2 Neoclassical Growth Theory

This chapter discusses the one-sector neoclassical growth model—the foundation for all the growth theory in the book. The primary focus of the chapter is growth via capital accumulation. We think of *capital* as man-made durable inputs to the production process. The first type of capital we include is *physical capital*. For our purposes, physical capital can be primarily thought of as plant and equipment that is produced in one period and then used in production in the following period.[1] To model production, we introduce *firms*, economic institutions that combine physical capital and labor to produce goods and services.

Physical capital introduces two fundamental features to the economy. First, the presence of physical capital allows the economy to expand its productive capacity over time—that is to experience economic growth through investment and capital accumulation. Second, the production of output generates *endogenous* income flows to households that own and rent physical capital to firms. In addition, physical capital will raise the productivity of the labor input and thereby raise the wage received by workers.

The accumulation of capital must be financed or funded by household saving. We use the life-cycle theory of household consumption as the basis for explaining saving behavior. In the pure life-cycle theory, households save to finance retirement consumption. Later in this chapter, we extend the life-cycle theory of the household to include transfers of assets across generations—including investments in children's *human capital* by their parents. Human capital, a second type of man-made durable input, is defined to be the embodied knowledge, skills, and health that affect a worker's productivity. Investments in human capital, just as with physical capital, allow an economy to experience economic growth.

[1] Definitions of physical capital will vary depending on the purpose at hand. In some cases, physical capital is defined to include inventories, software, land, and other inputs that extend beyond plant and equipment.

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Finally, we illustrate how the model with capital can be "estimated," or more precisely *calibrated*, to make quantitative analysis possible. In particular, we examine how well our simple models can replicate the economic growth in the USA from 1870 to 2000.

2.1 Firms, Production, and the Demand for Capital

The first step in developing a general equilibrium model of output and income is to introduce a production technology. We assume that production takes place in "firms"—organizations that hire labor and rent capital in order to produce output. Each firm's production knowledge or "technology" is represented by a *Cobb–Douglas production function*,

$$Y_t = AK_t^{\alpha} M_t^{1-\alpha}, \tag{2.1}$$

where Y denotes output, K denotes the capital stock rented, M denotes the hours of work hired, and where A and α are technological parameters. The production function is a technological "recipe" that relates the inputs hired and used by the firm to the output that the firm is capable of producing. The parameter A is sometimes referred to as *total factor productivity* (TFP). It captures a wide variety of unmeasured variables that affect the productivity of labor and capital; from climate and geography that determine natural resources available and the health environment of households to laws and regulations that restrict the way that production is carried out. The parameter α measures the relative importance of physical capital in the production process. This interpretation of α will become more clear as the theory of the firm is developed below.

The output produced by firms is a single "all-purpose" good that can either be consumed or invested as a physical asset (somewhat like corn that can be either consumed or stored and invested as a physical asset to plant and produce more corn in the future). This abstraction avoids the complication of having two distinct sectors of production, one producing consumer goods and the other capital goods. For some purposes, one may require this more elaborate two-sector model, but this is not the way to begin an analysis of a growing economy.

The Cobb–Douglas production function is a special case of what is called a "neoclassical" production function. All neoclassical production functions have three general properties: (i) positive marginal productivity, (ii) diminishing marginal productivity, and (iii) constant returns to scale. Economists believe that these properties are common to most production processes.

The marginal product of an input is the increase in output that results from an increase in the use of the input. Formally, it is the derivative of the production function with respect to a particular input, holding other inputs constant. For a Cobb–Douglas production function, the marginal product of labor and the marginal product of capital are $\frac{\partial Y_t}{\partial M_t} = (1-\alpha)AK_t^{\alpha}M_t^{-\alpha}$ and $\frac{\partial Y_t}{\partial K_t} = \alpha AK_t^{\alpha-1}M_t^{1-\alpha}$. The marginal productivity of increasing the level of either input is always positive—more output results when the firm hires either more labor or more capital.

Diminishing marginal productivity means the additional output, associated with adding an additional unit of an input, decreases as more of that input is used. So while output increases as the firm uses more of an input, the size of the increase gets smaller as the amount of the input used in production increases. This assumption is based on the intuitive notion of "input crowding." The increasing scarcity of the input held fixed, limits the production that results from adding more of the other input. For example, if there is given amount of capital, as more workers are hired the amount of capital that each worker can use decreases—serving to limit the rise in output. Note that the marginal product of labor expression above is decreasing in M_t, for a fixed value of K_t. The same reasoning applies to the marginal product of capital.

Constant returns to scale means that if *both* inputs were increased in the same proportion, then the ability to produce output would also increase by that proportion. This property makes sense because if the firm can simply duplicate its current plant, equipment, and workforce, it should be able to duplicate or double its output as well.

Finally, note that the properties we just established imply that the marginal product expressions can be simplified by combining M_t and K_t into the *capital–labor ratio*, also known as *capital intensity*, $k_t \equiv K_t/M_t$. The simplified expressions for the marginal products are, $\frac{\partial Y_t}{\partial M_t} = (1-\alpha)Ak_t^{\alpha}$ and $\frac{\partial Y_t}{\partial K_t} = \alpha A k_t^{\alpha-1}$. The marginal product of labor is increasing in capital intensity. The more capital per worker, the more productive an additional worker is. The marginal product of capital is decreasing in capital intensity. Higher capital intensity means that there are fewer workers available to work with any additional capital bought to the workplace.

The fact that the marginal products of capital and labor are both functions of the capital–labor ratio, k, and not the levels of K and M, is a consequence of the constant returns to scale assumption. This property implies that the *scale* of a firm is indeterminate, i.e., the optimal size of a firm cannot be pinned down by the theory. Firms are indifferent about the level of production, but they do want to hire capital and labor in a particular ratio that depends on the relative market prices of the inputs.

From the point of view of microeconomics, the indeterminacy of firm size can be seen as a disadvantage. One is forced to simply assume that firms are of a given size and that there are enough of them competing to justify the *perfect competition* assumption that is discussed below and used throughout the book. From a macroeconomic point of view, the indeterminate size of firms can be seen as a convenient simplification. The key expressions that characterize the production side of the economy apply both to the individual firm and to the collection of firms as a whole. This is why in many macroeconomic models, the distinction between the individual firms and production in the economy as a whole is not emphasized.

What makes (2.1) special in the class of neoclassical production functions is that the Cobb–Douglas functional form implies that the *shares* of national output that are paid to capital owners and workers are the *constant* output elasticity values α and $1-\alpha$. Data show that over the last century, income shares have in fact stayed roughly constant within and across countries. For this reason, many economists view the Cobb–Douglas functional form as a reasonable approximation to an economy's aggregate production technology. To explicitly see that (2.1) has the constant income share property, we next need to think about how capital owners and workers are paid.

We assume that markets are perfectly competitive in our production economy. As discussed in elementary economics, the notion of competitive markets applies not only to the markets for goods but also to the factor markets for labor and capital. The competitive assumption applied to the factor markets means that firms demand inputs to maximize profits taking as given the market prices of the inputs: the wage rate paid to labor (w) and rental rate for physical capital (r). No single firm is large enough to be able to influence market prices when they unilaterally change their production or input levels. The price of the economy's single output good is taken to be one. So we can think of output and revenue as being the same. Therefore, profit can then be written as $Y_t - w_t M_t - r_t K_t$.

Maximizing profits requires that firms hire capital and labor as long as the marginal benefit (marginal product) exceeds the marginal cost (factor price). Formally, the necessary conditions for profit maximization are

$$\alpha A k_t^{\alpha-1} = r_t \tag{2.2a}$$

$$(1-\alpha) A k_t^{\alpha} = w_t. \tag{2.2b}$$

Equation (2.2a, b) say that, in order to maximize profit, the marginal product of each input must be equated to its market price, just as in the theory of competitive factor markets from intermediate microeconomics.

From the perspective of an individual firm, that takes factor prices as given, it appears that there are two independent equations, (2.2a, b), to determine one unknown, k. In general, this situation leads to inconsistent solutions for k—i.e., different solutions for k from each equation. This is not the case here because of an important implication of competitive markets: economic profits are driven to zero. Competition between firms for the available resources will force factor prices to satisfy these equations, which in turn implies that economic profits are zero. Thus, (2.2a, b) also play a role in determining the market factor prices and not just k.

To think about this last point further, first notice that we can write the production function as $Y_t = A k_t^{\alpha} M_t$. Next, multiply each side of (2.2a, b) by M_t to get

$$\alpha Y_t = r_t K_t \tag{2.3a}$$

$$(1-\alpha) Y_t = w_t M_t \tag{2.3b}$$

Equation (2.3a) shows that the share of output and revenue paid to owners of capital (by each firm and in the economy as a whole) is the constant, α, an interpretation that was suggested above. Moreover, if αY_t goes to capital owners as a gross rent to capital, there is just enough revenue left over, $(1-\alpha)Y_t$, to pay workers the competitive wage, implying that economic profit is zero.

The connection made in (2.3a, b) allows us to refer to α and $1-\alpha$ as the capital and labor shares. The fact that the Cobb–Douglas technology, combined with competitive markets, implies constant factor shares is a strong prediction of the model. Remarkably, this prediction is approximately consistent with empirical evidence that shows little trend in factor shares as a country develops.

The two Eqs. (2.2a, b) are then profit-maximizing conditions that determine two variables: the firm's demand for capital relative to labor and, via the zero profit condition, one of the factor prices. To determine the remaining factor price, we need the final requirement of a competitive equilibrium: *market clearing*. The firm's demand for capital per worker must equal the supply of capital per worker coming from households. We will think of the rental rate on capital as the "price" that clears the capital market. Then interpreting (2.2a, b) as determining the demand for capital and the competitive wage rate that generates zero profit, we have three conditions to determine the three unknowns: r_t, w_t, and k_t.

The first step in developing the market clearing condition is to be more explicit about what we mean by the *demand* for capital in the production economy. Start by thinking of the capital–labor ratio on the left-hand side of (2.2a) as the capital–labor ratio *demanded* by firms at different rental rates for capital. Call the firm's demand for k, k_t^d. In period t, firms will enter the capital market to rent capital that they can use in production. Solving (2.2a) for k, we can write the demand for capital in period t as

$$k_t^d = \left[\frac{\alpha A}{r_t}\right]^{1/(1-\alpha)}. \tag{2.4}$$

Equation (2.4) indicates that as the rental rate required by the market rises, the firm's demand for capital declines. This is because, as the cost of capital rises, firms will shift toward using less capital and more labor in production.

The theory thus far gives us the firms' demand for capital intensity. Now we need to develop a theory for the supply of capital in period t. In other words, we need to discuss who owns the capital and how much capital they are willing to supply to the market.

2.2 Household Saving and the Supply of Capital

In our model, households purchase capital as an asset, a type of saving used to finance retirement consumption. The capital generates funds for retirement consumption purchases when the households rent the capital to firms. So, the supply of capital referred to at the end of Sect. 2.1 results from older households attempting to generate income for retirement consumption.

To capture a retirement motive for saving in the simplest way possible, we assume households live for two periods: one when they are young and working and one when they are old and retired. This means that in any one period, there are two household types from distinct generations: a young working household and an old retired household. Macroeconomic models where different generations operate as distinct decision makers in each period are called *overlapping generations* models.

Including the saving behavior of households is an important extension to the Solow model of capital accumulation that you may have seen in undergraduate macroeconomic courses. In the Solow model, saving is treated as an exogenous variable. The economy's saving rate is simply assumed to be a constant fraction of total income with no explanation provided.

2.2.1 The Supply of Labor and Capital

As just mentioned, the supply of capital that is rented to firms is owned by old retired households. They rent the capital to firms to generate income that finances their retirement consumption. Once the firms complete production using the capital, the retired households sell the capital to the young working households that are looking to save assets to finance their future retirement consumption. The sale of capital provides further resources for retirement consumption of the current old households.

Formally, the currently old households, who own the capital, purchased the capital as an asset during their working lives in the previous period. In period $t-1$, each young household supplied one unit of labor to firms and earned the wage, w_{t-1}. For now, there is no variation in household labor supply. We model the choice of how much to work in Chap. 5. With each household supplying one unit of labor, the aggregate supply of labor in each period is then just the number of young households. In period $t-1$, the total supply of labor to all firms is the total number of young households from that generation, $M_{t-1}^s = N_{t-1}$.

The capital supplied per unit of labor results from the household's saving behavior, s_{t-1}. Young households save in period $t-1$ by purchasing output and treating it like a physical asset that generates income during retirement by supplying or renting it to firms for use in production during period t. The firms use this physical capital to produce output and generate revenue in period t. The firms then return the capital, that has been depreciated by use in production, back to households and pay them the rental rate r_t. So, for every unit of capital that households purchase and rent to firms, they receive back in period t, $1-\delta+r_t$, as their return to saving, where δ is the fraction of capital that depreciates from use. We somewhat loosely refer to $r_t-\delta$ as both the "return to capital" and the "interest rate" on household saving.

The total supply of new capital to the market in period t is the total saving of young households in period $t-1$, $s_{t-1}N_{t-1}$. To match the demand concept in (2.4), we need an expression for the capital supplied *per worker* in period t. The supply of capital per worker in period t is $k_t^s \equiv s_{t-1}N_{t-1}/M_t^s = s_{t-1}N_{t-1}/N_t = s_{t-1}/n$, where n is the average number of children born in each young household. In Chap. 3, we make fertility endogenous by allowing parents to choose the number of children, but for now we treat n as an exogenous constant. The number of children each household has determines the relative population size of different generations. For example, if n = 1, then generations are of equal size and $N_t = N_{t-1}$. If n > 1, $N_t > N_{t-1}$ and there is positive population growth over time. Note that the rate of population growth is $(N_t/N_{t-1}) - 1 = n - 1$.

2.2.2 Household Saving

We now develop a theory of household saving. Households do not directly benefit from saving but rather use saving to create their desired lifetime consumption path. The consumption path that households prefer depends on their attitudes about consuming now rather than later in life. Household preferences are represented by a utility function. The utility function captures the household's preference for consuming at different points in their lifetime.

We assume that household preferences are represented by a *constant elasticity of substitution* (CES) utility function,

$$u_t = U(c_{1t}, c_{2t+1}) = \frac{(c_{1t}^{1-1/\sigma} - 1) + \beta(c_{2t+1}^{1-1/\sigma} - 1)}{(1 - 1/\sigma)}.$$

For a generation t household, consumption in the first and second periods, c_{1t} and c_{2t+1}, determine the value of lifetime utility. The CES utility function has the standard properties that the marginal utility of consumption in each period is positive but diminishing (try taking the first and second derivative with respect to consumption in any one period).

The two parameters used to reflect individual preferences about the timing of consumption are the pure time discount factor (β) and the intertemporal elasticity of substitution (σ). Typically, one assumes that $\beta < 1$ because people are generally viewed as being "impatient," i.e., they weigh utility gained from current consumption higher than utility gained from future consumption. The intertemporal elasticity of substitution is a measure of the individual's willingness to substitute current for future consumption when the relative price of future consumption falls, but this will not be made clear for a while. Subtracting 1 from each consumption term is done for purely technical reasons. It allows the commonly used logarithmic utility function, $\ln c_{1t} + \beta \ln c_{2t+1}$, to appear as a special case when σ = 1.

Households face constraints that restrict the consumption paths they can afford. In each period, there is a budget constraint that must be satisfied. In the first period of life, a generation t household has its wage (w_t) as a source of funds that can be used to purchase output for consumption (c_{1t}) or for saving (s_t). This gives the first-period budget constraint, $c_{1t} + s_t = w_t$. In the second period, consumption

(c_{2t+1}) is financed by the saving from the first period, $c_{2t+1} = R_t s_t$, where $R_t = 1 + r_{t+1} - \delta$ is the return from owning physical capital or what sometimes is called the "interest factor." The two single period budget constraints can be combined to form a single lifetime budget constraint that says the present value of consumption must equal the first-period wage, $c_{1t} + c_{2t+1}/R_t = w_t$.

Households maximize lifetime utility subject to the lifetime budget constraint. The solution to this problem gives us the optimal consumption and saving behavior of a household

$$c_{1t} = \Psi_{1t} w_t \tag{2.5a}$$

$$c_{2t+1} = \Psi_{2t} w_t \tag{2.5b}$$

$$s_t = \Psi_{1t} \beta^\sigma R_t^{\sigma-1} w_t, \tag{2.6}$$

where $\Psi_{1t} \equiv \frac{1}{1+\beta^\sigma R_t^{\sigma-1}} < 1$ and $\Psi_{2t} \equiv \frac{\beta^\sigma R_t^\sigma}{1+\beta^\sigma R_t^{\sigma-1}}$.

All choices are fractions of the wage rate that depends on the return to capital. The return to capital affects the fraction differently depending on the value of σ.

The effect of a change in R_t can be conceptually decomposed into an *income* effect and a *substitution* effect. The income effect refers to the fact that a higher return to capital increases the purchasing power of savers that own capital. Greater purchasing power increases the demand for consumption in both periods but lowers saving. The substitution effect refers to the fact that a higher return to capital raises the cost of consuming in the first period relative to consuming in the second period. This is because the forgone interest income from consuming, and not saving, in the first period has increased. The substitution effect from an increase in the return to capital lowers current consumption but raises saving and future consumption. Thus, the overall effect of an increase in the return to capital on current consumption and saving is ambiguous. The relative strength of the income and substitution effects is determined by σ. The greater the value of σ is, the stronger the substitution effect is and the weaker the income effect is. When $\sigma = 1$, the income and substitution effects cancel exactly and the saving rate become a constant fraction of wages, $\frac{\beta}{1+\beta}$. This convenient special case is similar to the Solow model in that the saving rate is a constant.

2.2.3 Supply of Capital Per Worker

Using Eq. (2.6), dated for a generation $t - 1$, and the definition of k_t^s that was introduced previously, we can now write the economy's supply of capital per worker as

$$k_t^s = \frac{w_{t-1}}{n}\left[\frac{1}{1+\beta^{-\sigma}(1+r_t-\delta)^{1-\sigma}}\right]. \tag{2.7}$$

The economy's supply of capital per worker next period is based on the saving per worker this period and the growth of the economy's workforce. An increase in this period's wage raises saving because a portion of the higher wage is consumed and a fraction is put aside to allow consumption in the future to rise as well. The interest rate has an ambiguous effect on saving because of conflicting income and substitution effects (with the relative strength of the substitution effect determined by σ).

The extent to which saving and capital supplied this period raises the capital–labor ratio next period depends on the growth in the workforce. Greater fertility implies a higher rate of population growth and a faster growing workforce. As the workforce next period rises relative to the current workforce, less saving and capital will be available per worker in the future. Thus, higher rates of population growth lower the capital–labor ratio by forcing the available capital to be spread over a larger workforce.

2.3 Competitive Equilibrium in a Growing Economy

Before moving to the determination of the market clearing condition in the capital market, let us summarize the key actions taken in period t by each agent.

Firms	hire labor, pay each worker w_t and rent physical capital per worker, k_t^d, pay owners r_t per unit supplied
Young households	supply one unit of labor, receive w_t, and purchase $s_t = nk_{t+1}^s$ units of physical capital
Old households	supply $s_{t-1} = nk_t^s$ units of physical capital and receive r_t per unit supplied

A market clearing equilibrium in the capital market requires that the firms' demand for capital per worker equals the supply of capital per worker by old households, i.e., $k_t^d = k_t^s$ for all values for t. As in other competitive markets, the market price is the mechanism for bringing the two sides of the market together. In the capital market, the market price is the rental rate on capital that is paid by those demanding capital and received by those supplying the capital. Market clearing requires finding a value of r_t that equates (2.4) and (2.7) in every period, as sketched in Fig. 2.1.[2]

[2] You can think of the value of r_t as actually determined in period $t-1$. In that period, households make their saving decision based on the firms' commitments to rent capital in period t and pay the rental rate r_t. In other words, r_t is determined in period $t-1$ based on the savings behavior of households and the *planned* investment demands of firms.

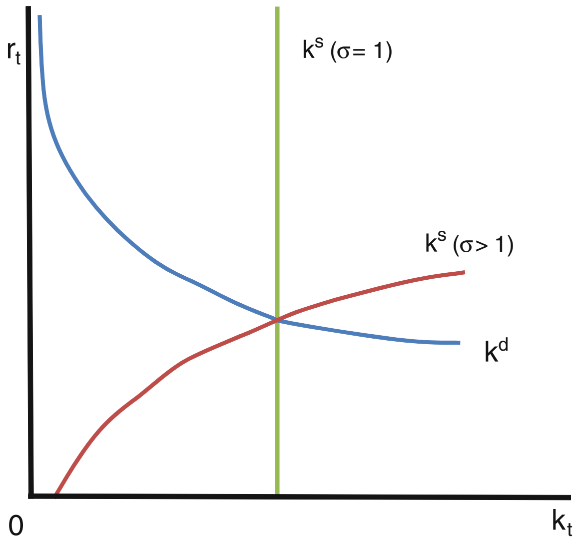


Fig. 2.1 Market clearing equilibria in the capital market

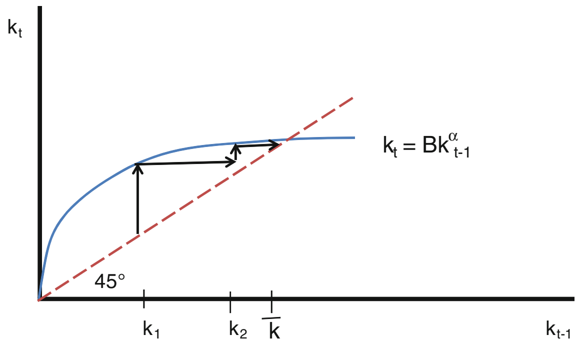
The period t equilibrium is sketched for two different "supply curves." The vertical supply curve assumes $\sigma = 1$ (exactly offsetting income and substitution effects) and the upward sloping supply curve assumes $\sigma > 1$ (a dominant substitution effect). Note that once the equilibrium values of r_t and k_t are determined, then the equilibrium value of w_t is determined by (2.2b).

The figure above is the standard demand–equals–supply way of thinking about how equilibrium is determined. While intuitive, it has limitations as an analytical tool. The problem is that it is a static snapshot of a dynamic economy. In general, a production economy will experience capital accumulation over time. In other words the k_t determined in the figure will be larger than k_{t-1}. This implies that, using (2.2b), w_t will be larger than w_{t-1}. The increase in wages over time will cause the supply curve in the figure to shift to the right each period. Thus, the diagram reveals that growth in the economy is due to the effect of capital on wages. As the capital stock increases, wages increase. The increase in wages increases saving and leads to further capital accumulation. However, there are important details of the growth process that are not revealed by this essentially static depiction.

Fortunately, there is a nice way of displaying the dynamics of the economy more explicitly. One can substitute the factor price equations from (2.2a, b) into (2.7) and impose the equilibrium condition $k_t^d = k_t^s$ to get

$$k_t = \frac{(1-\alpha)Ak_{t-1}^{\alpha}}{n}\left[\frac{1}{1+\beta^{-\sigma}\left(1+\alpha Ak_t^{\alpha-1}-\delta\right)^{1-\sigma}}\right]. \tag{2.8}$$

Fig. 2.2 Transitional growth



Mathematically, Eq. (2.8) is known as a *difference equation*, which is the discrete-time analog to the differential equation in continuous time. This equation highlights the underlying dynamics of the model that is driven by changes in the capital–labor ratio over time. In economics, Eq. (2.8) is referred to as a *transition equation* because it describes how the economy evolves over time.

Viewing (2.8), note that the variable we need to solve for, k_t, appears on both sides of the equation. In general, there is no way to use algebra to solve for the unknown value of k_t in terms of the parameters and the known value k_{t-1} (the value of k in the initial period is given as an initial condition of the economy). In this case, one needs to use a numerical approach to find an approximate solution to (2.8). The numerical approach will be discussed in Sect. 2.5.

The dynamic structure of the economy becomes more transparent when we assume $\sigma = 1$. In this special case, (2.8) simplifies to

$$k_t = Bk_{t-1}^{\alpha}, \tag{2.8′}$$

where $B \equiv \frac{\beta}{1+\beta}\frac{(1-\alpha)A}{n}$.

In this case, we have an explicit algebraic solution because the expression involving k_t on the right-hand side of (2.8) vanishes. As mentioned previously, this special case is similar to the Solow model because the saving rate of young households is a constant, with no dependence on the return to capital.

The dynamic features of (2.8′) can be easily sketched by plotting k_t against k_{t-1} as in Fig. 2.2.

Imagine that the economy begins at $k_{t-1} = k_1$. To find out what the capital–labor will be in period 2, move vertically up to the transition equation to find k_2. In period 2, k_2 will now be the initial capital–labor ratio, move horizontally from the transition equation to the 45-degree line and then back down vertically to the horizontal axis. The process then repeats itself until one reaches $k_t = \bar{k}$, where the transition equation crosses the 45-degree line.[3] At this point, the capital–labor ratio remains constant from period to period and the economy is said to have reached a

[3] The economy never literally reaches the steady state, although it will get arbitrarily close.

steady-state equilibrium. An algebraic solution for the steady state is found by setting $k_t = k_{t-1} = \bar{k}$ in (2.8) and then solving the equation for $\bar{k}$. The transition equation given by (2.8) is simple enough to allow an explicit solution for the steady-state capital–labor ratio, $\bar{k} = \mathrm{B}^{\frac{1}{1-\alpha}}$.

The transition diagram reveals an important prediction about economic growth via capital accumulation. In the early stages of growth, period-to-period changes in k_t are relatively large and the economy grows fast. Over time, the increases in k_t get smaller and the economy's growth rate slows down, until growth ceases altogether in the steady state. From the static demand and supply figure, we know that growth occurs due to the effect of capital accumulation on wages and saving. What the transition diagram makes clear is that the effect of capital accumulation on wages becomes weaker over time. There is a diminishing effect of k_t on w_t because α is less than one. When an economy is undeveloped and capital is scarce, the creation of new physical capital significantly raises worker productivity and wages. However, as the economy industrializes, the impact of further capital accumulation weakens.[4]

Notice two things about the steady state. First, as k_t grows in approaching the steady state, we know from (2.2a, b) that interest rates will be falling and wages will be rising. Once the steady state is obtained, because k_t is constant, interest rates and wages must also be constant. Thus, the steady state is characterized by constant interest rates and *zero* growth in labor productivity, real wages, and consumption. In many developed countries, the *average* values of interest rates and returns to capital have been relatively constant over long periods of time—suggesting that we might view the average position of the economy as being a steady state (with some annual business cycle fluctuations around the economy's typical or average position). However, these same economies are observed to experience *positive* growth rates in labor productivity and real wages *on average*. According to our model, if interest rates show no downward trend, then this positive growth cannot come from increases in the capital–labor ratio. Where does persistent, long-run growth come from after the steady-state capital–labor ratio is obtained?

2.3.1 Steady-State Growth—Technical Progress

One explanation for persistent economic growth is technical progress—that is increasing knowledge that improves productivity. Technical progress can be thought of as improved production designs or improved factories and equipment. To grow in the steady state with the same amount of capital per worker, we have to get smarter about how we use and design the capital. They are some attempts to explicitly model the research and development process that leads to technical

[4] The weakening effect of the capital–labor ratio on wages stems from the diminishing marginal product of capital. As capital accumulates relative to labor, the effect of further capital accumulation on output and wages gets smaller. Formally, note that the effect of an increase in k on the marginal product of labor is $(1-\alpha)\alpha Ak^{\alpha-1} = (1-\alpha) \times$ marginal product of capital.

process, but economists often treat technical progress as an exogenous variable, as we do here.

Think of technology as the current stock of *disembodied* blueprints for production methods and machine designs. The state of technology in period t affects the productivity of the workforce. We assume that there is an index number, D_t, that measures the extent to which the state of technology influences the *effective* workforce. The effective workforce in period t is defined as $H_t = D_t M_t$, which replaces M_t as an input in the Cobb–Douglas production function. When D_t increases, it raises the effective workforce proportionately. For example, if D_t doubles, and the number of workers remains the same, the effect on production will be the same as doubling the number of workers. We further assume that technical progress is such that D_t increases from one period to the next at the constant rate, d. Thus, $H_{t+1}/H_t = n(1+d)$, the effective workforce increases due to both population growth and technical progress.

We can model the firms as choosing H_t and paying a wage rate per unit of *effective* labor, w_t. The total wage payment received by an actual worker will now be $w_t D_t$. The factor price equations given by (2.2a, b) remain the same, except we now must interpret k as the capital-to-effective-labor ratio, i.e., $k = K/H$.

Now let us think about how the equilibrium and transition equation are altered by technical progress. The firm's demand for the capital, which we can think of as a demand for the ratio of capital to effective labor, will take the same form as (2.2a). On the household side, we need only adjust the saving function for the new concept of household wages to get $s_t = \Psi_{1t}\left(\beta^{\sigma} R_t^{\sigma-1} w_t D_t\right)$. The supply of capital per effective worker is defined as $k_t^S \equiv s_{t-1} N_{t-1}/D_t N_t$. Using the household saving function, the supply of capital per effective worker can be written as $k_t^s = \frac{w_{t-1}}{n(1+d)}\left[\frac{1}{1+\beta^{-\sigma}(1+r_t-\delta)^{1-\sigma}}\right]$. Finally, using the factor price equations, the adjusted transition equation becomes

$$k_t = \frac{(1-\alpha)Ak_{t-1}^{\alpha}}{n(1+d)}\left[\frac{1}{1+\beta^{-\sigma}\left(1+\alpha Ak_t^{\alpha-1}-\delta\right)^{1-\sigma}}\right], \tag{2.9}$$

which has the same form as (2.8), except for the presence of 1 + d in the denominator of the expression on the right-hand side of the equation.

Thus, the transitional dynamics of the economy are the same as before. However, now there is an *endogenous* source of growth (increasing physical capital intensity) and an *exogenous* source of growth (technical progress). When the steady state is reached, the transitional growth from increasing physical capital intensity is over and interest rates become constant. However, there will continue to be positive economic growth from exogenous technical progress. Labor productivity, real wages per worker $(w_t D_t)$, and the standard of living (measured by consumption per household) all increase at the rate $d > 0$ in the steady state.

2.4 Intergenerational Transfers

In his presidential address to the American Economic Association, Becker (1988) encouraged economists to pay greater attention to the role of the family in their thinking about macroeconomic issues. Becker pointed out how intergenerational transfers between family members are likely to influence economic growth and alter the effects of important fiscal policies such as social security and government borrowing. Understanding the causes and consequences of intergenerational transfers is one of the most important motivations for taking an overlapping generations approach that recognizes the generational structure of the economy.

We define intergenerational transfers to be a private transfer of resources from one generation to another generation. The transfers take two distinct forms: *human capital* investments (x) and transfers of physical or financial assets (b). The financial assets are identical to the assets used for life-cycle saving, but instead of being used to increase future income for the household, they are used to increase the future income of the household's children.

Human capital investments are *in-kind* transfers of goods and services designed to raise the recipient's productivity by increasing knowledge, skills, or health. The effect of these in-kind transfers on the child's market productivity is given by a human capital production function $H(x_{t+1})$. Here, we are treating effective labor supply as an endogenous function of parental investments. To focus on the human capital component of effective labor supply, we temporarily ignore the exogenous component of effective labor supply, D_t.

The only assumption about the human capital production function that we will use in this section is that the derivative of H with respect to x is positive but diminishing $(H' > 0, H'' < 0)$, i.e., H is increasing and strictly concave in x. This assumption means that while spending more on children's education this period will always increase their knowledge and productivity next period, when they enter the labor force, the marginal return to additional educational spending decreases with the level of expenditures. In other words, there are natural limits to the rate at which a child can accumulate skills and knowledge.

The distinction between the two types of transfers is important since there are situations where human and financial investments of equal dollar amounts are not equivalent in their effect on the recipient's behavior and wealth. For example, a dollar spent on a dependent child's primary and secondary education is generally not going to have the same economic effect as a dollar invested in a financial asset that is ultimately bequeathed to the child. In other circumstances, the distinction may be less clear. Is the payment of a child's college tuition by a parent equivalent to a cash transfer to the child? The answer is not obvious and will depend on the preferences and constraints of *both* the parent and the child. It may also be true that parents value the two transfers differently. For example, parents may value education for its own sake, beyond its effect on the child's future labor productivity in the market.

The accounting relationship between intergenerational transfers and the accumulation of wealth is made precise by examining the household budget constraints in our two-period model. Households have n children, an exogenous variable in this chapter, in the first period of adulthood. During the first period, along with the typical life-cycle consumption and saving choices, parents make human capital investments in their children and save to make financial transfers to their children when they become young adults in the next period. The first-period budget constraint is

$$c_{1t} + s_t + n(x_{t+1} + b_{t+1}) = W_t, \tag{2.10}$$

where $W_t \equiv w_t H_t + R_{t-1} b_t$ is initial wealth and recalls that $w_t H_t$ now represents lifetime earnings with w_t interpreted as the wage per unit of *effective* labor supply.[5] The other term on the right-hand side of the initial wealth expression is the physical or financial asset, plus interest, that the household receives from its parents during the first period of adulthood. The second-period constraint remains the same as before and given by

$$c_{2t+1} = R_t s_t. \tag{2.11}$$

In addition to the budget constraints, one important additional constraint is needed. It is assumed that parents cannot accumulate debt that their children are legally bound to pay (a legal restriction that exists in most societies). This means there is a nonnegativity constraint on financial transfers

$$b_{t+1} \geq 0. \tag{2.12}$$

We next turn to two different approaches to modeling *why* parents make intergenerational transfers. The first approach assumes parental altruism, parents care about the welfare of their children. The altruistic approach provides an interesting and rich theory, but also one that is technically demanding. Less advanced students should skip to Sect. 2.4.3, where a second theory of transfers is presented. This theory assumes that parents receive utility, a "warm glow," directly from the transfers they make to their children.

[5] Note that the acquisition of financial assets occurs in the first period of adulthood and the financial transfers are made in the second period of adulthood. Thus, the transfers are made when both generations are alive. Transfers of this type are called *intervivos* transfers, as opposed to *bequests* that are transferred at death. In our model, where we assume (i) perfect certainty, (ii) perfect life-cycle credit markets, and (iii) no strategic interactions between generations, the timing of financial transfers is irrelevant. However, the timing of transfers can matter when these conditions are not met [see, e.g., Bernheim et al. (1985) and Cox (1987)].

2.4.1 Altruism*

One natural way to explain *why* intergenerational transfers occur is to assume that parents are altruistic, i.e., that they care about their children's welfare as adults. An important way of modeling altruism assumes that parents care about their adult children's lifetime utility. We call this type of altruism *Barro–Becker* altruism named after the two economists who introduced the concept (Barro 1974; Becker 1974). As we shall see, this particular way of modeling altruism has very interesting implications, ones that have had a major effect on the development of macroeconomic theory.

Barro–Becker altruism implies a utility function for the generation t parent of the form, $U_t + \beta V_{t+1}(W_{t+1})$, where U_t is utility from the consumption of the generation t household, which we take to be of CES form throughout, and V is the *maximum attainable utility* of the next generation. The function V_{t+1} depends on W_{t+1} because it is the utility attained when adult children maximize their utility subject to their initial wealth and market prices.[6] We assume that parents cannot directly affect the choices of their adult children, but they can have a significant effect on their children's initial wealth and therefore on V. The value function is discounted by parents because the adult utility of their children is generated in the future and the parents have a positive rate of time preference. It is also possible that parents care more or less about their adult children's welfare than their own. In this case, the generational discount factor would differ from the pure time discount factor, but the qualitative conclusions would be the same.

We are now ready to formally lay out the optimization problem for the altruistic household. The generation t altruistic household chooses life-cycle consumption and intergenerational transfers to maximize

$$U_t + \beta V_{t+1}(W_{t+1}), \tag{2.13}$$

subject to $W_{t+1} \equiv w_{t+1} H(x_{t+1}, \cdot) + R_t b_{t+1}$ and the generation t lifetime budget constraint, formed by combining (2.10) and (2.11), along with (2.12). The problem can be solved in two steps. First, for *given* values of the transfer variables, the solution for life-cycle consumption follows as before

[6] In microeconomics, the maximum attainable utility function is called an *indirect utility function*. In the pure life-cycle version of our model, with no altruism, an indirect utility function is easily obtained. Take the optimal consumption choices of the household (2.5a, b) and substitute them back into the CES utility function. For example, if we do this for the case of $\sigma = 1$, we get the very simple indirect utility function $U_t^* = \beta \ln \beta + (1+\beta) \ln \frac{1}{1+\beta} + (1+\beta) \ln w_t + \beta \ln R_t$. With altruism, things are not nearly so simple. This is because generation t's utility depends on generation $t + 1$'s utility, which depends on generation $t + 2$'s utility, and so on. As we shall see, in this case V_t cannot be found directly. Instead, it is implicitly defined by a difference equation in the function V_t—what is known as a *Bellman equation*. In this case, V_t is called a *value function* and solving for this function is tricky business. Stokey and Lucas (1989) provide a general discussion of the conditions under which the value function exists, is unique, and is differentiable with respect to initial wealth.

$$c_{1t} = \Psi_{1t}(W_t - n(x_{t+1} + b_{t+1})) \tag{2.14a}$$

$$c_{2t+1} = [\beta R_t]^{\sigma} c_{1t}. \tag{2.14b}$$

The life-cycle consumption choices can then be substituted back into (2.13), making it solely a function of the intergenerational transfer choices.

Second, (2.13) can be maximized with respect to the transfer choices. The first-order conditions for the transfer choices are

$$n(W_t - n(x_{t+1} + b_{t+1}))^{-1/\sigma}\Psi_{1t}^{-/\sigma} = \beta V'_{t+1} w_{t+1} H'_{t+1} \tag{2.15a}$$

$$n(W_t - n(x_{t+1} + b_{t+1}))^{-1/\sigma}\Psi_{1t}^{-1/\sigma} \geq \beta V'_{t+1} R_t \tag{2.15b}$$

The left-hand side of each equation is the marginal cost of making a transfer to children measured in terms of forgone consumption and utility of the parent. The right-hand side is the marginal benefit of the transfer measured in terms of the resulting rise in lifetime wealth and utility of the adult child. The inequality in (2.15b) includes the possibility that a nonnegative financial transfer may not be optimal. If not, (2.12) binds and (2.15b) becomes a strict inequality.

Note that solving this maximization problem gives us the maximum attainable utility, or *value function*, of the current generation,

$$V_t(W_t) = \max\{U_t + \beta V_{t+1}(W_{t+1})\}.$$

This equation implicitly defines the value function and is known as the *Bellman equation*. The Bellman equation allows us to obtain the following result (see Appendix A)

$$V'_{t+1} = (W_{t+1} - n(x_{t+2} + b_{t+2}))^{-1/\sigma}\Psi_{t+1}^{-1/\sigma} = c_{1t+1}^{-1/\sigma}, \tag{2.16}$$

The derivative of the value function is just the marginal utility of consumption. Using (2.16), the first-order conditions given by (2.15a, b) can be presented and discussed intuitively for the two possible cases,

unconstrained transfer equations

$$w_{t+1} H'(x^*_{t+1}, \cdot) = R_t \tag{2.17a}$$

$$c_{1t+1} = [\beta R_t / n]^{\sigma} c_{1t} \tag{2.17b}$$

or

constrained transfer equations

$$b_{t+1} = 0, \tag{2.18a}$$

$$c_{1t+1} = [\beta w_{t+1} H'(x_{t+1}, \cdot)/n]^{\sigma} c_{1t}. \tag{2.18b}$$

If the optimal financial transfers are positive, then human capital investments are unconstrained. This allows the investments to be *productively efficient* in the sense that the return on human investment is equated to the return on physical investment. Parents invest in their children until the rate of return to human capital is driven down to the rate of return on physical capital. At that point, any further intergenerational transfers are in the form of physical or financial assets.

In the efficient case, human capital investment is completely determined by (2.17a), independent of consumption choices and household wealth. In addition, when the generations are linked by financial transfers, one gets the familiar Euler condition, (2.17b), where the growth of consumption over time (actually over generations in this case) is determined by the return to capital—perfectly analogous to the optimal pattern of life-cycle consumption growth in (2.14b).

Desired financial transfers may also be negative, i.e., parents may want to leave debt for their children to repay. In this case, (2.12) binds and there is a strict inequality in (2.15b). Now the human capital investment decision cannot be separated from the household's consumption choices; (2.14a, b), and (2.18b) must be solved simultaneously. Under these circumstances, the *wealth* of the current generation becomes a determinant of human capital investment. Combining (2.15a, b) shows that this situation is *productively inefficient*, because the rate of return on human capital investment exceeds the rate of return on financial assets. However, the current generation cannot "afford" to invest more in their children's education, which is why they want to raise both their own consumption and human capital investment in their children by leaving debt for their children to pay. The legal restrictions forbids this, so they are forced to set $b_{t+1} = 0$. If parents could invest more, by setting $b_{t+1} < 0$, then x_{t+1} would rise and H' would fall until (2.15a) is satisfied. This situation implies that the ratio of consumption of the next generation to the consumption of the current generation would also fall.

2.4.2 Explicit Household-Level Solutions in Some Special Cases*

To go further in uncovering the behavioral implications of adding altruistic transfers, we now solve the model for some special cases. For most of the special cases, we also assume that the economy is in a steady state with zero technical progress, so that $r_t = r$ and $w_t = w$. This assumption serves to simplify the notation considerably.

2.4.2.1 Exogenous Human Capital

To isolate the role of financial transfers, assume that human capital is exogenous and constant, and, for simplicity, set $H = 1$. Under this assumption, a household can only help its descendants by giving them financial transfers. In this case, the *constrained* solution, with $b_{t+1} = 0$ for all t, reverts back to the pure life-cycle solution—with no links between the generations. To solve the *unconstrained* case, with $b_{t+1} > 0$ for all t, we begin by writing out the lifetime budget constraints for the first two generations

$$c_{1t} + \frac{c_{2t+1}}{R} + nb_{t+1} = w + Rb_t \tag{2.19a}$$

$$c_{1t+1} + \frac{c_{2t+2}}{R} + nb_{t+2} = w + Rb_{t+1}. \tag{2.19b}$$

Solving for b_{t+1} in (2.19a), substituting into (2.19b), and rearranging gives

$$c_{1t} + \frac{c_{2t+1}}{R} + \left(\frac{n}{R}\right)\left[c_{1t+1} + \frac{c_{2t+2}}{R}\right] + \left(\frac{n}{R}\right)nb_{t+2} = Rb_t + w + \left(\frac{n}{R}\right)w. \tag{2.20}$$

Notice that we could solve (2.20) for b_{t+2} and substitute the solution into the generation $t + 2$ version of (2.19a, b) to get a version of (2.20) for *three* generations of the family.

Proceeding in this way, by successively substituting into the lifetime budget constraints of future generations, produces a budget constraint for the entire "family dynasty,"

$$c_{1t} + \frac{c_{2t+1}}{R} + \left(\frac{n}{R}\right)\left[c_{1t+1} + \frac{c_{2t+2}}{R}\right] + \left(\frac{n}{R}\right)^2\left[c_{1t+2} + \frac{c_{2t+3}}{R}\right] + \cdots = W_\infty, \tag{2.21}$$

where $W_\infty \equiv Rb_t + w + \left(\frac{n}{R}\right)w + \left(\frac{n}{R}\right)^2 w + \cdots$. Thus, W_∞ is the wealth of the entire family dynasty, a well-defined value provided that rate of return on assets exceeds the sum of population growth and the growth rate in wages (zero in this case) over time.

The left-hand side of (2.21) can be simplified by using the Euler equation for life-cycle consumption, (2.14b), to get

$$\Psi_1^{-1}c_{1t} + \left(\frac{n}{R}\right)\left[\Psi_1^{-1}c_{1t+1}\right] + \left(\frac{n}{R}\right)^2\left[\Psi_1^{-1}c_{1t+2}\right] + \cdots = W_\infty. \tag{2.22}$$

Next, use (2.17b), the Euler equation that applies to generations, to express consumption for *each* generation in terms of consumption for the *first* generation to get

$$\Psi_1^{-1}c_{1t}\left\{1 + \left(\frac{n}{R}\right)\left[\frac{\beta R}{n}\right]^\sigma + \left(\frac{n}{R}\right)^2\left[\frac{\beta R}{n}\right]^{2\sigma} + \cdots\right\} = W_\infty. \tag{2.23}$$

The geometric sum in the curly brackets is finite provided $\beta^\sigma\left(\frac{n}{R}\right)^{1-\sigma} < 1$ or $\beta^{\sigma/(1-\sigma)} \leq \frac{R}{n}$. If $R > n$, then this condition holds if $\sigma \leq 1$ and $\beta \leq 1$ Under these conditions, Eq. (2.22) then gives us the solution

$$c_{1t} = \Psi_1 \left(1 - \beta^{\sigma} \left(\frac{n}{R} \right)^{1-\sigma} \right) W_{\infty}, \tag{2.24}$$

which says the consumption of the current generation is a function of its wealth *and* the wealth of all future generations as well. Once the current generation makes its consumption choice, then by using (2.17b), we can find the consumption of *every* member of the family dynasty. Thus, the current generation determines the family consumption path into the indefinite future. In this sense, the current generation behaves "as if" it is "infinitely lived."

The *infinitely lived agent* model is perhaps the dominant model for studying macroeconomics because it allows one to avoid treating each generation as a distinct household. Being able to ignore the generational structure of the economy is a very handy simplification, especially when you have many generations coexisting in the same period. In Problem 10, you will see that the infinitely lived model allows the entire economy's behavior to be determined by a single *representative agent*.

However, the assumptions that must hold for the infinitely lived agent to be valid are strong. In particular, intergenerational transfers must be strictly motivated by Barro–Becker altruism and financial transfers must be nonzero for all generations. Evidence presented in Sect. 2.6 suggests that these assumptions do not generally hold. Still, for some purposes, the convenience of the infinitely lived agent model may be useful without doing obvious harm to the analysis. For example, to study business cycles, each period must represent a year or even a quarter of a year. Thus, you cannot aggregate time as we have in the two-period life-cycle model. If each period corresponds to a year, then the life cycle of an adult must contain 50–60 periods. This, in turn, means that *many* generations will be present in *each* period of the model. It simplifies the analysis greatly if this generational structure can be ignored. In addition, business cycles do not cause resources to be shifted across generations in any obvious systematic fashion. Here, the argument for the infinitely lived agent model is convincing. For other issues, long-run issues that involve significant transfers of resources across generations, use of the infinitely lived agent model can be very misleading.

It is possible to derive an explicit solution for the optimal financial transfers. This solution adds additional insights and gives a sense of the dramatic implications of assuming intergenerational altruism. Using (2.14b), (2.19a), and (2.24), we get

$$nb_{t+1} = W_t - \left(1 - \beta^{\sigma} \left[\frac{n}{R} \right]^{1-\sigma} \right) W_{\infty}. \tag{2.25}$$

Holding W_{∞} constant, an increase in W_t will raise transfers to children one for one. Note if W_{∞} is constant and W_t increases, then the wealth of future generations must have fallen in present value by the amount that W_t rose (so as to maintain W_{∞}). In this thought experiment, the rise in the wealth of the current generation is entirely at the expense of future generations. The current generation acts to exactly offset the exogenous reallocation of the dynasty's wealth by transferring the entire increase in W_t back to future generations. It does this by increasing transfers to its children. Thus, any exogenous reallocation of dynasty wealth is completely undone by endogenous intergenerational transfers in the opposite direction.

Next consider a rise in W_∞, holding W_t constant. In this thought experiment, intergenerational transfers fall. To see the intuition, if W_∞ increases, holding W_t constant, then one or more future generations in the dynasty must have experienced a rise in wealth. The only way that the current generation can share in the rise in dynastic wealth is by reducing transfers to the future generation, allowing its consumption to rise as indicated by (2.24).

Finally, consider a rise in W_t with no change in the wealth of future generations. In this case, W_∞ rises by the same amount as the rise in W_t. The change in transfers is $\beta^\sigma\left(\frac{n}{R}\right)^{1-\sigma} < 1$ times the change in W_t. Thus, part of the rise in the wealth of the current generation is consumed and part is transferred to future generations to allow their consumption to rise.

2.4.2.2 Endogenous Human Capital

Adjusting the *unconstrained* solution above to allow for endogenous human capital is straightforward. Begin by reintroducing (2.17a), the efficiency condition that requires equal rates of return on human and physical assets. Let us assume a specific form for the human capital production function, $H_{t+1} = \Theta x_{t+1}^\theta$, where $\Theta > 0, 0 < \theta < 1$. Now (2.17a) gives the following explicit solution for x_{t+1}

$$x_{t+1} = \left[\frac{\Theta\theta w}{R}\right]^{\frac{1}{1-\theta}}. \tag{2.26}$$

Notice again that, in the unconstrained case, the solution for human capital investment is simple. It involves equating the returns on the two assets and does not require any knowledge about the generational or dynastic wealth that determines consumption. Higher market wages raise the return to human capital investment, so investment must rise until the rate of return is driven back down to the interest rate. A rise in the interest rate causes a shift away from human capital investment, until its rate of return is driven up to the interest rate.

We complete the unconstrained solution by solving for consumption. Take the solution from (2.26) and define the new variable

$$\hat{w} \equiv w\Theta\left[\frac{\Theta\theta w}{R}\right]^{\frac{\theta}{1-\theta}} - n\left[\frac{\Theta\theta w}{R}\right]^{\frac{1}{1-\theta}}, \tag{2.27}$$

which is the productively efficient lifetime wage income of the current generation minus the investment needed to ensure that the next generation also receives the efficient level of lifetime wages. Next, just substitute $\hat{w}$ for w in the expression for W_∞ in (2.24). This substitution replaces the expression for wages due to exogenous human capital with an expression for wages due to endogenous human capital, net of the expenses of generating human capital across generations. After this substitution, the form of the solution for first-period consumption of the current generation is the same as in the exogenous human capital case. Consumption of subsequent generations then follows from the generational Euler conditions given by (2.17b). Optimal financial transfer behavior will again be given by (2.25).

The *constrained* case is more difficult due to the interaction between household wealth and human capital investment. However, if we specialize a bit more, by assuming $\sigma = 1$, then an explicit solution is still possible. To obtain a solution, we use what is known as a *guess-and-verify* method. First, an educated guess about the form of the solution is made and then the solution is verified to satisfy the Bellman equation. It is similar to a situation where you think you know the answer to an algebraic equation and then you substitute the answer back into the equation to make sure the equation is satisfied.

In order to see the full logic, and the recursive structure of the solution, we will relax the steady-state assumption and allow r_t and w_t to vary across generations. Experience with models assuming logarithmic preferences suggests that a good guess for the value function is

$$V_t(W_t) = E + \left[\frac{1+\beta}{1-\beta\theta}\right] \ln W_t + \beta\left(\Phi_t^R + \left[\frac{1+\beta}{1-\beta\theta}\right]\Phi_t^w\right), \tag{2.28}$$

where E is an undetermined constant, $\Phi_t^R \equiv \sum_{i=1}^{\infty} \beta^{i-1} \ln R_{t+i-1}$ and $\Phi_t^w \equiv \sum_{i=1}^{\infty} \beta^{i-1} \ln w_{t+i}$. Notice that (2.28) bears a family resemblance to the simple indirect utility function, associated with the pure life-cycle model without altruism, given in footnote 6. It is the experience in solving similar problems that form the basis for making guesses in more complicated settings.

To verify that this does indeed represent a solution, go through the steps given in Problem 11. In working out the problem, you will find that the optimal solution for human capital investment is

$$x_{t+1} = \frac{\beta\theta W_t}{n}. \tag{2.29}$$

The constrained solution provides a sharp contrast to the unconstrained solution given by (2.26) for two reasons. First, human capital investment in children now depends on wealth. Second, the relevant wealth concept is once again the wealth of the *current* generation of parents and *not* the wealth of the entire family dynasty. Thus, despite the presence of altruistically motivated investments in children, the family is not connected as completely as in the unconstrained case. The wealth of the current household constrains the investment in its children.

2.4.3 Warm Glow

Another approach to modeling intergenerational transfers is to assume that parents get utility, or a "warm glow," directly from *making* the transfers as opposed to getting utility from the *effects* of those transfers on their children's adult welfare. The warm-glow approach assumes that parents get *direct* satisfaction from taking care of their children and satisfying their sense of duty as good parents, or perhaps from the recognition they get from other adults in doing so. It also allows for the possibility that they enjoy giving one kind of transfer more than another. For example, they may prefer investing in their children's education more than giving them a saving bond, even when the effect on the child's wealth is the same in both cases.

An example of warm-glow preferences is given by the following utility function,

$$U_t = \ln c_{1t} + \beta \ln c_{2t+1} + \psi \ln H_{t+1} + \xi \ln b_{t+1}, \tag{2.30}$$

where ψ and ξ are nonnegative preference parameters. The household values the human capital of children and the financial transfers to children, directly and independently.[7]

Under the warm-glow approach, the household maximizes (2.30) subject to the lifetime budget constraint associated with (2.10) and (2.11), and the human capital production function $H_{t+1} = \Theta x_{t+1}^{\theta}$, to generate the following equations for optimal transfers

$$b_{t+1} = \frac{\xi}{1 + \beta + \theta\psi + \xi} \frac{W_t}{n} \tag{2.31a}$$

$$x_{t+1} = \frac{\theta\psi}{1 + \beta + \theta\psi + \xi} \frac{W_t}{n}. \tag{2.31b}$$

Both transfers are simply fractions of household wealth.

Contrasting (2.31a, b) to the transfer behavior when assuming altruism reveals several important differences. Households always make financial transfers—parents never find it optimal to give no financial transfers to their adult children. In addition, financial transfers are no longer affected by the household wealth of future generations in the family, as they were in (2.25). Under the warm-glow assumption, parents will not attempt to compensate for exogenous changes in their children's wealth.

The presence of financial transfers across generations does not imply that human capital investments are productively efficient as in (2.26). Depending on the wealth of the parents, human capital investments can be less than or *greater than* the efficient level. Some empirical researchers have concluded that marginal educational investments in most primary and secondary school students in the US yield rates of return, in terms of future adult earnings, that are far below the rate of return on financial assets. This observation is inconsistent with the altruistic model, but not

[7] Often, when the focus of the analysis is on intergenerational transfers, the life-cycle feature of the model is dropped. Households are modeled as living only a single period of adulthood in which they consume and make transfers to their children. We will see examples of these models in future chapters.

with the warm-glow model. As indicated, parents may value education for reasons beyond its impact on the wages of its adult children. If household wealth and ψ are sufficiently large, then the optimal human capital investment will exceed the productively efficient level, and the rate of return to educational investment will be below the rate of return on physical assets.

Finally, note that (2.29) and (2.31b) are quite similar. This is an important point. From the perspective of modeling human capital investments in children, assuming warm-glow preferences is similar to assuming altruism when the bequest constraint is binding. A binding bequest constraint is relevant because of the absence of intergenerational loan markets. Thus, warm-glow preferences are a convenient way of capturing the absence of intergenerational loans for human capital investment in children.

2.5 Quantitative Theory

Over the last 30 years, there has been an increasing tendency for macroeconomists to *quantify* their theoretical models. Quantifying a model means determining numerical values for the model's parameters, thereby enabling the model to generate numerical predictions that can be compared to real-world data. This healthy tendency to develop theories that can be quantified has greatly improved the understanding of many different phenomena and has created a progressive scientific paradigm within which to conduct macroeconomic research. Indeed, it is one of the main motivations for writing this book.

In this section, we quantify our simple growth model and compare its predictions to important qualitative patterns we commonly see in the data as economies grow. We are effectively repeating a version of the exercise conducted in the famous article by King and Rebelo (1993). They showed that the standard neoclassical model of physical capital accumulation is not consistent with the pattern of growth rates and interest rates experienced by the USA as it developed.

In most cases, it is not possible to use the traditional econometric approach of parameter estimation (due to a desire to limit the number of variables in the analysis, the nonlinear structure of the model, or the lack of appropriate data). Instead, the model is *calibrated*. That is, parameters are set so as to allow the model to *match* certain *targets*—observations or previously estimated behavioral responses.[8] Once calibrated, the model can generate predictions about the values of

[8] There are differences of opinion about what qualifies as an appropriate target. Some believe that calibration should *not* involve previous econometric estimation. According to this view, all parameters within a model should be set to match particular data points or statistical moments of a data set (sample means, variances, and covariances), but *not* to match econometric estimates found in the literature. Others broaden the targets to include previous statistical estimates of the model's parameters and behavioral responses, *even if the model used in the estimation is not the same as the one used in the calibration*. We are comfortable with either approach. The important point from our perspective is that all quantitative models, *however calibrated*, should be tested by comparing their predictions against observations or statistics *not* used in the calibration process.

variables that were *not* used in the calibration. The predicted values can then be compared against data to assess the model's ability to replicate the real world. Failures to replicate important real-world observations then lead to adjustments in the model, or an entirely new model, that provides a better approximation. The model currently providing the best approximation should be favored to conduct *policy analysis*, where the effects of current and proposed government policies are evaluated. Continually pursuing the best quantitative approximation is the best chance we have of improving our understanding of economies and policies.

Let us make these ideas more concrete by calibrating a simple neoclassical model of physical capital accumulation and then testing its predictions about economic growth. The transition Eq. (2.9) provides the basic model. The equation contains seven exogenous parameters: α, β, δ, σ, d, A, and n. To allow for endogenous growth through increasing physical capital intensity, we will have to start the economy in an initial position that is below its steady state. So, an initial value, k_1, will also have to be determined. Finally, the length of each time period in the model must be chosen. In fact, other parameter values will depend on the time-period choice.

To simplify things at first, let us assume $\sigma = 1$. This assumption yields an explicit transition equation solution for k_t, given k_{t-1} and allows us to compute the economy's growth path with a calculator instead of a computer (although a computer still comes in pretty handy even in this case). The transition equation is simplified to

$$k_t = \left[\frac{\beta}{1+\beta}\right]\frac{(1-\alpha)Ak_{t-1}^{\alpha}}{n(1+d)}. \tag{2.9$'$}$$

Part of the calibration typically involves matching the steady state of the model to certain observations (for example, the interest rate or return to capital). Since all variables in the neoclassical growth model can be related back to k, we will need the steady-state solution of (2.9′),

$$\bar{k} = \left[\frac{\beta A}{1+\beta}\frac{1-\alpha}{n(1+d)}\right]^{\frac{1}{1-\alpha}}. \tag{2.32}$$

In our two-period life-cycle model, the first period is designated the "work period" and the second period the "retirement period." In this setting, it is often assumed that each period lasts 30 years. In comparison with the real world, a 30-year period makes the working life too short and the retirement period too long.

(Footnote 8 continued)
The fact that these "tests" or comparisons are not as formal and refined as traditional hypothesis testing in statistics does not particularly concern us. At this stage in the profession's understanding of macroeconomics, models that even roughly approximate reality are difficult to find. Hopefully, as our approximations become more refined, we will need to worry about more formal testing procedure.

The more periods we allow in the life cycle, the more realistic the model becomes. For example, we could instead assume that *three* twenty-year periods represent a lifetime, with two working periods (40 years) and one retirement period (20 years). However, as you add periods, the model becomes more complicated for reasons that we highlight in Appendix B to this chapter. There, we also show that the complication of keeping track of different generations is a clear disadvantage of using an explicit overlapping generations approach rather than the infinitely lived agent simplification. However, advances in computing are lessening the disadvantage over time. For now, we stick with a two-period model.

With the time period selected, we can begin setting other parameter values. Our application will examine the model's ability to explain growth in the USA from the end of the Civil War through the end of the twentieth century. In applying the model, a useful way to proceed is to create a relatively simple *baseline* calibration and then do a *sensitivity analysis* by examining how results change as we deviate from the baseline calibration or model specification.

The annual rate of population growth actually fell over this historical period, from 2.3 % in the late nineteenth century to about 1 % by the end of the twentieth century (Barro 1997). For the baseline calibration, we set the annual rate of population growth to one percent throughout the entire period. Since our time periods last 30 years, the value for population growth in the model is the one percentage point annual rate of growth compounded for 30 years. The value of n is then chosen to satisfy the equation $n = (1.01)^{30} = 1.3478$.

The capital share of output and income has shown no systematic trend in US history or across countries at different stages of development today (Gollin 2002). We set α to a commonly estimated value of 1/3. The annual rate of depreciation on physical capital is estimated to be in a range between 5 and 10 % (e.g., Stokey and Rebelo 1995). We set the annual rate of depreciation to 7 %. To translate the annual depreciation rate into the depreciation rate over 30 years, think about how much capital remains each year after depreciation occurs. In any given year, the physical capital stock at year's end is 93 % of its value at the beginning of the year. If you start with one unit of capital today, then after 30 years, there would be $1 - \delta = (1 - 0.07)^{30} = 0.93^{30} = 0.1134$ units of capital. So, $\delta = 0.8866$.

Note that we can write worker productivity or output per worker as

$$\frac{Y_t}{M_t} = \frac{Ak_t^{\alpha} D_t M_t}{M_t} = Ak_t^{\alpha} D_t. \tag{2.33}$$

So we can write the ratio of worker productivity in 1990 to worker productivity in 1870 as

$$\frac{(Y/M)_{1990}}{(Y/M)_{1870}} = \left(\frac{k_{1990}}{k_{1870}}\right)^{\alpha} \frac{D_{1990}}{D_{1870}}. \tag{2.34}$$

.

For the baseline case, we arbitrarily set d so that exogenous technical progress explains "half" the economy's growth. The annual rate of growth in labor productivity from 1870 to 1990 was about 1.6 % (Rangazas 2002). If the economy grew at 1.6 % per year over 120 years, then labor productivity was 6.7180 times higher in 1990 than in 1870. In terms of a geometric mean, half of this growth is $6.7180^{1/2} = 2.5919$. The annual rate of technical progress needed to generate this much growth is 0.7968 %. This means that $1 + d = (1.007968)^{30} = 1.2688$, or $d = 0.2688$.

Finally, we set β to match the rate of return to capital. We take the rate of return to capital to be the rate of return on the Standard and Poor's 500 over the twentieth century. The annual real rate of return on this portfolio of stocks averaged 7 % over the twentieth century (Kocherlakota 1996). Due the absence of any trend in the annual rate of return over the twentieth century, we assume that the US economy was close to its steady by the end of the twentieth century. Thus, we have $1 + \bar{r} - \delta = 1.07^{30} = 7.6123$. Using (2.2a) and (2.32), we have

$$\frac{\beta}{1+\beta} = \frac{\alpha}{1-\alpha}\frac{n(1+d)}{\bar{r}}. \tag{2.35}$$

Plugging the calibrated values of the other parameters into (2.35) implies $\beta = 0.1287$.

We still have to set the initial value of k_t. The idea is to set k_1 so that half of the economy's growth is explained by capital accumulation (that portion not explained by technical progress). So, choose k_1 to satisfy

$$\left(\frac{k_5}{k_1}\right)^{\alpha} = 2.5919. \tag{2.36}$$

Since both k values in (2.36) are unknown before the simulation is run, we have to experiment with values for k_1 until we find one that satisfies (2.36).

By assuming that the economy is close to its steady state by 1990, we can get a good guess for k_1 by using (2.36) to write $k_1 = \bar{k}/17.41$. To determine the absolute values of $\bar{k}$ and k_1, we need to set a value for A. This parameter is different than the others because it just scales the level of production. There is no particular reason for us to replicate the *level* of production observed in the real world (even the real-world index numbers for GDP are arbitrary). So, we set A to be one. This implies $\bar{k} = 0.00937$ and, as an initial guess, $k_1 = 0.000538$.

We are now ready to do a historical simulation. Plug the guess for k_1 into (2.9′) and let the model generate values for k_t. Change the guess for k_1 until (2.36) is met. Once finding values for k_t that satisfy (2.36), then compute the predicted interest rates and labor productivity growth rates. The annualized values of predicted interest rates (solid line) and growth rates (dashed line) are displayed in Fig. 2.3.

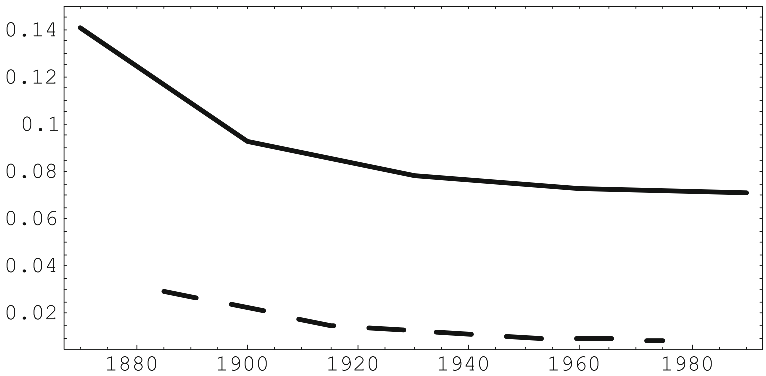


Fig. 2.3 Simulated US interest rates and growth rates: 1870–2000. *Notes* The *solid line* gives the annualized rate of return to capital and the *dashed line* gives the annualized growth rate of labor productivity. The annualized growth rates over 30-year periods were plotted above the midpoint of the intervals between the periods

The model predicts high interest rates (14 %) and growth rates (3 %) for the late nineteenth century and then a decline in both variables over the twentieth century. These predictions miss the mark for a number of reasons.

Returns to capital were probably higher in the late nineteenth century than during the twentieth century. We do not have returns on the Standard and Poor's 500 that go back as far as 1870, but the returns on other assets were 2–6 % points higher in 1870 than in the twentieth century. Wallis (2000, Figure 2), reports that real interest rates on national government debt averaged about 5 % in the first half of the nineteenth century and averaged about 2.5 % during the twentieth century. Barro (1997) reports that real interest rates on commercial paper were 9 % from 1840 to 1880, but averaged about 3 % during the twentieth century. The model predicts initial interest rates out of this range, 7 % points higher than in the twentieth century. Also by 1900, interest rates showed no trend, while the model predicts a downward trend throughout the twentieth century, especially in the first third of the century.

The growth rate predictions are even less accurate. Table 2.1 presents estimates of US labor productivity growth rates for two centuries (Mourmouras and Rangazas 2009). Growth rates showed little trend from 1840 to 2000. In contrast, the model predicts high growth rates in the nineteenth century and then a steady decline.

The fundamental problem with the standard neoclassical growth model is clear. In order to satisfy (2.36), the capital–labor ratio must be set well below its steady-state value in 1870. The relatively low capital–labor ratio produces relatively high returns to capital. The fact that the capital–labor ratio is well below its steady-state value generates high and declining growth rates, as indicated qualitatively by the transition equation diagram in Fig. 2.2.

Table 2.1 Growth rate in output per worker

1820	0.31
1840	1.82
1860	1.32
1880	1.84
1900	1.53
1920	1.40
1940	1.72
1960	2.45
1980	1.58
2000	1.62

Notes The table gives annual growth rates in worker productivity over two centuries of US historical. See Mourmouras and Rangazas (2009) for sources

Sensitivity analysis that allows for minor variations on the theme, such as declining population growth rates or setting $\sigma < 1$, helps a little but not enough to change the basic pattern in Fig. 2.3. These variations help to increase the capital–labor ratio at later stages of development and therefore smooth out growth rates a bit, but we still get a clear declining pattern in growth rates and interest rates (see Problem 13). The fundamental conclusion is that growth due to increased physical capital intensity was not a major cause of growth in the USA over the twentieth century—otherwise, the data would exhibit a clear declining pattern in growth rates and interest rates.

One reaction to this conclusion has been to introduce human capital into growth models. Human capital investments increased dramatically over the twentieth century in the USA and many economists view human capital as an important source of economic growth that is ignored in the standard model. Table 2.2 presents two measures of human capital investment in children: real spending per child in primary and secondary school (x_t) and the fraction of the year spent in school by children ages 0–19 years (e_t). School spending per pupil expanded more than 25-fold since 1870 and time spent in school expanded more than threefold.

The growth implications of increasing human capital investment can be determined by generalizing the human capital production we used in discussing intergenerational transfers. Our previous modeling either treated human capital as an exogenous variable or as an endogenous variable that was a function solely of

Table 2.2 Human capital investments in the USA

Investment	1870	1900	1930	1960	1990
x_{t+1}	1.0	1.8	4.6	9.5	25.2
e_{t+1}	0.09	0.11	0.215	0.29	0.30

Notes School spending is defined as total expenditures per pupil. The first row gives the real spending per pupil for each year divided by the real spending for the 1870 (Rangazas 2002). The time investment data are from Lord and Rangazas (2006) school spending per pupil. The more general specification given below includes student time as an additional input in the human capital production function. It is defined to include the previous treatments of human capital as special cases. The more general human capital production function is given by

$$H_t = D_t^{1-\theta_1} x_t^{\theta_1} e_t^{\theta_2}, \tag{2.37}$$

where θ_1 and θ_2 are constant parameters that capture the effect of increasing school spending and student time on human capital and wages. We assume the parameters satisfy the restriction $0 \leq \theta_1 + \theta_2 \leq 1$. Note that the model of exogenous effective labor supply from Sect. 2.3 can be obtained by setting $\theta_1 = \theta_2 = 0$. The simple model of intergenerational transfers from Sect. 2.4 assumed no technical change $(D_t \equiv 1)$ and a full school-time input in every period $(e_t \equiv 1)$.

To carry out a growth simulation, we need to derive the physical capital transition equation with human capital in the model. First, write (2.37) as $H_t = D_t \tilde{x}_t^{\theta_1} e_t^{\theta_2}$, where $\tilde{x}_t \equiv x_t / D_t$. Next, treat H_{t+1} just as D_{t+1} was treated in deriving the transition equation at the end of Sect. 2.3. The adjusted transition equation, with $\sigma = 1$, is

$$k_t = \left[\frac{\beta}{1+\beta}\right] \frac{(1-\alpha)Ak_{t-1}^{\alpha}}{n(1+d)} \left(\frac{\tilde{x}_{t-1}}{\tilde{x}_t}\right)^{\theta_1} \left(\frac{e_{t-1}}{e_t}\right)^{\theta_2}, \tag{2.38}$$

where $k \equiv K/H$.

For this experiment, we will treat school spending and school time as exogenous inputs (government mandated compulsory public schooling).[9] So we will simply take the values from Table 2.1 and plug them into (2.37) and (2.38). In order to proceed, we still need estimates of θ_1 and θ_2. Based on econometric evidence, and consistency with empirical estimates of the rates of return to human capital investments, reasonable settings for these parameters are $\theta_1 = 0.10$ and $\theta_2 = 0.40$ (see a complete discussion in Rangazas 2002).

The experiment is conducted in the same manner as in the model without human capital. The initial value of k is determined by

$$\left(\frac{k_5}{k_1}\right)^{\alpha} \left(\frac{\tilde{x}_5}{\tilde{x}_1}\right)^{\theta_1} \left(\frac{e_5}{e_1}\right)^{\theta_2} = 2.5919 \tag{2.39}$$

.

Notice a few things about (2.39). As was true for Eq. (2.36), this expression accounts for "half" the growth from 1870 to 1990, with the other half accounted for by exogenous technical progress. The fact that (2.39) contains $\tilde{x}$, rather than x,

---

9 One can think of the financing of public expenditures as coming from a tax on capital or capital income (similar to the property tax used to finance schooling in the USA). When $\sigma = 1$, a capital tax has no effect on saving and the transition equation.

means that we are giving credit to technical progress for explaining some of the rise in school spending and the resulting effect on economic growth. This is a relatively conservative approach to gauge the *causal* effect of school spending on growth, because it is only school spending increases that exceed technical progress that are given credit for *causing* economic growth. Second, physical capital intensity is now getting help from human capital in explaining half the economy's growth over this period. As a result, k_1 will not have to be as far below k_5 in order to satisfy (2.39) as in our first experiment. This feature should help in eliminating the counterfactual predictions given in Fig. 2.3. Finally, note that while human capital investments rose sharply over the period, both types of investments are subject to diminishing returns because θ_1 and θ_2 are both significantly less than one. Thus, there will be two opposing forces on the economy's growth rates—rising rates of human capital investment versus diminishing returns to those investments.

Figure 2.4 presents the model's historical predictions regarding interest rates and growth rates. Both time series have flattened considerably compared to the series in Fig. 2.3. Growth rates now show no trend over the entire period and interest rates no trend over the twentieth century—making both series more consistent with the data.

The slow growth in human capital inputs caused human capital growth to be slow before 1900. As a result, much of the model economy's growth before 1900 was due to rising physical capital intensity. This is clear from the approximately 2 % age point drop in interest rates from 1870 to 1900—within the range of the decline seen in the historical interest rate data. During the twentieth century, physical capital intensity and interest rates showed no trend—in the model and in the data. The explained growth over this period was due to rising human capital investments. However, growth rates in the model showed no trend during the

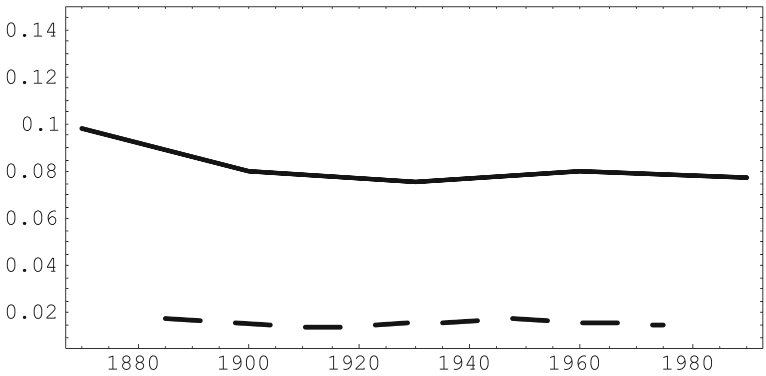


Fig. 2.4 Simulated US interest rates and growth rates: 1870 to 2000—with human capital. *Notes* The *solid line* gives the annualized rate of return to capital, and the *dashed line* gives the annualized growth rate of labor productivity. The annualized growth rates over 30-year periods were plotted above the midpoint of the intervals between the periods

twentieth century, also consistent with the data, because the diminishing returns to human capital investments roughly offset the effect of the rising human capital investments on growth rates.

2.6 Related Literature

The first two-period overlapping generations model with physical capital and production was due to Diamond (1965). It has become one of the two workhorse models of macroeconomics. Unfortunately, most students are not exposed to this model, or any other model built on microeconomic foundations, as undergraduates. If the model does not sound familiar, you may want to read an undergraduate text along side of this book. An excellent intermediate undergraduate textbook treatment of the overlapping generations model is provided by Auerbach and Kotlikoff (1998). Farmer and Schelnast (2013) provide a nice introduction to a graduate treatment of the overlapping generations model, with a special focus on international trade. More advanced and more general treatments of the overlapping generations model can be found in Azariadis (1993) and de la Croix and Michel (2002). These are important books for graduate students who want to concentrate on theoretical work.

The overlapping generation model was extended to include altruistic intergenerational transfers by Barro (1974), Drazen (1978), and Becker (1981, 1988). If the nonnegativity constraint on financial transfers is ignored, altruism provides a justification for infinitely lived agent model, the other workhorse model of macroeconomics. Barro (1997) provides an intermediate undergraduate treatment of macroeconomics using the infinitely lived agent model. A more advanced undergraduate textbook that covers the overlapping generations model, and the extension to include intergenerational transfers is Lord (2001). Graduate treatments of the infinitely lived agent model include Romer (2001) and Acemoglu (2009).

A large empirical literature has developed to test the implications of altruistically motivated transfers. Most econometric tests indicate that either financial transfers are not motivated by altruism alone or altruistic financial transfers are not operative for significant fractions of the population (see, e.g., the papers by Wilhelm (1996) and Altonji et al. (1997). The finding that the intergenerational credit market is imperfect due to binding nonnegativity constraints on financial transfers is consistent with empirical results suggesting that parent's lifetime resources affect the level of investments in their young children (Heckman and Cunha 2007).

There are other reasons, not examined here, for intergenerational transfers. Some transfers are unintended or accidental. Individuals save for their own retirement, but die earlier than expected. The remaining wealth is often left to children. For analysis and applications of models with unintended transfers, start with articles by Davies (1981), Abel (1985), and Hurd (1989). In addition, some transfers may be intentionally made for strategic reasons. In particular, parents might accumulate wealth that is potentially bequeathed to children in order to elicit services from them when they are adults. For more reading on strategic bequests, begin with Bernheim et al. (1985) and Cox (1987).

Calibrated dynamic general equilibrium models were first used in public finance to examine the effects of tax reform (Summers 1981; Auerbach et al. 1983) and in macroeconomics to explain business cycles (Kydland and Prescott 1982). Calibration methods have since been extended to every area of macroeconomics. For a general discussion of the approach, including additional applications, see Kydland and Prescott (1996), followed by a critique from Hansen and Heckman (1996).

The calibration experiment we presented in Sect. 2.5, that uses the model with physical capital to explain economic growth, is based on King and Rebelo (1993). The extension to include human capital is based on Rangazas (2000, 2002). Córdoba and Ripoll (2013) and Manuelli and Seshadri (2014) offer quantitative theories where human capital accumulation is the dominant determinant of an economy's labor productivity. An alternative approach that stresses the connection between TFP and *broad notions* of capital, emphasizing the knowledge embodied in firms rather than individuals, has been developed by Parente and Prescott (2000).

2.7 Exercises

Questions

1. Define the following concepts and give an example of each.
   (a) *technology*
   (b) *capital*
   (c) *physical capital*
   (d) *human capital*
2. Explain the meaning of (2.2a, b). What variables are determined by these two equations?
3. Think of the physical capital stock as fixed, as in the short-run model of the competitive firm from introductory and intermediate microeconomics. Sketch the marginal product of labor as a function of the employment level of an individual firm. Next, add the competitive market wage rate to the diagram. Finally, locate the firm's profit-maximizing employment level. How does the profit-maximizing solution change if there an increase in the firm's capital stock?
4. Explain the differences between *rental rate*, *rate of return on capital*, and *interest rate*.
5. Explain how each of the following affects current consumption, future consumption, and saving: (a) wages, (b) return to capital, and (c) the preference parameter β.
6. Sketch Eq. (2.7) with k on the horizontal axis and r on the vertical axis. Explain what determines the sign of the slope of the sketch. How is the sketch affected by an increase in w? An increase in n? Explain

7. Explain why an increase in this period's capital stock causes an increase in next period's capital stock. Why does the linkage become weaker as the economy accumulates more capital?
8. As the economy moves toward the steady state from below what happens to its growth? What happens to the growth rate if it heads to the steady state from above?
9. Suppose that economy A and economy B have identical structures (production technologies, preferences, and population growth rates), but economy A has a higher capital–labor ratio. Which country is "richer"? Which country grows faster? Your answer explains what is known as *conditional convergence*. Show, by example, that if two countries do *not* have identical structures that *absolute* or *unconditional convergence* is not guaranteed.
10. Explain why a high rate of population growth will make a country poor.
11. How does an increase in exogenous technological change affect capital intensity, interest rates, and labor productivity? In what way is it similar to an increase in population growth and in what way is it different?
12. Give an example of a human capital transfer and a financial transfer. When would a parent's payment of college tuition be equivalent to a financial transfer to the child? Do government guaranteed college loans alleviate the effects of the legal restriction that parents cannot leave debt to their children?
13. If intergenerational transfers are altruistically motivated, explain why human capital investments are productively efficient if financial transfers are positive, but are inefficiently low if financial transfers are zero.
14. Suppose that intergenerational transfers are unconstrained. Intuitively guess what happens to c_{1t} and c_{1t+1} when R_t increases. To make your guess note that (2.17b) has exactly the same form as the expression for the optimal ratio of future to current consumption over the life cycle (see Problem 4 or (14)). Suppose now that households are constrained. Use (2.18b) to guess what happens to c_{1t}, c_{1t+1} and x_{t+1} when w_{t+1} increases.
15. Under special assumptions, Eq. (2.29) gives an explicit solution for constrained human capital investments. Use your answer to question 14 to explain why w_{t+1} does not enter the Eq.
16. Assume that intergenerational transfers are altruistically motivated and that financial transfers are positive. Explain what happens to human capital investments in children and financial transfers if
    (a) W_t increases by one unit and W_∞ is held constant
    (b) W_∞ increases by one unit and W_t is held constant.

Repeat the exercise when (i) financial transfers are zero and then again when (ii) transfers are motivated by warm glow.

17. Explain why (2.29) and (2.31b) are similar despite the fact that they are generated by two different assumptions about intergenerational transfers.
18. Under each of the motives for intergenerational transfers, explain if and when investments in human capital can be greater than the productively efficient level.

19. Explain what it means to calibrate a model. Briefly describe the calibration of the model of physical capital accumulation.
20. Discuss the design and the results of the calibration experiment when the model of physical capital accumulation was used to historically replicate growth in the USA from 1870 to 2000.
21. Discuss the design and the results of the calibration experiment where the model of physical capital accumulation was extended to include human capital.

Problems

1. Show that (2.1) exhibits the neoclassical properties of diminishing marginal productivity and constant returns to scale.
2. Derive Eqs. (2.2a, b) and (2.3a, b). How are they related?
3. What is total income in the model with capital and production? Show that the value of output is equal to the value of income.
4. Derive the optimal life-cycle behavior given by (2.5a, b) and (2.6).
5. Starting from the definition of k^S, derive Eq. (2.7).
6. Starting with the adjusted definition of k^S, derive the transition Eq. (2.9).
7. If $\sigma = 1$, how do we know that the transition equation will be concave in Fig. 2.2? The concavity of the transition function establishes three crucial properties of the steady-state equilibrium: (i) existence (there is a steady state), (ii) uniqueness (there is only one steady state), and (iii) dynamic stability (if you start away from the steady state you will always move toward it). Use the diagram to explain this.
8. Let $\delta = 1$ and $\sigma = \dfrac{2-\alpha}{1-\alpha}$ in (2.9). Derive an explicit transition equation for k_t in terms of k_{t-1}. Sketch the transition equation as accurately as you can.
9. Derive the financial transfer equations under the two motives for intergenerational transfers. In particular, use (2.14b), (2.19a) and (2.24) to derive (2.25). Next, maximize (2.30) subject to the household lifetime budget constraint to get (2.31a).
10. Assuming that (i) intergenerational transfers are altruistically motivated and (ii) the nonnegativity constraint on financial transfers does not bind, derive *aggregate* consumption in period t in terms of the behavior of a "representative agent," whose consumption behavior is a function of the dynasty's wealth. First note that aggregate consumption in period t is $C_t \equiv N_t c_{1t} + N_{t-1} c_{2t} = N_t \left[c_{1t} + \dfrac{c_{2t}}{n} \right]$. Next use (2.14b) and (2.17b) to get $C_t = \kappa N_t c_{1t}$, where $\kappa \equiv 1 + n^{\sigma-1}$ and c_{1t} are given by (2.24). This shows that aggregate consumption is proportional to the consumption of a single representative agent, the current young adult, of the family dynasty.
11. Take a deep breath and complete the following steps to verify that (2.28) and (2.29) are indeed the solution to the constrained problem with human capital when $\sigma = 1$.

(i) Write (2.28) for generation $t + 1$ instead of generation t.
(ii) Substitute your answer to (i) for $V(W_{t+1})$ in (2.13).
(iii) Use (2.5a, b) to write U_t solely in terms of c_{1t} and then in terms of W_t and x_{t+1} by using the lifetime budget constraint associated with (2.10) and (2.11) (remember that $\sigma = 1$ and $b_{t+1} = 0$)
(iv) Now you have the objective function in (2.13) written solely in terms of the choice variable x_{t+1}. Show the first-order condition for maximizing (2.13) with respect to x_{t+1} yields (2.29).
(v) Finally, substitute (2.29) back into the objective function from (iv) and show that $V(W_t)$ is in fact (2.28). This is done by organizing all the constant terms into a term labeled E and then forming three other expressions that group terms involving W_t, human capital rental rates, and interest rates. Careful, this is probably the most difficult part.

12. Using a calculator or computer, reproduce the historical simulations displayed in Fig. 2.3.
13. Use a computer to generate historical simulations for model of physical capital accumulation when $\sigma = 0.5$. You will need a program that includes a nonlinear equation solver to find a numerical solution for k_t given k_{t-1} in each period.
14. An alternative way of thinking about equilibrium is to focus on the goods market rather than the capital market. In the simple model of physical capital accumulation, with no human capital or technological progress, the goods market clearing condition just says that the supply of goods or output, Y_t, must equal the demand for consumption and investment goods, $C_t + K_{t+1} - (1 - \delta)K_t$, so that

$$C_t + K_{t+1} - (1 - \delta)K_t = Y_t,$$

where, as in Problem 10, you must think of C_t as the total consumption of all generations. We can write this as a transition equation in the capital–labor ratio by dividing by M_t and rearranging a bit to get

$$nk_{t+1} = Ak_t^{\alpha} - (1 - \delta)k_t - \frac{C_t}{M_t}.$$

Show that in the two-period overlapping generation model, this condition is equivalent to the transition Eq. (2.8) that was derived from the capital market equilibrium condition.

Appendix*

A: Derivative of the Value Function

Substituting (2.14a, b) back into U_t gives

$$U_t = \frac{[\Psi_{1t}(W_t - n(x_{t+1} + b_{t+1}))]^{1-1/\sigma}}{1 - 1/\sigma} + \beta \frac{[(\beta R_t)^{\sigma}\Psi_{1t}(W_t - n(x_{t+1} + b_{t+1}))]^{1-1/\sigma}}{1 - 1/\sigma}$$
$$= \Psi_{1t}^{-1/\sigma} \frac{(W_t - n(x_{t+1} + b_{t+1}))^{1-1/\sigma}}{1 - 1/\sigma}.$$

Next, denote the optimal choices of the intergenerational transfers with an asterisk so that we can write

$$V_t(W_t) = \Psi_{1t}^{-1/\sigma} \frac{(W_t - n(x^*_{t+1} + b^*_{t+1}))^{1-1/\sigma}}{1 - 1/\sigma} + \beta V_{t+1}\left(w_{t+1}H\left(x^*_{t+1}\right) + (1 + r_{t+1})b^*_{t+1}\right).$$

Note that the optimal intergenerational transfers are in general functions of W_t (in the constrained case, financial transfers are zero and thus do *not* depend on W_t). However, because the derivative of the right-hand side with respect to these choices is zero, then we may ignore these indirect effects of W_t when differentiating V_t with respect to W_t (in the constrained case, the derivative is taken only with respect to human capital investments). Thus, the derivative of the value function with respect to W_t *includes only the direct effect*, working through U_t in the first expression above. This result is a general one and is known as the *envelope theorem*. Equation (2.16) is the envelope theorem applied to $V_{t+1}(W_{t+1})$.

B: Many-Period Models

In models that extend beyond two periods, it is easier to proceed by thinking of the equilibrium or market clearing condition in terms of goods rather than capital. The two ways of thinking about things are actually equivalent, but the exposition is easier if we conduct the discussion in terms of goods (see Problem 14). This is especially true if one wants to contrast the infinitely lived agent approach with the overlapping generations approach. To reduce notation, we will limit the discussion to the simple model of physical capital accumulation with no human capital or technical progress. In addition, the key points can be made for the case with $\sigma = 1$ and $n = 1$.

As shown in Problem 14, using the goods market approach allows the transition equation for the economy to be written as

$$nk_{t+1} = Ak_t^{\alpha} - (1-\delta)k_t - \frac{C_t}{M_t}.$$

In the overlapping generations model with two-period lifetimes, this equation reduces nicely to a first-order difference equation (Problem 14). This is because the right-hand side can be reduced to the consumption behavior of a single generation whose consumption depends only on period t wages and thus only on k_t. The old generation consumes all its income. As a result, their income and their consumption are equal and cancel from the right-hand side of the transition equation.

However, with more periods of life there will be more generations on the right-hand side, whose income is less than their consumption, i.e., who save by purchasing capital. The consumption behavior of these generations will depend on wages earned *before* period t. For example, think of a model with five periods of life—four working periods and one retirement period. The aggregate consumption behavior on the right-hand side sums over five different generations. In general, it is only the generation who is retired that will "cancel out." The saving of other generations will generally not be zero, so all variables that affect their consumption behavior will appear in the transition equation.

Consider the next oldest generation, who is in the last period of its working life. This generation's working life began three periods ago. Their consumption behavior in period t will depend on the wages they earned in each of those periods. Thus, wages as far back at period t-3 will appear in the transition equation. The wages in each period are determined by the capital–labor ratio from that period. This implies that the transition equation will be a *fifth-order* difference equation, including the variables, $k_{t+1}, k_t, k_{t-1}, k_{t-2}, k_{t-3}$. The important point is that the *state variables*, here the capital–labor ratios, characterizing the economy increase as the number of periods of the life-cycle increase. This *curse of dimensionality* raises the computational complexity of the model when the number of periods in the life cycle is large.

In contrast, if one assumes that the generations are linked by intergenerational financial transfers, the transition equation of the economy is a *second-order* difference equation, *no matter how many periods in the life cycle* are included. To see this, first note that as long as financial intergenerational transfers link the generations together, then we can write $C_t = \kappa N_t c_{1t}$ (see Problem 10). As periods in the life cycle are added, only the value of κ changes. For example, $\kappa = 2$, with two periods of life, and $\kappa = 5$, if there are five periods of life.

Next, solve for c_{1t} using the transition equation to get

$$c_{1t} = \frac{M_t}{N_t\kappa}\left[Ak_t^{\alpha} - (1-\delta)k_t - nk_{t+1}\right].$$

Now substitute into (2.17b) to get

$$\left[Ak_{t+1}^{\alpha} - (1-\delta)k_{t+1} - nk_{t+2}\right] = \left[\frac{\beta\left(1 + \alpha A k_{t+1}^{\alpha-1} - \delta\right)}{n}\right]\left[Ak_t^{\alpha} - (1-\delta)k_t - nk_{t+1}\right],$$

This is a *second-order* difference equation in k_t that is completely *independent of the number of periods in the life cycle*. Thus, the infinitely lived agent simplification is able to avoid the dreaded curse of dimensionality. A very nifty and useful result.

The lesson is that if you want to do computations, use the infinitely lived agent approach whenever you can get away with it (e.g., business cycle analysis). Unfortunately, the evidence suggests that for the issues we examine in this book, those that directly focus on private intergenerational transfers and government policies that transfer resources across generations, one must stick with the overlapping generations approach.

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