# A General Geometric Representation of Sphere-Sphere Interactions

**Ho-Kei Chan, Eric B. Lindgren, Anthony J. Stace and Elena Bichoutskaia**

**Abstract** A general geometric representation of sphere-sphere interactions is derived using the bispherical coordinate system. It presents a dimensionless, scaled surface-to-surface separation parameter $s^*$, which is valid for all possible combinations of sphere size and separation distance. The proposed geometric description is not limited to sphere-sphere interactions, but also describes interactions that involve a point particle or a plane. The surface-to-surface separation parameter approaches the limit of $s^* = 1$ if the radii of both spheres are much smaller than the actual surface-to-surface separation distance $s$, i.e. in the limit of two point particles. On the other hand, the geometric limit of $s^* = 0$ corresponds to two planes, namely when the radii of both spheres are much larger than $s$.

**Keywords** Sphere-sphere interactions · Bispherical coordinates · Geometric description · Surface-to-surface separation · Inverse points

## 1 Introduction

Much research has been carried out on pairwise interactions, notably in the area of classical electrostatics [1–7], in which each interacting body is geometrically described as a point particle, a sphere or a plane. As shown in Fig. 1, the geometries include the interaction between a pair of point particles (as described by Coulomb's law in the case of two point charges), the interaction between two spheres, as well

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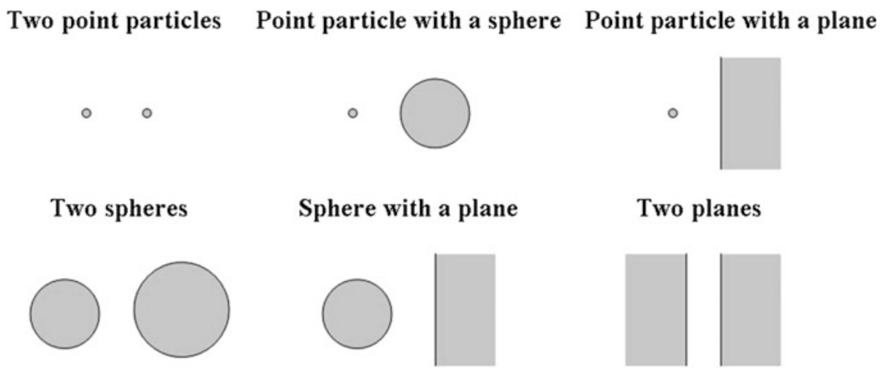


**Fig. 1** Different geometric combinations for pairwise interactions involving point particles, spheres or planes

as four other geometric combinations drawn from the set of a point particle, a sphere and a plane. Note that a point particle and a plane are defined as a sphere of infinitely small and infinitely large radius, respectively.

While the *absolute* geometry of an indiviual point particle, sphere or plane is mathematically well defined, the geometry of a pair of interacting objects can only be described with respect to their surface-to-surface separation, which is denoted here as a distance $s$. Note that an alternative description based on the centre-to-centre separation distance would be ambiguous if one of the interacting objects is a plane. We therefore introduce a dimensionless, length-scale independent parameter $s^*$ that describes all possible combinations of sphere size and surface-to-surface separation distance for a two-body system. Generally, a dimensionless separation distance $s^*$ can be obtained by dividing the surface-to-surface separation $s$ by a characteristic length, $l$, that depends on the sizes of the interacting bodies and their separation, i.e. $s^* = s/l$. For any given surface-to-surface separation $s$, a suitable choice of the length $l$ will allow one to determine, from $s^* = s/l$, whether a pair of interacting objects is geometrically close to the limit of two point particles, the limit of two planes, or neither of these limits. For $a_1$ and $a_2$ being the radii of the interacting objects, any linear combination (or some other simple functions) of $a_1 a_2$ or $(a_1 + a_2)$ is not a suitable form for the length $l$, because as $a_\mathrm{i}$ (i = 1, 2) approaches infinity (one of the interacting bodies approaches the planar limit) the value of $s^*$ would approach zero if the value of the other $a_i$ is non-zero. This implies that a system close to the geometric limit of two interacting planes ($a_1 \gg s$ and $a_2 \gg s$) cannot be distinguished from a system containing only one plane. In this paper, it is shown that a suitable choice of the length $l$ can be derived from the bispherical coordinate system [8, 9], which has recently been employed for a study of electrostatic sphere-sphere [2, 10] and sphere-plane interactions [2].

## 2 Introduction to the Bispherical Coordinate System

In the bispherical coordinate system [8, 9] shown in Fig. 2 for a two-sphere system, the position of any point $X$ in space is described with reference to a pair of foci, which are separated by a distance of $2a$. The foci are defined as inverse points of each other such that

$$d_1 c_1 = a_1^2 \tag{1}$$

and

$$d_2 c_2 = a_2^2. \tag{2}$$

The centre-to-centre separation $h > 0$ is given by

$$h = s + a_1 + a_2 = 2a + c_1 + c_2 \tag{3}$$

where $s \geq 0$ is the surface-to-surface separation. The bispherical coordinates are often denoted as $(\eta, \xi, \phi)$, where

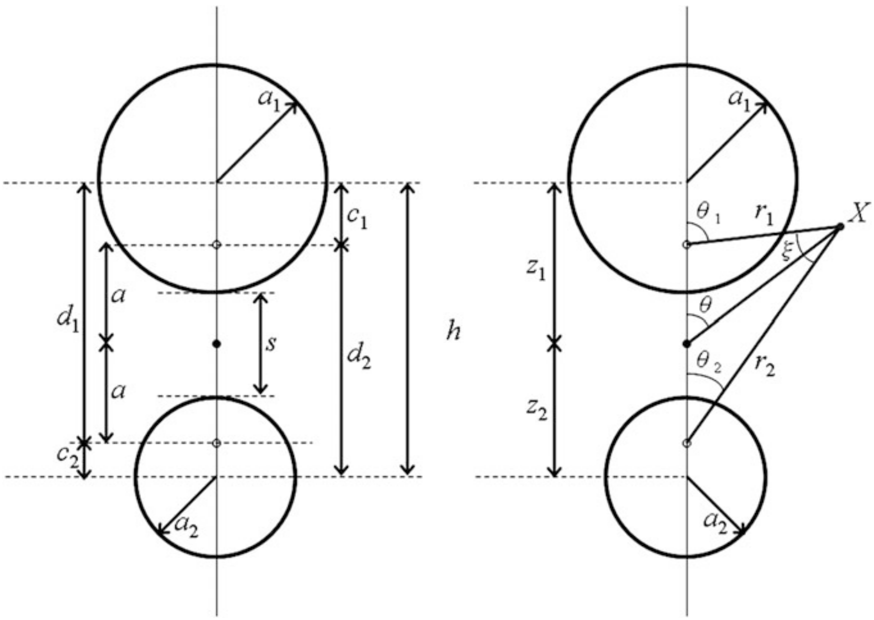


**Fig. 2** Schematic diagrams of the bispherical coordinate system. The diagram on the *left* illustrates how the length quantities $c_1$, $c_2$, $d_1$ and $d_2$ are related to the inter-focal separation $2a$ and to the centre-to-centre separation $h$, and the diagram on the *right* illustrates how the position of an arbitrary point $X$ in space is described with reference to the foci

$$\eta \equiv -\ln\left(\frac{r_1}{r_2}\right), \tag{4}$$

$$\xi \equiv \theta_1 - \theta_2 \tag{5}$$

and $\phi$ is an azimuthal angle about the axis that joins the centres of the spheres. They are related to the Cartesian coordinates ($x$, $y$, $z$) as follows

$$x = \frac{a \sin \xi \cos \phi}{\cosh \eta - \cos \xi}; y = \frac{a \sin \xi \sin \phi}{\cosh \eta - \cos \xi}; z = \frac{a \sinh \eta}{\cosh \eta - \cos \xi}. \tag{6}$$

The surface of each sphere is a surface of constant $\eta$, where the parameters

$$\eta = \eta_1 > 0 \tag{7}$$

and

$$\eta = -\eta_2 < 0 \tag{8}$$

represent the surfaces of sphere 1 and sphere 2, respectively. In general, $\eta$ is positive for the upper half plane occupied by sphere 1 ($z \geq 0$ or $0 \leq \theta \leq \pi/2$) and negative for the lower half plane occupied by sphere 2 ($z \leq 0$ or $\pi/2 \leq \theta \leq \pi$).

## 3 Derivation of the Scaled Surface-to-Surface Separation

The interfocal separation $2a$ can be expressed as a function of $h$, $a_1$ and $a_2$. Using Eqs. (1) and (2), together with

$$d_1 = c_1 + 2a \tag{9}$$

and

$$d_2 = c_2 + 2a, \tag{10}$$

two quadratic equations for $c_1 > 0$ and $c_2 > 0$, respectively, are obtained:

$$c_1^2 + 2ac_1 - a_1^2 = 0 \tag{11}$$

and

$$c_2^2 + 2ac_2 - a_2^2 = 0, \tag{12}$$

with solutions

$$c_1 = -a + \sqrt{a^2 + a_1^2} \geq 0 \tag{13}$$

and

$$c_2 = -a + \sqrt{a^2 + a_2^2} \geq 0, \tag{14}$$

respectively. Substituting Eqs. (13) and (14) into Eq. (3) yields

$$h = \sqrt{a^2 + a_1^2} + \sqrt{a^2 + a_2^2} \tag{15}$$

which implies

$$h^2 - 2a^2 - (a_1^2 + a_2^2) = 2\sqrt{a^4 + a^2(a_1^2 + a_2^2) + a_1^2 a_2^2}. \tag{16}$$

A further rearrangement of terms leads to the arrival of an expression for the inter-focal separation:

$$\begin{aligned} 2a &= \frac{1}{h}\sqrt{h^4 + (a_1^2 - a_2^2)^2 - 2h^2(a_1^2 + a_2^2)} \\ &= \frac{1}{(s + a_1 + a_2)}\sqrt{(s + a_1 + a_2)^4 + (a_1^2 - a_2^2)^2 - 2(s + a_1 + a_2)^2(a_1^2 + a_2^2)} \end{aligned} \tag{17}$$

which leads to

$$\begin{aligned} \frac{2a}{s} &= \frac{1}{s(s + a_1 + a_2)}\sqrt{(s + a_1 + a_2)^4 + (a_1^2 - a_2^2)^2 - 2(s + a_1 + a_2)^2(a_1^2 + a_2^2)} \\ &= \frac{1}{(1 + a_1' + a_2')}\sqrt{(1 + a_1' + a_2')^4 + (a_1'^2 - a_2'^2)^2 - 2(1 + a_1' + a_2')^2(a_1'^2 + a_2'^2)} \end{aligned} \tag{18}$$

where $a_1' \equiv a_1/s$ and $a_2' \equiv a_2/s$ are relative measures of the radii of the interacting spheres with respect to their surface-to-surface separation $s$. Equation (18) can be written in the following form

$$\frac{2a}{s} = \frac{1}{\left(1 + \frac{1}{a_1' + a_2'}\right)}\sqrt{\frac{1}{(a_1' + a_2')^2} + 4\left[1 + \frac{\left(\frac{1}{a_2'} + 2a_1'\right)}{\left(1 + \frac{a_1'}{a_2'}\right)} + \frac{\left(\frac{a_1'}{a_2'}\right)}{\left(1 + \frac{a_1'}{a_2'}\right)^2}\right]} \tag{19}$$

which implies

$$\lim_{a'_2 \to \infty} \frac{2a}{s} = 2\sqrt{(1 + 2a'_1)}. \tag{20}$$

According to Eq. (20), the ratio $2a/s$ diverges only if both $a_1'$ and $a_2'$ approach infinity, which suggests that it can be used as a parameter to distinguish between geometries of sphere-plane and plane-plane interactions. If a scaled surface-to-surface separation $s^* \equiv s/2a$ is considered, a normalized parameter that applies to all possible combinations of sphere size and separation distance can be obtained, where it is

$$\lim_{\substack{a'_1 \to 0 \\ a'_2 \to 0}} s^* = 1 \tag{21}$$

for the interaction between two point particles, according to Eq. (18). Furthermore,

$$\lim_{\substack{a'_1 \to 0 \\ a'_2 \to \infty}} s^* = \frac{1}{2} \tag{22}$$

corresponds to the interaction of a point particle with a plane, according to Eq. (20); and

$$\lim_{\substack{a'_1 \to \infty \\ a'_2 \to \infty}} s^* = 0 \tag{23}$$

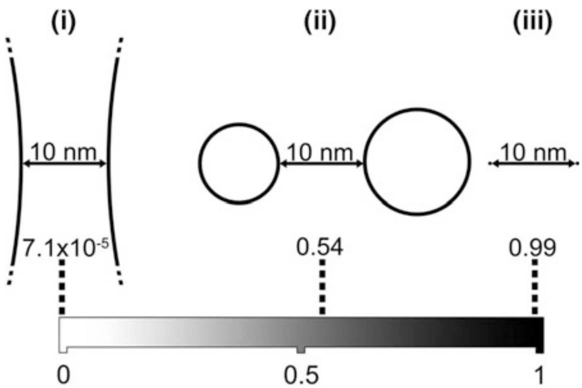


**Fig. 3** Schematic illustration of various geometries between the limits $s^* = 0$ to $s^* = 1$, for $s^* \equiv s/2a$. The sphere radii in the given examples, which all correspond to a surface-to-surface separation of $s^* = 10\,\text{nm}$, are: (i) $a_1 = a_2 = 0.5$ m, (ii) $a_1 = 5$ nm and $a_2 = 7.5$ nm, and (iii) $a_1 = a_2 = 0.05$ nm. At $s^* \to 0$, the interacting system is close to the geometric limit of two planes, and at $s^* \to 1$, the system is close to the geometric limit of two point particles. The range of values of $s^*$ from 0 to 1 corresponds to a continuum of all possible combinations of sphere size and separation distance

corresponds to the interaction between two planes. As illustrated in Fig. 3, for any possible combination of sphere size and separation distance, the value of $s^*$ lies within the range [0, 1]. A value of $s^*$ close to unity indicates that the system is close to the geometric limit of two point particles, and a value of $s^*$ close to zero indicates that it is close to the limit of two planes.

## 4 Graphical Representation of the Scaled Surface-to-Surface Separation

Figure 4 illustrates how the scaled surface-to-surface separation $s^*$ depends on $a_1'$ and $a_2'$. As shown in Fig. 4a, each value of $s^*$ for $0 < s^* < 1$ does not correspond to a unique geometry, but rather to a range of possible combinations of $a_1'$ and $a_2'$. If, for example, $a_2'$ increases while $a_1'$ remains unchanged, the value of $s^*$ would decrease. To return to the original value of $s^*$, one can move in the direction of decreasing $a_1'$ until the contour line of the original value of $s^*$ is reached. Figure 4b is a ln-ln plot of the same contour map, which illustrates a difference in the dependence of $s^*$ on $a_1'$ and $a_2'$ between cases of $s^* < 0.5$ and cases of $s^* > 0.5$. Consider the regime of $s^* < 0.5$: At any given value of $s^*$, if $a_1'$ increases indefinitely, $a_2'$ will decrease towards a finite value. If $a_2'$ decreases towards zero while $a_1'$ increases indefinitely, the value of $s^*$ will instead approach 0.5 which describes, among many others, a point-plane geometry. But if $a_2'$ remains unchanged for increasing $a_1'$, the value of $s^*$ will decrease towards a finite value, which describes a particular range of sphere-plane geometries.

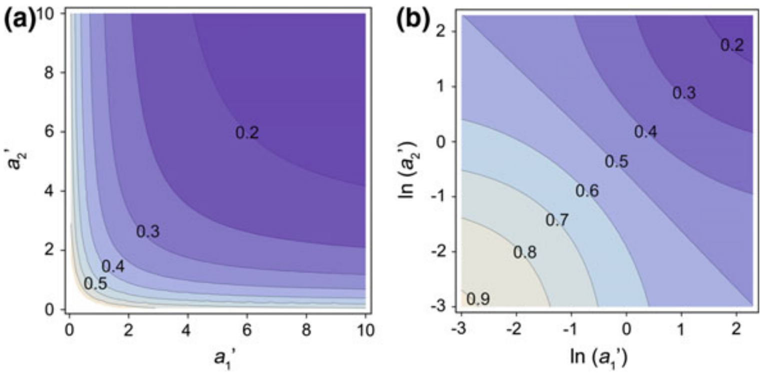


**Fig. 4** Contour maps of the scaled surface-to-surface separation $s^* \equiv s/2a$, showing its dependence on $a_1'$ and $a_2'$ in (**a**), and on $\ln(a_1')$ and $\ln(a_2')$ in (**b**). The dimensionless parameter $s^*$ approaches unity if the system is close to the geometric limit of two point particles at $a_1' = a_2' = 0$. The values of $s^*$ ranging between 0 and 1 correspond to particular combinations of $a_1'$ and $a_2'$. The ln-ln plot in (**b**) illustrates a difference in the behaviour of $s^*$ between cases of $s^* < 0.5$ and cases of $s^* > 0.5$

Now consider the regime of $s^* > 0.5$. At any given value of $s^*$, if $a_1'$ decreases towards zero, $a_2'$ will increase towards a finite value. If $a_2'$ increases indefinitely while $a_1'$ decreases towards zero, the value of $s^*$ will approach 0.5, again for a point-plane geometry. But if $a_2'$ remains unchanged while $a_1'$ decreases, the value of $s^*$ will increase towards a limit which is less than unity, because in this case the parameter $s^*$ describes only a particular range of point-sphere geometries but not the geometric limit of a pair of point particles.

## 5 Conclusions

A dimensionless, scaled surface-to-surface separation distance $s^* \in [0, 1]$ has been derived from the bispherical coordinate system to describe geometries of sphere-sphere interactions. It serves as a measure of how close a system of interacting spheres is to the geometric limit of two point particles or two planes. A value close to unity indicates that the system is close to the limit of two point particles, and a value close to zero indicates that the system is close to that of two planes. This approach applies to all possible combinations of sphere size and separation distance, including ambiguous cases where a description of the interacting bodies as spheres becomes questionable.

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