Numerical Experiments with a New Dynamic Mixed Subgrid-Scale Model

P. Lampitella, F. Inzoli and E. Colombo

1 Introduction

One of the main drawbacks of the classical LES approach [1] is the lack of connection with its practical implementation in numerical solvers and the consequent limits in the derivation of proper subgrid-scale (SGS) models. A well known example of this comes from the implicit filtering approach; indeed, it can provide full adherence to the continuous classical LES model if performed with certain fully spectral methods, but also to a completely different one if performed with low order finite difference/volume methods. Both perform a projection of the relevant fields into a lower dimensional phase space, but for low order numerical methods the resulting LES attractor is substantially distorted by numerical artifacts. In this case, a possibility is to resort to the explicit filtering approach, but then one has to face the additional burden of the non commutation between the numerical filtering and the numerical differentiation operators. A possible approach to deal with commutation errors (CE) is to adopt higher order commuting filters, e.g., [2]. However, the divergence of the resulting SGS stress tensor is also affected and the two maintain the same scaling with respect to the filter cutoff length, showing the pivotal need for CE modeling within the classical LES approach [3]. While direct CE modeling has found various contributors, very few authors [4, 5] have considered removing CE ab initio by a proper interpretation of the LES problem.

The aim of the present work is twofold. First, previous works [4, 5] are generalized and unified in a single, more flexible, formulation allowing implicit and explicit filtering approaches; multilevel features are also exploited to extend its general applicability. Finally, a Taylor series analysis of the SGS stress tensor arising in the new framework is used to derive a consistent form of scale-similar model. Then, this is combined with a classical eddy viscosity term in a new form of dynamic procedure which is consistent with the proposed LES methodology. Model performances are assessed in the simulation of the turbulent channel flow at $Re_\tau = 590$ [6].

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2 LES Framework

As in classical LES, let us introduce the filtering operator G^i:

$$\bar{\phi}\left(\mathbf{x}, t, \Delta^i\right) = G^i * \phi(\mathbf{x}, t) = \int_{\Omega} G\left[\mathbf{x} - \xi, \Delta^i(\mathbf{x})\right] \phi(\xi, t)\, \mathrm{d}\xi \tag{1}$$

where Ω is the generally finite fluid domain and G^i is a spatially varying filter kernel with width $\Delta^i(\mathbf{x})$. For future reference, we also introduce the conventions:

$$\begin{gathered} \bar{\phi}^n = G^n * \bar{\phi}^{n-1} = G^n * G^{n-1} * \cdots * G^1 * G^0 * \bar{\phi}^0 \\ G^0 * \bar{\phi}^0 = \bar{\phi}^0 = \phi \;; \quad G^1 * \bar{\phi}^0 = \bar{\phi} \;; \quad \tilde{\phi}^n = \overline{\rho\phi}^n / \bar{\rho}^n \end{gathered} \tag{2}$$

where the ith filter kernel G^i has an associated filter width $\Delta^i \geq \Delta^{i-1}$, which is generally different from the cutoff length of the actual filter determining $\bar{\phi}^n$, Δ_n. To remain general, we also introduced the classical Favre filtering with ρ the density of the fluid. With this notation, it is a matter of simple manipulations to express the Navier-Stokes equations at a generic filter level $n \geq 0$; limiting the discussion to the momentum equations, for the sake of conciseness, we get:

$$\begin{gathered} \frac{\partial\left(\bar{\rho}^n \tilde{u}_i^n\right)}{\partial t} + \overline{\frac{\partial}{\partial x_j}\left(\bar{\rho}^n \tilde{u}_i^n \tilde{u}_j^n + \tilde{\sigma}_{ij}^n\right)}^n = \overline{\frac{\partial}{\partial x_j}\left(\tau_{ij}^{n-0}\right)}^n \\ \tilde{\sigma}_{ij}^n = \bar{p}^n \delta_{ij} - 2\mu\left(\tilde{S}_{ij}^n - \frac{1}{3}\tilde{S}_{gg}^n \delta_{ij}\right) \\ \tilde{S}_{ij}^n = \frac{1}{2}\left(\frac{\partial \tilde{u}_i^n}{\partial x_j} + \frac{\partial \tilde{u}_j^n}{\partial x_i}\right) \\ \tau_{ij}^{n-m} = \left(\bar{\rho}^n \tilde{u}_i^n \tilde{u}_j^n - \bar{\rho}^m \tilde{u}_i^m \tilde{u}_j^m\right) + \left(\tilde{\sigma}_{ij}^n - \tilde{\sigma}_{ij}^m\right) \end{gathered} \tag{3}$$

where τ_{ij}^{n-m} with $m = 0$ is the SGS stress tensor arising in the formulation (3). With respect to classical LES formulations, we notice that: (a) no commutation with spatial derivatives is ever required, (b) there is full correspondence with the implicit filtering approach if the filter (1) is interpreted as a finite volume discretization, (c) an explicit filtering approach can be adopted if the filter (1) is effectively applied trough a numerical procedure. A classical criticism with the proposed formulation is the lack of Galilean invariance. However, it is worth noting that this is a property of any quantity which is non-uniformly filtered in space, hence it is natural that equations based on such quantities also inherit this property.

3 SGS Tensor Analysis

The main effect of reformulating the LES problem is the appearance of the SGS tensor in a different form; hence, the question of what form of SGS model should be used, naturally arises. In order to attempt an answer, we analyze the Taylor series development for τ_{ij}^{m-n} assuming that, for $m > n$, the following hold [7]:

$$\bar{\phi}^m = \bar{\phi}^n + \Delta_m^2 M_k \frac{\partial^2 \bar{\phi}^n}{\partial x_k^2} + O\left(\Delta_m^4\right) \quad \Delta_m^2 = \sum_{i=0}^{m-n} \left(\Delta^{m-i}\right)^2 \tag{4}$$

where M_k are filter dependent coefficients. Under this circumstance, and for sufficiently smooth variations of Δ_m, the following estimates are valid:

$$\bar{\rho}^m \tilde{u}_i^m \tilde{u}_j^m - \bar{\rho}^n \tilde{u}_i^n \tilde{u}_j^n = \Delta_m^2 M_k \frac{\partial^2 \bar{\rho}^n \tilde{u}_i^n \tilde{u}_j^n}{\partial x_k^2} - 2\Delta_m^2 M_k \bar{\rho}^n \frac{\partial \tilde{u}_i^n}{\partial x_k} \frac{\partial \tilde{u}_j^n}{\partial x_k} + O\left(\Delta_m^4\right) \tag{5}$$

$$\bar{p}^m - \bar{p}^n = \Delta_m^2 M_k \frac{\partial^2 \bar{p}^n}{\partial x_k^2} + O\left(\Delta_m^4\right) \tag{6}$$

$$\begin{aligned}\tilde{S}_{ij}^m - \tilde{S}_{ij}^n = {} & \frac{1}{2}\left[\frac{\partial}{\partial x_j}\left(\Delta_m^2 M_k \frac{\partial^2 \tilde{u}_i^n}{\partial x_k^2}\right) + \frac{\partial}{\partial x_i}\left(\Delta_m^2 M_k \frac{\partial^2 \tilde{u}_j^n}{\partial x_k^2}\right)\right] \\ & + \frac{1}{2}\left[\frac{\partial}{\partial x_j}\left(2\frac{\Delta_m^2 M_k}{\bar{\rho}^n} \frac{\partial \bar{\rho}^n}{\partial x_k} \frac{\partial \tilde{u}_i^n}{\partial x_k}\right) + \frac{\partial}{\partial x_i}\left(2\frac{\Delta_m^2 M_k}{\bar{\rho}^n} \frac{\partial \bar{\rho}^n}{\partial x_k} \frac{\partial \tilde{u}_j^n}{\partial x_k}\right)\right] \\ & + O\left(\Delta_m^4\right)\end{aligned} \tag{7}$$

The resulting approximation for τ_{ij}^{m-n} can then be summarized as follows:

$$\tau_{ij}^{m-n} = \Delta_m^2 f\left(\bar{\rho}^n, \tilde{u}_i^n, \tilde{u}_j^n\right) + \frac{\partial \Delta_m^2}{\partial x_j} h\left(\bar{\rho}^n, \tilde{u}_i^n\right) + \frac{\partial \Delta_m^2}{\partial x_i} h\left(\bar{\rho}^n, \tilde{u}_j^n\right) + O\left(\Delta_m^4\right) \tag{8}$$

where f and h are functional forms easily derived from Eqs. (5–7). However, when the same analysis is performed for the true SGS stress tensor τ_{ij}^{n-0}, recalling that $\bar{\phi}^m = \bar{\phi}^n + O\left(\Delta_m^2\right)$, the resulting estimate is:

$$\tau_{ij}^{n-0} = \Delta_n^2 f\left(\bar{\rho}^n, \tilde{u}_i^n, \tilde{u}_j^n\right) + \frac{\partial \Delta_n^2}{\partial x_j} h\left(\bar{\rho}^n, \tilde{u}_i^n\right) + \frac{\partial \Delta_n^2}{\partial x_i} h\left(\bar{\rho}^n, \tilde{u}_j^n\right) + O\left(\Delta_n^4\right) \tag{9}$$

Hence, even under the most simplifying assumptions, e.g., Eq. (4), a scale-similar model based on τ_{ij}^{m-n}, as originally proposed in [5], has a second order difference with respect to the true SGS stress tensor τ_{ij}^{n-0}. In order to overcome such deficiency, we propose to adopt a scaled version of scale-similar term according to the following estimate:

$$\frac{\Delta_n^2}{\Delta_m^2}\tau_{ij}^{m-n} - \tau_{ij}^{n-0} = \Delta_n^2\left[h\left(\bar{\rho}^n, \tilde{u}_j^n\right)\beta_i + h\left(\bar{\rho}^n, \tilde{u}_i^n\right)\beta_j\right] + O\left(\Delta_m^2\Delta_n^2\right) + O\left(\Delta_n^4\right)$$
$$\beta_i = \frac{1}{\Delta_m^2}\frac{\partial\Delta_m^2}{\partial x_i} - \frac{1}{\Delta_n^2}\frac{\partial\Delta_n^2}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\log\left(\frac{\Delta_m^2}{\Delta_n^2}\right)\right] \tag{10}$$

From which it follows that, for any two couples of filter levels satisfying the scale-similarity hypothesis and the assumption (4), the proposed scale-similar tensor $\left(\Delta_n^2/\Delta_m^2\right)\tau_{ij}^{m-n}$ approximates the convective/pressure part of the tensor τ_{ij}^{n-0} with $O\left(\Delta_n^4\right)$ accuracy; if, in addition, the ratio Δ_n^2/Δ_m^2 is constant in space, the accuracy is restored also for the diffusive part.

4 Dynamic SGS Modeling

As in the computational practice some of the assumptions concerning the model derivation might be violated, we implemented the previous model in a dynamic version trough the following Germano identity, consistent with the formulation (3):

$$\tau_{ij}^{m-k} - \tau_{ij}^{n-k} = \tau_{ij}^{m-n} \tag{11}$$

It is worth noting that the lack of commutation between the filter and derivative operators results in Eq. (11) not involving any test filtered tensor. This, in turn, produces two major advantages: there is no arbitrary extraction of model constants from the test filter and no commutation property is required for the test filter. An additional advantage is the lack of filtered products of variables: for the dynamic mixed model presented below only 6 scalars need to be test filtered while for a classical dynamic Smagorinsky model the required number of filter applications is 15. Besides Eq. (11) and the cited advantages, the proposed dynamic procedure follows the classical approach. Two-parameter, mixed SGS models are introduced for the basic (n) and test (m) filter levels:

$$\begin{aligned}\tau_{ij}^{n-0} &= C_{ev}2\bar{\rho}^n\left|\tilde{S}^n\right|^\theta\left(k_{SGS}^{n-m}\right)^{\frac{1-\theta}{2}}\Delta_n^{1+\theta}\left(\tilde{S}_{ij}^n - \tfrac{1}{3}\tilde{S}_{gg}^n\delta_{ij}\right) + C_{ss}\left(\tfrac{\Delta_n}{\Delta_m}\right)^2\tau_{ij}^{m-n}\\ \tau_{ij}^{m-0} &= C_{ev}2\bar{\rho}^m\left|\tilde{S}^m\right|^\theta\left(k_{SGS}^{m-r}\right)^{\frac{1-\theta}{2}}\Delta_m^{1+\theta}\left(\tilde{S}_{ij}^m - \tfrac{1}{3}\tilde{S}_{gg}^m\delta_{ij}\right) + C_{ss}\left(\tfrac{\Delta_m}{\Delta_r}\right)^2\tau_{ij}^{r-m}\end{aligned} \tag{12}$$

with the additional test filter level r ($\Delta_r > \Delta_m > \Delta_n$) and:

$$\begin{aligned}k_{SGS}^{n-m} &= \tfrac{1}{2}\left(\tilde{u}_j^n - \tilde{u}_j^m\right)\left(\tilde{u}_j^n - \tilde{u}_j^m\right)\\ \tilde{S}^n &= \sqrt{2\tilde{S}_{ij}^n\tilde{S}_{ij}^n}\end{aligned} \tag{13}$$

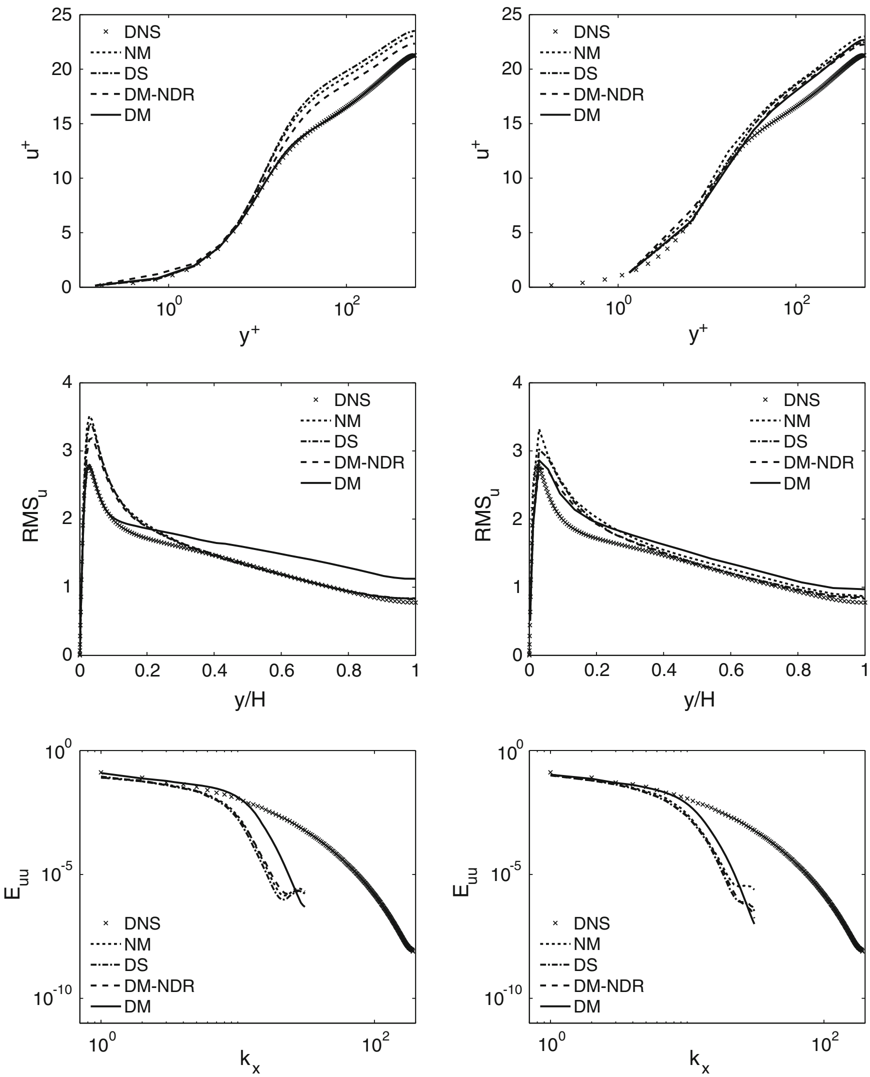

Fig. 1 Results for $\theta = 0$. *Left* fine grid, *Right* coarse grid. *Top* mean velocity, *Center* r.m.s. stream-wise velocity fluctuations, *Bottom* stream-wise spectra of stream-wise velocity at y/H=1

The dynamic constants C_{ev} and C_{ss} are then computed as in classical two parameter dynamic models [8].

5 Numerical Experiments and Discussion

The model is implemented in a commercial, unstructured, finite volume solver [9] by exploiting the implicit filtering features of the formulation. Numerical tests are performed on the turbulent channel flow at $\mathrm{Re}_\tau = 590$ [6] on a domain with extensions $L_x = 2\pi H$, $L_y = 2H$ and $L_z = \pi H$, in the stream-wise, wall-normal and span-wise directions, with H the channel half-height. Two different grids are adopted, both having 64 cells in the homogeneous directions and 33 (coarse) or 99 (fine) cells in the wall-normal direction, distributed according to a sin stretching law.

Grid effects are investigated in Fig. 1 for $\theta = 0$ (DM); results for the same model without the proposed scaling (DM-NDR), a classical dynamic Smagorinsky model (DS) and a no model computation (NM) are also reported. Major differences are evident on the fine grid. The DM model is the only one capable of recovering the mean velocity profile and the peak of the velocity fluctuation which, however, is over-predicted in the core of the channel. Notably, the DM-NDR model fails in reproducing the mean velocity profile at the first few points near the wall, where the grid stretching is higher, somehow showing the incorrect scaling. The differences are mitigated for the coarse grid, possibly because of the higher influence of the numerical error and the reduced effect of the scarcely resolved near-wall region, but the DM model is still the only one correctly reproducing the logarithmic slope of the velocity profile. Finally, on both grids, velocity spectra highlight that the present scale-similar formulation is necessary in order to recover a substantial part of the energy in the smallest resolved scales; in contrast, no actual difference is found in the spectra for the remaining modeling options. This energy recovery also allows the DM model to suppress the energy pile-up observed for the other models. This effect has been observed for different eddy-viscosity parameterizations (θ) with no substantial differences in the remaining quantities.

In conclusion, despite some grid-effect limitations arising from the implicitly filtered approach, the proposed model is found effective and necessary in removing major drawbacks of SGS closures not considering the LES framework in their derivation. Some pitfalls emerged as well, like the over-prediction of stream-wise velocity fluctuations in the core of the channel and of span-wise energy spectra (not shown); the model dependence on the Reynolds number and some preliminary tests suggest that a tensorial scale-similar dynamic constant might alleviate the problem.

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