots\}$ implies the independence of N and $\{X_n, n = 1, 2, \ldots\}$ and the independence of N and $\{C_n, n = 1, 2, \ldots\}$.

Hence (2.17.1) and (2.17.2) imply that the distribution function of the random variable $T = \min(X_1, X_2, \ldots, X_N)$ is

$$F_T(t) = 1 - P_N(1 - F_X(t)) \tag{2.17.4}$$

and (2.17.1), (2.17.3) imply that the distribution function of the random variable $L = \min(C_1, C_2, \ldots, C_N)$ is

$$F_L(\ell) = 1 - P_N(1 - F_C(\ell)). \tag{2.17.5}$$

Let $F_{T,L}(t, \ell)$ be the distribution function of the vector (T, L).We have

$$F_{T,L}(t, \ell) = P(T \leq t, L \leq \ell). \tag{2.17.6}$$

Since

$$P(T \leq t) = P(T \leq t, L \leq \ell) + P(T \leq t, L > \ell)$$

or equivalently

$$P(T \leq t, L \leq \ell) = P(T \leq t) - P(T \leq t, L > \ell) \tag{2.17.7}$$

and

$$P(L \geq \ell) = P(T \geq t, L \geq \ell) + P(T < t, L \geq \ell)$$

or equivalently

$$P(T < t, L \geq \ell) = P(L \geq \ell) - P(T \geq t, L \geq \ell) \tag{2.17.8}$$

then from (2.17.7) and (2.17.8) we get that

$$P(T \leq t, L \leq \ell) = P(T \leq t) - P(L \geq \ell) + P(T \geq t, L \geq \ell). \tag{2.17.9}$$

From (2.17.4), (2.17.5), (2.17.6) and (2.17.9) it follows that

$$\begin{aligned} F_{T,L}(t, \ell) = 1 &- P_N(1 - F_X(t)) - P_N(1 - F_C(\ell)) \\ &+ \sum_{n=1}^{\infty} P[\min(X_1, \ldots, X_N) \geq t, \min(C_1, \ldots, C_N) \geq \ell | N = n] P(N = n). \end{aligned}$$

Hence

$$F_{T,L}(t,\ell) = 1 - P_N(1 - F_X(t)) - P_N(1 - F_C(\ell)) + \sum_{n=1}^{\infty} P[\min(X_1,\ldots,X_n) \geq t, \min(C_1,\ldots,C_n) \geq \ell | N = n]P(N = n). \tag{2.17.10}$$

From (2.17.10) it follows that

$$F_{T,L}(t,\ell) = 1 - P_N(1 - F_X(t)) - P_N(1 - F_C(\ell)) + \sum_{n=1}^{\infty} P(X_1 \geq t,\ldots,X_n \geq t, C_1 \geq \ell,\ldots,C_n \geq \ell | N = n)P(N = n). \tag{2.17.11}$$

The independence of N, $\{X_n, n = 1,2,\ldots\}$ and $\{C_n, n = 1,2,\ldots\}$ implies the independence of the random variables $N, X_1,\ldots,X_n,\ C_1,\ldots,C_n$. Hence (2.17.11) has the form

$$F_{T,L}(t,\ell) = 1 - P_N(1 - F_X(t)) - P_N(1 - F_C(\ell)) + \sum_{n=1}^{\infty} P(X_1 \geq t,\ldots,X_n, C_1 \geq \ell,\ldots,C_n \geq \ell)P(N = n). \tag{2.17.12}$$

Since the independence of the random variables $N,\ X_1,\ldots,X_n,\ C_1,\ldots,C_n$ implies the independence of the random variables $X_1,\ldots,X_n,\ C_1,\ldots,C_n$ then (2.17.12) has the form

$$F_{T,L}(t,\ell) = 1 - P_N(1 - F_X(t)) - P_N(1 - F_C(\ell)) + \sum_{n=1}^{\infty} P(X_1 \geq t)\ldots P(X_n \geq t)P(C_1 \geq \ell)\ldots P(C_n \geq \ell)P(N = n).$$

Since the random variables of the sequence $\{X_n, n = 1,2,\ldots\}$ are equally distributed with the random variable X having distribution function $F_X(x)$ and the random variables of the sequence $\{C_n, n = 1,2,\ldots\}$ are equally distributed with the random variable C having distribution function $F_C(c)$ then (2.17.12) has the form

$$F_{T,L}(t,\ell) = 1 - P_N(1 - F_X(t)) - P_N(1 - F_C(\ell)) + \sum_{n=1}^{\infty} (1 - F_X(t))^n (1 - F_C(\ell))^n P(N = n)$$

or equivalently

$$F_{T,L}(t,\ell) = 1 - P_N(1 - F_X(t)) - P_N(1 - F_C(\ell)) - P_N[(1 - F_X(t))(1 - F_C(\ell))].$$

An interpretation of the vector (T, L), where $T = \min(X_1, X_2,\ldots,X_N)$ and $L = \min(C_1, C_2,\ldots,C_N)$, in risk management is the following.

We consider a firm at the time point 0. The random variable N denotes the number of risk threatening the firm. The random variable X_n denotes the occurrence time of the nth risk. Hence the random variable $T = \min(X_1, X_2, \ldots, X_N)$ denotes the minimum risk occurrence time. The random variable C_n denotes the severity of the nth risk. Hence the random variable $L = \min(C_1, C_2, \ldots, C_N)$ denotes the minimum severity of risks. The vector (T, L) is a strong analytical tool for investigating the evolution of the firm under a random number N of independent competing risks. The independence of N, $\{X_n, n = 1, 2, \ldots\}$ and $\{C_n, n = 1, 2, \ldots\}$ permits the evaluation of the distribution function

$$F_{T,L}(t, \ell) = 1 - P_N(1 - F_X(t)) - P_N(1 - F_C(\ell)) + P_N[(1 - F_X(t))(1 - F_C(\ell))]$$

of the vector (T, L).

In this case the practical applicability in risk management of the above random vector is substantially extended. If the N risks threatening the firm at the time point 0 are catastrophic then the random variable $T = \min(X_1, X_2, \ldots, X_N)$, the random variable $L = \min(C_1, C_2, \ldots, C_N)$, the vector (T, L), and the corresponding distribution function

$$F_{T,L}(t, \ell) = 1 - P_N(1 - F_X(t)) - P_N(1 - F_C(\ell)) + P_N[(1 - F_X(t))(1 - F_C(\ell))]$$

constitute structural factors for investigating the evolution of the firm. □

2.18 Pair of Systems Under Competing Risks

Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ are equally distributed with the random variable X having distribution function $F_X(x)$.

We consider the discrete random variable N with values in the set $\mathbf{N} = \{1, 2, \ldots\}$ and probability generating function $P_N(z)$, and we set $T = \min(X_1, X_2, \ldots, X_N)$.

Let $\{C_s, s = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence $\{C_s, s = 1, 2, \ldots\}$ are equally distributed with the random variable C having distribution function $F_C(c)$. We consider the discrete random variable S with values in the set $\mathbf{N} = \{1, 2, \ldots\}$ and probability generating function $P_S(z)$, and we set $L = \min(C_1, C_2, \ldots, C_S)$.

We consider the vector (T, L).

The purpose of the present section is the establishment of properties and applications in risk management of the above vector.

The following result establishes sufficient conditions for evaluating the distribution function of the vector (T, L).

Theorem 2.18.1 *Let* $\{X_n, n = 1, 2, \ldots\}$ *be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence* $\{X_n, n = 1, 2, \ldots\}$ *are equally distributed with the random variable* X *having distribution function*

$$F_X(x). \tag{2.18.1}$$

We consider the discrete random variable N *with values in the set* $\mathbf{N} = \{1, 2, \ldots\}$ *and probability generating function*

$$P_N(z) \tag{2.18.2}$$

and we set $T = \min(X_1, X_2, \ldots, X_N)$. *Let* $\{C_s, s = 1, 2, \ldots\}$ *be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence* $\{C_s, s = 1, 2, \ldots\}$ *are equally distributed with the random variable* C *having distribution function*

$$F_C(c). \tag{2.18.3}$$

We consider the discrete random variable S *with values in the set* $\mathbf{N} = \{1, 2, \ldots\}$, *and probability generating function*

$$P_S(z) \tag{2.18.4}$$

and we set $L = \min(C_1, C_2, \ldots, C_S)$.

If $\{X_n, n = 1, 2, \ldots\}$, N, $\{C_s, s = 1, 2, \ldots\}$ *and* S *are independent then the random variables* $T = \min(X_1, X_2, \ldots, X_N)$ *and* $L = \min(C_1, C_2, \ldots, C_S)$ *are independent and*

$$F_{T,L}(t, \ell) = [1 - P_N(1 - F_X(t))][1 - P_S(1 - F_C(\ell))]$$

is the distribution function of the vector (T, L).

Proof The independence of $\{X_n, n = 1, 2, \ldots\}$, N, $\{C_s, s = 1, 2, \ldots\}$ and S implies the independence of $\{X_n, n = 1, 2, \ldots\}$ and N, and the independence of $\{C_s, s = 1, 2, \ldots\}$ and S.

Hence (2.18.1) and (2.18.2) imply that the distribution function of the random variable $T = \min(X_1, X_2, \ldots, X_N)$ is

$$F_T(t) = 1 - P_N(1 - F_X(t)) \tag{2.18.5}$$

and (2.18.3), (2.18.4) imply that the distribution function of the random variable $L = \min(C_1, C_2, \ldots, C_S)$ is

$$F_L(\ell) = 1 - P_S(1 - F_C(\ell)). \tag{2.18.6}$$

Let $F_{T,L}(t, \ell)$ be the distribution function of the vector (T, L).

The establishment of the independence of the random variables $T = \min(X_1, X_2, \ldots, X_N)$ and $L = \min(C_1, C_2, \ldots, C_S)$ requires the establishment of the relationship $F_{T,L}(t, \ell) = F_T(t)F_L(\ell)$. We have

$$F_{T,L}(t,\ell) = P(T \le t, L \le \ell). \tag{2.18.7}$$

Since

$$P(T \le t) = P(T \le t, L \le \ell) + P(T \le t, L > \ell)$$

or equivalently

$$P(T \le t, L \le \ell) = P(T \le t) - P(T \le t, L > \ell) \tag{2.18.8}$$

and

$$P(L \ge \ell) = P(T \ge t, L \ge \ell) + P(T < t, L \ge \ell)$$

or equivalently

$$P(T < t, L \ge \ell) = P(L \ge \ell) - P(T \ge t, L \ge \ell) \tag{2.18.9}$$

then (2.18.8) and (2.18.9) imply that

$$P(T \le t, L \le \ell) = P(T \le t) - P(L \ge \ell) + P(T \ge t, L > \ell) \tag{2.18.10}$$

From (2.18.5), (2.18.6), (2.18.7) and (2.18.10) it follows that

$$F_{T,L}(t,\ell) = 1 - P_N(1 - F_X(t)) - P_S(1 - F_C(\ell)) + P(T \ge t, L \ge \ell) \tag{2.18.11}$$

Moreover (2.18.11) implies that

$$\begin{aligned} F_{T,L}(t,\ell) &= 1 - P_N(1 - F_X(t)) - P_S(1 - F_C(\ell)) \\ &\quad + \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} P\left[\min(X_1, \ldots, X_N) \ge t, \min(C_1, \ldots, C_S) \ge \ell | N = n, S = s\right] P(N = n, S = s). \end{aligned}$$

Hence

$$\begin{aligned} F_{T,L}(t,\ell) &= 1 - P_N(1 - F_X(t)) - P_S(1 - F_C(\ell)) \\ &\quad + \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} P\left[\min(X_1, \ldots, X_n) \ge t, \min(C_1, \ldots, C_s) \ge \ell | N = n, S = s\right] P(N = n, S = s). \end{aligned} \tag{2.18.12}$$

From (2.18.12) it follows

$$F_{T,L}(t,\ell) = 1 - P_N(1 - F_X(t)) - P_S(1 - F_C(\ell)) + \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} P\left(X_1 \geq t, \ldots, X_n \geq t, C_1 \geq \ell, \ldots, C_s \geq \ell | N = n, S = s\right) P(N = n, S = s). \tag{2.18.13}$$

The independence of $\{X_n, n = 1, 2, \ldots\}$, N, $\{C_s, s = 1, 2, \ldots\}$ and S implies the independence of the random variables $X_1, \ldots, X_n$, N, $C_1, \ldots, C_s$, S and the independence of the random variables N, S.

Hence (2.18.13) has the form

$$F_{T,L}(t,\ell) = 1 - P_N(1 - F_X(t)) - P_S(1 - F_C(\ell)) + \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} P(X_1 \geq t, \ldots, X_n \geq t, C_1 \geq \ell, \ldots, C_s \geq \ell) P(N = n) P(S = s). \tag{2.18.14}$$

Since the independence of the random variables $X_1, \ldots, X_n$, N, $C_1, \ldots, C_s$, S implies the independence of the random variables $X_1, \ldots, X_n$, $C_1, \ldots, C_s$, then (2.18.14) has the form

$$F_{T,L}(t,\ell) = 1 - P_N(1 - F_X(t)) - P_S(1 - F_C(\ell)) + \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} P(X_1 \geq t) \ldots P(X_n \geq t) P(C_1 \leq \ell) \ldots P(C_s \geq \ell) P(N = n) P(S = s).$$

or equivalently the form

$$F_{T,L}(t,\ell) = 1 - P_N(1 - F_X(t)) - P_S(1 - F_C(\ell)) + \sum_{n=1}^{\infty} P(X_1 \geq t) \ldots P(X_n \geq t) P(N = n) \sum_{s=1}^{\infty} P(C_1 \geq \ell) \ldots P(C_s \geq \ell) P(S = s). \tag{2.18.15}$$

Since the random variables of the sequence $\{X_n, n = 1, 2, \ldots\}$ are equally distributed with the random variable X having distribution function $F_X(x)$ and the random variables of the sequence $\{C_s, s = 1, 2, \ldots\}$ are equally distributed with the random variable C having distribution function $F_C(c)$ then (2.18.15) has the form

$$F_{T,L}(t,\ell) = 1 - P_N(1 - F_X(t)) - P_S(1 - F_C(\ell)) + \sum_{n=1}^{\infty} (1 - F_X(t))^n P(N = n) \sum_{s=1}^{\infty} (1 - F_C(\ell))^s P(S = s).$$

or equivalently

$$F_{T,L}(t,\ell) = 1 - P_N(1 - F_X(t)) - P_S(1 - F_C(\ell)) + P_N(1 - F_X(t)) P_S(1 - F_C(\ell)). \tag{2.18.16}$$

From (2.18.16) it follows that

$$F_{T,L}(t,\ell) = [1 - P_N(1 - F_X(t))][1 - P_S(1 - F_C(\ell))] \tag{2.18.17}$$

Moreover (2.18.5), (2.18.6) and (2.18.17) imply that $F_{T,L}(t,\ell) = F_T(t)F_L(\ell)$.

Hence the random variable $T = \min(X_1, X_2, \ldots, X_N)$ is independent of the random variable $L = \min(C_1, C_2, \ldots, C_S)$ and the distribution function $F_{T,L}(t,\ell)$ of the vector (T, L) is

$$F_{T,L}(t,\ell) = [1 - P_N(1 - F_X(t))]\ [1 - P_S(1 - F_C(\ell))]$$

An interpretation of the vector (T, L), with $T = \min(X_1, X_2, \ldots, X_N)$ and $L = \min(C_1, C_2, \ldots, C_S)$ in risk management is the following.

We consider two systems at the time point 0. The random variable N denotes the number of risks threatening the first system and the random variable S denotes the number of systems threatening the second system. The random variable X_n denotes the occurrence time of the nth risk threatening the first system. Hence the random variable $T = \min(X_1, X_2, \ldots, X_N)$ denotes the minimum risk occurrence time for the first system. The random variable C_s denotes the occurrence time of the sth risk threatening the second system. Hence the random variable $L = \min(C_1, C_2, \ldots, C_S)$ denotes the minimum risk occurrence time for the second system. The vector (T, L) is a strong analytical tool for investigating the evolution of the above mentioned systems. The independence of $\{X_n, n = 1, 2, \ldots\}$, N, $\{C_s, s = 1, 2, \ldots\}$ and S permits the evaluation of the distribution function

$$F_{T,L}(t,\ell) = [1 - P_N(1 - F_X(t))]\ [1 - P_S(1 - F_C(\ell))]$$

of the vector (T, L).

In this case the applicability of the above vector in risk management is substantially extended. □

2.19 Time of First Major Damage and Asset Liquidation

Let N be a discrete random variable with values in the set $\mathbf{N} = \{1, 2, \ldots\}$.We suppose that the random variable N follows the geometric type II distribution with probability generating function $P_N(z) = \frac{pz}{1-qz}$ and $\{C_n, n = 1, 2, \ldots\}$ is a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence are equally distributed with the random variable C having characteristic function $\varphi_C(u)$. We set $Y = C_1 + C_2 + \cdots + C_N$. Let $\{\Pi_n, n = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence are equally distributed with the random variable Π having characteristic function $\varphi_\Pi(\xi)$ and we set $V = \Pi_1 + \Pi_2 + \cdots + \Pi_N$. We consider the vector (Y, V).

The purpose of the present section is the establishment of properties and applications in risk management of the vector (Y, V). The following result establishes sufficient conditions for evaluating the characteristic function $\varphi_{Y,V}(u, \xi)$ of the vector (Y, V).

Theorem 2.19.1 *Let N be a discrete random variable with values in the set $\mathbf{N} = \{1, 2, \ldots\}$. We suppose that the random variable N follows the geometric type II distribution with probability generating function $P_N(z) = \frac{pz}{1-qz}$ and $\{C_n, n = 1, 2, \ldots\}$ is a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence are equally distributed with the random variable C having characteristic function $\varphi_C(u)$. We set $Y = C_1 + C_2 + \cdots + C_N$.*

Let $\{\Pi_n, n = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence are equally distributed with the random variable Π having characteristic function $\varphi_\Pi(\xi)$ and we set $V = \Pi_1 + \Pi_2 + \cdots + \Pi_N$. If N, $\{C_n, n = 1, 2, \ldots\}$ and $\{\Pi_n, n = 1, 2, \ldots\}$ are independent then the characteristic function of the vector (Y, V) is

$$\varphi_{Y,V}(u, \xi) = \frac{p\varphi_C(u)\varphi_\Pi(\xi)}{1 - q\varphi_C(u)\varphi_\Pi(\xi)}$$

Proof The proof of Theorem 2.19.1 follows from the proof of Theorem 2.12.1. An interpretation of vector (Y, V), where $Y = C_1 + C_2 + \cdots + C_N$ and $V = \Pi_1 + \Pi_2 + \cdots + \Pi_N$, in risk management is the following.

A firm faces a risk. We suppose that $\{X_n, n = 1, 2, \ldots\}$ is a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence are equally distributed with the random variable X having distribution function

$$F_X(x). \tag{2.19.1}$$

The random variable X_n denotes the size of the damage due to the nth occurrence of the risk threatening the firm.

Let θ be a positive real number. If $X_n > \theta$ then the damage due to the nth risk occurrence is considered as a major one. If p is the probability of the event "the damage due the nth risk occurrence is major", then $p = P(X_n > \theta)$ or equivalently $p = 1 - P(X_n \leq \theta)$.

Hence (2.19.1) implies that $p = 1 - F_X(\theta)$. If the random variable N denotes the number of risk occurrences required for the first appearance of a major damage then the random variable N follows the geometric type II distribution with probability generating function $P_N(z) = \frac{pz}{1-qz}$.

We consider the sequence of random variables $\{C_n, n = 1, 2, \ldots\}$ and we suppose that the random variable C_n denotes the time between the nth and the $(n-1)$th risk occurrence. Hence the random variable $Y = C_1 + C_2 + \cdots + C_N$ denotes the time of the first appearance of a major damage.

We consider the sequence of random variables $\{\Pi_n, n = 1, 2, \ldots\}$ and we suppose that the random variable Π_n denotes the income of the firm from asset liquidation at the time point of the nth risk occurrence. Hence the random variable $V = \Pi_1 + \Pi_2 + \cdots + \Pi_N$ denotes the total income of the firm from asset liquidation in the time interval $[0, Y]$.

In this case the vector (Y, V) is a strong analytic tool for investigating the evolution of the firm under conditions of the first appearance of a major damage $Y = C_1 + C_2 + \cdots + C_N$ and the total income $V = \Pi_1 + \Pi_2 + \cdots + \Pi_N$ obtained by asset liquidation until the time point $Y = C_1 + C_2 + \cdots + C_N$.

The independence for N, $\{C_n, n = 1, 2, \ldots\}$ and $\{\Pi_n, n = 1, 2, \ldots\}$ permits the evaluation of the characteristic function

$$\varphi_{Y,V}(u, \xi) = \frac{p\varphi_C(u)\varphi_\Pi(\xi)}{1 - q\varphi_C(u)\varphi_\Pi(\xi)}$$

of the vector (Y, V).

In this case the practical applicability of the above vector in risk management is substantially extended. □

2.20 Time of First Major Damage and Loan Portfolio

Let $\{C_n, n = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence are equally distributed with the random variable C having characteristic function $\varphi_C(u)$.

We suppose that N is a discrete random variable with values in the set $\mathbf{N} = \{1, 2, \ldots\}$ and the random variable N follows the geometric type II distribution with probability generating function $P_N(z) = \frac{pz}{1-qz}$ and we set $Y = C_1 + C_2 + \cdots + C_N$.

Let $\{\Pi_s, s = 1, 2, \ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence are equally distributed with the random variable Π having characteristic function $\varphi_\Pi(\xi)$.

We suppose that S is a discrete random variable with values in the set $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ and probability generating function $P_S(z)$ and we set $V = \Pi_1 + \Pi_2 + \cdots + \Pi_S$.

We consider the vector (Y, V).

The purpose of the present section is the establishment of properties and applications in risk management of the vector (Y, V).

The following result establishes sufficient conditions for evaluating the characteristic function $\varphi_{Y,V}(u,\xi)$ of the vector (Y,V).

Theorem 2.20.1 *Let $\{C_n, n=1,2,\ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence are equally distributed with the random variable C.*

We suppose that N is a discrete random variable with values in the set $\mathbf{N}=\{1,2,\ldots\}$ and the random variable N follows the geometric type II distribution with probability generating function $P_N(z)=\frac{pz}{1-qz}$ and we set $Y=C_1+C_2+\cdots+C_N$.

Let $\{\Pi_s, s=1,2,\ldots\}$ be a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence are equally distributed with the random variable Π having characteristic function $\varphi_\Pi(\xi)$. We suppose that S is a discrete random variable with values in the set $\mathbf{N}_0=\{0,1,2,\ldots\}$ and probability generating function $P_S(z)$ and we set $V=\Pi_1+\Pi_2+\cdots+\Pi_S$. If $\{C_n, n=1,2,\ldots\}$, N, $\{\Pi_s, s=1,2,\ldots\}$, and S are independent then the random variables $Y=C_1+C_2+\cdots+C_N$ and $V=\Pi_1+\Pi_2+\cdots+\Pi_S$ are independent and

$$\varphi_{Y,V}(u,\xi)=\frac{p\varphi_C(u)}{1-q\varphi_C(u)}P_S(\varphi_\Pi(\xi))$$

is the characteristic function of the vector (Y,V).

Proof The proof of Theorem 2.20.1 follows from the proof of Theorem 2.13.1. An interpretation of the vector (Y,V), where $Y=C_1+C_2+\cdots+C_N$ and $V=\Pi_1+\Pi_2+\cdots+\Pi_S$, in risk management is the following.

A bank faces a risk. We suppose that $\{X_n, n=1,2,\ldots\}$ is a sequence of continuous, positive, independent, and identically distributed random variables. The random variables of the sequence are equally distributed with the random variable X having distribution function

$$F_X(x) \tag{2.20.1}$$

The random variable X_n denotes the damage due to the nth occurrence of the risk threatening the bank.

Let θ be a positive real number. If $X_n>\theta$ then the damage due to the nth risk occurrence is considered as major one. If p is the probability of the event "the damage due to the nth risk occurrence is major", then $p=P(X_n>\theta)$ or equivalently $p=1-P(X_n\le\theta)$.

Hence (2.20.1) implies that $p=1-F_X(\theta)$.

If the random variable N denotes the number of risk occurrences required for the first appearance of a major damage then the random variable N follows the geometric type II distribution with probability generating function $P_N(z)=\frac{pz}{1-qz}$.

We consider the sequence of random variables $\{C_n, n = 1, 2, \ldots\}$ and we suppose that the random variable C_n denotes the time between the nth and the $(n-1)$th risk occurrence. Hence the random variable $Y = C_1 + C_2 + \cdots + C_N$ denotes the time of the first appearance of a major damage. We consider the discrete random variable S denoting the number of loans in the portfolio of loans of the bank at the time point $Y = C_1 + C_2 + \cdots + C_N$.

We consider the sequence of random variables $\{\Pi_s, s = 1, 2, \ldots\}$ and we suppose that the random variable Π_s denotes the size of the sth loan of the portfolio of loans of the bank at the time point $Y = C_1 + C_2 + \cdots + C_N$. Hence the random variable $V = \Pi_1 + \Pi_2 + \cdots + \Pi_S$ denotes the total size of the portfolio of loans of the bank at the time point $Y = C_1 + C_2 + \cdots + C_N$. In this case the vector (Y, V) is a strong analytical tool for investigating the evolution of the bank under the time of the first appearance of a major damage $Y = C_1 + C_2 + \cdots + C_N$ and the total size of the portfolio of loans of the bank at the time point $Y = C_1 + C_2 + \cdots + C_N$.

The independence for N, $\{C_n, n = 1, 2, \ldots\}$, S and $\{\Pi_s, s = 1, 2, \ldots\}$ permits the evaluation of the characteristic function

$$\varphi_{Y,V}(u, \xi) = \frac{p\varphi_C(u)}{1 - q\varphi_C(u)} P_S(\varphi_\Pi(\xi))$$

of the vector (Y, V).

In this case the practical applicability in risk management of the above vector is substantially extended. □

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