Chapter 2
The Space-Time Structure of Extreme Current and Activity Events in the ASEP

Gunter M. Schütz

Abstract A fundamental question in the study of extreme events is whether during a rare and strong fluctuation a system exhibits phenomena that are qualitatively different from its typical behaviour. We answer this question quantitatively for the asymmetric simple exclusion processes (ASEP) on a ring, conditioned to an atypically large particle current or an atypical large hopping activity. We show that this classical problem is related to the integrable quantum Heisenberg ferromagnet. For strongly atypical fluctuations we show that the equal-time density correlations decay algebraically, as opposed to the typical stationary correlations which are short-ranged. We compute the exact dynamical structure factor which shows that that the dynamical exponent in the extreme regime is $z = 1$ rather than the KPZ exponent $z = 3/2$ for typical behaviour. An open problem is the transition point from typical to extreme.

2.1 Introduction

In a many-body system with noisy dynamics intrinsic fluctuations may occur that drive characteristic properties of the system far away from their typical values. An example of this problem, that has attracted great attention in the last decade, are fluctuations of the entropy production and related thermodynamic quantities such as heat and work [1, 2]. Of interest in this context are not only the tails of the probability distribution or the the statistics of extreme events, but particularly the space-time structure of the system undergoing such a rare and intrinsic fluctuation.

Generally, in equilibrium systems, time-reversal symmetry implies that the fluctuation out of an extreme event is the mirror image of the fluctuation that led into it. Unfortunately, little more can be said generally. In systems that are driven permanently out of equilibrium, even less is known. The distribution of the entropy

G.M. Schütz (✉)
Institute for Complex Systems II, Forschungszentrum Jülich, 52425 Jülich, Germany
e-mail: g.schuetz@fz-juelich.de

G.M. Schütz
Interdisziplinäres Zentrum für Komplexe Systeme, Universität Bonn, Brühler Straße 7, 53119 Bonn, Germany

B. Aneva and M. Kouteva-Guentcheva (eds.),
Nonlinear Mathematical Physics and Natural Hazards,
Springer Proceedings in Physics 163, DOI 10.1007/978-3-319-14328-6_2

production satisfies the Gallavotti-Cohen symmetry (or similar relations [1, 2]). However, the absence of time-reversal symmetry does not allow for generally valid predictions of temporal behaviour.

A notable exception from this unfortunate state of affairs are driven diffusive systems, i.e., lattice gas models for stochastic interacting particle systems [5–8]. For example, it could be demonstrated for a specific lattice gas model, the zero-range process with open boundary conditions [9, 10], that a failure of the celebrated Gallavotti-Cohen symmetry [3, 4] of the distribution function for entropy production can arise from a real-space condensation phenomenon [11, 12]. A macroscopic fluctuation theory, based on the seminal papers [13, 14] allows for the computation of macroscopic density profiles during a long event of strongly atypical particle current or hopping activity. Interestingly, for a particular model system, the asymmetric simple exclusion process (ASEP, see below) a dynamical phase transition occurs from a macroscopically flat density profile to a travelling shock/antishock wave (atypically low current in the driven case [15]) or a phase separation arises (low activity in the undriven system [16]). Microscopic information about atypically low currents has recently been obtained for the ASEP [17, 18] by making use of the mapping of the generator of the ASEP to the Heisenberg quantum ferromagnet.

Following [19, 20] we use this approach to consider here the microscopic space-time structure of the ASEP for *large* atypical current, and, going beyond our earlier work, also for atypical activity. We derive detailed information about equal-time correlations, relaxation times and the dynamical structure function, which indicate a qualitative change of the typical dynamics in the universality class of the Kardar-Parisi-Zhang equation with dynamical exponent $z = 3/2$ [21] to a ballistic universality class with $z = 1$ [22] during extreme events of strong current or activity.

2.2 Grandcanonical Conditioning for the ASEP

We proceed to define the model and to exhibit its relationship to the ferromagnetic Heisenberg quantum spin chain. Then we define the conditioned dynamics and illustrate the setting for independent particles.

2.2.1 The Asymmetric Simple Exclusion Process

The asymmetric simple exclusion process (ASEP) [6–8] with periodic boundary conditions is a lattice gas model for a driven diffusive system where each site k on a ring of L sites can be occupied by at most one particle. Particles hop randomly to empty nearest neighbour sites after an exponentially distributed random time with mean $1/(p + q)$. A jump to the right (in clockwise direction) is attempted with probability $p/(p + q)$ and to the left (anticlockwise) with probability $q/(p + q)$. If the target site is already occupied, the jump attempt is rejected. Physically, this models an on-site excluded-volume interaction. We shall set $p = we^{\phi}$ and $q = we^{-\phi}$. Here w plays the role of an attempt frequency of jumps and ϕ is proportional to a driving force that acts on the particles. Without loss of generality we shall assume $\phi > 0$ throughout this article. We denote a microscopic configuration of the ASEP η and the local occupation number by $\eta(k) \in \{0, 1\}$. The total number of particles $N = \sum_k \eta(k)$ is conserved. Originally the model was introduced in a biophysics context to describe the kinetics of biopolymerization on RNA [23, 24] and independently in the probabilistic literature to study the emergence of large scale hydrodynamic behaviour [25].

2.2.1.1 Master Equation

This Markovian jump dynamics can be described in terms of a master equation for the time evolution for the probability $P(\eta, t)$ to find the configuration η at time $t \geq 0$. It is convenient to introduce a column vector $|P(t)\rangle$ which has these 2^L probabilities as components. To this end we assign to a configuration η a canonical basis vector $|\eta\rangle = |\eta(1))\otimes|\eta(1))\otimes\cdots\otimes|\eta(L)) \in (\mathbb{C}^2)^{\otimes L}$ where $|0) = (1, 0)^T$ and $|1) = (0, 1)^T$ are the canonical basis vectors of $\mathbb{C}^2$ and the superscript T denotes transposition [7]. By introducing also a dual basis of row vectors $\langle\eta|$ and inner product $\langle\eta|\eta'\rangle = \delta_{\eta,\eta'}$ we can write $P(\eta, t) = \langle\eta|P(t)\rangle$ with $|P(t)\rangle := \sum_\eta P(\eta, t)|\eta\rangle$. The master equation then takes the form

$$\frac{d}{dt}|P(t)\rangle = -H|P(t)\rangle \tag{2.1}$$

where the off-diagonal matrix elements $H_{\eta',\eta}$ of the generator H are the negative transition rates for transitions from η to η' and the diagonal elements are the inverse sojourn times of a configuration η, i.e., $H_{\eta,\eta} = \sum'_\eta H_{\eta',\eta}$. Notice that this construction implies that the summation vector $\langle s| := \sum_\eta \langle\eta|$ is a left eigenvector of H with eigenvalue 0. This property expresses conservation of probability $d/dt \sum_\eta P(\eta, t) = d/dt\langle s|P(t)\rangle = -\langle s|H|P(t)\rangle = 0$.

The corresponding right eigenvector with eigenvalue 0 is a stationary distribution $|P^*\rangle$ of the process. For periodic boundary conditions and fixed number of particles N this is the uniform distribution that gives equal probability to all microscopic configurations with N particles, independently of the driving force ϕ which is a direct consequence of pairwise balance [26]. From these uniform canonical distributions one can construct also a grand canonical distribution which is uncorrelated, i.e., on each lattice on finds a particle with probability ρ, independent of the occupation of other sites. In the thermodynamic limit these two stationary distributions become equivalent for $\rho = N/L$. The stationary current takes the form $j^* = 2w \sinh(\phi)\rho(1 - \rho)$. The apparent (and unphysical) divergence of the current with the driving force stems from the fact that for convenience we have chosen the time scale of the process to be given by p and q. As will be seen below a physically more natural choice is a normalization by the inverse mean time $p + q = 2w \cosh(\phi)$ of jump attempts of a single particle.

The time-dependent solution of (2.1) has the simple form $|P(t)\rangle = \exp(-Ht)|P_0\rangle$ for an initial distribution $|P_0\rangle := |P(0)\rangle$. In particular, we have for the transition probability $P(\eta_2, t|\eta_1, 0)$ into a configuration η_2, starting from η_1,

$$P(\eta_2, t|\eta_1, 0) = \langle \eta_2 | \mathrm{e}^{-Ht} | \eta_1 \rangle. \tag{2.2}$$

For the expectation of a function $f(\eta)$ we obtain

$$\langle f(t) \rangle := \sum_{\eta} F(\eta) P(\eta, t) = \langle \eta | \hat{f} \mathrm{e}^{-Ht} | P_0 \rangle \tag{2.3}$$

where $\hat{f} = \sum_\eta f(\eta)|\eta\rangle\langle\eta|$ is a diagonal matrix with the values $f(\eta)$ on its diagonal. The real part of eigenvalues of the generator are the inverse relaxation times of the system.

2.2.1.2 Link to Quantum Systems

The point behind choosing the tensor basis is the fact that the generator H of the ASEP takes the form

$$H = -w \sum_{k=1}^{L} \left[\mathrm{e}^{\phi} (\sigma_k^+ \sigma_{k+1}^- - \hat{n}_k (\mathbb{1} - \hat{n}_{k+1})) + \mathrm{e}^{-\phi} (\sigma_k^- \sigma_{k+1}^+ - (\mathbb{1} - \hat{n}_k) \hat{n}_{k+1}) \right] \tag{2.4}$$

with the matrices

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \hat{n} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.5}$$

the two-dimensional unit matrix $\mathbb{1}$ and the notation $\hat{x}_k := \mathbb{1} \otimes \cdots \otimes \hat{x} \otimes \ldots \mathbb{1}$ indicating that the two-dimensional matrix $\hat{x}$ acts non-trivially on the factor k in tensor space, corresponding to site k in the ring. One recognizes in (2.4) the quantum Hamiltonian of the spin-1/2 Heisenberg ferromagnet with an imaginary Dzyaloshinsky-Moriya interaction term [27, 28]. This is an integrable model that can be solved with the Bethe ansatz. The form of the master equation (2.1) and of the generator (2.4) has given this tensor basis approach the name quantum Hamiltonian formalism. It allows the application of mathematical techniques borrowed from quantum mechanics to treat this problem of classical stochastic dynamics.

For future purposes we split H into three parts

$$H = H^+ + H^- + H^0 \tag{2.6}$$

where $H^+ = -w\mathrm{e}^{\phi}\sum_k \sigma_k^+\sigma_{k+1}^-$ generates jumps to the right, $H^- = -w\mathrm{e}^{-\phi}\sum_k \sigma_k^-$ σ_{k+1}^+ generates jumps to the left and H^0 is the diagonal part for the conservation of probability.

2.2.1.3 Non-interacting Particles

For reference purposes we also consider non-interacting particles. In this case each lattice site can be occupied by an arbitrary integer number $\eta(k) \in \mathbb{N}$ of particles. The generator H takes form [29, 30]

$$H = -w\sum_{k=1}^{L}\left[\mathrm{e}^{\phi}(\hat{a}_k^-\hat{a}_{k+1}^+ - \hat{n}_k) + \mathrm{e}^{-\phi}(\hat{a}_k^+\hat{a}_{k+1}^- - \hat{n}_{k+1})\right] \tag{2.7}$$

where the infinite-dimensional local hopping matrices have matrix elements $(\hat{a}^+)_{ij} = \delta_{i,j+1}$, $(\hat{a}^-)_{ij} = i\delta_{i+1,j}$ (with $i, j \in \mathbb{N}$) and with the diagonal number operator $(\hat{n})_{ij} = i\delta_{i,j}$. Notice that here a^+ (a^-) creates (annihilates) a particle. In the single-site basis one has $\hat{a}^+|k) = |k+1)$ $\forall k \in \mathbb{N}$ and $\hat{a}^-|0) = 0$, $\hat{a}^-|k) = k|k-1)$ $\forall k \geq 1$. The number operator is given by $\hat{n} = \hat{a}^+\hat{a}^-$. These operators commute at different sites and satisfy the harmonic oscillator algebra $[\hat{a}_k^-, \hat{a}_l^-] = [\hat{a}_k^+, \hat{a}_l^+] = 0$, $[\hat{a}_k^-, \hat{a}_l^+] = \delta_{k,l}$ for the same site. In quantum language (2.7) is the Hamiltonian for non-interacting bosons hopping on a lattice under the influence of an driving field with imaginary amplitude, analogous to the Dzyaloshinsky-Moriya interaction. Also the generator (2.7) can be split naturally into three parts analogous to (2.6).

The ground state with eigenvalue 0, corresponding to the stationary distribution of the system, is the projection on N particles of the grand canonical factorized distribution where on each lattice site the number of particles is Poisson distributed with parameter ρ. Here ρ is the average particle density. The factorization property of the grand canonical distribution implies the absence of density correlations between different sites. The dynamics of fluctuations can be studied by considering the dynamical structure function $S(r, t) = \langle\eta(k + r, t)\eta(k, 0)\rangle - \rho^2$ where the expectation is taken in the stationary distribution. Since the particles are non-interacting, the dynamical structure function satisfies a lattice diffusion equation with a constant drift term. On large space and time scales its solution is the Gaussian which is invariant under dynamical scaling $r \to ar$, $t \to a^z t$ where $z = 2$ is the dynamical exponent of the diffusive universality class.

2.2.2 Grandcanonically Conditioned Dynamics

The master equation describes the evolution of the probability distribution of the configurations η, but does not provide any information about the number of jumps that have occurred to reach a given final configuration at some time T. To describe also these properties of the process we introduce $J^{\pm}(T)$ as the number of jumps to the right (left) up to time T and also the integrated current $J(T) := J^{+}(T) - J^{-}(T)$ and the integrated activity $A(T) := J^{+}(T) + J^{-}(T)$. These are random numbers with initial value 0 at time 0 that depend on the particular realization of the stochastic dynamics.

2.2.2.1 Joint Generating Function

Following [1, 31] the joint generating function $Y(\lambda, \mu, T) = \langle \exp(\lambda J(T) + \mu A(T)) \rangle$ for the distribution of A and J is given by

$$Y(\lambda, \mu, T) = \langle s | \mathrm{e}^{-\tilde{H}(\lambda,\mu)T} | P_0 \rangle. \tag{2.8}$$

Here

$$\tilde{H}(\lambda, \mu) = \mathrm{e}^{\lambda+\mu} H^{+} + \mathrm{e}^{-\lambda+\mu} H^{-} + H^{0} \tag{2.9}$$

which in the case of the ASEP is also an integrable quantum Heisenberg ferromagnet with imaginary Dzyaloshinsky-Moriya interaction. For non-interacting particles one has a similar expression with the generator (2.7).

Notice that the generating function is by definition the average over all final microscopic configurations η and all realizations of the process with final values $J(T) = J$ and $A(T) = A$. This generating function is formally analogous to a grandcanonical partition function where the intensive variables λ and μ are conjugate to the extensive variables J and A (proportional to time T and length L).

Analogously we can study grandcanonically conditioned expectations of functions $f(\eta)$ of a configuration η. These are the quantities

$$\langle f(T) \rangle_{P_0}^{\lambda,\mu} := \langle s | \hat{f} \mathrm{e}^{-\tilde{H}(\lambda,\mu)T} | P_0 \rangle / Y(\lambda, \mu, T) \tag{2.10}$$

In particular, for $f(\eta) = \mathbf{1}_{\eta}$ which is represented by the projector $\hat{f} = |\eta\rangle\langle\eta|$ we find for the grand-canonically conditioned probability distribution $P^{\lambda,\mu}(\eta, T) := \langle \eta | \mathrm{e}^{-\tilde{H}(\lambda,\mu)T} | P_0 \rangle / Y(\lambda, \mu, T)$. Therefore the fundamental quantity of interest is the weighted probability distribution

$$|P^{\lambda,\mu}(T)\rangle := \mathrm{e}^{-\tilde{H}(\lambda,\mu)T} | P_0 \rangle. \tag{2.11}$$

In the limit $T \to \infty$ we have asymptotically

$$\mathrm{e}^{-\tilde{H}(\lambda,\mu)T} \sim \frac{|g\rangle\langle g|}{\langle g|g\rangle} \mathrm{e}^{-g(\lambda,\mu)T} \tag{2.12}$$

where $g(\lambda, \mu)$ is the lowest eigenvalue of $\tilde{H}(\lambda, \mu)$ and $|g\rangle$ ($\langle g|$) is the corresponding right (left) eigenvector whose components we denote by $g^R_{\lambda,\mu}(\eta)$ ($g^L_{\lambda,\mu}(\eta)$). Since $\tilde{H}(\lambda, \mu)$ is in general not symmetric the notion of the lowest eigenvalue refers to the lowest real part.

2.2.2.2 Optimal Paths

With this approach one can study also how the particle configuration behaves at intermediate times t within the conditioning time interval $[0, T]$. This yields the answer to the question how the *typical* time evolution of an *untypical* fluctuation is realized, or, in other words, what is the optimal path that a random variable takes under conditioned dynamics. The conditional expectation of a one-time observables $f(\eta)$ at time t is given by

$$\langle f(t)\rangle^{\lambda,\mu,T}_{P_0} := \langle s|\mathrm{e}^{-\tilde{H}(\lambda,\mu)(T-t)}\hat{f}\mathrm{e}^{-\tilde{H}(\lambda,\mu)t}|P_0\rangle/Y(\lambda, \mu, T) \tag{2.13}$$

For long conditioning period $T \to \infty$ we define edge intervals $[0, u]$ and $[T - v, T]$ and consider $t \in [u, T - v]$. In the limit $u, v \to \infty$ we use (2.12) to find that $\langle f(t)\rangle^{\lambda,\mu,T}_{P_0} \to \langle g|\hat{f}|g\rangle/\langle g|g\rangle =: \sum_\eta f(\eta)P^*_{\lambda,\mu}(\eta)$ is independent of t inside the observation window and also independent of the initial distribution. The interpretation is that between an initial transient period and a final transient period the conditioned system is in a stationary state with stationary conditional distribution

$$P^*_{\lambda,\mu}(\eta) = g^R_{\lambda,\mu}(\eta)g^L_{\lambda,\mu}(\eta)/Z(\lambda, \mu). \tag{2.14}$$

Here $Z(\lambda, \mu) = \langle g|g\rangle = \sum_\eta g^R_{\lambda,\mu}(\eta)g^L_{\lambda,\mu}(\eta)$ is the normalization factor.

For two observables $f_a(\eta)$, $f_b(\eta)$ at different times $t_1, t_2 \in [u, T - v]$ with $t_1 \le t_2$ one finds in the limit $u, v \to \infty$

$$\langle f_b(t_2) f_a(t_1)\rangle^{\lambda,\mu,T}_{P_0} \to \langle g|\hat{f}_b\mathrm{e}^{-[\tilde{H}(\lambda,\mu)-g(\lambda,\mu)]\tau}\hat{f}_a|g\rangle/Z(\lambda, \mu) \tag{2.15}$$

with $\tau = t_2 - t_1$. As expected from a stationary process, the two-time correlation function depends only on the time difference τ.

2.2.2.3 Effective Dynamics

Defining the diagonal matrix $\Delta(\lambda, \mu)$ with the components $g^L_{\lambda,\mu}(\eta)$ of the *left* eigenvector on the diagonal and defining the transformed Hamiltonian

$$G = \Delta\tilde{H}\Delta^{-1} - g \tag{2.16}$$

we can rewrite (2.15) as

$$\langle f_b(t_2) f_a(t_1)\rangle_{P_0}^{\lambda,\mu,T} \to \langle s|\hat{f}_b \mathrm{e}^{-G(\lambda,\mu)\tau} \hat{f}_a|P^*_{\lambda,\mu}\rangle. \tag{2.17}$$

In other words, the conditioned two-time correlation function turns into the stationary correlation function of an effective process given by G. This effective process is a dynamics under which the atypical, conditioned dynamics of the original process become unconditioned, typical dynamics [32]. It realizes an optimal path (in the sense described above) as typical path. The stationary distribution of this effective process is given by (2.14).

2.2.3 Conditioned Dynamics in the Noninteracting Case

It is instructive to apply the grandcanonical conditioning to the case of non-interacting particles. Since in this case H^+, H^- and H^0, defined in (2.7) through (2.9), all mutually commute, all eigenvectors are independent of μ and λ. Because of the harmonic oscillator algebra all terms can be diagonalized simultaneously by Fourier transformation (see e.g. [7] for details). In terms of the Fourier modes p one obtains $\tilde{H} = \sum_p \varepsilon(p)\hat{b}_p^+\hat{b}_p^-$ where the momenta p are of the form $p = 2\pi m/L$ with $m \in \{0, 1, 2, \ldots, L-1\}$ and the summation over all p amounts to a summation over all m. For the single-particle energy one has

$$\varepsilon(p) = w\left[2\cosh\phi - \mathrm{e}^{\mu}(\mathrm{e}^{\lambda+\phi}\mathrm{e}^{ip} + \mathrm{e}^{-\lambda-\phi}\mathrm{e}^{-ip})\right] \tag{2.18}$$

The N-particle eigenstates are of the form $\hat{b}_{p_1}^+ \ldots \hat{b}_{p_N}^+ |0\rangle$ where $|0\rangle$ is the vacuum state with no particles. The corresponding eigenvalues are the sum of the single particle energies with momenta p_i. Hence the lowest eigenvalue in the N-particle sector is obtained for choosing all momenta to be 0 which yields

$$g(\lambda,\mu) = Nw\left[2\cosh\phi - \mathrm{e}^{\mu+\lambda+\phi} - \mathrm{e}^{\mu-\lambda-\phi}\right]. \tag{2.19}$$

This result allows us to describe the effective conditioned dynamics. Since the ground state does not depend on μ and λ we have that the transformation matrix Δ is the unit operator. Hence

$$G = \tilde{H} - g = -w\mathrm{e}^{\mu}\sum_{k=1}^{L}\left[\mathrm{e}^{\lambda+\phi}(\hat{a}_k^-\hat{a}_{k+1}^+ - \hat{n}_k) + \mathrm{e}^{-\lambda-\phi}(\hat{a}_k^+\hat{a}_{k+1}^- - \hat{n}_{k+1})\right] \tag{2.20}$$

which is similar to the original process (2.7), but with renormalized hopping rates

$$\tilde{p} = \mathrm{e}^{\lambda+\mu}p, \quad \tilde{q} = \mathrm{e}^{-\lambda+\mu}q. \tag{2.21}$$

Therefore conditioning on higher than typical activity ($\mu > 0$) corresponds to a higher frequency $\tilde{w} = we^{\mu}$ for jumps. Conditioning on higher than typical current ($\lambda > 0$) corresponds to a stronger driving force $\tilde{\phi} = \phi + \lambda$. All other fluctuations in the dynamics remain unchanged. Therefore, non-interacting particles conditioned on high activity and/or current behave essentially like under typical conditions, except that jumps occur with higher frequency and the hopping bias is stronger. Phrased differently, one can generate the effective dynamics, where the untypical extreme behaviour of the original dynamics becomes typical, just by changing the jump frequency and the driving field. Conditioning on extreme behaviour does not lead to any change in universal properties of the dynamics. Long-range correlations in the stationary distribution remain absent and one has diffusive relaxation with dynamical exponent $z = 2$.

It is natural to define the intrinsic time scale of the process by normalizing by the inverse sum of the hopping rates, i.e., the mean sojourn time of a particle. Then the normalized effective dynamics becomes independent of μ, i.e., conditioning on untypical activity does not change the normalized dynamics. In the limit of high current ($\lambda \to \infty$) the hopping becomes totally asymmetric.

2.3 Results for the ASEP

Even though the Hamiltonian (2.6) is exactly solvable via Bethe ansatz it is very hard to extract for general λ and μ explicit results for the weighted distribution (2.11) for finite time T or the stationary correlations (2.17) in the infinite-time limit. Nevertheless, some special cases can be studied in some detail.

2.3.1 Bethe Ansatz Equations

In order to obtain the Bethe ansatz equations for the spectrum of $\tilde{H}$ we introduce new notation. Instead of labelling basis vectors by occupation number we choose the particle positions which we shall denote by k_i mod L for the ith particle and by $\mathbf{k} = \{k_1, \ldots, k_N\}$ the ordered set of all coordinates. The particle label $i \in \{1, 2, \ldots, N\}$ is associated with the particles whose order remains preserved in the time evolution. We also introduce $\mathbf{z} = \{z_1, \ldots, z_N\}$ where the z_i can be thought of as exponentials of (possibly complex) pseudomomenta and the quantities

$$a_{ij} = \tilde{p} + \tilde{q} z_i z_j - (p + q) z_i. \tag{2.22}$$

Notice the appearance of the modified rates (2.21) in this definition.

The Bethe ansatz for the right eigenvectors $|\mathbf{z}\rangle$ (see e.g. [19, 28, 31] for the present context) is given by

$$|\mathbf{z}\rangle = \sum_{1\leq k_1<k_2<k_3\leq L} \sum_{\sigma\in S_N} A_\sigma\, z_{\sigma(1)}^{k_1} z_{\sigma(2)}^{k_2} \ldots z_{\sigma(N)}^{k_N} |\mathbf{k}\rangle = \sum_{1\leq k_1<k_2<k_3\leq L} Y(\mathbf{k})|\mathbf{k}\rangle \tag{2.23}$$

where the second sum is over all permutations σ of the N particle labels and the coefficients A_σ are given by

$$\frac{A_{\ldots\sigma(ij)\ldots}}{A_{\ldots ij\ldots}} = -\frac{a_{ji}}{a_{ij}} = -\frac{p\mathrm{e}^{\lambda+\mu} + q\mathrm{e}^{-\lambda+\mu} z_i z_j - (p+q) z_j}{p\mathrm{e}^{\lambda+\mu} + q\mathrm{e}^{-\lambda+\mu} z_i z_j - (p+q) z_i}. \tag{2.24}$$

Periodic boundary conditions leads to a quantization condition: The Bethe roots z_j satisfy the Bethe ansatz equations

$$z_k^L = (-1)^{N-1} \prod_{i=1}^{N} \frac{p\mathrm{e}^{\lambda+\mu} + q\mathrm{e}^{-\lambda+\mu} z_i z_k - (p+q) z_k}{p\mathrm{e}^{\lambda+\mu} + q\mathrm{e}^{-\lambda+\mu} z_i z_k - (p+q) z_i} \tag{2.25}$$

for arbitrary N. The eigenvalue $\varepsilon(\mathbf{z})$ of a Bethe eigenvector is a sum of single-particle excitation energies

$$\varepsilon(\mathbf{z}) = \sum_{i=1}^{N} \varepsilon(z_i) \tag{2.26}$$

where $\varepsilon(z_i) = -z_i a_{ii}(z^{-1})$ (cf. (2.18) with the identification $z = \mathrm{e}^{ip}$). The rescaled single-particle energies read

$$\tilde{\varepsilon}(z) = \frac{2\mathrm{e}^{-\mu}\cosh(\phi) - \mathrm{e}^{\phi+\lambda} z - \mathrm{e}^{-\phi-\lambda} z^{-1}}{\mathrm{e}^{\phi+\lambda} + \mathrm{e}^{-\phi-\lambda}}. \tag{2.27}$$

For typical behaviour $\lambda = \mu = 0$ the Bethe ansatz equations (2.25) have been analyzed in [27, 28]. It turns out that the real part of the energy gap, which yields the inverse of the longest relaxation time, scales with system size as L^{-z} with the dynamical exponent $z = 3/2$ of the Kardar-Parisi-Zhang (KPZ) universality class [21]. Therefore the exclusion interaction changes the dynamical universality class from diffusive (in the non-interacting case) to KPZ. The stationary distribution, however, is uncorrelated, as is the case for non-interacting particles.

2.3.2 Stationary State for High Activity or High Current

In the limit of high activity $\mu \to \infty$ or high current $|\lambda| \to \infty$ the Bethe equations (2.25) simplify considerably. The right hand side reduces to the factor $(-1)^{N-1}$ which means that the Bethe roots are of the form $z_k = \mathrm{e}^{2\pi i m_k/L}$ where m_k is either integer (N odd) or half integer (N even). As pointed out in [19] the model becomes a free fermion system. The N-particle wave function $Y(\mathbf{k})$ becomes a Slater determinant

$$Y(\mathbf{k}) = \det \begin{vmatrix} z_1^{k_1} & z_1^{k_2} & \dots & z_1^{k_N} \\ z_2^{k_1} & z_2^{k_2} & \dots & z_2^{k_N} \\ \dots & \dots & \dots & \dots \\ z_N^{k_1} & z_N^{k_2} & \dots & z_N^{k_N} \end{vmatrix}. \tag{2.28}$$

Following [19] the ground state corresponds to the choice of Bethe roots $m_k = k - (N-1)/2$ with $k = 0, 1, 2, \dots, N-1$. The stationary distribution can then be expressed in the form of a double product

$$P_L(\mathbf{k}) = \frac{2^{N(N-1)}}{L^N} \prod_{1 \le i < j \le N} \sin^2 \left(\pi \frac{k_i - k_j}{L} \right) \tag{2.29}$$

From this one obtains the well-known expression [33] for the static two-point density correlation for the particle occupation numbers

$$S(r) := \langle \eta(k)\eta(k+r) \rangle - \rho^2 = -\frac{\sin^2 r\pi\rho}{r^2\pi^2}. \tag{2.30}$$

Remarkably, the correlations decay algebraically, unlike for typical dynamics where the stationary distribution is uncorrelated, or in the non-interacting case where also the conditioned stationary distribution is uncorrelated.

For the stationary current per site we find after rescaling of the time scale

$$j^* = \frac{\tanh(\tilde{\phi}) \sin(\pi\rho)}{L \sin(\pi/L)} \tag{2.31}$$

with $\tilde{\phi} = \phi + \lambda$. In this quantity another interesting feature appears: The finite-size corrections are of order $1/L^2$ rather than of order $1/L$ which is expected from typical behaviour in systems with short-range interactions. In the thermodynamic limit $L \to \infty$ we get

$$j^* = \frac{1}{\pi} \tanh(\tilde{\phi}) \sin(\pi\rho) \tag{2.32}$$

2.3.3 Dynamical Properties

The relaxational behaviour is encoded in the spectrum of $\tilde{H}$, which can be computed using the free fermion structure of the process conditioned on large activity $\mu \to \infty$.

In order to do so we rescale time by the effective single-particle sojourn time and adapt the approach of [19] to the present case where

$$\tilde{H} = -\frac{1}{2\cosh\tilde{\phi}} \sum_{k=1}^{L} \left[e^{\tilde{\phi}} \sigma_k^+ \sigma_{k+1}^- + e^{-\tilde{\phi}} \sigma_k^- \sigma_{k+1}^+ \right] \tag{2.33}$$

In the limit $\lambda \to \infty$ we recover the case of large current studied in [19, 20].

2.3.3.1 Longest Relaxation Time

From the ground state choice of the Bethe roots one obtains the lowest excited state by exchanging the root with $m = 0$ with $m = -1$. The real part of the spectral gap, i.e., the inverse of the longest relaxation time τ_L in a finite system of size L is independent of $\tilde{\phi}$ and given by

$$1/\tau_L = 2\sin(\pi\rho)\sin\left(\frac{\pi}{L}\right) \propto 1/L \tag{2.34}$$

where $\rho = N/L$ is the particle density.

For large L the gap is inversely proportional to the system size, unlike in the unconstrained ASEP where the real part of the spectrum gap scales as $O(1/L^{3/2})$ [27, 28] with the dynamical exponent $z = 3/2$ of the KPZ universality class. We conclude that the conditioned dynamics is in a different dynamical universality class, characterized by a dynamical exponent $z = 1$ and first studied by Spohn [22] in the context of the related model with long-range interactions. Indeed, generalizing the work of [19] it is readily seen that the generator G of the effective process in the symmetric case $\tilde{\phi} = 0$ is identical with the quantum Hamiltonian of Spohn. As pointed out in that work this symmetric case can be interpreted classically as a system of non-intersecting random walks or quantum mechanically as a lattice model of Dyson's Brownian motion of the eigenvalues of a random matrix [34]. Non-intersecting random walks appear also in the study of diffusive pair annihilation processes and many different techniques (see e.g. [35–38] and references therein) allow for a detailed analysis of this problem.

2.3.3.2 Dynamical Structure Function

This ballistic universality class can be studied in terms of the dynamic structure factor which is defined as the Fourier transform of the time-dependent stationary correlation function $S_{L,N}(r,t) = \langle \eta(k+r,t)\eta(k,0)\rangle - \rho^2$. In order to compute this quantity we follow the approach of [20]. We introduce the Fourier transform

$$\hat{S}_{L,N}(p,t) = \sum_{r=0}^{L-1} \mathrm{e}^{-2\pi i p n/L} S_L(r,t) \tag{2.35}$$

which has the particle-hole symmetry, i.e., $\hat{S}_{L,L-N}(p,t) = \hat{S}_{L,N}(-p,t)$. Therefore we can restrict the computation to the case $0 \leq \rho \leq 1/2$. After some computation one finds from the free-fermion property

$$\begin{aligned}\hat{S}_{L,N}(p,t) = &\frac{1}{L}\sum_{k=0}^{N-1}\left[\mathrm{e}^{(\varepsilon_k - \varepsilon_{k+p})t} - \mathrm{e}^{-(\varepsilon_k - \varepsilon_{k-p})t}\right] \\ &+ \frac{1}{L}\sum_{k=0}^{N-1}\sum_{l=N}^{L-1} \mathrm{e}^{-(\varepsilon_k - \varepsilon_l)t}\delta_{p,k-l}\end{aligned} \tag{2.36}$$

where in contrast to [20]

$$\varepsilon_k = -\frac{1}{2\cosh\tilde{\phi}}\left(\mathrm{e}^{\tilde{\phi}}\mathrm{e}^{-i\alpha_k} + \mathrm{e}^{-\tilde{\phi}}\mathrm{e}^{i\alpha_k}\right) \tag{2.37}$$

with

$$\alpha_k = \frac{2\pi}{L}\left(k - \frac{N-1}{2}\right). \tag{2.38}$$

This yields

$$\begin{aligned}\varepsilon_k - \varepsilon_{k-p} &= -\frac{\mathrm{e}^{\tilde{\phi}}\left(1 - \mathrm{e}^{\frac{2\pi i p}{L}}\right)\mathrm{e}^{-i\alpha_k} + \mathrm{e}^{-\tilde{\phi}}\left(1 - \mathrm{e}^{\frac{-2\pi i p}{L}}\right)\mathrm{e}^{i\alpha_k}}{\cosh\tilde{\phi}} \\ &= \left(1 - \cos\left(\frac{2\pi p}{L}\right)\right)\cos\alpha_k + \sin\left(\frac{2\pi p}{L}\right)\sin\alpha_k \\ &\quad + i\tanh\tilde{\phi}\left[\sin\left(\frac{2\pi p}{L}\right)\cos\alpha_k - \left(1 - \cos\left(\frac{2\pi p}{L}\right)\right)\sin\alpha_k\right]\end{aligned} \tag{2.39}$$

Taking the thermodynamic limit $L \to \infty$ with density $\rho = N/L$ fixed turns the sums into integrals as in [20] and thus yields an exact expression valid for all $p \in [-\pi, \pi]$ and $t \geq 0$. In order to explore the large-scale behaviour of the dynamic structure factor we study the behaviour for small momentum p and large times t. To this end we define the scaling variable $u = p^z t$ and the limit $t \to \infty$ with u fixed. Inspection of (2.39) shows that non-trivial scaling behaviour is obtained for $z = 1$, as expected from the scaling of the energy gap (2.34). In this scaling we have $t(1 - \mathrm{e}^{ip}) = -iut$ and therefore

$$\hat{S}(u) = \frac{|u|}{2\pi t}\mathrm{e}^{-iu\tanh\tilde{\phi}\cos\rho\pi - |u|\sin\rho\pi} \tag{2.40}$$

which is valid for all $\rho \in [0, 1]$. We read off the collective velocity

$$v_c = \tanh \tilde{\phi} \cos \rho\pi \tag{2.41}$$

of the lattice gas. Hence for conditioning on high activity we recover the universal scaling function of the ASEP conditioned on high current even for finite current. Conditioned on an atypical current amounts only to a shift in the driving field $\phi \to \tilde{\phi}$. By comparing with (2.32) one sees that one has $v_c = \partial j^*/\partial\rho$ as in lattice gases with static short-range correlations. To our knowledge this is the first verification of this relation for a lattice gas with long-range correlations. The validity is in agreement with the notion [39] that this relation should remain generally valid for static correlations that decay faster than $1/r$.

2.4 Conclusions and Open Questions

The perhaps most significant results of our studies are the emergence of long-range stationary correlations and the change of the dynamical universality class from KPZ to ballistic as one goes from typical to high activity or current. Hence, in a state of extremely high current or activity the ASEP does not behave essentially like "normal", with just upscaled parameter values as is the case for non-interacting particles. It is important to understand whether this is specific for the ASEP (where it can be traced to the underlying non-intersecting random walks) or whether this is a generic phenomenon for driven diffusive systems.

A more specific open problem concerns the location of the phase transition point. Does the ballistic universality class arise for any finite deviation from the typical activity or current, or is some threshold required? This question can be addressed by a careful analysis of the Bethe ansatz equations (2.25) along the lines of [27, 28], since the finite-size scaling of the spectral gap of the generator will reveal the dynamical exponent. Also the answer to this question could be of interest beyond the ASEP.

Acknowledgments Much of what is presented here results from joint work with V. Popkov and D. Simon to whom the author is indebted for many fruitful discussions. This work was supported by Deutsche Forschungsgemeinschaft.

References

1. R.J. Harris, G.M. Schütz, Fluctuation theorems for stochastic dynamics. J. Stat. Mech. P07020 (2007)
2. U. Seifert, Stochastic thermodynamics, fluctuation theorems, and molecular machines. Rep. Prog. Phys. **75**, 126001 (2012)

3. G. Gallavotti, E.G.D. Cohen, Dynamical ensembles in stationary states. J. Stat. Phys. **80**, 931–970 (1995)
4. J.L. Lebowitz, H. Spohn, A gallavotti-cohen-type symmetry in the large deviation functional for stochastic dynamics. J. Stat. Phys. **95**, 333–365 (1999)
5. B. Schmittmann, R.K.P. Zia, in *Phase Transitions and Critical Phenomena*, ed. by C. Domb, J. Lebowitz (Academic Press, London, 1995)
6. T.M. Liggett, *Stochastic Interacting Systems: Contact Voter and Exclusion Processes* (Springer, Berlin, 1999)
7. G.M. Schütz, in *Phase Transitions and Critical Phenomena*, vol. 19, ed. by C. Domb, J. Lebowitz (Academic Press, London 2001), pp. 1–251
8. A. Schadschneider, D. Chowdhury, K. Nishinari, *Stochastic Transport in Complex Systems* (Elsevier, Amsterdam, 2010)
9. M.R. Evans, T. Hanney, 2005 Nonequilibrium statistical mechanics of the zero-range process and related models. J. Phys. A: Math. Gen. **38**, R195–R240 (2005)
10. E. Levine, D. Mukamel, G.M. Schütz, Zero-range process with open boundaries. J. Stat. Phys. **120**(5–6), 759–778 (2005)
11. R.J. Harris, A. Rakos, G.M. Schütz, Breakdown of Gallavotti-Cohen symmetry for stochastic dynamics. Europhys. Lett **75**(2), 227–233 (2006)
12. R.J. Harris, V. Popkov, G.M. Schütz, Dynamics of instantaneous condensation in the ZRP conditioned on an atypical current. Entropy **15**(11), 5065–5083 (2013)
13. B. Derrida, J.L. Lebowitz, E.R. Speer, Large deviation of the density profile in the steady state of the open symmetric simple exclusion process. J. Stat. Phys. **107**, 599–634 (2002)
14. L. Bertini, A. De Sole, D. Gabrielli, G. Jona Lasinio, C. Landim, Macroscopic fluctuation theory for stationary non-equilibrium states. J. Stat. Phys. **107**, 635–675 (2002)
15. T. Bodineau, B. Derrida, Distribution of current in non-equilibrium diffusive systems and phase transitions. Phys. Rev. E **72**, 066110 (2005)
16. V. Lecomte, J.P. Garrahan, F. Van Wijland, Inactive dynamical phase of a symmetric exclusion process on a ring. J. Phys. A: Math. Theor. **45**, 175001 (2012)
17. V. Belitsky, G.M. Schütz, Microscopic structure of shocks and antishocks in the ASEP conditioned on low current. J. Stat. Phys. **152**, 93–111 (2013)
18. V. Belitsky, G.M. Schütz, Antishocks in the ASEP with open boundaries conditioned on low current. J. Phys. A: Math. Theor. **46**, 295004 (2013)
19. V. Popkov, D. Simon, G.M. Schütz, ASEP on a ring conditioned on enhanced flux. J. Stat. Mech. P10007 (2010)
20. V. Popkov, G.M. Schütz, Transition probabilities and dynamic structure factor in the ASEP conditioned on strong flux. J. Stat. Phys. **142**(3), 627–639 (2011)
21. M. Kardar, G. Parisi, Y.-C. Zhang, Dynamic scaling of growing interfaces Phys. Rev. Lett. **56**, 889 (1986)
22. H. Spohn, Bosonization, vicinal surfaces, and hydrodynamic fluctuation theory. Phys. Rev. E **60**, 6411–6420 (1999)
23. J.H. MacDonald, Gibbs and AC Pipkin. Kinetics of biopolymerization on nucleic acid templates, Biopolymers **6**, 1–25 (1968)
24. G.M. Schütz, The Heisenberg chain as a dynamical model for protein synthesis—some theoretical and experimental results. Int. J. Mod. Phys. B **11**(1–2), 197–202 (1997)
25. F. Spitzer, Interaction of Markov processes. Adv. Math. **5**, 246–290 (1970)
26. G.M. Schütz, R. Ramaswamy, M. Barma, Pairwise balance and invariant measures for generalized exclusion processes. J. Phys. A: Math. Gen. **29**(4), 837–843 (1996)
27. L.H. Gwa, H. Spohn, Six-vertex model, roughened surfaces, and an asymmetric spin Hamiltonian. Phys. Rev. Lett. **68**, 725 (1992)
28. D. Kim, Bethe ansatz solution for crossover scaling functions of the asymmetric XXZ chain and the KPZ-type growth model. Phys. Rev. E **52**, 3512–3524 (1995)
29. M. Doi, Second quantization representation for classical many-particle system J. Phys. A: Math. Gen. **9** 1465–1477 (1976)

30. L. Peliti, Path integral approach to birth-death processes on a lattice. J. Phys. France **46**, 1469–1483 (1985)
31. B. Derrida, C. Appert, Universal large deviation function of the Kardar-Parisi-Zhang equation in one dimension. J. Stat. Phys. **94**, 1–30 (1999)
32. D. Simon, Construction of a coordinate Bethe ansatz for the asymmetric simple exclusion process with open boundaries. J. Stat. Mech. P07017 (2009)
33. E. Lieb, T. Schultz, D. Mattis, Two soluble models of an antiferromagnetic chain. Ann. Phys. **16**, 407–466 (1961)
34. F.J. Dyson, A brownian motion model for the eigenvalues of a random matrix. J. Math. Phys. **3**, 1191–1198 (1962)
35. F. Colomo, A.G. Isergin, V.E. Korepin, V. Tognetti, Temperature Correlation functions in the XX0 Heisenberg chain. Theor. Math. Phys. **94**(1), 11–38 (1993)
36. G.M. Schütz, Diffusion-annihilation in the presence of a driving field. J. Phys. A: Math. Gen. **28**(12), 3405–3415 (1995)
37. A. Ayyer, K. Mallick, Exact results for an asymmetric annihilation process with open boundaries. J. Phys. A: Math. Theor. **43**(4), 045003 (2010)
38. M. Katori, H. Tanemura, Complex Brownian motion representation of the Dyson model, Electron. Commun. Probab. **18**(4), 1–16 (2013)
39. R.M. Grisi, G.M. Schütz, Current symmetries for particle systems with several conservation laws. J. Stat. Phys. **145**(6), 1499–1512 (2011)

Springer

http://www.springer.com/978-3-319-14327-9

Nonlinear Mathematical Physics and Natural Hazards
Selected Papers from the International School and Workshop held in Sofia, Bulgaria, 28 November – 02 December, 2013
Aneva, B.; Kouteva-Guentcheva, M. (Eds.)
2015, XXVI, 141 p. 41 illus., 36 illus. in color., Hardcover
ISBN: 978-3-319-14327-9