Preface

Subfactors (unital inclusions of von Neumann algebras with trivial centre) became a thriving focus of research interest after Vaughan Jones discovered in 1983 the quantization of the index below four. The associated principal graph was immediately identified as an important combinatorial invariant beyond the index, controlling the induction and restriction of bimodules of and between the two factors; more detailed information is encoded in the "planar algebra".

There is a close similarity with the theory of superselection sectors in relativistic quantum field theory (QFT), which was developed in the late 1960s and early 1970s by Sergio Doplicher, Rudolf Haag and John E. Roberts in Algebraic Quantum Field Theory (DHR). Especially, the method to obtain the quantization of the index closely resembled the argument for the quantization of the "statistical dimension" in the DHR theory. This involves (independent) works of two of us in 1989: R.L. found the direct link between the statistical dimension and the Jones index, and K.-H.R. (with K. Fredenhagen and B. Schroer) studied the braided tensor categorical superselection structure in low-dimensional quantum field theory.

While the details of the two theories differ (Jones theory addresses type II factors, whereas the local algebras of QFT are generically type III factors), the underlying mathematical structure is in both cases a C* tensor category (or 2-category). In the first case, its objects are bimodules, in the latter case they are endomorphisms (or homomorphisms); but as abstract structures, one deals with "the same" categories. A main purpose of the present work is in fact to "transfer" concepts from abstract tensor categories (going beyond just subfactor theory) into the language of von Neumann algebras and their endomorphisms, see below.

A main aim of the DHR theory in four spacetime dimensions (finally achieved in 1990 by S. Doplicher and J.E. Roberts) was to establish the identification of the category of DHR endomorphisms with a dual of a compact group, which is then the global gauge group of an extended quantum field theory (the "field algebra") containing the original QFT as its gauge fixed points. Crucial for this identification was the existence of a braiding, which is in fact maximally degenerate (i.e., a "permutation symmetry") in the DHR category.

This is markedly different in low-dimensional quantum field theory, notably in chiral and two-dimensional conformal QFT, which also experienced a research boost in the mid-1980s after the complementary breakthrough discoveries of A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov (minimal models) and D. Friedan, Z. Qiu, S. Shenker (classification of positive-energy representations of the Virasoro algebra). Again, the sectors of these theories are described by a braided C* tensor category, but the braiding turned out to be non-degenerate (modular) in most models of interest; a structural argument about why (and when) this is the case was given in 2001 by two of us (Y.K. and R.L.) in collaboration with M. Müger.

Not least for the reason that these structures had been discovered both in the physics context and in connection with quantum groups at about the same time, the focus of mathematical interest concentrated on modular tensor categories, which appear to describe generalized symmetries akin to group symmetries, but placed "at the other end of the range of possibilities" (tensor categories with modular braiding vs. tensor categories with symmetric braiding). Not only classification results were obtained, but relations with different fields of mathematics (vertex operator algebras, algebraic topology, elliptic functions) were discovered and explored.

On the physics side, the idea was put forward to hinge the axiomatic definition of a conformal QFT on its modular tensor category. While the present authors do not entirely conform to this idea (because it would exclude important models), it was certainly very fruitful for the discussion of a large class of interesting models.

A most important insight emerged from the formulation of "topological quantum field theory" (TQFT) in terms of the data of a given modular category, promoted by Jürgen Fuchs, Jürg Fröhlich, Ingo Runkel, Christoph Schweigert et al. (FFRS) in the late 1990s until today. This is the insight that the effect of representation-changing spacetime boundaries is entirely controlled by structures within the modular category: notably modules and bimodules of Frobenius algebras.

Frobenius algebras in a C* tensor category of endomorphisms had also been discovered—under the name of Q-systems—by one of us (R.L.) in 1994 as a complete invariant for type III subfactors $N \subset M$ of finite index. A crucial aspect is that the relevant category is the category of endomorphisms of the *smaller* factor N, so that the *larger* factor M is characterized in terms of data pertaining to N. This changes the perspective from *subfactors* ("N is embedded into a given M") to *extensions* ("M extends a given N"). In fact, for single subfactors, there is a duality (related to the Jones tower) by which an extension $N \subset M$ is equivalently described by a subfactor $\gamma(M) \subset N$ (where γ is a canonical endomorphism of M with values in N), so that this change in perspective seems to be just a matter of taste. However, it becomes crucial in the application to QFT, where N and M have a direct physical meaning while $\gamma(M)$ has not.

It is in this field of research where the interests of the four of us have eventually converged.

Two of us had noticed the relevance of subfactor theory, and in particular the characterization of extensions in terms of Q-systems, for questions like, "Which quantum field theories possibly share the same stress-energy tensor" (or some other common sub-theory)? In chiral conformal QFT, the stress-energy tensor is described by the Virasoro algebra, whose positive-energy representations are known and give rise to well-studied modular C* tensor categories (provided the central charge is $c < 1$). Indeed, full classifications have been obtained along this line.

Especially, the formulation of boundaries and boundary conditions in relativistic conformal QFT, and its relation to the remarkable findings in the TQFT approach by FFRS, have intrigued us. The present work, along with several research papers, is an outcome of the endeavour to gain a better understanding of these connections.

A main difference with the TQFT approach is that a TQFT is essentially *defined* in terms of a modular tensor category (which need not be a C* tensor category), whereas in our feeling, a conformal QFT is in the first place a relativistic quantum field theory with an enhanced symmetry, subject to well-established axioms among which the C* structure (crucial for quantum observables) and local commutativity (Einstein causality) are most essential. In this vein, the presence of the modular category has to be *derived* (via the DHR theory), and its role in the formulation of QFT with boundaries has to be established.

In the present work, we focus on the theory of (modular) C* tensor categories, only keeping the applications to QFT in the back of our minds, and devoting the final chapter to a review of these applications. Large parts of the abstract theory were originally developed by FFRS, involving more recently also L. Kong; our main contribution is to clarify the "transfer" of these results into the language of endomorphisms of von Neumann algebras (which then facilitates the intended application to QFT).

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Nashville, October 2014	Marcel Bischoff
Tokyo	Yasuyuki Kawahigashi
Rome	Roberto Longo
Göttingen	Karl-Henning Rehren

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