Chapter 2
Elliptic Equations and Elliptic Boundary Value Problems

In this chapter, we prove main theorems of the general theory of "smooth" elliptic equations and problems. Generality is somewhat minimized; in particular, we dwell on scalar equations and problems. More general facts (with similar proofs) are only stated or mentioned. We rarely touch on pseudodifferential operators in this chapter; their theory will be expounded in [26], where the material of this chapter will be substantially supplemented.

In Section 6, the basic ideas of the general theory are explained in the case of a scalar elliptic partial differential equation on a closed C^∞ manifold. In this case, there are no boundary conditions. A particular case is an equation (in fact, on the torus) with periodic boundary conditions. The coefficients in the equation are assumed to be C^∞.

Our main theorems are on the equivalence of ellipticity and the Fredholm property in the spaces H^s, on the smoothness of solutions of equations with smooth right-hand sides, and on the unique solvability of elliptic equations with a parameter (we consider only the case of a linear dependence on the parameter). Their proofs use the method of freezing coefficients. The necessary material related to abstract Fredholm operators is collected in Section 18.1. In particular, all needed definitions are given there.

In Section 7, similar theorems are proved for general scalar boundary value problems in a smooth bounded domain.

In these two sections, 6 and 7, we also outline an explanation of the spectral theory of elliptic equations and problems. Some preliminaries related to abstract notions of spectral theory are given in Section 18.3.

Analogues of the main theorems for matrix equations and problems are stated.

Section 8 is devoted to basic variational boundary value problems, namely, the Dirichlet and Neumann problems. We first consider these problems for second-order strongly elliptic equations and then briefly outline their generalization to higher-order systems. These problems, which have a much longer history, are particularly close to applications.

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The Sobolev spaces are also very useful in the theory of parabolic and hyperbolic equations and boundary value problems. Parabolic problems will be mentioned in Section 7.1 (and Section 17.6), while hyperbolic problems are beyond the scope of this book.

6 Elliptic Equations on a Closed Smooth Manifold

6.1 Definitions

Consider an mth-order linear partial differential operator on $\mathbb{R}^n$:

$$Au(x) = a(x,D)u(x) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha u(x). \tag{6.1.1}$$

Here $u(x)$ is a scalar function. The coefficients are generally complex-valued functions; we suppose them to be infinitely differentiable and uniformly bounded together with all their derivatives:

$$|D^\beta a_\alpha(x)| \leq C_{\alpha,\beta}. \tag{6.1.2}$$

We know that such an operator acts boundedly from $H^s(\mathbb{R}^n)$ to $H^{s-m}(\mathbb{R}^n)$ for any $s \in \mathbb{R}$:

$$\|Au\|_{H^{s-m}(\mathbb{R}^n)} \leq C_s\|u\|_{H^s(\mathbb{R}^n)}. \tag{6.1.3}$$

Its *symbol* is the polynomial $a(x,\xi)$ obtained from $a(x,D)$ by replacing all D_j by real numbers ξ_j. The *principal symbol* $a_0(x,\xi)$ is the leading homogeneous part of the symbol:

$$a_0(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha. \tag{6.1.4}$$

Before the term "principal symbol" became conventional, this quantity was referred to as the *characteristic polynomial*.

The operator A is said to be *elliptic at a point* x if

$$a_0(x,\xi) \neq 0 \quad (0 \neq \xi \in \mathbb{R}^n). \tag{6.1.5}$$

This operator is said to be *elliptic on a set* $X \subseteq \mathbb{R}^n$ if it is elliptic at each point $x \in X$, and *uniformly elliptic on* X if there is a positive constant C such that

$$|a_0(x,\xi)| \geq C|\xi|^m \tag{6.1.6}$$

for all $x \in X$. For example, the Laplace operator

$$\Delta = -(D_1^2 + \ldots + D_n^2) \tag{6.1.7}$$

is uniformly elliptic on $\mathbb{R}^n$. If X is a compact (i.e., closed and bounded) set, then any operator elliptic on X is automatically uniformly elliptic on X, because, by continuity, an inequality of the form (6.1.6) with some constant holds on the compact set of points (x,ξ) with $x \in X$ and $|\xi| = 1$.

Instead of saying that the operator A is elliptic, we may say that the equation $Au = f$ is elliptic.

Now we define ellipticity with parameter. Let $\varLambda$ be a closed sector, or angle, in the complex plane with vertex at the origin; e.g., this may be a ray starting at the origin (and containing it). The operator A is said to be *elliptic with parameter in* $\varLambda$ *at a point* x if

$$a_0(x,\xi) - \lambda \neq 0 \quad (|\xi| + |\lambda| \neq 0,\ \lambda \in \varLambda); \tag{6.1.8}$$

A is said to be *elliptic with parameter in* $\varLambda$ on a set X if condition (6.1.8) holds at each point $x \in X$, and *uniformly elliptic with parameter in* $\varLambda$ *on* X if there is a positive constant C such that

$$|a_0(x,\xi) - \lambda| \geq C(|\xi|^m + |\lambda|) \tag{6.1.9}$$

for all $x \in X$ and $\lambda \in \varLambda$. For example, the operator $-\varDelta$ is uniformly elliptic with parameter on $\mathbb{R}^n$ in any closed sector with vertex at the origin not containing the positive half-axis.

Obviously, ellipticity follows from ellipticity with parameter: it suffices to set $\lambda = 0$ in the definition of the latter.

In the situation considered here, instead of A, the operator $A - \lambda I$ may be referred to as elliptic with parameter.

Remark 6.1.1. Obviously, the ellipticity of an operator A implies that the *coefficients of all higher-order derivatives* D_j^m *in* A *are nonzero*. Indeed, this follows from (6.1.5) with the vectors ξ one of whose coordinates, ξ_j, is 1 and the remaining coordinates are zero.

Next, *for* $n > 2$*, ellipticity implies the evenness of the order* m. Indeed, consider, e.g., the equation

$$a_0(x,\xi',\zeta) = 0 \tag{6.1.10}$$

with respect to ζ; here ζ is written instead of ξ_n. By virtue of ellipticity, for real $\xi' \neq 0$, the roots of this equation are nonreal. If $n > 2$, then we can pass continuously from any point $\xi' \neq 0$ to the point $-\xi'$ along the hyperplane $\xi_n = 0$ avoiding the origin. Under such a passage, the numbers of roots ζ in the upper and the lower half-plane are preserved, and *these numbers are equal*, because, when we replace ξ' by $-\xi'$, each root ζ is mapped to the root $-\zeta$.

If the operator A *is elliptic with parameter, then the coincidence of the numbers of the roots in the upper and the lower half-plane and evenness are obtained for* $n = 2$ *in a similar way*, because we can pass from any point $(\xi',0) \neq 0$ in $\mathbb{R}^2 \times \varLambda$, $\xi \neq 0$, to $(-\xi',0)$ avoiding $(0,0)$.

If m is even, we set $m = 2l$.

Now consider an mth-order partial differential operator A on a closed manifold M. Generally, it can be written in the form (6.1.1) only locally, in local coordinates.

A convenient exception is the standard torus $\mathbb{T}^n = [0, 2\pi]^n$, on which global 2π-periodic coordinates $x = (x_1, \dots, x_n)$ can be used.

If the coefficients in A are infinitely differentiable, then this is a bounded operator from $H^s(M)$ to $H^{s-m}(M)$ for any s.

An examination of the behavior of the principal symbol $a_0(x, \xi)$ under transformations of coordinates shows that $a_0(x, \xi)$ can be treated globally as a function on the cotangent bundle T^*M (see [132]). Therefore, on a manifold, the definitions of ellipticity and ellipticity with parameter still make sense; we mean now global ellipticity and ellipticity with parameter everywhere. They are automatically uniform by the compactness of the manifold.

A classical example of an elliptic operator on a manifold is the Beltrami–Laplace operator Δ on a Riemannian manifold M. Let us recall its expression. If the metric is locally written in the form $ds^2 = \sum g_{j,k}\, dx_j\, dx_k$, where $g_{j,k} = g_{k,j}$, $(g^{j,k})$ is the matrix inverse to $(g_{j,k})$, and g denotes the determinant of $(g_{j,k})$, then

$$\Delta = -\frac{1}{\sqrt{g}} \sum D_j(\sqrt{g}\, g^{j,k} D_k). \tag{6.1.11}$$

The basic statements of the theory of elliptic operators on a closed manifold are the equivalence of the ellipticity of an operator A to the Fredholm property of A (in particular, to the presence of a parametrix; see Section 18.1) as an operator from $H^s(M)$ to $H^{s-m}(M)$ for any s and the equivalence of its ellipticity with a parameter in Λ to the invertibility of the operator $A - \lambda I$ for $\lambda \in \Lambda$ with sufficiently large absolute values. Some results are also obtained for uniformly elliptic operators on $\mathbb{R}^n$, but in this case, there is no equivalence to the invertibility or the Fredholm property of A. The reason for this difference is that, for $s_1 < s_2$, the space $H^{s_2}(M)$ is embedded in $H^{s_1}(M)$ compactly (see Theorem 2.3.1), while for $\mathbb{R}^n$ instead of M, this is not true.

6.2 Main Theorems

Let A be an mth-order partial differential operator on M with C^∞ coefficients.

Theorem 6.2.1. *The following conditions are equivalent.*

1°. *The operator A is elliptic on M.*

2°. *This is a Fredholm operator from $H^s(M)$ to $H^{s-m}(M)$ for any s.*

3°. *The operator A has a two-sided parametrix B acting boundedly from $H^{s-m}(M)$ to $H^s(M)$ for which $T_1 = BA - I$ and $T_2 = AB - I$ are bounded operators from $H^s(M)$ to $H^{s+1}(M)$ for any s.*

4°. *The a priori estimate*

$$\|u\|_{H^s(M)} \leq C_s(\|Au\|_{H^{s-m}(M)} + \|u\|_{H^{s-1}(M)}) \tag{6.2.1}$$

with a constant not depending on u holds.

The most important assertion is the equivalence of 1° and 2°, and the theorem is usually proved by the scheme

$$1^\circ \Rightarrow 3^\circ \Rightarrow 2^\circ \Rightarrow 4^\circ \Rightarrow 1^\circ. \tag{6.2.2}$$

In comparison with the abstract situation considered in Section 18.1, we additionally have the *scales* of spaces $X_s = H^s(M)$ and $Y_s = H^{s-m}(M)$; the latter is obtained from the former by shifting the index. Moreover, the parametrix has stronger properties than in our abstract case: the operators T_1 and T_2 are smoothing, i.e., increase smoothness; namely, they take functions in $H^s(M)$ to functions in $H^{s+1}(M)$ (in particular, these operators are compact in each $H^s(M)$). We refer to such parametrices as *qualified*.

We also mention that the above a priori estimate turns out to be *two-sided*: its right-hand side is dominated by the left-hand side. This shows that the spaces H^s are adequate to the operators under consideration.

As mentioned in Proposition 18.1.6, if the kernel of A is trivial, then the term $\|u\|_{s-1}$ on the right-hand side of the estimate can be omitted. In the general case, this norm can be replaced by the norm of any order lower than m.

In [26], a similar theorem will be proved for more general pseudodifferential elliptic operators by means of the *calculus* of these operators, which is constructed in advance. All analytical work is concentrated in the construction of this calculus. Instead, here we give an outline of the proof by a classical method of the theory of elliptic equations, which was developed long before this calculus. Its key ingredients are localization by using a partition of unity on M and "freezing coefficients." This method is known as the *method of freezing coefficients*.

Proof of Theorem 6.2.1. Let us verify that 1° ⇒ 3°. We describe the construction of the right parametrix in detail.

Step 1. Consider the operator A on $\mathbb{R}^n$. We remove the lower-order terms and freeze the coefficients at a point x_0, obtaining the homogeneous operator

$$A_0 = a_0(x_0, D) = \sum_{|\alpha|=m} a_\alpha D^\alpha \tag{6.2.3}$$

with constant coefficients. We set

$$B_0 = F^{-1} \frac{|\xi|^m}{(|\xi|^m + 1) a_0(x_0, \xi)} F, \tag{6.2.4}$$

where F is the Fourier transform in the sense of distributions. The numerator cancels the singularity in the denominator at the origin. The fraction $|\xi|^m/(|\xi|^m + 1)$ tends to 1 as $|\xi| \to \infty$. Obviously, we have $A_0 B_0 = I + T_0$, where

$$T_0 = -F^{-1} \frac{1}{|\xi|^m + 1} F$$

is a smoothing operator (it increases the order of smoothness by m, acting boundedly from $H^s(\mathbb{R}^n)$ to $H^{s+m}(\mathbb{R}^n)$). Of course, this operator is not compact in $H^s(\mathbb{R}^n)$, but its compactness is not required. We refer to the operator B_0 as a *quasi-parametrix* for A_0.

Step 2. Consider the operator $a(x,D)$ on $\mathbb{R}^n$ with any lower-order terms (of order less than m) and leading coefficients close to constants equal to their values at x_0. We write it in the form

$$a(x,D) = a_0(x_0,D) + a_1(x,D) + a_2(x,D), \tag{6.2.5}$$

where $a_1(x,D)$ is the sum of the lower-order terms of $a(x,D)$ and $a_2(x,D)$ is obtained from the leading part of $a(x,D)$ by subtracting $a_0(x_0,D)$. Now we shall show that

$$a(x,D)B_0 = I + T_1 + T_2, \tag{6.2.6}$$

where T_1 is again a smoothing operator and T_2 is an operator with norm less than 1 for fixed s, provided that the values of the leading coefficients in $a(x,D)$ are close enough to their values at x_0. The operator $I+T_2$ is invertible, and the required quasi-parametrix for $a(x,D)$ is obtained in the form

$$B_0(I+T_2)^{-1}. \tag{6.2.7}$$

Let us verify the assertion concerning the right-hand side of (6.2.6). As we have seen, $a_0(x_0,D)B_0$ is the sum of the identity and a smoothing operator. Consider $a_2(x,D)B_0$. This operator consists of products of the coefficients in $a_2(x,D)$ (whose absolute values have small upper bounds) and operators of order zero. According to Corollary 1.9.4, this operator decomposes into the sum of a smoothing operator and the operator T_2, whose norm is small if the coefficients in $a_2(x,D)$ are close enough to zero. Gathering all smoothing terms in T_1, we obtain the quasi-parametrix (6.2.7).

Step 3. Take two systems of infinitely differentiable functions on the manifold, $\{\varphi_j(x)\}$ and $\{\psi_j(x)\}$. The former is a sufficiently fine finite partition of unity: $\sum\varphi_j = 1$. The latter consists of functions $\psi_j(x)$ such that, for each j, $\psi_j(x) = 1$ in a neighborhood of the support of φ_j and the support of ψ_j is contained in some coordinate neighborhood U_j on M. The operator A can be written in the form

$$A = \sum \psi_j A\varphi_j \cdot . \tag{6.2.8}$$

Suppose that we can pass to local coordinates in the jth summand and replace the operator A by an elliptic operator A_j on $\mathbb{R}^n$ with leading coefficients close to constants which has the quasi-parametrix B_j (constructed at Step 2). Then

$$A = \sum \psi_j A_j \varphi_j \cdot, \tag{6.2.9}$$

and we construct a right parametrix for A in the form

$$B = \sum \varphi_k B_k \psi_k \cdot . \tag{6.2.10}$$

Obviously,

$$ABf = \sum \psi_j A_j \varphi_j \varphi_k B_k \psi_k f. \tag{6.2.11}$$

The only nonzero terms in this sum are those in which $\varphi_j \varphi_k \neq 0$. In these terms, we pass from the expression for A in the form A_j to the expression in the form A_k. This can be done if the supports of all φ_j are sufficiently small. The commutator $A_k(\varphi_j \varphi_k) - (\varphi_j \varphi_k) A_k$ is a partial differential operator of order at most $m-1$. We obtain

$$ABf = \sum \varphi_j \varphi_k f + Tf = f + Tf,$$

where T is an operator increasing the smoothness of functions in $H^{s-m}(M)$ by 1 and, therefore, compact in $H^{s-m}(M)$. We have reached our goal.

Actually, this construction is performed for all s in a finite but arbitrarily long interval simultaneously (this is a little less than promised in the statement of the theorem; the truth of this theorem in full completeness will be proved in [26]).

The construction of a left parametrix is similar; the only difference is in obvious permutations in the expressions written above. We leave this construction to the reader. Both parametrices turn out to be two-sided (see Proposition 18.1.4).

$3° \Rightarrow 2° \Rightarrow 4°$: see Propositions 18.1.3 and 18.1.6.

The implication $4° \Rightarrow 1°$ is proved by contradiction. Suppose that ellipticity is violated at some point (x_0, ξ_0) of the cotangent bundle, where $\xi_0 \neq 0$. Then the a priori estimate turns out to be false: it is disproved by substituting the function $\varphi(x)\exp(\lambda x \cdot \xi_0)$ in local coordinates, where φ is a smooth function that has small support containing x_0 and takes the value 1 near this point and λ is a positive parameter tending to infinity. After reducing the exponential we see that if the support of φ is small enough, then the left-hand side grows somewhat faster than the right-hand one. We leave the verification of this assertion to the reader. $\square$

The index of an elliptic operator does not depend on s. This will be explained in Section 6.3.

Remark 6.2.2. The proof can be changed as follows. Instead of constructing the left parametrix, we can derive an a priori estimate (again in three steps), which will also imply the finite dimensionality of the kernel and the closedness of the range; see Proposition 18.1.7.

Theorem 6.2.3. *Suppose that A is an elliptic operator and $Au = f$, where $u \in H^s(M)$ but $f \in H^{s-m+\tau}(M)$, $\tau > 0$. Then $u \in H^{s+\tau}(M)$.*

We refer to this theorem as the *theorem on the regularity of solutions*, or *on increasing the smoothness of solutions*. It is proved by applying the parametrix to our equation on the left:

$$BAu = u + Tu = Bf. \tag{6.2.12}$$

We see that $u \in H^{\min(s+1, s+\tau)}(M)$; if $\tau > 1$, then the parametrix is applied again as many times as needed.

Corollary 6.2.4. *The kernel* Ker A *of an elliptic operator on M in any space $H^s(M)$ consists of infinitely differentiable functions and, therefore, does not depend on s.*

We add that Theorem 6.2.3 has a local version, which asserts that if the right-hand side has higher smoothness in some domain on the manifold, then the smoothness of the solution is accordingly higher on this domain. To prove this, we apply the parametrix on the left to the equality

$$A\psi u = (A\psi u - \psi Au) + \psi f,$$

where ψ is a smooth function supported in the domain under consideration and taking the value 1 in a smaller subdomain.

Theorem 6.2.5. *If an operator A on M is elliptic with a parameter in a sector Λ, then the equation*

$$(A - \lambda)u = f \tag{6.2.13}$$

with $f \in H^0(M)$ is uniquely solvable in $H^m(M)$ for $\lambda \in \Lambda$ with sufficiently large $|\lambda|$. Moreover, the a priori estimate

$$\|u\|_{H^m(M)} + |\lambda|\|u\|_{H^0(M)} \leq C\|f\|_{H^0(M)} \tag{6.2.14}$$

with a constant not depending on λ holds for these λ. The condition of ellipticity with parameter is necessary for the validity of this estimate.

For simplicity, we give here the statement only for $s = m$.

Proof. The proof of this theorem is similar to but simpler than that of Theorem 6.2.1. We consider the solution in the norm $|||u|||_{H^m(M)}$, which depends on the parameter and equals the left-hand side of (6.2.14). Already at the first step, instead of the quasi-parametrix, we obtain the inverse operator in the form

$$B_0 = F^{-1}\frac{1}{a_0(x_0,\xi) - \lambda}F; \tag{6.2.15}$$

it is sufficient to assume $\lambda \in \Lambda$ to be nonzero. At the second step we obtain a right inverse operator for λ with sufficiently large absolute values. Even larger absolute values of λ should be taken at the third step, and again a right inverse operator is obtained. □

For the remaining values of λ, we obtain the Fredholm property for $A - \lambda$ with index zero.

6.3 Adjoint Operators

Let $(u, v)_M$ denote an inner product in $L_2(M)$, and let A and A^* be partial differential operators of order m related by

$$(Au, v)_M = (u, A^*v)_M \tag{6.3.1}$$

for infinitely differentiable functions u and v. Then these operators are said to be *formally adjoint*, and if $A = A^*$, then A is a *formally self-adjoint operator*. For example, the Beltrami–Laplace operator on a Riemannian manifold is formally self-adjoint. Of course, the adjointness relation between A and A^* depends on the choice of the inner product. If the local coordinates are compatible with the density on M determining the inner product (see Section 2.1), then, in these coordinates,

$$A^* = \sum_{|\alpha|\leq m} D^\alpha[\overline{a_\alpha(x)\cdot}] \quad \text{if and only if} \quad A = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha. \tag{6.3.2}$$

In particular, this is true for operators on the standard torus with periodic coordinates.

Clearly, $A^{**} = A$.

Suppose that the operator A is elliptic. Then so is the operator A^*, because its principal symbol is the function complex conjugate to the principal symbol of A. Both these operators have finite-dimensional kernels consisting of infinitely differentiable functions. For A and A^* considered as operators from $H^m(M)$ to $H^0(M) = L_2(M)$, relation (6.3.1) remains valid (it is transferred to functions in $H^m(M)$ by the passage to the limit). This means that *A and A^* are adjoint as unbounded operators in the Hilbert space $H^0(M)$* (and have common domain $H^m(M)$). Their ranges $R(A)$ and $R(A^*)$ are closed in $H^0(M)$.

Proposition 6.3.1. *The following relations hold*:

$$H^0(M) = R(A) \oplus \operatorname{Ker} A^* = R(A^*) \oplus \operatorname{Ker} A. \tag{6.3.3}$$

Proof. It is seen from (6.3.1) that if a function v belongs to $\operatorname{Ker} A^*$, then it is orthogonal to $R(A)$ in $L_2(M)$. Let us verify the converse. Relation (6.3.1) remains valid for $u \in H^m(M)$ and $v \in H^0(M)$; in this case, we consider A^*v in $H^{-m}(M)$, extending the inner product on the right to $H^m(M) \times H^{-m}(M)$. This means that *$A\colon H^m(M) \to H^0(M)$ and $A^*\colon H^0(M) \to H^{-m}(M)$ are adjoint as operators in Banach spaces* (which are Hilbert here). Suppose that v is orthogonal to $R(A)$. Then $(u, A^*v)_M = 0$ for all $u \in H^m(M)$, so that $A^*v = 0$ in $H^{-m}(M)$; cf. Proposition 18.1.8. But we know that the kernel of an elliptic operator consists of C^∞ functions, in particular, $v \in \operatorname{Ker} A^* \subset H^m(M)$. We have proved the first equality for $H^0(M)$ in (6.3.3); the proof of the second is similar. □

Corollary 6.3.2. *The codimension of the range of an elliptic operator $A\colon H^m(M) \to H^0(M)$ coincides with the dimension of the kernel of A^*. Therefore, for the index $\varkappa(A)$ of A, the formula*

$$\varkappa(A) = \dim \operatorname{Ker} A - \dim \operatorname{Ker} A^* \tag{6.3.4}$$

is true, and

$$\varkappa(A^*) = -\varkappa(A). \tag{6.3.5}$$

Now the index of an operator $A\colon H^s(M) \to H^{s-m}(M)$ can be defined by (6.3.4) for any s. We see that it does not depend on s.

Let us make the following additional remark. For any s, relation (6.3.1) remains valid if $u \in H^s(M)$ and $v \in H^{m-s}(M)$. This means that, *for any s, the operators $A\colon H^s(M) \to H^{s-m}(M)$ and $A^*\colon H^{m-s}(M) \to H^{-s}(M)$ are mutually adjoint as operators in Banach spaces*. It follows again that a function belongs to $R(A)$ in $H^{s-m}(M)$ if and only if it is orthogonal to $\operatorname{Ker} A^*$ with respect to the extension of the form $(\cdot,\cdot)_M$. Therefore, the codimension of the range of A in $H^{s-m}(M)$ does not depend on s; thus, the assertion that *the index $\varkappa(A)$ does not depend on s* remains valid under the original definition of index (given in Section 18.1). Moreover, the dimensions of the kernels $\operatorname{Ker} A$ and $\operatorname{Ker} A^*$, whose difference equals the index $\varkappa(A)$, do not depend on s either.

Properties of index will be discussed in more detail in [26]. We shall verify that the index of an operator does not depend on the lower-order terms of this operator and is homotopy invariant, that is, does not change under an ellipticity-preserving continuous variation of the coefficients in the principal symbol (cf. Proposition 18.1.12 (3)). The problem of calculating the index of a general (matrix) elliptic operator was essentially stated in Gel'fand's celebrated paper [164]. This paper has exerted a strong influence on the development of the theory of elliptic equations. In particular, it gave rise to the theory of elliptic pseudodifferential operators. The problem of index calculation was solved in topological terms for operators on a closed manifold by Atiyah and Singer [45].

6.4 Some Spectral Properties of Elliptic Operators

Let A be an mth-order elliptic operator on a manifold M. Consider it as an unbounded operator in $L_2(M)$ with domain $H^m(M)$. Suppose that the resolvent set of A is nonempty. (A necessary condition for this is the vanishing of the index.) Then this is an operator with discrete spectrum, because its resolvent is a bounded operator from $L_2(M)$ to the space $H^m(M)$, which is compactly embedded in $L_2(M)$.

It follows from Theorem 6.2.3 that all generalized eigenfunctions (root functions) of such an operator, that is, all eigenfunctions or eigen- and associated functions, belong to all spaces $H^s(M)$ and, hence, are infinitely smooth. They remain generalized eigenfunctions in all these spaces.

Any formally self-adjoint elliptic operator turns out to be self-adjoint in $L_2(M)$. An example of such an operator is the Beltrami–Laplace operator on a closed Riemannian manifold. Numerous mathematicians, beginning with H. Weyl (1912), studied the asymptotic behavior of eigenvalues of a self-adjoint elliptic operator. Its principal symbol (in the scalar case) is real and of constant sign. If this sign is plus, then all but possibly finitely many eigenvalues of the operator are positive and have a unique accumulation point, $+\infty$. Let us number them with positive integers in nondecreasing order with multiplicities taken into account. Then the following (Weyl's) asymptotic formula is valid:

$$\lambda_j \sim c j^q, \quad q = \frac{m}{n}, \tag{6.4.1}$$

where c is a positive constant expressed in terms of the principal symbol. We do not give this expression here. The difficult problem of estimating the remainder in (6.4.1), i.e., the difference between the left- and right-hand sides, has been studied by many authors. Hörmander showed that an optimal (in the general case) estimate for semibounded scalar (pseudodifferential) operators has the form $O(j^{(m-1)/n})$ (see [184]); under certain conditions, this estimate can be somewhat improved.

Eigenfunctions of a self-adjoint elliptic operator form an orthonormal basis $\{e_j\}$ in the space $L_2(M)$. Any function $f \in L_2(M)$ can be expanded in a series of the form

$$f = \sum c_j e_j, \quad c_j = (f, e_j)_M, \tag{6.4.2}$$

which unconditionally converges in the norm of this space. But the eigenfunctions belong to all spaces $H^s(M)$ and provide unconditional bases in all of them (with account of isomorphisms). Therefore, for $f \in H^s(M)$, the series unconditionally converges in $H^s(M)$.

By a *nearly self-adjoint elliptic operator* (or a *weak perturbation of a self-adjoint elliptic operator*) of order m we mean an operator which differs from a self-adjoint one by a term of order at most $m-1$. According to Theorem 18.3.1, the system of its generalized eigenfunctions is complete in $L_2(M)$ provided that its principal symbol is positive; as a consequence, this system is complete in all $H^s(M)$. Theorem 18.3.2 also applies to non-self-adjoint elliptic operators with $p = n/m$.

The directions of the most rapid decay of the resolvent (if they exist) are precisely the directions of ellipticity with parameter.

The spectral properties of elliptic operators will be discussed in more detail in [26].

The self-adjoint operator $-\Delta + I$ on a Riemannian manifold has positive eigenvalues. Therefore, its real powers $(-\Delta + I)^t$ are defined. This operator isomorphically maps $H^s(M)$ to $H^{s-2t}(M)$ for any s and t. This can be verified by means of interpolation theory (see Section 13.8.1).

6.5 Generalizations

Here we only list some generalizations of the theory of elliptic operators, without going into details.

1. The theory can be generalized to *matrix* operators, which act on vector-valued functions. This generalization is simplest when the leading parts of all elements of the matrix

$$a(x, D) = (a_{j,k}(x, D)) \tag{6.5.1}$$

are considered as having the same order m. In this case, from the symbols of these higher-order parts a (matrix) principal symbol $a_0(x,\xi)$ is composed. Some of its elements may be zero. The ellipticity of the operator (6.5.1) means that the determinant of the principal symbol does not vanish for $\xi \neq 0$. A self-adjoint matrix elliptic operator has Hermitian principal symbol with real nonzero eigenvalues. If there are eigenvalues of both signs, then the eigenvalues of the operator have the accumulation points $\pm\infty$ with asymptotics of type (6.4.1). For an optimal estimate of the remainder (in the case of pseudodifferential operators), see Ivrii [188].

A more general definition of ellipticity is as follows. Suppose that the matrix $a(x,D)$ has size $d\times d$. We fix two systems $m_1,\dots,m_d$ and $t_1,\dots,t_d$ of nonnegative integers and assume that the leading order in $a_{j,k}(x,D)$ is m_j+t_k. The principal symbol is composed of the symbols of these higher-order parts (some of which may vanish), and the ellipticity condition is again the assumption that the determinant of the principal symbol is nonzero. This property is known as *Douglis–Nirenberg ellipticity* [131]. The corresponding operator can be treated as an operator from the direct product of the spaces $H^{s+t_k}(M)$ to the direct product of the spaces $H^{s-m_j}(M)$. Note that it is not easy to define ellipticity with parameter in this case if the orders m_j+t_j of diagonal elements depend on j.

2. Matrix operators can be considered *on sections of vector bundles.* Roughly speaking, this means that not only the local coordinate system for the independent variables but also the coordinate system in which the vector of the scalar functions under consideration is written change as a point moves along the manifold; see, e.g., [185, vol. 3, Sec. 18].

3. We can define ellipticity with parameter for operators polynomially depending on a parameter; see, e.g., [10] and [34]. The parameter is then considered as having a fixed weight with respect to differentiation. For example, the operator may have the form

$$\sum_{j=0}^{m}\lambda^{m-j}A_j(x,D), \tag{6.5.2}$$

where each A_j is an operator of order j; here the weight of the parameter equals 1, while in Eq. (6.2.13) the weight equals m. If $a_{0,j}(x,\xi)$ is the principal symbol of A_j, then for (6.5.2), ellipticity with parameter in a sector Λ means that

$$\det\sum\lambda^{m-j}a_{0,j}(x,\xi)\neq 0,\quad (\xi,\lambda)\neq 0,\ \lambda\in\Lambda. \tag{6.5.3}$$

Such operators are the subject of the more general spectral theory of *operator pencils*, that is, operators polynomially depending on the spectral parameter; see, e.g., [249].

A relationship between problems elliptic with parameter and parabolic equations is described in the next section.

4. Relaxing the smoothness assumptions, we can consider elliptic operators on spaces $H^s(M)$ with s ranging in a finite interval. We shall have an occasion to use this possibility in Section 12 (see Theorem 12.1.1).

7 Elliptic Boundary Value Problems in Smooth Bounded Domains

In this section we consider only spaces H^s with nonnegative s, i.e., the Sobolev–Slobodetskii L_2-spaces.

7.1 Definitions and Statements of Main Theorems

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ. Suppose that we have a scalar linear partial differential operator

$$A = a(x,D) = \sum_{|\alpha|\le 2l} a_\alpha(x)D^\alpha \tag{7.1.1}$$

on Ω of even order $2l$ with coefficients infinitely differentiable on $\overline{\Omega}$.

Suppose also given l boundary operators

$$B_j = b_j(x,D) = \sum_{|\beta|\le r_j} b_{j\beta}(x)D^\beta \quad (j = 1,\dots,l) \tag{7.1.2}$$

on Γ of nonnegative orders r_j with coefficients infinitely differentiable on Γ. The boundary value problem which we shall consider in this section is

$$a(x,D)u(x) = f(x) \text{ in } \Omega, \tag{7.1.3}$$

$$b_j(x,D)u(x) = g_j(x) \text{ on } \Gamma \quad (j = 1,\dots,l). \tag{7.1.4}$$

With this boundary value problem we associate the operator $\mathfrak{A}u = (f, g_1,\dots,g_l)$ taking each solution to the corresponding set of right-hand sides.

In the simplest functional setting of the problem, the spaces for the functions u, f, and g_j are chosen as follows:

$$u \in H^s(\Omega), \quad f \in H^{s-2l}(\Omega), \quad g_j \in H^{s-r_j-1/2}(\Gamma). \tag{7.1.5}$$

Here we assume for simplicity that

$$s \ge 2l, \quad s > \max r_j + 1/2. \tag{7.1.6}$$

The following proposition must be obvious for those who have read the preceding sections.

Proposition 7.1.1. *The operator $\mathfrak{A}$ acts boundedly from the space $H^s(\Omega)$ to the direct product*

$$H^s(\Omega,\Gamma) = H^{s-2l}(\Omega) \times \prod_{j=1}^{l} H^{s-r_j-1/2}(\Gamma). \tag{7.1.7}$$

In other words, the estimate

$$\|f\|_{H^{s-2l}(\Omega)} + \sum_{j=1}^{l} \|g_j\|_{H^{s-r_j-1/2}(\Gamma)} \leq C_s \|u\|_{H^s(\Omega)} \tag{7.1.8}$$

with a constant not depending on u *is valid.*

Below we give the definition of ellipticity for this problem.

The *first ellipticity condition* is that the operator A is elliptic on $\overline{\Omega}$, that is, for its main symbol $a_0(x,\xi)$, we have

$$a_0(x,\xi) \neq 0 \quad (x \in \overline{\Omega},\ 0 \neq \xi \in \mathbb{R}^n). \tag{7.1.9}$$

The *second ellipticity condition*, which is called the *regular ellipticity* of A, is that the equation $a_0(x,\xi',\zeta) = 0$ with $\xi' \neq 0$ has the same number of roots ζ in the upper and lower half-planes (and this number equals l). It was mentioned in Remark 6.1.1 that this condition holds automatically if $n > 2$ or $n = 2$ and the operator A is elliptic with parameter. For $n = 2$, it eliminates examples of the type $(D_1 + iD_2)^2$. By continuity, regular ellipticity is preserved by any rotation of the coordinate system, and if it holds at some point x, then it holds everywhere.

The *third condition* is known as the *Lopatinskii condition*. It is imposed on the operators of the problem at each point x_0 of the boundary Γ. Its statement has the simplest form when the origin is transferred to a point x_0 and the coordinate system is rotated so that the $t = x_n$ axis is directed along the inner normal to the boundary at this point. Suppose that the operators of the problem are rewritten in this coordinate system. Consider the following problem on the ray $\mathbb{R}_+ = \{t\colon t > 0\}$ for fixed $\xi' = \xi_0' \neq 0$:

$$\begin{aligned} &a_0(x_0,\xi_0',D_t)v(t) = 0 && (t > 0),\\ &b_{j0}(x_0,\xi_0',D_t)v|_{t=0} = h_j && (j = 1,\dots,l), \end{aligned} \tag{7.1.10}$$

where $b_{j0}(x,\xi)$ are the principal symbols of the operators b_j:

$$b_{j,0}(x,\xi) = \sum_{|\beta|=r_j} b_{j\beta}(x)\xi^\beta. \tag{7.1.11}$$

Problem (7.1.10) is required to have precisely one solution in $L_2(\mathbb{R}_+)$ for any $\xi_0' \neq 0$ and any numbers h_j.

Note that, under the above assumptions, the space of those solutions of the equation $a_0(x_0,\xi_0',D_t)v(t) = 0$ which belong to $L_2(\mathbb{R}_+)$ consists of functions with absolute value decreasing exponentially as $t \to +\infty$; the dimension of this space is l, and the number of boundary conditions equals this dimension (cf. Remark 7.1.2, 1 below). Problem (7.1.10) is obtained from the original problem by freezing the coefficients at the point x_0, removing the lower-order terms, and applying the formal Fourier transform with respect to the tangent variables.

The Lopatinskii condition first appeared in its full generality in Lopatinskii's paper [242]. It is sometimes called the *Shapiro–Lopatinskii condition*, because similar considerations were performed by Shapiro [340, 341] under more special assumptions. This condition is also referred to by saying that the boundary operators *cover* the given elliptic operator.

If all of the three conditions stated above hold, then the problem is said to be *elliptic*.

The simplest example of an elliptic problem is the Dirichlet problem for the Poisson equation:

$$-\Delta u = f \text{ on } \Omega, \quad u = g \text{ on } \Gamma. \tag{7.1.12}$$

The second example is the Neumann problem for the same equation:

$$-\Delta u = f \text{ on } \Omega, \quad \partial_\nu u = g \text{ on } \Gamma, \tag{7.1.13}$$

where ∂_ν is the inner (for definiteness) normal derivative.

Problem 1. Verify the ellipticity of these problems.

Remark 7.1.2. The Lopatinskii condition can be reformulated as follows.

1. If the operator $a(x,D)$ is regularly elliptic, then, at each $\xi' \neq 0$, the polynomial $a(\zeta) = a_0(x_0, \xi', \zeta)$ can be factorized as

$$a_0(\zeta) = a_0^+(\zeta) a_0^-(\zeta), \tag{7.1.14}$$

where the roots of the polynomials $a_0^+(\zeta)$ and $a_0^-(\zeta)$ belong to the upper and lower half-planes, respectively. The coefficients in these polynomials depend smoothly on (x_0, ξ'). All solutions of the equation $a_0^+(D_t)v(t) = 0$ decrease exponentially in absolute value as $t \to \infty$; these are all solutions of the first equation in (7.1.10) whose absolute values decrease as $t \to \infty$. We set $b_{j,0}(\zeta) = b_{j,0}(x_0, \xi', \zeta)$. Given any $\xi' \neq 0$, the Lopatinskii condition at the point under consideration can be written as follows: *The problem*

$$a_0^+(D_t)v(t) = 0 \quad (t > 0), \quad b_{j,0}(D_t)v(t)|_{t=0} = h_j \quad (j = 1, \ldots, l) \tag{7.1.15}$$

is uniquely solvable for any h_j.

It is easy to see from here that the Lopatinskii condition is also equivalent to the condition that *the remainders* $\widetilde{b}_{0,j}(\zeta)$ *of the division of the polynomials* $b_{0,j}(\zeta)$ *by* $a_0^+(\zeta)$ *are linearly independent*. In (7.1.15), the polynomials $b_{j,0}$ can be replaced by $\widetilde{b}_{j,0}$.

2. This also implies the equivalence of the Lopatinskii condition to the uniqueness for problem (7.1.10) or (7.1.15) in $L_2(\mathbb{R}_+)$: if $h_j = 0$ for all j, then the solution belonging to $L_2(\mathbb{R}_+)$ is trivial.

3. On the other hand, the Lopatinskii condition is equivalent to that obtained by replacing zero on the right-hand side of the first equation by any function $f(t) \in L_2(\mathbb{R}_+)$. Indeed, extending this function by zero to $t < 0$, we can easily construct a

solution of the equation $a_0(D_t)v_0(t) = f(t)$ in $L_2(\mathbb{R}_+)$ by using the Fourier transform as

$$v_0(t) = F^{-1}[a_0(\xi)]^{-1}(Ff)(\xi)\,d\xi.$$

Subtracting $v_0(t)$ from the solution $v(t)$ of the equation $a_0(D_t)v(t) = f(t)$, we obtain problem (7.1.10).

The *Dirichlet problem* for a regularly elliptic equation of order $2l$ is the problem with boundary conditions

$$\partial_\nu^j u = g_j \quad (j = 0, \dots, l-1). \tag{7.1.16}$$

As a consequence of the first remark, we obtain the following generalization of the first example: *The Dirichlet problem for any* (*scalar*) *regularly elliptic equation of order* $2l$ *is elliptic.* Indeed, in this case, the remainders are the polynomials $b_{j,0}(\zeta) = \zeta^j$ themselves.

In particular, the Dirichlet problem for the equation

$$\Delta^l u = f \tag{7.1.17}$$

is elliptic.

The "mainest" theorem of the theory of the elliptic problems is similar to Theorem 6.2.1 and is stated as follows.

Theorem 7.1.3. *The following conditions are equivalent.*

1°. *The boundary value problem* (7.1.3)–(7.1.4) *is elliptic.*

2°. *The operator* $\mathfrak{A}\colon H^s(\Omega) \mapsto H^s(\Omega, \Gamma)$ *is Fredholm.*

3°. *The operator* $\mathfrak{A}$ *has a two-sided parametrix* $\mathfrak{R}$ *acting boundedly from* $H^s(\Omega,\Gamma)$ *to* $H^s(\Omega)$ *and such that the operators* $\mathfrak{A}\mathfrak{R} - \mathcal{I}$ *and* $\mathfrak{R}\mathfrak{A} - I$ *act boundedly from* $H^s(\Omega,\Gamma)$ *to* $H^{s+1}(\Omega,\Gamma)$ *and from* $H^s(\Omega)$ *to* $H^{s+1}(\Omega)$*, respectively.* Here $\mathcal{I}$ and I denote the corresponding identity operators.

4°. *The a priori estimate*

$$\|u\|_{H^s(\Omega)} \le C'_s\Big[\|f\|_{H^{s-2l}(\Omega)} + \sum_{j=1}^{l} \|g_j\|_{H^{s-r_j-1/2}(\Gamma)} + \|u\|_{H^0(\Omega)}\Big] \tag{7.1.18}$$

holds, where the constant does not depend on u*.*

We again have two scales of spaces. As to the estimate, it is again two-sided: the right-hand side is dominated by the left-hand side (see (7.1.8)). This is yet another evidence that the Sobolev–Slobodetskii spaces are adequate to the problems under consideration. In the case of uniqueness, the last term on the right-hand side in (7.1.18) is unnecessary. The parametrix is again qualified.

The next theorem is on increasing the smoothness of solutions, or on the regularity of solutions.

Theorem 7.1.4. *Suppose that the boundary value problem* (7.1.3)–(7.1.4) *is elliptic and a number* s *satisfies conditions* (7.1.6)*. Suppose also that* $\tau > 0$*,*

$$u \in H^s(\Omega), \quad f \in H^{s-2l+\tau}(\Omega), \quad \textit{and} \quad g_j \in H^{s-r_j-1/2+\tau}(\Gamma) \quad \textit{for} \quad j = 1,\dots,l.$$

Then $u \in H^{s+\tau}(\Omega)$.

In particular, if $f \in C^\infty(\overline{\Omega})$ and $g_j \in C^\infty(\Gamma)$, then $u \in C^\infty(\overline{\Omega})$.

This theorem has a local version, which asserts that if the smoothness of the right-hand sides is enhanced near some point, then the smoothness of the solution near this point is enhanced accordingly.

Next, there are conditions of *ellipticity with parameter*, which guarantee the unique solvability of the problem at large absolute values of the parameter. Below we state the simplest result of this kind. Instead of the equation $a(x,D)u = f$, we consider the equation with parameter

$$a(x,D)u - \lambda u = f \tag{7.1.19}$$

in Ω. For simplicity, we assume that the boundary conditions do not depend on λ. The parameter λ varies within a closed sector Λ in the complex plane with vertex at the origin. The definition of the ellipticity of a problem with parameter is similar to that of ellipticity. The changes are that the principal symbol $a_0(x,\xi)$ is replaced by $a_0(x,\xi) - \lambda$, it is assumed that this difference does not vanish for $(\xi,\lambda) \neq 0$ with $\xi \in \mathbb{R}^n$ and $\lambda \in \Lambda$, and in the Lopatinskii condition it is assumed that the problem on the ray for the equation

$$[a_0(x_0,\xi_0',D_t) - \lambda]v(t) = 0 \tag{7.1.20}$$

has a unique solution decreasing as $t \to +\infty$ for $(\xi_0',\lambda) \neq 0$, $\lambda \in \Lambda$. The conditions of ellipticity with parameter contain the ellipticity conditions (which correspond to $\lambda = 0$).

For example, the Dirichlet problem for the equation $\Delta u + \lambda u = f$ is elliptic with parameter along any ray except the positive half-axis. The same is true for the Neumann problem.

Problem 2. Prove this. Prove also that, if a regularly elliptic equation of order $2l$ is elliptic with parameter along some rays, then the Dirichlet problem for this equation is elliptic with parameter along the same rays.

Theorem 7.1.5. *If the boundary value problem* (7.1.19), (7.1.4) *is elliptic with parameter in* Λ, *then, for any* s *satisfying condition* (7.1.6), *this problem is uniquely solvable for* $\lambda \in \Lambda$ *with sufficiently large absolute values.*

Moreover, there is an a priori estimate uniform in the parameter. The simplest estimate under the assumptions $s = 2l > \max r_j$ and $g_j = 0$ is

$$\|u\|_{H^{2l}(\Omega)} + |\lambda|\|u\|_{H^0(\Omega)} \leq C\|f\|_{H^0(\Omega)}, \tag{7.1.21}$$

where the constant does not depend on u and λ. The conditions of ellipticity with parameter are necessary for this estimate to hold.

For other λ, the problem turns out to be Fredholm with index zero.

Note that problems with parameter are obtained from nonstationary mixed problems in the cylindrical domain $\Omega \times (0,\infty)$ with coefficients not depending on time $t \in (0,\infty)$ (above, the letter t had a different meaning). For example, consider the heat equation

$$\Delta U(x,t) - \partial_t U(x,t) = F(x,t) \tag{7.1.22}$$

in such a domain with Dirichlet boundary condition

$$U(x,t) = G(x,t) \quad (x \in \Gamma) \tag{7.1.23}$$

on the lateral surface and homogeneous (for simplicity) initial condition

$$U(x,0) = 0. \tag{7.1.24}$$

The formal Laplace transform

$$u(x,\lambda) = \int_0^\infty U(x,t)e^{-\lambda t}dt \tag{7.1.25}$$

reduces this problem to the problem with parameter for the Laplace operator, which was mentioned before the statement of Theorem 7.1.5. Here Λ is the right half-plane. In fact, this is a way to study nonstationary, "parabolic," problems; see, e.g., [10] and [34]. There is also a direct way—the study of mixed problems with coefficients depending in addition on t by the method of freezing coefficients (see, e.g., Solonnikov's paper [357] and Eidel'man's survey [145]).

Problem 3. Verify that the Dirichlet problem for the equation $\Delta^l u - \lambda u = 0$ is elliptic with parameter along any ray except $\mathbb{R}_+$ or $\mathbb{R}_-$. Derive from this that, for $s \geq 2l$, a bounded operator which maps any set of functions $v_j \in H^{s-j-1/2}(\Gamma)$ $(j = 0,\dots,l-1)$ to a function $u \in H^s(\Omega)$ with Cauchy data $v_0,\dots,v_{l-1}$ can be defined independently of s. (Cf. the statement after Theorem 5.1.9.)

Now we outline the proofs of Theorems 7.1.3–7.1.5. To simplify calculations, we assume that $r_j < 2l$ for all j and $s = 2l$. We dwell on only important points and omit some technical details similar to those considered in the preceding section but a little more cumbersome.

7.2 Proofs of Main Theorems

As in Section 6, the proof of Theorem 7.1.3 consists of three steps. We begin with the first one. Consider the problem in the half-space $\mathbb{R}^n_+$ for operators without lower-order terms and assume that the coefficients do not depend on x. The problem has the form

$$a_0(D)u(x) = f(x) \qquad (x_n > 0), \tag{7.2.1}$$

$$b_{j,0}(D)u(x)|_{x_n=0} = g_j(x') \quad (j = 1,\dots,l). \tag{7.2.2}$$

Applying the (formal) Fourier transform F' with respect to the tangent variables and setting $t = x_n$, we obtain the problem

$$a_0(\xi', D_t)v(\xi', t) = h(\xi', t) \quad (t > 0), \tag{7.2.3}$$

$$b_{j,0}(\xi', D_t)v(\xi', t)|_{t=0} = h_j(\xi') \quad (j = 1,\dots,l), \tag{7.2.4}$$

where $v = F'u$, $h = F'f$, and $h_j = F'g_j$. We denote the space of solutions of the equation $a_0(\xi', D_t)v(t) = 0$ which decrease as $t \to \infty$ by $\mathfrak{M} = \mathfrak{M}(\xi')$ and refer to any basis $\omega_1(t),\dots,\omega_l(t)$ in this space as a *stable basis*.

7.2.1 A Canonical Basis

In (7.2.3)–(7.2.4) with $h = 0$, we can replace the polynomial $a_0(\xi', \zeta)$ in ζ by $a_0^+(\xi', \zeta)$ and the polynomials $b_j(\xi', \zeta)$ in ζ by the remainders $\widetilde{b}_{j,0}(\xi', \zeta)$ after dividing them by $a_0^+(\xi', \zeta)$ (the variables ξ' play the role of parameters):

$$a_0^+(\xi', D_t)v(\xi', t) = 0 \quad (t > 0), \tag{7.2.5}$$

$$\widetilde{b}_{j,0}(\xi', D_t)v(\xi', t)|_{t=0} = h_j(\xi') \quad (j = 1,\dots,l). \tag{7.2.6}$$

Without loss of generality, we assume that the leading coefficient in a_0^+ is equal to 1. Note that the functions $\widetilde{b}_{j,0}(\xi', \zeta)$ are positive homogeneous in (ξ', ζ) of degree r_j for $\xi' \neq 0$.

A stable basis can be constructed in the form of the contour integrals

$$\omega_k(\xi', t) = \frac{1}{2\pi} \int\limits_\gamma \frac{e^{i\zeta t}\zeta^{k-1}}{a_0^+(\xi', \zeta)}\, d\zeta \quad (k = 1,\dots,l), \tag{7.2.7}$$

where $\gamma = \gamma(\xi')$ is a closed contour in the upper half-plane enclosing all roots ζ of the polynomial $a_0^+(\xi', \zeta)$, i.e., all roots of the polynomial $a_0(\xi', \zeta)$ in this half-plane. Obviously, these are solutions of Eq. (7.2.5) decreasing in absolute value as $t \to +\infty$, and these solutions are linearly independent. The contour may be changed, but locally, near the chosen point ξ_0', it can be assumed to be independent of ξ'. We define a *canonical basis* of $\Omega_k(\xi', t)$ in $\mathfrak{M}(\xi')$ by the conditions

$$\widetilde{b}_{j,0}(\xi', D_t)\Omega_k(\xi', t)|_{t=0} = \delta_{j,k} \quad (j, k = 1,\dots,l). \tag{7.2.8}$$

It exists by virtue of the Lopatinskii condition, and each function $\Omega_k(\xi', t)$ can be determined by substituting a linear combination of the functions $\omega_1(\xi', t)$, ..., $\omega_l(\xi', t)$ into condition (7.2.8) with fixed k. We obtain a linear system of equations with nonzero determinant, which uniquely determines the coefficients. Moreover, the $\Omega_k(\xi', t)$ have the form (7.2.7) but with the ζ^{k-1} replaced by some polynomials $N_k(\xi',\zeta)$ in ζ. These polynomials are determined not uniquely, but they can be assumed to satisfy the conditions

$$\widetilde{b}_{j,0}(\xi',\zeta)N_k(\xi',\zeta) = \delta_{j,k}\zeta^{l-1}.$$

Clearly, each $N_k(\xi',\zeta)$ is a polynomial of degree at most $l-1$ in ζ, which is positive homogeneous in (ξ',ζ) of degree $l-r_k-1$ for $\xi' \neq 0$. We have proved the following assertion.

Proposition 7.2.1. *The canonical basis has the form*

$$\Omega_k(\xi',t) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{e^{i\zeta t}N_k(\xi',\zeta)}{a_0^+(\xi',\zeta)}\,d\zeta \quad (k = 1,\dots,l), \tag{7.2.9}$$

where the $N_k(\xi',\zeta)$ are polynomials in ζ of degree at most $l-1$ in which all coefficients are C^∞ functions with respect to ξ' at $\xi' \neq 0$. Furthermore, the N_k are positive homogeneous of degree $l-r_k-1$ with respect to (ξ',ζ) at these ξ'.

Explicit expressions for N_k are given in [9] in a slightly different notation.

Proposition 7.2.2. *The following estimates with a constant not depending on ξ' are valid*:

$$\int\limits_0^\infty |D_t^j\Omega_k(\xi',t)|^2dt \leq C_1|\xi'|^{2(j-r_k)-1} \quad (j = 0,\dots,2l;\ k = 1,\dots,l). \tag{7.2.10}$$

Proof. Suppose that the contour γ does not depend on ξ' with $|\xi'| = 1$, that is, $\gamma(\xi') = \gamma_0$ for such ξ'. Suppose also that, for other ξ', $\gamma(\xi')$ is obtained from γ_0 by a homothety with coefficient $|\xi'|$, that is, $\gamma(\xi') = |\xi'|\gamma_0$. Clearly, this assumption can be made. Then, differentiating the integrand in (7.2.9) j times with respect to t and treating $\zeta/|\xi'|$ as a new variable, we obtain

$$|D_t^j\Omega_k(\xi',t)| \leq C|\xi'|^{j-r_k}e^{-\varepsilon|\xi'|t} \quad (j = 0,\dots,2l,\ k = 1,\dots,l)$$

with positive C and ε not depending on ξ' and t. Therefore,

$$\int\limits_0^\infty |D_t^j\Omega_k(\xi',t)|^2dt \leq C|\xi'|^{2(j-r_k)}\int\limits_0^\infty e^{-2\varepsilon|\xi'|t}dt.$$

Taking $|\xi'|t$ for a new variable, we obtain the required estimate. □

7.2.2 The A Priori Estimate

We assume for simplicity that $s = 2l > r_k$. At the first step, we want to obtain the estimate

$$\|u\|^2_{H^{2l}(\mathbb{R}^n_+)} \le C\Big[\|f\|^2_{H^0(\mathbb{R}^n_+)} + \sum_{k=1}^{l} \|g_k\|^2_{H^{2l-r_k-1/2}(\mathbb{R}^{n-1})} + \|u\|^2_{H^0(\mathbb{R}^n_+)}\Big] \tag{7.2.11}$$

for problem (7.2.1), (7.2.2). For convenience, we have replaced all norms by their squares. First, suppose that $f = 0$. Then the solution of problem (7.2.5), (7.2.6) can be expressed as

$$v(\xi', t) = \sum_{k=1}^{l} \Omega_k(\xi', t) h_k(\xi'). \tag{7.2.12}$$

Now it suffices to prove the estimate

$$\sum_{j=0}^{2l} |\xi'|^{2(2l-j)} \int_0^\infty |D_t^j v(\xi', t)|^2 dt \le C' \sum_{k=1}^{l} |\xi'|^{2(2l-r_k)-1} |h_k(\xi')|^2 \tag{7.2.13}$$

with a constant not depending on ξ' and the functions under consideration. Indeed, integrating this estimate with respect to ξ', we shall obtain an estimate which differs from the required estimate (7.2.11) in that it contains seminorms instead of norms. It will remain to add the zeroth-order squared norms of the right-hand sides g_j of the boundary conditions on the right and the squared zeroth norm of the solution on the right and left.

But estimate (7.2.13) follows directly from (7.2.10).

Now suppose that $f \neq 0$. This case is reduced to that considered above as follows. We continue $h(\xi', t)$ by zero to $t < 0$ and set

$$v_0(\xi', t) = F_n^{-1} a_0^{-1}(\xi) F_n h(\xi', t) \tag{7.2.14}$$

for $\xi' \neq 0$. Twice applying Parseval's identity with respect to the last variable, we obtain

$$|\xi'|^{2(2l-j)} \int_0^\infty |D_t^j v_0(\xi', t)|^2 dt$$
$$\le C_1 \int \frac{|\xi'|^{2(2l-j)} \xi_n^{2j}}{|a_0(\xi)|^2} |(F_n h)(\xi)|^2 \, d\xi_n \le C_2 \int_0^\infty |h(\xi', t)|^2 dt.$$

This implies the required estimate, because the integration of the integral on the right-hand side with respect to ξ' yields $\|f\|^2_{H^0(\mathbb{R}^n_+)}$.

Thus, we have obtained estimate (7.2.11) and, thereby, completed the first step. Let us rewrite the result for the operators without lower-order terms with coefficients frozen at x_0:

$$\|u\|^2_{H^{2l}(\mathbb{R}^n_+)}$$
$$\leq C\Big[\|a_0(x_0,D)u\|^2_{H^0(\mathbb{R}^n_+)} + \sum_{k=1}^{l} \|b_{k,0}(x_0,D)u\|^2_{H^{2l-r_k-1/2}(\mathbb{R}^{n-1})} + \|u\|^2_{H^0(\mathbb{R}^n_+)}\Big].$$

At the second step we consider the problem in the half-space for operators containing lower-order terms:

$$a(x,D) = a_0(x,D) + a_1(x,D) \quad \text{and} \quad b_k(x,D) = b_{k,0}(x,D) + b_{k,1}(x,D).$$

Here we assume that all coefficients in the higher-order terms $a_0(x,D)$ and $b_{k,0}(x,D)$ are close to those at the boundary point x_0. We obtain

$$\|u\|^2_{H^{2l}(\mathbb{R}^n_+)} \leq C\Big[\|a(x,D)u\|^2_{H^0(\mathbb{R}^n_+)} + \sum_{k=1}^{l} \|b_k(x,D)u\|^2_{H^{2l-r_k-1/2}(\mathbb{R}^{n-1})}$$
$$+ \|u\|^2_{H^0(\mathbb{R}^n_+)} + T_0 + \sum_{k=1}^{l} T_k\Big],$$

where

$$T_0 = \|a_1(x,D)u\|^2_{H^0(\mathbb{R}^n_+)} + \|[a_0(x_0,D) - a_0(x,D)]u\|^2_{H^0(\mathbb{R}^n_+)}$$

and

$$T_k = \|b_{k,1}(x,D)u\|^2_{H^{2l-r_k-1/2}(\mathbb{R}^{n-1})} + \|[b_{k,0}(x_0,D) - b_{k,0}(x,D)]u\|^2_{H^{2l-r_k-1/2}(\mathbb{R}^{n-1})}.$$

To complete the second step, it remains to obtain an estimate of the form

$$C\Big[T_0 + \sum_{k=1}^{l} T_k\Big] \leq \frac{1}{2}\|u\|^2_{H^{2l}(\mathbb{R}^n_+)} + C_1\|u\|^2_{H^0(\mathbb{R}^n_+)}$$

for higher-order coefficients close enough to constants. This can be done thanks to the workpieces which we have prepared: Theorem 3.3.1 about trace on the boundary of the half-space, Theorem 3.2.3 on a bound for the norm of the operator of multiplication by a smooth function on the half-space with using the upper bound for the absolute value of this function, inequality (3.2.3) for intermediate norms on the half-space, and similar results of Section 1 for $\mathbb{R}^{n-1}$ instead of $\mathbb{R}^n$ (Theorem 1.9.2 and Proposition 1.8.1).

At the third step, we multiply a function $u \in H^{2l}(\Omega)$, after it is extended, by the elements of a partition of unity on the closure of a neighborhood of the domain Ω:

$$u = \sum_{0}^{N} \varphi_j u,$$

where the φ_j are smooth functions, the support of φ_0 is contained strictly inside the domain, and each function φ_j with $j > 0$ vanishes outside a small neighborhood of some boundary point. Here we assume that the functions are replaced by their restrictions to Ω. We also assume that, in this neighborhood, it is possible to rectify the boundary and apply results of the second step, because the leading coefficients are close to constants. An estimate for $\varphi_0 u$ is taken from Section 6. Now we can assume that, in the initial coordinates, we have

$$\|u\|^2_{H^{2l}(\Omega)} \leq C\Big[\sum_{j=0}^{N} \|a(x,D)(\varphi_j u)\|^2_{H^0(\Omega)} + \sum_{j=1}^{N}\sum_{k=1}^{l} \|b_k(x,D)(\varphi_j u)\|^2_{H^{2l-r_j-1/2}(\Gamma)} + \|u\|^2_{H^0(\Omega)}\Big].$$

Next, we interchange the differential operators and the functions φ_j; then there arise additional terms, which can be estimated via $\varepsilon\|u\|^2_{H^{2l}(\Omega)}$, where ε is a small coefficient, and $\|u\|^2_{H^0(\Omega)}$ by using estimates for the intermediate norms and the trace theorem for a domain. At the last step we get rid of the functions φ_j by using the boundedness of the operators of multiplication by them.

7.2.3 A Right Parametrix

In deriving the a priori estimate we supposed given a solution. Now we are given right-hand sides and must construct an approximate solution.

We only outline the first step. We set

$$\mathfrak{R}(f,g) = R_0 f + \sum_{1}^{l} R_j(g_j - B_j R_0 f). \tag{7.2.15}$$

Here

$$R_0 f = F^{-1}\frac{|\xi|^{2l}}{1+|\xi|^{2l}} a_0^{-1}(\xi) F\mathcal{E} f, \tag{7.2.16}$$

$\mathcal{E}$ is a bounded operator of extension of the function to the entire space, and

$$R_j g_j = F'^{-1}\frac{|\xi'|^{r_j+1}}{1+|\xi'|^{r_j+1}} \Omega_j(\xi',t) F' g_j. \tag{7.2.17}$$

The fractions are aimed at canceling the singularities of the function $a_0^{-1}(\xi)$ at the point $\xi = 0$ and of the functions in the canonical basis at the point $\xi' = 0$.

The necessity of the algebraic ellipticity conditions is proved by contradiction by substituting special families of functions depending on an additional parameter into the a priori estimate.

In the case of a problem with a parameter, an exact right inverse of the problem operator is constructed in this way. For a problem in the half-space with constant leading coefficients without lower-order terms, such an operator can be constructed explicitly by using the Fourier transform with respect to the tangent variables. At the next two steps, we use the invertibility of an operator close in norm to an invertible one. This proximity is ensured by increasing the absolute value of the parameter λ.

Details can be found, e.g., in [34].

After these proofs appeared, the calculus of boundary value problems was developed [68, 69]. To boundary value problems with, generally, pseudodifferential (rather than differential) operators, matrix operators are assigned, which form an algebra; the operators have matrix symbols, and each elliptic problem has a two-sided parametrix within this algebra. This theory was expounded in the book [311]. There is also calculus of problems with a parameter [175].

7.3 Normal Systems of Boundary Operators and Formally Adjoint Boundary Value Problems. Boundary Value Problems with Homogeneous Boundary Conditions

Consider the system of boundary operators

$$B_j = b_j(x,D) = \sum_{|\beta|\le r_j} b_{j,\beta}(x)D^\beta \quad (j = 1,\dots,k) \tag{7.3.1}$$

of orders r_j. In this subsection, we assume that their coefficients are infinitely differentiable in a neighborhood of the boundary and all r_j are less than $2l$, where $2l$ is the order of the operator $a(x,D)$; we also assume that k does not exceed $2l$. We denote the leading parts of these operators by $B_{j,0} = b_{j,0}(x,D)$.

Definitions. System (7.3.1) is said to be *normal* if the orders r_j are pairwise different and the boundary $\varGamma$ is noncharacteristic for each of the operators B_j at each point. The latter means that if the B_j are written in coordinates in which the x_n axis is normal to the boundary, then the coefficient of the leading derivative with respect to x_n (of order r_j) is a nonvanishing function.

If there is a ray of ellipticity with parameter for problem (7.1.19), (7.1.4), then the boundary operators of this problem form a normal system. This can be verified by setting $\xi' = 0$ and $\lambda \ne 0$ in the Lopatinskii condition.

A *Dirichlet system* is, by definition, a normal system of $2l$ operators.

An example is the system of consecutive normal derivatives

$$1, D_\nu, \dots, D_\nu^{2l-1}. \tag{7.3.2}$$

Obviously, if $k < 2l$, then any normal system can be completed to a Dirichlet system by adding, e.g., the normal derivatives of missing orders.

Note also that any two Dirichlet systems can be linearly expressed in terms of each other by using matrix partial differential operators in the tangent variables with C^∞ coefficients. For example, suppose that (7.3.1) is a Dirichlet system, $k = 2l$, and $r_j = j-1$. We decompose the operator $b_j(x,D)$ in powers of the normal derivative:

$$b_j(x,D) = b_{j,0}(x,D') + b_{j,1}(x,D')D_\nu + \ldots + b_{j,j-1}(x)D_\nu^{j-1}. \tag{7.3.3}$$

This is the expression for the operators of system (7.3.1) in terms of the operators of system (7.3.2). Here each $b_{j,k}(x,D')$ is a differential operator of order at most $j-1-k$ with C^∞ coefficients containing differentiations only along the tangent directions, in which the last coefficient $b_{j,j-1}(x)$ is a nowhere vanishing numerical function. Thus, the matrix of the passage from system (7.3.2) to (7.3.1) is a lower triangular nonsingular matrix of differential operators. The orders of its elements increase under each shift from right to left and from top to bottom. The inverse matrix has a similar structure. As a consequence, any Dirichlet system can be linearly expressed in terms of any other Dirichlet system in a similar way.

Recall that the operator *formally adjoint* to A is defined as

$$A^*v = \sum_{|\alpha|\le 2l} D^\alpha[\overline{a_\alpha(x)}v(x)]. \tag{7.3.4}$$

This operator is elliptic and, as is easy to verify, regularly elliptic together with A. The operators A and A^* are related by

$$(Au,v)_\Omega = (u,A^*v)_\Omega \tag{7.3.5}$$

for functions $u,v \in \mathcal{D}(\Omega) = C_0^\infty(\Omega)$ (which is verified by integration by parts). Just this relation means that the operators A and A^* are formally adjoint on Ω.

We return to system (7.3.1); now we shall assume that $k = l$, the system is normal, and the problem for the operator A with given boundary operators is elliptic.

Let us complete the system $\{B_1,\ldots,B_l\}$ to a Dirichlet system $\{B_1,\ldots,B_{2l}\}$.

Theorem 7.3.1. *There exists another Dirichlet system $\{C_1,\ldots,C_{2l}\}$ such that, for any functions $u,v \in H^{2l}(\Omega)$, the Green identity*

$$(Au,v)_\Omega - (u,A^*v)_\Omega = \sum_1^l (B_{l+j}u, C_jv)_\Gamma - \sum_1^l (B_ju, C_{l+j}v)_\Gamma \tag{7.3.6}$$

is valid.

Here all operators are assumed to have C^∞ coefficients. The sum of the orders of B and C in each summand equals $2l-1$. The nonuniqueness of the construction is contained in the choice of $B_{l+1},\ldots,B_{2l}$; as soon as these operators are chosen, the second system is determined uniquely, as we shall see later on. Note that if functions $u,v \in H^{2l}(\Omega)$ satisfy the conditions

$$B_1u = \ldots = B_lu = 0, \quad C_1v = \ldots = C_lv = 0, \tag{7.3.7}$$

then the Green identity (7.3.6) takes the form (7.3.5).

If (7.3.6) holds, then the problems with the operators $A, B_1, \dots, B_l$ and with the operators $A^*, C_1, \dots, C_l$ are said to be *formally adjoint*. In particular, any problem coinciding with a problem formally adjoint to it is said to be *formally self-adjoint*.

Let us outline the proof of the theorem. We can assume that the functions $u(x)$ and $v(x)$ are infinitely differentiable and have small supports near boundary points. Rectifying the boundary, we assume that the functions are defined on the half-space $\{x\colon x_n > 0\}$. Let us decompose the operator A in powers of the derivative D_n:

$$A = A_0 + A_1 D_n + \ldots + A_{2l} D_n^{2l}.$$

Here the A_j are differential operators with respect to the tangent variables of order at most $2l - j$, and the last coefficient is a nonvanishing function (see Remark 6.1.1). It is easy to trace the procedure of integration by parts, which leads to the relation

$$(Au, v)_{\mathbb{R}^n_+} - (u, A^* v)_{\mathbb{R}^n_+} = \sum_{k=1}^{2l} (D_n^{k-1} u, N_k v)_{\mathbb{R}^{n-1}},$$

where each N_k is a partial differential operator of order $2l - k$ in which the coefficient of D_n^{2l-k} coincides with A^*_{2l} up to sign. Therefore, these operators form a normal system.

We write the sum of boundary terms in the form of the inner product $[Du, Nv]$ of the columns

$$Du = (u, \ldots, D_n^{2l-1} u)' \quad \text{and} \quad Nv = (N_1 v, \ldots, N_{2l} v)'.$$

For a while, we rearrange the $B_j u$ so that their orders decrease and denote the resulting column by $\widetilde{B}u$. We have $Du = \mathcal{B}\widetilde{B}u$, where $\mathcal{B}$ is a lower triangular matrix of partial differential operators with respect to x' whose main diagonal consists of nonvanishing functions, and

$$[Du, Nv] = [\mathcal{B}\widetilde{B}u, Nv] = [\widetilde{B}u, \mathcal{B}^* Nv].$$

The orders of the operators N_k decrease from top to bottom, and $\mathcal{B}^*$ is an upper triangular matrix. The verification that v is here subject to the action of a Dirichlet system and that an appropriate change of notation yields the required formula (7.3.6) is left to the reader.

Theorem 7.3.2. *If boundary value problems with operators $A, B_1, \dots, B_l$ and A^*, $C_1, \dots, C_l$ are formally adjoint and one of them is elliptic, then so is the other.*

For a detailed proof we refer the reader to the book [237, Chap. 2, Sec. 2]. (A different, purely algebraic and more formal, proof is given in [330].) Here we only explain the main idea. Performing localization, freezing the coefficients at a boundary point, omitting lower-order terms, and applying the Fourier transform with respect to the tangent variables, we arrive at the following situation. There are two problems on the ray $\mathbb{R}_+ = \{t\colon t > 0\}$ for ordinary differential operators of order $2l$ with constant coefficients:

$$a_0(D)u(t) = f(t) \quad (t > 0), \qquad b_{j,0}u(t)|_{t=0} = 0 \quad (j = 1,\dots,l),$$
$$a_0^*(D)v(t) = g(t) \quad (t > 0), \qquad c_{j,0}u(t)|_{t=0} = 0 \quad (j = 1,\dots,l).$$

We assume that the right-hand sides and solutions of these problems belong to $L_2(\mathbb{R}_+)$. For functions u and v satisfying the given boundary conditions, we have

$$\int_0^\infty a_0(D)u(t) \cdot \overline{v(t)}\,dt = \int_0^\infty u(t) \cdot \overline{a_0^*(D)v(t)}\,dt.$$

Since the initial problem is elliptic, the first problem with right-hand side in $L_2(\mathbb{R}_+)$ is uniquely solvable in this space. As to the second problem, it suffices to show that if the function $g(t)$ is identically zero, then this problem has only the trivial solution in this space (see item 2 in Remark 7.1.2). But, for such $g(t)$, the last relation implies

$$\int_0^\infty a_0(D)u(t) \cdot \overline{v(t)}\,dt = 0.$$

Now it suffices to find a solution $u(t)$ of the first problem with $f(t) = v(t)$. We obtain

$$\int_0^\infty |v(t)|^2 dt = 0,$$

whence $v(t) = 0$, as required.

Let $H_B^{2l}(\Omega)$ denote the subspace of functions in $H^{2l}(\Omega)$ which satisfy the homogeneous (i.e., zero) boundary conditions $B_j u = 0$ $(j = 1,\dots,l)$, and let A_B denote the operator $H_B^{2l}(\Omega) \to L_2(\Omega)$ defined by $A_B u = Au$. It corresponds to the problem

$$Au = f \text{ on } \Omega, \quad B_1 u = \ldots = B_l u = 0 \text{ on } \Gamma. \tag{7.3.8}$$

Theorem 7.3.3. *Under the above assumptions, the operator A_B is Fredholm. Its kernel consists of C^∞ functions (possibly corrected on a set of measure zero).*

Proof. This follows from similar properties of the operator corresponding to the initial problem with inhomogeneous boundary conditions. Indeed, the kernel of A_B coincides with that of the operator corresponding to the initial problem; therefore, it is finite-dimensional and consists of infinitely differentiable functions. The a priori estimate, which we already know, takes the form

$$\|u\|_{H^{2l}(\Omega)} \le C[\|A_B u\|_{L_2(\Omega)} + \|u\|_{L_2(\Omega)}]. \tag{7.3.9}$$

This implies the closedness of the range of A_B. The codimension of the range remains finite. □

Note that the domain $H_B^{2l}(\Omega)$ of A_B is dense in $L_2(\Omega)$. The obtained estimate implies also the closedness of the operator A_B in $L_2(\Omega)$: if $u_j \to u$ and $A_B u_j \to f$ in $L_2(\Omega)$, then $u \in H_B^{2l}(\Omega)$ and $A_B u = f$.

The situation with the operator $(A^*)_C$ corresponding to the adjoint problem is similar.

The operators A_B and $(A^*)_C$ are related by

$$(A_B u, v)_\Omega = (u, (A^*)_C v)_\Omega \tag{7.3.10}$$

on their domains, and they are mutually adjoint in $L_2(\Omega)$. The situation is similar to that considered in Proposition 6.3.1, and the following assertion similar to this proposition is valid.

Corollary 7.3.4. *Under the same assumptions, the orthogonal complement to the range of the operator A_B in $L_2(\Omega)$ coincides with the kernel of $(A^*)_C$, and the orthogonal complement to the range of $(A^*)_C$ coincides with the kernel of A_B.*

7.4 Spectral Boundary Value Problems

The simplest spectral boundary value problem for an elliptic equation with a spectral parameter inside a domain (i.e., contained in the equation) has the form

$$Au(x) = \lambda u(x) \ \text{ on } \ \Omega, \tag{7.4.1}$$

$$B_j u(x) = 0 \ \text{ on } \ \Gamma \quad (j = 1, \dots, l). \tag{7.4.2}$$

We use the same notation as in the preceding section and assume that the same assumptions hold, that is, the problem under consideration is an elliptic problem for the operator $A = a(x, D)$ of order $2l$ with normal system of boundary conditions of orders $r_j < 2l$. To this problem we assign an unbounded operator A_B in $L_2(\Omega)$ with domain $D(A_B) = H_B^{2l}(\Omega)$. This operator can also be regarded as a bounded Fredholm operator from $H_B^{2l}(\Omega)$ to $L_2(\Omega)$. If the problem is uniquely solvable for some $\lambda = \lambda_0$, then this operator has discrete spectrum, and all generalized eigenfunctions turn out to be infinitely differentiable.

The simplest are self-adjoint problems, and they are particularly important for applications. The corresponding operator A_B is then self-adjoint in $L_2(\Omega)$, i.e.,

$$(A_B u, v)_\Omega = (u, A_B v)_\Omega \quad (u, v \in D(A_B)). \tag{7.4.3}$$

The eigenvalues of A_B are real, and their asymptotic behavior is known. Describing it has turned out to be a very difficult problem and long been taking much effort of many mathematicians. If the principal symbol $a_0(x, \xi)$ is positive, then the eigenvalues (which are positive in this case, at least beginning with some number) numbered by positive integers in nondecreasing order with multiplicities taken into account satisfy the asymptotic relation

$$\lambda_k \sim ck^{2l/n} \tag{7.4.4}$$

as $k \to \infty$, where the constant c is calculated in terms of the principal symbol. In the general case, the optimal estimate of the remainder term in this asymptotics, i.e., the difference between the left- and right-hand sides, has the form $O(k^{(2l-1)/n})$. See Safarov and Vassiliev [321].

In $L_2(\Omega)$, there is an orthonormal basis of eigenfunctions, and it remains a basis in the domain $H^{2l}_B(\Omega)$ of the operator A_B.

If the operator A_B is non-self-adjoint, then $(A^*)_C$ is adjoint to it.

We say that the operator A_B is *nearly self-adjoint* (or is a *weak perturbation of the self-adjoint operator* $\frac{1}{2}(A_B + A_B^*)$) if A and A^* have the same principal symbol and A_B and A_B^* have the same system of boundary operators. In this case, an arbitrarily narrow sector symmetric with respect to $\mathbb{R}_+$ contains the spectra of the operators A_B and A_B^*, possibly except finitely many eigenvalues. (Moreover, all but finitely many eigenvalues belong to some "parabolic neighborhood" of the ray $\mathbb{R}_+$.) We mean the case of a scalar problem with A having positive principal symbol. The eigenvalue asymptotics specified above remains valid. Ellipticity with parameter takes place along any ray except $\mathbb{R}_+$.

If A_B is non-self-adjoint, then there arises the problem of *conditions for the completeness of the system of generalized eigenfunctions* of the operator A_B in $L_2(\Omega)$ and in $D(A_B)$, i.e., for the density of the linear combinations of generalized eigenfunctions in these spaces. A sufficient completeness condition (in these spaces) consists in the existence of rays of ellipticity with parameter such that the angles between any two neighboring rays is less than $2l\pi/n$. In particular, this condition holds for nearly self-adjoint operators.

Completeness also holds in some other spaces, including the intermediate spaces obtained from $L_2(\Omega)$ and $D(A_B)$ by complex interpolation (this will be explained in Section 13). In the self-adjoint case, the eigenfunctions form a basis in these spaces. See also Section 17.2.

Examples. Consider the spectral Dirichlet problem for the Laplace equation

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma. \tag{7.4.5}$$

Let us denote the corresponding operator by $-\Delta_D$. This is a self-adjoint operator with discrete spectrum consisting of positive eigenvalues of finite multiplicity, which tend to $+\infty$ as $ck^{2/n}$. Self-adjointness follows from the Green identity

$$-\int_\Omega \Delta u \cdot \overline{v}\, dx = \int_\Omega \nabla u \cdot \nabla \overline{v}\, dx - \int_\Gamma \partial_\nu u \cdot \overline{v}\, dS, \tag{7.4.6}$$

or, more precisely, from the second Green identity for the Laplace operator

$$\int_\Omega \Delta u \cdot \overline{v}\, dx - \int_\Omega u \cdot \Delta \overline{v}\, dx = \int_\Gamma \partial_\nu u \cdot \overline{v}\, dS - \int_\Gamma u \cdot \partial_\nu \overline{v}\, dS. \tag{7.4.7}$$

Relation (7.4.6) implies also the positivity of all eigenvalues. Indeed, setting $v = u$, we see that if $\lambda \le 0$ and the boundary value vanishes, then $\nabla u = 0$ almost everywhere and, hence, everywhere in Ω, so that $u \equiv \text{const}$ on Ω, which implies $u \equiv 0$, because $u = 0$ on the boundary.

A similar situation holds in the case of the Neumann spectral problem for the Laplace operator, that is, the problem

$$-\Delta u = \lambda u \text{ in } \Omega, \quad \partial_\nu u = 0 \text{ on } \Gamma. \tag{7.4.8}$$

We denote the corresponding operator in $L_2(\Omega)$ by $-\Delta_N$. This is a self-adjoint operator with discrete spectrum consisting of nonnegative eigenvalues, which tend to $+\infty$ and have the same asymptotics. The number 0 is an eigenvalue, and the corresponding space of eigenfunctions is one-dimensional and contains only constants. Self-adjointness and the absence of negative eigenvalues again follow from the Green identity.

Spectral elliptic boundary value problems with a spectral parameter in boundary conditions are interesting and useful as well. For example, consider the problem

$$Lu = 0 \text{ in } \Omega, \quad \partial_\nu u = \lambda u \text{ on } \Gamma \tag{7.4.9}$$

for a second-order elliptic equation (e.g., for the Laplace equation). This is the *Poincaré–Steklov spectral problem* (cf. [306] and [362]).

Suppose that the Dirichlet problem for the equation $Lu = 0$ is uniquely solvable. Then the Poincaré–Steklov problem reduces to a spectral equation on the boundary Γ. Let D_Γ denote the operator which takes the right-hand side of the Dirichlet boundary condition for solutions of the homogeneous equation to the right-hand side of the Neumann boundary condition:

$$u|_\Gamma \mapsto u \mapsto \partial_\nu u. \tag{7.4.10}$$

The operator D_Γ is known as the *Dirichlet-to-Neumann operator*. It acts boundedly from $H^{s-1/2}(\Gamma)$ to $H^{s-3/2}(\Gamma)$ for $s > 3/2$.

If the Neumann problem is uniquely solvable, then we can introduce the operator N_Γ which maps the right-hand side of the Neumann boundary condition to the right-hand side of the Dirichlet condition. This is the *Neumann-to-Dirichlet operator*. It acts boundedly from $H^{s-3/2}(\Gamma)$ to $H^{s-1/2}(\Gamma)$ and is compact in $H^{s-3/2}(\Gamma)$, because so is the embedding of the latter space in the former. If both problems, Dirichlet and Neumann, are uniquely solvable, then the operators D_Γ and N_Γ are mutually inverse. These operators play an important role in the theory of inverse problems for elliptic equations (see, e.g., [380]).

It has been proved in the theory of pseudodifferential operators that D_Γ is an *elliptic pseudodifferential operator of order* 1. In particular, we can set $s = 3/2$ and consider D_Γ as an *unbounded* operator in $L_2(\Gamma)$ with domain $H^1(\Gamma)$. Problem (7.4.9) reduces to the equation

$$(D_\Gamma - \lambda I)\varphi = 0, \tag{7.4.11}$$

where $\varphi = u|_\Gamma$.

In the case of the equation $-\Delta u = \mu u$ with real μ different from the eigenvalues of the Dirichlet and Neumann problems for the Laplace operator, both operators turn out to be self-adjoint in $L_2(\Gamma) = H^0(\Gamma)$; this follows from the relation

$$\int_\Gamma \partial_\nu u \cdot \overline{v}\, dS = \int_\Gamma u \cdot \partial_\nu \overline{v}\, dS, \tag{7.4.12}$$

which is a consequence of the second Green identity. The spectrum of D_Γ is discrete and consists of eigenvalues of finite multiplicity, which tend to $+\infty$ as $c' k^{1/(n-1)}$. The eigenfunctions (solutions of the homogeneous equation $(D_\Gamma - \lambda I)\varphi = 0$) belong to $C^\infty(\Gamma)$ (provided that the boundary is smooth). They form an orthonormal basis in $L_2(\Gamma)$, which can be shown to remain a basis in all spaces $H^s(\Gamma)$.

We shall return to similar problems in Section 11 in the context of the theory of second-order strongly elliptic systems in Lipschitz domains.

A more detailed information about elliptic spectral problems and bibliography can found in the surveys [12,64,315].

7.5 Generalizations

1. General elliptic boundary value problems can be considered for Douglis–Nirenberg elliptic systems (see, e.g., [9] and [183,185]). In principle, this does not involve anything essentially new; it is only required to choose adequate spaces for solutions and right-hand sides.

The Dirichlet problem for an even-order elliptic system with leading part homogeneous with respect to differentiation may be nonelliptic and, therefore, non-Fredholm. The first example of a non-Fredholm Dirichlet problem was given by Bitsadze [67]. This is the problem for the 2×2 system in the plane with matrix

$$\begin{pmatrix} \partial_1^2 - \partial_2^2 & 2\partial_1\partial_2 \\ -2\partial_1\partial_2 & \partial_1^2 - \partial_2^2 \end{pmatrix}. \tag{7.5.1}$$

There are simplified versions of the Lopatinskii condition for the matrix Dirichlet problem, which are essentially due to Lopatinskii [244], in terms of the principal symbol $a_0(x,\xi)$ of the system; see also [146] and [11]. In particular, a necessary and sufficient condition for the ellipticity of the Dirichlet problem is the factorizability of the principal symbol:

$$a_0(x,\xi',\zeta) = a_-(x,\xi',\zeta)a_+(x,\xi',\zeta). \tag{7.5.2}$$

Here x is a boundary point, the x_n axis is normal to the boundary, and the zeros ζ of the determinants of the matrices a_+ and a_- at $\xi' \neq 0$ belong to the upper and lower half-planes, respectively. There is also another, equivalent, condition; for a second-order system, it consists in that the matrix-integral

$$\int_{\gamma} e^{i\zeta t} a_0^{-1}(x, \xi', \zeta)\, d\zeta \tag{7.5.3}$$

over a contour enclosing all roots ζ of the determinant of $a_0(x, \xi', \zeta)$ in the upper half-plane is nonsingular for $\xi' \neq 0$.

2. Boundary value problems for systems can be considered on a compact smooth manifold with smooth boundary, and vector-valued functions can be replaced by sections of vector bundles; see, e.g., [183, 185].

3. Boundary value problems elliptic with parameter can be considered in a more general setting, with all operators polynomially depending on a parameter; see, e.g., [10] and [34]. Of interest are also the corresponding spectral problems, which have an extensive literature; see, e.g., [249].

4. Relaxing the smoothness assumptions, we can consider boundary value problems in spaces with index s varying in a finite interval.

5. There is an extensive theory of differential elliptic operators on a manifold with conical, edge, and similar singularities and of boundary elliptic problems in a domain with boundary singularities of these types. Outside the singularities, the manifold or the boundary is assumed to be smooth. This theory is substantially more complicated. It introduces and uses spaces related in a special way to the singularities and studies the asymptotic behavior of solutions near the singularities; see the initial paper [216] by Kondrat'ev and books [117] by Dauge, [284] by Nazarov and Plamenevsky, [220, 221] by Kozlov, Maz'ya, and Rossman, and [253] by Maz'ya and Rossman.

8 Strongly Elliptic Equations and Variational Problems

In the theory of elliptic problems, there is a different approach, which was developed before the general theory of elliptic problems outlined in the preceding section. As applied to the simplest problems, it is explained in textbooks on mathematical physics, such as [225] and [349]; see also [265]. This approach is less general in the sense that it requires the given elliptic equation to be associated with an "energy" quadratic form with positive definite real part. This is the strong ellipticity condition. It holds for many equations arising in applications.

In the case of smooth boundary and coefficients, this approach provides a faster way to theorems on the unique solvability or the Fredholm property of problems, but in spaces of lower smoothness (which is interesting by itself), and proving the smoothness of solutions to equations with smooth right-hand sides requires more effort. This case is considered below.

The approach in question is particularly effective in the case of nonsmooth coefficients (see Remark 8.1.6 below) and a nonsmooth (e.g., Lipschitz) boundary. Second-order strongly elliptic systems in Lipschitz domains are considered in Sections 11, 12 and 16, 17.

In the context of this book, of most interest in this approach is the choice of function spaces.

We begin with a detailed consideration of the Dirichlet and Neumann problems for a second-order scalar equation and then mention generalizations to higher-order systems.

8.1 The Dirichlet and Neumann Problems for a Second-Order Scalar Equation

Let Ω be a bounded domain with boundary Γ. For simplicity, we first assume it to be C^∞. In Ω, consider the second-order scalar equation with leading part written in the *divergence form*

$$Lu = f, \tag{8.1.1}$$

where

$$Lu(x) = -\sum_{j,k=1}^{n} \partial_j a_{j,k}(x)\partial_k u(x) + \sum_{j=1}^{n} b_j(x)\partial_j u(x) + c(x)u(x). \tag{8.1.2}$$

The coefficients are generally complex-valued functions, which are assumed for simplicity to be infinitely differentiable on $\overline{\Omega}$. We can also assume (although this is not necessary) that

$$a_{j,k}(x) = a_{k,j}(x) \quad (j \neq k). \tag{8.1.3}$$

This condition means that, after differentiation, the coefficients of similar terms in the first sum become equal. If, in addition, $a_{j,k} = \overline{a_{k,j}}$ (so that the $a_{j,k}$ are real), $b_j = 0$, and $c = \overline{c}$, then we have a formally self-adjoint equation. The first sum can be rewritten in the form

$$\nabla a(x)\nabla u(x) = \mathrm{div}[a(x)\,\mathrm{grad}\,u(x)], \tag{8.1.4}$$

where $a(x)$ is a symmetric (but not necessarily real) $n \times n$ matrix with elements $a_{j,k}(x)$.

On the higher-order part we impose the *strong ellipticity* condition introduced by Vishik in [388]. This is the requirement that the form

$$a(x,\xi) = \sum a_{j,k}(x)\xi_j\xi_k \tag{8.1.5}$$

with real $\xi = (\xi_1, \dots, \xi_n)$ have positive definite real part, i.e., the form

$$\operatorname{Re} a(x,\xi) = \sum \frac{a_{j,k}(x) + \overline{a_{j,k}(x)}}{2}\, \xi_j \xi_k \tag{8.1.6}$$

be positive definite. To be more precise, we assume that the uniform condition

$$\operatorname{Re} a(x,\xi) \geq C_0 |\xi|^2 \quad (x \in \overline{\Omega}), \tag{8.1.7}$$

where C_0 is a positive constant, is satisfied.

Given a complex vector $\zeta = \xi + i\eta$, consider the form $a(x,\zeta)$ defined by

$$a(x,\zeta) = \sum a_{j,k}(x) \zeta_k \overline{\zeta_j}. \tag{8.1.8}$$

Under condition (8.1.3), we have

$$a(x,\zeta) = a(x,\xi) + a(x,\eta),$$

so that inequality (8.1.7) is generalized to complex numbers ζ:

$$\operatorname{Re} a(x,\zeta) \geq C_0 |\zeta|^2. \tag{8.1.9}$$

For real coefficients $a_{j,k}(x) = a_{k,j}(x)$, strong ellipticity follows from the ellipticity condition $a(x,\xi) > 0$. Conversely, ellipticity always follows from strong ellipticity. We again emphasize that we assume the strong ellipticity condition to hold here.

The most important problems for Eq. (8.1.1) are the Dirichlet problem with boundary condition

$$u^+(x) = g(x) \text{ on } \Gamma \tag{8.1.10}$$

and the Neumann problem. From now on, we use the superscript + to denote boundary values on Γ. To write the Neumann boundary condition, we introduce the so-called *conormal derivative*. Let $\nu = \nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ be the unit outer normal vector to the boundary Γ at a boundary point x. If the function $u(x)$ is smooth (it suffices to assume that it belongs to $H^s(\Omega)$ with $s > 3/2$), then we set

$$T^+ u(x) = \sum_{j=1}^{n} \nu_j(x) a_{j,k}(x) \gamma^+ \partial_k u(x) \tag{8.1.11}$$

on Γ. This is what is called the conormal derivative for Eq. (8.1.1). The Neumann boundary condition has the form

$$T^+ u(x) = h(x) \text{ on } \Gamma. \tag{8.1.12}$$

The conormal derivative is related to the equation closer than the usual normal derivative (see the Green identity below). In the case of the Laplace equation, it coincides with the normal derivative.

Both problems are elliptic: we have already mentioned the ellipticity of the Dirichlet problem in Section 7.1, and the ellipticity of the Neumann problem is easy to verify (by rotating the coordinate system so that ∂_n becomes the normal derivative).

Moreover, *these problems are elliptic with parameter in the sector of opening* $\pi + \varepsilon$ *with bisector* $\mathbb{R}_-$, *where* $\varepsilon > 0$ *is sufficiently small*. The sector of ellipticity with parameter is larger than the left half-plane because, by continuity, the strong ellipticity condition is preserved under the replacement of $a_{j,k}(x)$ by $e^{i\theta}a_{j,k}(x)$ with sufficiently small $|\theta|$.

Let us introduce the form

$$\Phi_\Omega(u,v) = \int_\Omega \Big[\sum_{j,k} a_{j,k}\partial_k u \cdot \partial_j \overline{v} + \sum_j b_j \partial_j u \cdot \overline{v} + cu\overline{v} \Big] dx. \tag{8.1.13}$$

The first sum can be rewritten in the form

$$a(x)\nabla u(x) \cdot \nabla \overline{v(x)} = a(x)\,\mathrm{grad}\,u(x) \cdot \mathrm{grad}\,\overline{v(x)}. \tag{8.1.14}$$

If $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, then, integrating by parts, we obtain the *first Green identity*

$$(Lu,v)_\Omega = \Phi_\Omega(u,v) - (T^+u, v^+)_\Gamma. \tag{8.1.15}$$

Here $(u,v)_\Omega$ and $(\varphi,\psi)_\Gamma$ are standard inner products in $L_2(\Omega)$ and $L_2(\Gamma)$, respectively. It is convenient for what follows to denote the operator formally adjoint to L by $\widetilde{L}$; it can be written as

$$\widetilde{L}v = -\sum_{j,k=1}^n \partial_j \overline{a_{k,j}(x)} \partial_k v(x) - \sum \overline{b_j(x)} \partial_j v(x) + [\overline{c(x)} - \sum \partial_j \overline{b_j(x)}] v(x). \tag{8.1.16}$$

We want to have the first Green identity for $\widetilde{L}$ with the same form $\Phi_\Omega(u,v)$:

$$(u, \widetilde{L}v)_\Omega = \Phi_\Omega(u,v) - (u^+, \widetilde{T}^+ v)_\Gamma. \tag{8.1.17}$$

Then the corresponding conormal derivative of a function $v \in H^s(\Omega)$, $s > 3/2$, must be

$$\widetilde{T}^+ v(x) = \sum_{j,k} \nu_j(x) \overline{a_{k,j}(x)} \gamma^+ \partial_k v(x) + \sum \nu_j(x) \overline{b_j(x)} v^+(x). \tag{8.1.18}$$

Relation (8.1.17) is obtained for $u \in H^1(\Omega)$ and $v \in H^2(\Omega)$. The first Green identities (8.1.15) and (8.1.17) with $u, v \in H^2(\Omega)$ imply the *second Green identity*

$$(Lu,v)_\Omega - (u, \widetilde{L}v)_\Omega = (u^+, \widetilde{T}^+ v)_\Gamma - (T^+u, v^+)_\Gamma. \tag{8.1.19}$$

For $s > 3/2$, passing to the limit, we can extend relation (8.1.15) to $u \in H^s(\Omega)$ and $v \in H^1(\Omega)$, relation (8.1.17) to $u \in H^1(\Omega)$ and $v \in H^s(\Omega)$, and relation (8.1.19) to $u, v \in H^s(\Omega)$. For this purpose, we approximate functions from $H^s(\Omega)$ by functions from $H^2(\Omega)$.

Although we assume the boundary and the coefficients to be smooth in this section, of great interest are equations with nonsmooth right-hand sides, whose solutions are functions of low smoothness. It is possible to consider them in the context of the weak setting of the Dirichlet and Neumann problems, which we describe below, and weak solutions.

First, consider these problems with homogeneous boundary conditions, i.e., zero right-hand sides.

In this case, the Dirichlet problem is written as

$$(f,v)_\Omega = \Phi_\Omega(u,v). \tag{8.1.20}$$

Here v is any test function,

$$u,v \in \widetilde{H}^1(\Omega), \quad \text{and} \quad f \in H^{-1}(\Omega). \tag{8.1.21}$$

We assume that the form $(f,v)_\Omega$ is extended to the direct product $H^{-1}(\Omega)\times\widetilde{H}^1(\Omega)$. The choice of the space containing u and v is caused by the homogeneity of the Dirichlet condition: recall that $\widetilde{H}^1(\Omega)$ is identified with $\mathring{H}^1(\Omega)$.

The Neumann problem with homogeneous boundary condition is written in the same form (8.1.20) but with

$$u,v \in H^1(\Omega) \quad \text{and} \quad f \in \widetilde{H}^{-1}(\Omega). \tag{8.1.22}$$

Thus, we need spaces with negative indices. In both cases, f and u, v belong to spaces dual to each other with respect to the (extended) form $(f,v)_\Omega$. Note that the form $\Phi_\Omega(u,v)$ is bounded on $H^1(\Omega)$ and, in particular, on $\widetilde{H}^1(\Omega)$:

$$\begin{aligned} |\Phi_\Omega(u,v)| &\le C_1\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \quad (u,v\in H^1(\Omega)),\\ |\Phi_\Omega(u,v)| &\le C_1\|u\|_{\widetilde{H}^1(\Omega)}\|v\|_{\widetilde{H}^1(\Omega)} \quad (u,v\in \widetilde{H}^1(\Omega)). \end{aligned} \tag{8.1.23}$$

In the case of the Dirichlet problem with homogeneous boundary condition, L is bounded as an operator from the space $\widetilde{H}^1(\Omega)$ of functions u to the space $H^{-1}(\Omega)$ of functions f. Indeed, in this case we have

$$\|f\|_{H^{-1}(\Omega)} \le C_2 \sup_{v\neq 0}\frac{|(f,v)_\Omega|}{\|v\|_{\widetilde{H}^1(\Omega)}} = C_2 \sup_{v\neq 0}\frac{|\Phi_\Omega(u,v)|}{\|v\|_{\widetilde{H}^1(\Omega)}} \le C_3\|u\|_{\widetilde{H}^1(\Omega)}.$$

In the case of the Neumann problem with homogeneous boundary condition, a similar argument shows that L is a bounded operator from $H^1(\Omega)$ to $\widetilde{H}^{-1}(\Omega)$.

At first sight, it seems strange that the functions u and f belong to spaces of different classes, H and $\widetilde{H}$. But this is natural for problems under consideration, in which the right-hand sides belong to spaces with negative indices dual to the solution spaces; cf. the Lax–Milgram theorem in Section 18.2. Recall also that the spaces $H^s(\Omega)$ and $\widetilde{H}^s(\Omega)$ can be identified if $|s| < 1/2$ (see Section 5.1). In Section 13.8, we shall see that, for this reason, the spaces containing u and f belong to the same interpolation scale.

First, consider the Dirichlet problem in more detail.

Theorem 8.1.1. *For a strongly elliptic operator of the form* (8.1.2)*, there exist constants* $C_4 > 0$ *and* $C_5 \geq 0$ *such that all functions in* $\widetilde{H}^1(\Omega)$ *satisfy the inequality*

$$\|u\|^2_{\widetilde{H}^1(\Omega)} \leq C_4 \operatorname{Re} \Phi_\Omega(u,u) + C_5 \|u\|^2_{H^0(\Omega)}. \tag{8.1.24}$$

This inequality is known as the *Gårding inequality*. If it holds, the form $\Phi_\Omega(u,u)$ is said to be *coercive on* $\widetilde{H}^1(\Omega)$, and (8.1.24) is also called the *coercivity condition* for Φ_Ω on $\widetilde{H}^1(\Omega)$.

Proof. In the scalar case, under condition (8.1.9) (which follows, as we saw, from (8.1.7) and (8.1.3)), the proof is quite simple. The substitution of $\zeta_j = \partial_j u(x)$ into (8.1.9) and integration with respect to x yield the inequality

$$\sum \|\partial_j u\|^2_{H^0(\Omega)} \leq C_4 \operatorname{Re} \Phi_{0,\Omega}(u,u),$$

where $\Phi_{0,\Omega}$ is the principal part of the form Φ_Ω:

$$\Phi_{0,\Omega}(u,v) = \int_\Omega a(x) \nabla u(x) \cdot \nabla \overline{v(x)}\, dx. \tag{8.1.25}$$

If all coefficients b_j vanish, then we immediately obtain the required assertion. If some of the coefficients b_j are nonzero, then we apply (5.1.9). □

If $\operatorname{Re} c(x)$ is sufficiently large (or $c(x)$ is replaced by $c(x) - \lambda$ and $\mu = -\operatorname{Re}\lambda$ is sufficiently large), then estimate (8.1.24) with $C_5 = 0$ is valid:

$$\|u\|^2_{\widetilde{H}^1(\Omega)} \leq C_4 \operatorname{Re} \Phi_\Omega(u,u). \tag{8.1.26}$$

In what follows, we usually assume for simplicity that this estimate does hold. We call it the *strong Gårding inequality*, or the *strong coercivity condition for* Φ_Ω *on* $\widetilde{H}^1(\Omega)$. As we shall see below, this condition implies the unique solvability (rather than the Fredholm property) of the problem. Relation (8.1.20) implies

$$\|u\|^2_{\widetilde{H}^1(\Omega)} \leq C_6 \|f\|_{H^{-1}(\Omega)} \|u\|_{\widetilde{H}^1(\Omega)};$$

therefore, we have the *a priori estimate*

$$\|u\|_{\widetilde{H}^1(\Omega)} \leq C_6 \|f\|_{H^{-1}(\Omega)}, \tag{8.1.27}$$

which implies uniqueness for the Dirichlet problem.

As mentioned above, the reverse inequality is valid as well. Thus, we have the *two-sided estimate*

$$\|u\|_{\widetilde{H}^1(\Omega)} \leq C_6 \|f\|_{H^{-1}(\Omega)} \leq C_7 \|u\|_{\widetilde{H}^1(\Omega)}. \tag{8.1.28}$$

This shows that the chosen spaces are adequate to the problem under consideration.

Next, estimate (8.1.26) implies the existence of a solution of the problem for any right-hand side f. In the important special case where the equation is formally self-adjoint, which means that

$$\overline{\Phi_\Omega(u,v)} = \Phi_\Omega(v,u), \tag{8.1.29}$$

this is a consequence of F. Riesz' theorem, according to which the form $\Phi_\Omega(u,v)$ has properties of inner product, so that we can represent the functional $(f,v)_\Omega$ in the form $\Phi_\Omega(u,v)$. But even without this assumption, the form $\Phi_\Omega(u,v)$ with a fixed function u is a general continuous antilinear functional on $\widetilde{H}^1(\Omega)$, which uniquely determines u by virtue of the Lax–Milgram theorem (see Section 18.2). Thus, we have proved the following existence and uniqueness theorem.

Theorem 8.1.2. *If inequality* (8.1.26) *holds, then the Dirichlet problem* $Lu = f$ *in* Ω, $u^+ = 0$ *has precisely one weak solution in* $\widetilde{H}^1(\Omega)$ *for any right-hand side* $f \in H^{-1}(\Omega)$*, and the a priori estimate* (8.1.27) *holds.*

Remark 8.1.3. To have inequality (8.1.26), it is not necessary to assume the presence of a zero-order term in the equation. For example, in the case of the Poisson equation $-\Delta u = f$, we can use the *Friedrichs inequality* (see, e.g., [225] or [358]): for functions vanishing on Γ,

$$\|u\|^2_{L_2(\Omega)} \le C_\Omega \|\nabla u\|^2_{L_2(\Omega)}. \tag{8.1.30}$$

It is well known that the Dirichlet problem in this case is uniquely solvable. The Friedrichs inequality is also valid in the L_p-norms.

To extend the theorem on the unique solvability of the Dirichlet problem to the case of an inhomogeneous boundary condition $u^+ = g \in H^{1/2}(\Gamma)$ (in which the solution is sought in $H^1(\Omega)$), it suffices to subtract a function $u_0 \in H^1(\Omega)$ with given boundary value g from the solution, which yields the problem considered above for the difference $u - u_0$. This is explained in detail in Section 11.1.

Considering the equation with parameter

$$Lu - \lambda u = f \tag{8.1.31}$$

instead of (8.1.1), we obtain, as shown below, the estimate

$$\|u\|_{\widetilde{H}^1(\Omega)} + \mu\|u\|_{H^{-1}(\Omega)} \le C\|f\|_{H^{-1}(\Omega)} \tag{8.1.32}$$

for sufficiently large $\mu = -\operatorname{Re}\lambda$, which is uniform in the parameter and resembles (7.1.21). This can be compared with the remark on the ellipticity of problems with parameter before (8.1.13).

Let us derive estimate (8.1.32). Note that if we add $-(\lambda u, u)_\Omega$ to $\Phi_\Omega(u,u)$ under the Re sign, inequality (8.1.26) remains valid. But

$$\Phi_\Omega(u,u) - \lambda(u,u)_\Omega = (f,u)_\Omega.$$

Therefore,

$$\|u\|^2_{\widetilde{H}^1(\Omega)} \le C\|f\|_{H^{-1}(\Omega)}\|u\|_{\widetilde{H}^1(\Omega)},$$

which gives the required estimate of the first term on the left-hand side in (8.1.32). Next, we have

$$\mu\|u\|_{H^{-1}(\Omega)} \le \|f\|_{H^{-1}(\Omega)} + \|Lu\|_{H^{-1}(\Omega)},$$

and the last term in this expression can be estimated via the first term on the left-hand side in (8.1.32), which has just been estimated. This gives the estimate of the second term on the left-hand side in (8.1.32).

Moreover, an estimate of the form (8.1.32) with $|\lambda|$ instead of μ, that is,

$$\|u\|_{\widetilde{H}^1(\Omega)} + |\lambda|\|u\|_{H^{-1}(\Omega)} \le C\|f\|_{H^{-1}(\Omega)}, \tag{8.1.33}$$

holds outside any angle with vertex at the origin which encloses an angle containing the values of the quadratic form $\Phi_\Omega(u,u)$. The proof of this estimate uses the possibility of multiplying the form $\Phi_\Omega(u,u)$ by $e^{i\theta}$ with sufficiently small $|\theta|$.

We proceed to the Neumann problem. In this case, we need an inequality for functions in $H^1(\Omega)$ similar to (8.1.24):

$$\|u\|^2_{H^1(\Omega)} \le C_4 \operatorname{Re}\Phi_\Omega(u,u) + C_5\|u\|^2_{H^0(\Omega)}. \tag{8.1.34}$$

In the case of a scalar equation satisfying condition (8.1.3), which we consider in this section, inequality (8.1.34) is obtained in precisely the same way as above. We say that the *form $\Phi_\Omega(u,v)$ is coercive on $H^1(\Omega)$*. For sufficiently large $\operatorname{Re} c(x)$, this inequality holds with $C_5 = 0$:

$$\|u\|^2_{H^1(\Omega)} \le C_4 \operatorname{Re}\Phi_\Omega(u,u). \tag{8.1.35}$$

We refer to (8.1.35) as the *strong coercivity condition for the form Φ_Ω on $H^1(\Omega)$*. Having (8.1.35), we can repeat the above considerations of the Dirichlet problem for the Neumann problem almost without changes. As a result, we obtain the following theorem.

Theorem 8.1.4. *If inequality* (8.1.35) *holds, then the Neumann problem $Lu = f$ in Ω, $T^+u = 0$ on Γ has precisely one weak solution in $H^1(\Omega)$ for any right-hand side $f \in \widetilde{H}^{-1}(\Omega)$, and this solution satisfies the a priori estimate*

$$\|u\|_{H^1(\Omega)} \le C_6\|f\|_{\widetilde{H}^{-1}(\Omega)}. \tag{8.1.36}$$

We again have a two-sided estimate. Moreover, an estimate with a parameter similar to (8.1.33) holds.

For the weak setting of the Neumann problem with inhomogeneous boundary condition, we need the complete Green identity

$$(f,v)_\Omega = \Phi_\Omega(u,v) - (h,v^+)_\Gamma \quad (v \in H^1(\Omega)). \tag{8.1.37}$$

Here $v^+ \in H^{1/2}(\Gamma)$; therefore, $h \in H^{-1/2}(\Gamma)$ and $(h, v^+)_\Gamma$ is a continuous antilinear functional on $H^{1/2}(\Gamma)$ and, hence, on $H^1(\Omega)$.

Probably somewhat unexpectedly, for functions $u, v \in H^1(\Omega)$, the Green identities (8.1.15) and (8.1.37) cannot be proved. The point is that, in the framework of the trace theorem which we know, the expression (8.1.11) does not make sense for a function in $H^1(\Omega)$. Moreover, the right-hand side of the equation $Lu = f$ is uniquely determined by u as a distribution only inside the domain Ω. As an element of the space of continuous (anti)linear functionals on $H^1(\Omega)$, this right-hand side may contain a component concentrated on Γ, namely, a functional of the form $(w, v^+)_\Gamma$, where $w \in H^{-1/2}(\Gamma)$. But such a component can be transferred (or not transferred) to the boundary term in (8.1.37).

The way out of this situation generally accepted in the literature (see, e.g., [258]) is to *postulate* the Green identity (8.1.37). *Given* $u \in H^1(\Omega)$ *and* $f \in \widetilde{H}^{-1}(\Omega)$, *this identity is taken for the definition of the conormal derivative.* Generally, the conormal derivative is no longer expressed by (8.1.11). We also take relation (8.1.37) for the definition of *a solution of the Neumann problem with inhomogeneous boundary condition, i.e., with given* $f \in \widetilde{H}^{-1}(\Omega)$ *and the conormal derivative* $T^+u = h \in H^{-1/2}(\Gamma)$. Thus, there is an arbitrariness in the statement of the problem, since f and h are not independent. But if f or T^+u is given, then T^+u or f, respectively, is determined uniquely. In particular, the two Neumann problems with $f = 0$ and with $h = 0$ are of independent significance.

Note also that, in the case where f is known to belong to $L_2(\Omega)$, it is usually this function f extended by zero outside Ω which is considered as the right-hand side of the equation $Lu = f$ in $\widetilde{H}^{-1}(\Omega)$; see [258, p. 117]. Under this convention, the conormal derivative is determined uniquely. In particular, this relates to the case where $f = 0$ on Ω.

We shall return to this point in Section 11. In Section 11.2, we shall propose a method for eliminating the arbitrariness mentioned above.

However, it is useful to know that the functional $(T^+u, v^+)_\Gamma$ can be represented in the form $(f_1, v)_\Omega$ with $f_1 \in \widetilde{H}^{-1}(\Omega)$. Thus, we can always reduce the general Neumann problem to the problem with homogeneous boundary condition, which was considered above, by changing the right-hand side of the equation. Therefore, the unique solvability theorem remains valid in the case of an inhomogeneous boundary condition.

We refer to the conormal derivative defined by (8.1.11) as the *smooth conormal derivative*.

A feature of the general Neumann problem which is of particular interest for us in this section is that its setting allows specifying the right-hand side of the boundary condition in the space with negative index $-1/2$.

Now let us explain the term "variational problem." *This term refers to the case where the problem is formally self-adjoint, or, equivalently, satisfies condition* (8.1.29)*, and the form* $\Phi_\Omega(v, v)$ *is nonnegative*. Consider the functional

$$\Psi(v) = \mathrm{Re}[\Phi_\Omega(v, v) - 2(f, v)_\Omega]. \tag{8.1.38}$$

This functional attains its minimum value at a solution. Indeed, we have

$$\Psi(u+v)-\Psi(u)=\Phi_\Omega(v,v)+2\,\mathrm{Re}[\Phi_\Omega(u,v)-(f,v)_\Omega],$$

and if u is a solution, then the expression in brackets vanishes, so that the left-hand side is nonnegative.

For this reason, the weak settings of the Dirichlet and Neumann problems considered in this section are also called *variational settings*. Using this name, we do not exclude the case of the absence of formal self-adjointness.

For studying the smoothness of the solution under additional smoothness conditions on the right-hand sides and the boundary, there is Nirenberg's method of difference quotients [290]; see our Section 16.5. For example, it turns out that if the right-hand side of the equation belongs to $L_2(\Omega)$, then the solution belongs to $H^2(\Omega')$ on any interior subdomain Ω', and if the boundary is C^2, then the solution of the Dirichlet problem belongs to $H^2(\Omega)$.

But particularly simple is the proof of the following theorem in the case of smooth coefficients and boundary; it follows from the consistency of usual and variational elliptic theories.

Theorem 8.1.5. *Let $s \geq 2$. Then the solution of the Dirichlet problem with $f \in H^{s-2}(\Omega)$ and $g \in H^{s-1/2}(\Gamma)$ belongs to $H^s(\Omega)$ and is a solution in the sense of general theory* (see Section 7). *Therefore, it belongs to $H^s(\Omega)$. The same is true for the solution of the Neumann problem with $f \in H^{s-2}(\Omega)$ and $h \in H^{s-3/2}(\Gamma)$.*

Proof. It suffices to verify that a usual solution is a variational solution for $s = 2$, which is performed elementarily by integration by parts. Note that, in the case under consideration, f has no components supported on Γ and the conormal derivative is smooth and determined uniquely. Since the variational solution is unique, it follows that this is also a solution in the usual sense, and we can apply the familiar theorem on the smoothness of solutions of usual elliptic problems. □

Remark 8.1.6. Of independent interest is the question of what results are valid for equations with nonsmooth coefficients. Theorem 8.1.1 remains true when the coefficients in (8.1.2) are only bounded measurable functions. This assumption is also sufficient for proving Theorems 8.1.2 and 8.1.4 if only weak solutions are considered.

Remark 8.1.7. Now consider the Dirichlet problem in the case where strong ellipticity is present but the form $\Phi_\Omega(u,v)$ is not strongly coercive on $\widetilde{H}^1(\Omega)$. Let L be the corresponding operator. Then the operator $L_\tau = L+\tau$ satisfies the strong coercivity condition for sufficiently large τ. The operator $\mathcal{L}_D\colon \widetilde{H}^1(\Omega) \to H^{-1}(\Omega)$ corresponding to the original Dirichlet problem with homogeneous boundary condition turns out to be a weak perturbation of the invertible operator $\mathcal{L}_{\tau,D}$. To be more precise, their difference is a bounded operator in $\widetilde{H}^1(\Omega)$, which is, of course, compact as an operator from $\widetilde{H}^1(\Omega)$ to $H^{-1}(\Omega)$. Therefore, $\mathcal{L}_D$ is a *Fredholm operator with index zero*; see Proposition 18.1.12. Next, by virtue of the second Green identity, the operators $\mathcal{L}_D$ and $\widetilde{\mathcal{L}}_D$, where $\widetilde{\mathcal{L}}_D$ corresponds to the Dirichlet problem for a formally adjoint operator, turn out to be adjoint as operators in Banach (Hilbert in the case under consideration) spaces, i.e.,

$$(\mathcal{L}_D u, v)_\Omega = (u, \widetilde{\mathcal{L}}_D v)_\Omega \quad (u, v \in \widetilde{H}^1(\Omega)). \tag{8.1.39}$$

The operator $\widetilde{\mathcal{L}}_D$ is Fredholm and has index zero as well as $\mathcal{L}_D$, so that, e.g., *the equation* $\mathcal{L}_D u = f \in H^{-1}(\Omega)$ *is solvable if and only if the right-hand side* f *satisfies the condition*

$$(f, v)_\Omega = 0 \tag{8.1.40}$$

for all solutions of the homogeneous equation $\widetilde{\mathcal{L}}_D v = 0$; see Proposition 18.1.8.

A similar situation occurs in the case of the Neumann problem if the form $\Phi_\Omega(u, v)$ is coercive but not strongly coercive on $H^1(\Omega)$. In this case, *the operator* $\mathcal{L}_N : H^1(\Omega) \to \widetilde{H}^{-1}(\Omega)$ *corresponding to the Neumann problem with homogeneous boundary condition is Fredholm and has index zero, and the equation* $\mathcal{L}_N u = f$ *is solvable if and only if* f *satisfies condition* (8.1.40) *for all solutions of the adjoint homogeneous equation* $\widetilde{\mathcal{L}}_N v = 0$.

Our Theorem 8.1.5 can be extended to the Fredholm situation as well. For example, it suffices to rewrite the equation $\mathcal{L}_D u = f$ in the form $\mathcal{L}_{\tau,D} u = f + \tau u$ with an invertible operator on the left.

We must also mention that an important role in the theory of strongly elliptic equations is played by *surface potentials*. If $E(x, y)$ is a fundamental solution, i.e., a solution of the equation

$$L_x E(x, y) = \delta(x - y), \tag{8.1.41}$$

then the classical *single-layer potential* is defined by

$$u(x) = \mathcal{A}\psi(x) = \int_\Gamma E(x, y)\psi(y)\, dS_y, \tag{8.1.42}$$

and the *double-layer potential* is defined by

$$u(x) = \mathcal{B}\varphi(x) = \int_\Gamma [\partial_{\nu_y} E(x, y)]\varphi(y)\, dS_y \quad (x \notin \Gamma). \tag{8.1.43}$$

Here for simplicity we wrote the last expression only for a formally self-adjoint operator L. The functions ψ and φ must be regular in a certain sense. Both operators, (8.1.42) and (8.1.43), map functions given on the boundary to solutions of the homogeneous equation outside the boundary. Of great importance are also the restrictions of these operators to Γ and the conormal derivatives of these functions on Γ. In the case of a smooth boundary and smooth coefficients in L, the last four operators are pseudodifferential operators on Γ, and we can study them by means of the calculus of these operators. In the nonsmooth case, there is a simplified approach to derive properties of these operators and the so-called hypersingular operator $H = -T^+\mathcal{B}$, which is based on the assumption that the Dirichlet and Neumann problems are uniquely solvable. This approach is explained in Section 12 in a more general situation, namely, for second-order strongly elliptic systems on Lipschitz domains. But the explanation begins with a revision of the definitions of these operators.

Potential-type operators are convenient for solving problems in the case of a homogeneous equation and an inhomogeneous boundary condition; we shall see this in Section 12.

Operators acting on the boundary include also the Neumann-to-Dirichlet operator N taking Neumann data for a solution of the homogeneous equation to Dirichlet data and the inverse Dirichlet-to-Neumann operator D. For uniquely solvable Dirichlet and Neumann problems, the operator D acts boundedly from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ and is invertible, and the operator N has similar properties in the reverse direction. These two operators, which have already been mentioned in Section 7.4 in a different context, are considered in more detail in Section 11.

8.2 Generalizations

The definition of strong ellipticity is first generalized to second-order systems. They will be considered in Sections 11 and 12; see also Sections 16 and 17. Then it is generalized to higher-order systems with the Douglis–Nirenberg structure (see our Section 6.5). Take a set of positive (for simplicity) integers $m_1,\dots,m_d$. In the notation of Section 6.5, we consider the case $t_j = m_j$. Let us write the system with leading part singled out:

$$Lu(x) = L_0u(x) + \dots = f(x). \tag{8.2.1}$$

The vector-valued functions $u(x)$ and $f(x)$ are columns of height d:

$$u(x) = (u_1(x),\dots,u_d(x))', \quad f(x) = (f_1(x),\dots,f_d(x))'.$$

The operators L and L_0 are $d\times d$ matrix operators. Let $L = (L^{r,s})$, and let $L_0 = (L_0^{r,s})$. Each of the scalar operators $L^{r,s}$ can be written in divergence form as

$$L^{r,s} = L^{r,s}(x,\partial) = \sum_{|\alpha|\le m_r, |\beta|\le m_s} (-1)^{|\alpha|}\partial^\alpha[a^{r,s}_{\alpha,\beta}(x)\partial^\beta\cdot\,]. \tag{8.2.2}$$

The matrix L_0 consists of the leading parts of these operators:

$$L_0^{r,s} = L_0^{r,s}(x,\partial) = \sum_{|\alpha|= m_r, |\beta|= m_s} (-1)^{m_r}\partial^\alpha[a^{r,s}_{\alpha,\beta}(x)\partial^\beta\cdot\,]. \tag{8.2.3}$$

The strong ellipticity condition has the form

$$\mathrm{Re}\sum_{r,s=1}^{d}\sum_{|\alpha|=m_r,|\beta|=m_s} a^{r,s}_{\alpha,\beta}(x)\xi^{\alpha+\beta}\zeta_s\overline{\zeta}_r \ge C_0\sum_{j=1}^{d}|\xi|^{2m_j}|\zeta_j|^2, \tag{8.2.4}$$

where C_0 is a positive constant. The coordinates of the vector $\xi = (\xi_1, \dots, \xi_n)$ are real, while the coordinates of $\zeta = (\zeta_1, \dots, \zeta_d)'$ are complex. The definition of strongly elliptic systems in this generality is given by Nirenberg in [290]. So far we assume the functions $a^{r,s}_{\alpha,\beta}(x)$ to be infinitely differentiable on $\overline{\Omega}$.

First, consider the inner product $(Lu, v)_\Omega$ of compactly supported infinitely differentiable functions on Ω. Integration by parts yields

$$(f, v)_\Omega = \Phi_\Omega(u, v), \tag{8.2.5}$$

where Φ_Ω is the form

$$\Phi_\Omega(u, v) = \int_\Omega \sum_{r,s=1}^{d} \sum_{|\alpha| \le m_r, |\beta| \le m_s} a^{r,s}_{\alpha,\beta}(x) \partial^\beta u_s(x) \overline{\partial^\alpha v_r(x)}\, dx \tag{8.2.6}$$

with leading part

$$\int_\Omega \sum_{r,s=1}^{d} \sum_{|\alpha| = m_r, |\beta| = m_s} a^{r,s}_{\alpha,\beta}(x) \partial^\beta u_s(x) \overline{\partial^\alpha v_r(x)}\, dx. \tag{8.2.7}$$

Of greatest importance and interest are the Dirichlet and Neumann problems in the weak setting with homogeneous boundary conditions. In the Dirichlet problem, we have

$$u_j, v_j \in \widetilde{H}^{m_j}(\Omega), \quad f_k \in H^{-m_k}(\Omega). \tag{8.2.8}$$

By the *Neumann problem* we now mean the problem in which

$$u_j, v_j \in H^{m_j}(\Omega), \quad f_k \in \widetilde{H}^{-m_k}(\Omega). \tag{8.2.9}$$

It is required that relation (8.2.5) hold for u, f, and any test functions v.

Let us introduce the space

$$H^{\boldsymbol{m}}(\Omega) = H^{m_1}(\Omega) \times \ldots \times H^{m_d}(\Omega). \tag{8.2.10}$$

The norm on this space is

$$\|u\|_{H^{\boldsymbol{m}}(\Omega)} = \Big(\sum_1^d \|u_j\|^2_{H^{m_j}(\Omega)}\Big)^{1/2}. \tag{8.2.11}$$

Possibly, $m_1 = \ldots = m_d = m$. The form Φ_Ω is said to be *coercive* on the space (8.2.10) if all its elements satisfy the inequality

$$\|u\|^2_{H^{\boldsymbol{m}}(\Omega)} \le C_1 \operatorname{Re} \Phi_\Omega(u, u) + C_2 \|u\|^2_{H^0(\Omega)} \tag{8.2.12}$$

with constants $C_1 > 0$ and $C_2 \ge 0$. This inequality with $C_2 = 0$ is called the *strong coercivity condition*. In a similar way, coercivity and strong coercivity on other solution spaces V are defined.

The most important of these spaces is

$$\widetilde{H}^{\boldsymbol{m}}(\Omega) = \widetilde{H}^{m_1}(\Omega) \times \ldots \times \widetilde{H}^{m_d}(\Omega), \tag{8.2.13}$$

which is identified with

$$\mathring{H}^{\boldsymbol{m}}(\Omega) = \mathring{H}^{m_1}(\Omega) \times \ldots \times \mathring{H}^{m_d}(\Omega). \tag{8.2.14}$$

It corresponds to the Dirichlet problem. More general solution spaces V are assumed to be subspaces of $H^{\boldsymbol{m}}(\Omega)$ containing $\widetilde{H}^{\boldsymbol{m}}(\Omega)$:

$$\widetilde{H}^{\boldsymbol{m}}(\Omega) \subset V \subset H^{\boldsymbol{m}}(\Omega). \tag{8.2.15}$$

In this case, we seek a solution u of the system $Lu = f$ in V and assume that the right-hand side f belongs to the space V' dual to V with respect to the extension of the inner product $(f, v)_\Omega$. Coercivity of the form Φ_Ω on $H^{\boldsymbol{m}}(\Omega)$ implies coercivity on all subspaces V.

Theorem 8.2.1. *The strong ellipticity of the operator L implies coercivity on $\widetilde{H}^{\boldsymbol{m}}(\Omega)$.*

This result is essentially due to Gårding [163], although in the generality considered here, this fact was mentioned in [290]. Unlike in the scalar case studied in Section 8.1, it needs to be proved, but the proof is simple and based on the method of freezing coefficients. For $m_1 = \ldots = m_d = 1$, i.e., in the case of a second-order system, it is given in Section 11; the general case is handled in a similar way.

The coercivity condition becomes the strong coercivity condition

$$\|u\|^2_{H^{\boldsymbol{m}}(\Omega)} \leq C_1 \operatorname{Re} \Phi_\Omega(u, u) \tag{8.2.16}$$

if the real part of the form $(cu, u)_\Omega$ of the lower-order term is large enough.

Theorem 8.2.2. *If the form Φ_Ω is strongly coercive on the space $\widetilde{H}^{\boldsymbol{m}}(\Omega)$, then the Dirichlet problem is uniquely solvable.*

This follows from the Lax–Milgram theorem.

If the form is only coercive (not strongly), i.e., the operator is only strongly elliptic, then, instead of unique solvability, the Fredholm property is obtained.

The situation with the Neumann and other problems is substantially more complicated. Sufficient conditions for coercivity on $H^{\boldsymbol{m}}(\Omega)$ (and other subspaces V) have a very extensive literature. We give references in Section 19. The books most useful to read first are [8] and [286]. In Section 11 we give a convenient sufficient condition for the coercivity of second-order systems on $H^1(\Omega)$.

Theorem 8.2.3. *In the case of strong coercivity on $H^{\boldsymbol{m}}(\Omega)$, the Neumann problem is uniquely solvable.*

This follows from the same Lax–Milgram theorem. In the case of nonstrong coercivity, we again obtain the Fredholm property.

The infinite differentiability of the coefficients is not required in these theorems. It suffices that the higher-order coefficients be continuous and the other coefficients be measurable and bounded. But if the coefficients, the boundary, and the function f are smooth, we can investigate the smoothness of the solution. As in Section 8.1, this can be done in two ways, by using Nirenberg's method of difference quotients [290] (see our Section 16.5) and by identifying the problem with the corresponding problem of the general theory of smooth elliptic problems. It should only be checked that the boundary operators of a variational problem satisfy the Lopatinskii condition.

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