Contents

Preface . . . ix
Preliminaries . . . xiii

1 The spaces H^s . . . 1
1 The spaces $H^s(\mathbb{R}^n)$ . . . 1
1.1 Definition and simplest properties . . . 1
1.2 Embedding theorems . . . 3
1.3 The spaces $H^s(\mathbb{R}^n)$ of negative order . . . 7
1.4 Isometric isomorphisms Λ^t . . . 8
1.5 Dense subsets . . . 8
1.6 Continuous linear functionals on $H^s(\mathbb{R}^n)$ . . . 9
1.7 Norms of positive fractional order . . . 11
1.8 Estimates of intermediate norms . . . 13
1.9 Multipliers . . . 13
1.10 Traces on hyperplanes . . . 16
1.11 Mollifications and shifts . . . 18
1.12 A compactness theorem . . . 19
1.13 Changes of coordinates . . . 20
1.14 Discrete norms and discrete representation of functions in $H^s(\mathbb{R}^n)$ . . . 21
1.15 Embedding of the spaces $H^s(\mathbb{R}^n)$ in $L_p(\mathbb{R}^n)$ . . . 23
2 The spaces $H^s(M)$ on a closed smooth manifold M . . . 24
2.1 Closed smooth manifolds . . . 24
2.2 The spaces $H^s(M)$ . . . 27
2.3 Basic properties of the spaces $H^s(M)$ . . . 29
2.4 Manifolds of finite smoothness . . . 32
3 The spaces $H^s(\mathbb{R}^n_+)$ . . . 32
3.1 Definitions . . . 32
3.2 Properties of the spaces $H^s(\mathbb{R}^n_+)$ . . . 36
3.3 Boundary values . . . 38
3.4 Extension by zero . . . 41

3.5 Gluing together functions in $H^s(\mathbb{R}^n_+)$ and $H^s(\mathbb{R}^n_-)$ 46
3.6 Decomposition of the space $H^s(\mathbb{R}^n)$ with $|s| < 1/2$ into the sum of two subspaces 46
4 The spaces $\widetilde{H}^s(\mathbb{R}^n_+)$ and $\mathring{H}^s(\mathbb{R}^n_+)$ 48
4.1 The spaces $\widetilde{H}^s(\mathbb{R}^n_+)$ 48
4.2 Duality between the spaces $H^s(\mathbb{R}^n_+)$ and $\widetilde{H}^{-s}(\mathbb{R}^n_+)$ 50
4.3 The spaces $\mathring{H}^s(\mathbb{R}^n_+)$ 52
4.4 The spaces $\widetilde{H}^{-s}(\mathbb{R}^n_+)$ and $H^{-s}(\mathbb{R}^n_+)$ with non-half-integer $s > 1/2$ 56
5 The spaces H^s on smooth bounded domains and manifolds with boundary 57
5.1 The spaces $H^s(\Omega)$ 57
5.2 The spaces $H^s(M)$ 63

2 Elliptic equations and elliptic boundary value problems 65
6 Elliptic equations on a closed smooth manifold 66
6.1 Definitions 66
6.2 Main theorems 68
6.3 Adjoint operators 72
6.4 Some spectral properties of elliptic operators 74
6.5 Generalizations 75
7 Elliptic boundary value problems in smooth bounded domains 77
7.1 Definitions and statements of main theorems 77
7.2 Proofs of main theorems 82
7.3 Normal systems of boundary operators and formally adjoint boundary value problems. Boundary value problems with homogeneous boundary conditions 88
7.4 Spectral boundary value problems 92
7.5 Generalizations 95
8 Strongly elliptic equations and variational problems 96
8.1 The Dirichlet and Neumann problems for a second-order scalar equation 97
8.2 Generalizations 107

3 The spaces H^s and second-order strongly elliptic systems in Lipschitz domains 111
9 Lipschitz domains and Lipschitz surfaces 111
9.1 Specifics of Lipschitz domains and surfaces 111
9.2 The spaces H^s on Lipschitz domains and Lipschitz surfaces 118
9.3 Integration by parts 121
10 Discrete Norms, Discrete Representation of Functions, and a Universal Extension Operator 124
10.1 Discrete norms on $H^s(\mathbb{R}^n_x)$ 124
10.2 Discrete representation of functions on $\mathbb{R}^n$ 127

10.3 Discrete representation of functions and norms on a special Lipschitz domain . . . 128
10.4 The extension operator . . . 132
10.5 Construction of φ_0 with $N = \infty$ . . . 134
11 Boundary value problems in Lipschitz domains for second-order strongly elliptic systems . . . 136
11.1 Basic definitions and results . . . 136
11.2 The Weyl decomposition of the space $H^1(\Omega)$ and the choice of f and h . . . 147
11.3 The Poincaré–Steklov operators . . . 152
11.4 The mixed problem . . . 153
11.5 Duality relations on Γ . . . 155
11.6 Spectral problems . . . 156
11.7 Examples . . . 161
11.8 Other problems . . . 164
11.9 Two classical operator approaches to variational problems . . 165
12 Potential operators and transmission problems . . . 170
12.1 A system on the torus . . . 170
12.2 Definition of single- and double-layer potentials . . . 174
12.3 Solution representation and its consequences. The transmission problem . . . 177
12.4 Operators on Γ and the Calderón projections . . . 179
12.5 Strong coercivity of forms and invertibility of the operators A and H . . . 181
12.6 Relations between operators on the boundary . . . 184
12.7 Duality relations on Γ . . . 184
12.8 Problems with boundary conditions on a nonclosed surface . 185
12.9 Problems with a spectral parameter in transmission conditions . . . 190
12.10 More general transmission problems . . . 191

4 More general spaces and their applications . . . 193
13 Elements of interpolation theory . . . 193
13.1 Contents of the section. The spaces L_p . . . 193
13.2 Basic definitions and the complex interpolation method 195
13.3 The real method . . . 205
13.4 Retractions and coretractions . . . 208
13.5 Duality . . . 210
13.6 Iterated interpolation . . . 211
13.7 Interpolation and extrapolation of invertibility . . . 212
13.8 Further results . . . 217
13.9 Positive operators and their fractional powers . . . 220
14 The spaces W_p^s, H_p^s, and B_p^s . . . 222
14.1 Spaces of functions on $\mathbb{R}^n$ . . . 222
14.2 Spaces on smooth manifolds . . . 227

14.3 Function spaces on $\mathbb{R}^n_+$ and on smooth domains . . . 227
14.4 Remarks to embedding theorems . . . 229
14.5 The spaces H^s_p and B^s_p on Lipschitz domains and Lipschitz surfaces . . . 230
14.6 Spaces with $p = 1$ and $p = \infty$ . . . 232
14.7 The Triebel–Lizorkin spaces and the general Besov spaces . 232
15 Applications to the general theory of elliptic equations and boundary value problems . . . 234
15.1 General elliptic problems in the Sobolev–Slobodetskii spaces 234
15.2 Generalizations . . . 236
16 Applications to boundary value problems in Lipschitz domains. 1 . . 236
16.1 Main boundary value problems in more general H^s spaces . 236
16.2 The operators A and H in more general H^s spaces . . . 240
16.3 The Rellich identity and its consequences . . . 241
16.4 Savaré's generalized theorem . . . 243
16.5 Nirenberg's method and the regularity of solutions inside a domain . . . 250
16.6 Fractional powers of the operators corresponding to problems in Lipschitz domains and the Kato problem . . . 252
17 Applications to boundary value problems in Lipschitz domains. 2 . . 257
17.1 Further generalizations of the settings of the Dirichlet and Neumann problems . . . 257
17.2 New corollaries of Shneiberg's theorem . . . 260
17.3 Examples . . . 266
17.4 The optimal resolvent estimate . . . 267
17.5 Elementary facts about semigroups . . . 273
17.6 Parabolic problems in a Lipschitz cylinder . . . 275

18 Appendix: Definitions and facts from operator theory . . . 276
18.1 Fredholm operators . . . 276
18.2 The Lax–Milgram theorem . . . 283
18.3 A few definitions and facts from spectral theory . . . 287
18.4 Pseudodifferential operators . . . 291
19 Additional remarks and literature comments . . . 296
19.1 To Chapter 1 . . . 296
19.2 To Chapter 2 . . . 297
19.3 To Chapter 3 . . . 301
19.4 To Chapter 4 . . . 304
19.5 To Section 18 . . . 310

References . . . 313

Index . . . 329

Springer

http://www.springer.com/978-3-319-14647-8

Sobolev Spaces, Their Generalizations and Elliptic Problems in Smooth and Lipschitz Domains
Agranovich, M.S.
2015, XIII, 331 p., Hardcover
ISBN: 978-3-319-14647-8