Preface

This book consists of four chapters and two additional sections.

The first two chapters contain introductory courses. Chapter 1 presents the theory of Sobolev-type spaces H^s ($s \in \mathbb{R}$) on $\mathbb{R}^n$, on a smooth closed manifold, and on a smooth bounded domain. Chapter 2 outlines the foundations of the theory of general linear elliptic equations on smooth closed manifolds and the theory of general elliptic boundary value problems in smooth bounded domains in the spaces H^s.

Chapter 3 presents the theory of basic boundary value problems for second-order strongly elliptic systems in the same spaces in a bounded Lipschitz domain; its boundary can generally be very nonsmooth.

Chapter 4 is of review character. Here we briefly consider the more general Bessel potential spaces H_p^s and the Besov spaces $B_p^s = B_{p,p}^s$, $1 < p < \infty$, and describe their applications to the same problems.

The Sobolev spaces W_p^s are the spaces H_p^s for integer $s \geq 0$. The spaces B_p^s coincide with the Slobodetskii spaces W_p^s for noninteger $s > 0$. For $p = 2$, the spaces H_2^s and B_2^s coincide with H^s.

This is a great honor for the author that the Springer publishing company has decided to include this book in the series "Springer Monographs in Mathematics." But I must mention that the book is based on lectures delivered by the author at the Independent University of Moscow in 2005–2006, substantially extended and finalized. As well as [18], this book was originally mainly intended for beginning mathematicians and is accessible in Russia to four-year students (familiar, in particular, with the basic notions of functional analysis, such as Lebesgue measure, Hilbert and Banach spaces, compact operators and their spectra, etc.).

References to the required material of undergraduate mathematical courses are usually not given.

First of all, this book may be useful to students and postgraduate students specializing in partial differential equations and functional analysis. But it may also interest those specializing in other areas of mathematics, including geometry and applied mathematics, and in physics.

We now describe the contents of the book in more detail.

In the first chapter we consider, in particular, the spaces H^s (the simplest Bessel potential spaces) of negative order, which are not Sobolev–Slobodetskii spaces; they are used in Chapter 3. Even in the case of the half-space $\mathbb{R}^n_+$, some statements of Sections 3 and 4 require fairly delicate proofs. The construction of a bounded universal extension operator for functions defined on a domain to the entire space, proposed by Rychkov in [320], is postponed to Section 10, because this operator is constructed for Lipschitz domains.

In the second chapter, we proceed to the general theory of elliptic equations and elliptic boundary value problems. It is developed in detail in Sections 6 and 7 with minimal generality, but the main generalizations are stated or, at least, mentioned and explained. Note that the theory of elliptic equations and problems constructed on the basis of the theory of Sobolev-type spaces, in turn, substantially influenced the latter by providing, first, important questions and, second, some convenient results.

Our point of view on what topics form the foundations of the theory of elliptic equations is, possibly, not quite canonical. As essential elements of this theory we regard the theory of equations elliptic with parameter and the "more classical" theory of strongly elliptic equations, in which the key role is played by uniquely solvable (rather than Fredholm) equations and problems. The classical variational Dirichlet and Neumann problems for strongly elliptic equations are first considered in smooth domains in Section 8. In our view, strongly elliptic equations presently constitute a rich of content section of the theory of elliptic equations rather than a past stage of its development.

The third chapter begins with Section 9, in which we first discuss the specifics of Lipschitz domains and surfaces. In particular, we show, following Stepanov's paper [363], that any Lipschitz function is differentiable almost everywhere; we also show that any convex function is Lipschitz. Then we explain the basic facts of the theory of H^s spaces on Lipschitz domains and Lipschitz surfaces. In Section 10 we, following [320], discuss discrete norms and a discrete representation of functions, which are popular in the current literature (preliminaries on these norms and representation are given in Section 1.14) and construct a universal extension operator (but only for the H^s spaces, with some simplifications in comparison with [320]).

Sections 11 and 12 present, against the background of the already described theory of general elliptic problems, the theory of basic boundary value problems in Lipschitz domains in the same simplest spaces for second-order strongly elliptic systems. It is possibly a central place in the book. These sections develop and supplement the material of the excellently written book [258]. Much progress in this theory has been made during the past 15–20 years. The author's personal achievements are mentioned in Section 19. In our exposition of this theory, its technical details were methodologically purged. This theory can hardly be considered quite complete. Sections 11 and 12 and their continuation in Sections 16 and 17 may be interesting for specialists in the theory of partial differential equations, functional analysis, and probably other fields of mathematics or physics.

Some decades ago the author had an occasion to participate in the development of the general theory of (pseudodifferential) elliptic boundary value problems and the corresponding spectral problems, and in recent years, of the theory of boundary value problems in Lipschitz domains. The latter theory impelled him to reevaluate, to some extent, classical things, which was one of the stimuli for writing this book. Hopefully, it will also be interesting to the reader to compare these theories.

The priority for the author was to present the exposition accessible. For this reason, statements and proofs are not always given in full generality.

A beginning mathematician deserves not only a self-contained exposition of the simplest version of a theory but also explanations of what lies further, as far as possible without unnecessary details. For this reason, our book contains the survey fourth chapter devoted to generalizations of the material of the first three chapters to the Bessel potential spaces H_p^s and the Besov spaces B_p^s. This is an important but not easy material, and its complete exposition with all proofs would require too much space. Certainly, the exposition is accompanied by literature references, in particular, to many monographs. This chapter begins with an essay on interpolation theory in Section 13, which is written from the standpoint of a "consumer" of this remarkable theory. Although this section is of review character, it contains proofs of several very useful theorems. In Sections 16 and 17 important theorems supplementing the material of Chapter 3 are proved by using elements of interpolation theory.

We deviate from the tradition of proving everything, which is usually followed by the authors of mathematical books, but try to comment all definitions and facts presented in the book. The character of the exposition is closer to that of a lecture course than of a monograph. For a lecturer, the informal clarity of exposition is more important than its formal completeness.

This book cannot be used as a handbook on Sobolev-type spaces and elliptic problems.

We use notions of distribution theory; a brief introduction to this theory is contained in the author's preceding book [18], but it has not been translated into English. The reader may use other books, e.g., classical books by L. Schwartz [335] and Gel'fand and Shilov [165]. On the other hand, the author is planning to write a continuation of the present book, the third book [26]. There, in particular, the calculus of pseudodifferential elliptic operators will be touched on, while here we restrict ourselves to the classical method of freezing coefficients, which has not lost its significance, is transparent, and deserves to be mastered. In the author's opinion, for a beginning analyst, it is more useful to learn this method before studying the calculus of pseudodifferential operators, not to mention the more complicated calculus of pseudodifferential elliptic boundary value problems. The method of freezing coefficients is used not only to study "smooth" elliptic problems but also to obtain the Gårding inequality in Lipschitz domains.

However, pseudodifferential operators are repeatedly mentioned in the book; therefore, the main definitions and a few basic facts of their theory are given in Section 18.

To orient the reader, we touch on spectral problems at several places, but we do not give detailed proofs of theorems on which the study of these problems is based. We plan to present these theorems in [26]. In particular, asymptotic formulas for eigenvalues are given without proof.

Section 18 contains reference information from the general theory of linear operators. First, statements concerning Fredholm operators and the Lax–Milgram theorem are presented with proofs. Then some basic facts of the spectral theory of operators (Section 18.3) and results related to pseudodifferential operators (Section 18.4) are given without proof.

Section 19 is devoted to comments on the preceding sections. Here references to the literature are detailed and some directions in the theory of elliptic equations and related problems are mentioned, which border on the material of the preceding sections but are not touched upon there.

Sections 18 and 19 are not included in Chapter 4.

The author again thanks his listeners at the Independent University of Moscow, especially Polina Vytnova, Nikolai Gorev, Vasilii Novikov, and Mikhail Surnachev; discussions with them have been of great help in selecting the material and searching for an accessible form of exposition.

Special thanks are due to Prof. V. I. Ovchinnikov for reading and criticizing the initial version of Section 13 and to Prof. S.E. Mikhailov for useful remarks to the initial version of Sections 11.1 and 11.2.

It was the greatest luck for the author that Prof. T. A. Suslina agreed to take on the scientific editing of the book. She was the first reader of the yet raw Russian text. Thanks to her highest mathematical level, exceptional scrupulousness, and interest in the topic, many refinements and substantial improvements were made.

In the English version, Chapters 3 and 4 are somewhat reorganized. The number of sections has increased by one. The text of the book has underwent new refinements and improvements; in particular, Sections 17.4–17.6 are added. Again, of invaluable help were Suslina's comments and remarks.

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The author would be grateful for any remarks and comments; the readers are kindly asked to send them to the address `magran@orc.ru`

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