Chapter 1
Gas of Point Particles

Abstract In this chapter we study the problem of viscous friction in the framework of microscopic models of classical point particles. The system body/medium is modeled by the dynamics of a heavy particle (the body), subjected to a constant force and interacting with infinitely many identical particles (the medium). We discuss conditions on the body/medium interaction that are necessary for the body to reach a finite limiting velocity. Rigorous results are given in the case of quasi-one-dimensional and one-dimensional systems.

1.1 General Equations and Results

A reasonable microscopic model of viscous friction can be detailed as follows. A heavy particle of mass M and position $\mathbf{r} \in \mathbb{R}^d$ freely moves under the action of an external constant force $\mathbf{F}$ and interacts via a two-body potential Ψ with N point particles of equal mass m and positions $\mathbf{r}_i \in \mathbb{R}^d$. These particles, hereafter denoted as the *background* (particles), mutually interact by means of a two-body potential Φ. We are interested in the long time behavior of the heavy particle when the number N is huge and the background particles are initially distributed to describe a real gas.

From a mathematical point of view, it is natural to consider the limiting problem when $N \to \infty$ and the density remains bounded (a sort of thermodynamic limit). Otherwise stated, the heavy particle interacts with a background of infinitely many particles and the equations of motion read,

$$\begin{cases} M\ddot{\mathbf{r}}(t) = \mathbf{F} - \sum_j \nabla\Psi(\mathbf{r}(t) - \mathbf{r}_j(t)) \ , \\ \\ m\ddot{\mathbf{r}}_i(t) = -\nabla\Psi(\mathbf{r}_i(t) - \mathbf{r}(t)) - \sum_{j \neq i} \nabla\Phi(\mathbf{r}_i(t) - \mathbf{r}_j(t)) \ , \quad i \in \mathbb{N} \ . \end{cases} \tag{1.1}$$

We have to explain the precise meaning of this limit $N \to \infty$, i.e., of the time evolution defined by (1.1) and hereafter called *infinite dynamics*. We fix an initial datum,

$$(\mathbf{r}(0), \dot{\mathbf{r}}(0)) \ , \quad \{(\mathbf{r}_i(0), \dot{\mathbf{r}}_i(0))\}_{i \in \mathbb{N}} \ , \tag{1.2}$$

P. Buttà et al., *Mathematical Models of Viscous Friction*, Lecture Notes in Mathematics 2135, DOI 10.1007/978-3-319-14759-8_1

such that the particles distribution in space is locally finite, i.e., the number of particles inside any bounded region of $\mathbb{R}^d$ is finite. Without loss of generality we also assume $\mathbf{r}(0) = \mathbf{0}$. For each $n \in \mathbb{N}$ we introduce the so called *n-partial dynamics*, obtained by neglecting those particles initially outside the sphere of radius n and center the origin. More precisely, setting $I_n := \{i \in \mathbb{N}\colon |\mathbf{r}_i(0)| \le n\}$, the n-partial dynamics is the solution to

$$\begin{cases} M\ddot{\mathbf{r}}(t) = \mathbf{F} - \displaystyle\sum_{j\in I_n} \nabla\Psi(\mathbf{r}(t) - \mathbf{r}_j(t)) \ , \\ m\ddot{\mathbf{r}}_i(t) = -\nabla\Psi(\mathbf{r}_i(t) - \mathbf{r}(t)) - \displaystyle\sum_{j\in I_n: j\neq i} \nabla\Phi(\mathbf{r}_i(t) - \mathbf{r}_j(t)) \ , \quad i \in I_n \ , \end{cases} \tag{1.3}$$

with the same initial conditions (1.2) but restricted to $i \in I_n$ (we tacitly assume that such a global solution exists for any $n \in \mathbb{N}$). Of course, such a solution depends on n and is denoted by $\mathbf{r}^{(n)}(t)$, $\{\mathbf{r}_i^{(n)}(t)\}_{i\in I_n}$. A natural candidate solution to the Cauchy problem (1.1)–(1.2) is given by

$$\mathbf{r}(t) = \lim_{n\to\infty} \mathbf{r}^{(n)}(t) \ , \qquad \mathbf{r}_i(t) = \lim_{n\to\infty} \mathbf{r}_i^{(n)}(t) \ , \quad i \in \mathbb{N} \ , \tag{1.4}$$

provided that the above limits exist. It is quite obvious that we cannot expect the convergence in (1.4) to be uniform with respect to $i \in I_n$. Instead, we must fix a single index i and then perform the limit $n \to \infty$. Clearly, the existence of this limit means that the motion of the ith particle is not very much influenced by the presence of particles very far away from it.

The question is now under which conditions the infinite dynamics exists. We observe that the limiting procedure described above suggests that such a question is essentially equivalent to another one: whether or not the time evolution via the partial dynamics remains local.

The answer is nontrivial because the evolution could bring in a finite time infinitely many particles in a bounded region of the space, as we can see in this simple example in dimension $d = 1$. Consider a system of free particles of unitary mass moving on the real line with the initial condition $r_i(0) = i$, $\dot{r}_i(0) = -i$, $i \in \mathbb{N}$. It is evident that at time $t = 1$ all the particles are in the origin. Of course, in this example of free motion it is easy to extend the evolution to times $t > 1$, but in presence of mutual interactions the forces become infinite at time $t = 1$ and the Newton's law looses meaning.

To avoid this kind of "collapses" we must restrict the allowed initial conditions, but we cannot be too drastic. In fact, for the model to be meaningful, the class of admissible initial conditions must contain all the data compatible with the physical experiment we want to describe, which can be summarized as follows. At time $t = 0$ the heavy particle is located at the origin and is surrounded by a gas at thermal equilibrium (or in some non-equilibrium status very close to equilibrium); we then switch on a constant force $\mathbf{F}$ acting on the heavy particle and look at its asymptotic motion.

Our goal is to show the following *conjecture*: a necessary condition for the heavy particle to reach a bounded asymptotic velocity is that its interaction with the background be *singular*. It would be nice to prove such a result for a generic system of infinitely many particles in $\mathbb{R}^3$, but, as we shall see, it is too difficult at the present stage of knowledge.

Instead, we rigorously prove the conjecture in two specific models. In the first one, we consider the particles posed in an infinitely extended tube of $\mathbb{R}^3$ and the external force $\mathbf{F}$ is parallel to the symmetry axis of the tube. We then show that a bounded interaction cannot give rise to a finite limit velocity if the intensity of $\mathbf{F}$ is sufficiently large. As a corollary, we obtain that if the medium is initially at thermal equilibrium then the average velocity of the heavy particle diverges as time goes to infinite. To extend the result to the case of $\mathbf{F}$ with any intensity, we need to give up this more realistic geometry and consider, as second model, the genuine one-dimensional case. These results are the content of Sect. 1.3.

Let us goes back to the choice of the initial conditions. By the above discussion, our first requirement on the model is that the infinite time evolution (1.1) has to be defined for all the initial microscopic configurations of the gas (i.e., positions and velocities of the particles) which are compatible with any reasonable thermodynamic (equilibrium or non-equilibrium) state. For the convenience of the reader, we first summarize in the next subsection some basic results from rigorous equilibrium statistical mechanics.

1.1.1 Infinite Volume Gibbs States

The microscopic explanation of the thermodynamic properties of matter is the content of the equilibrium statistical mechanics, a very well established branch of theoretical and mathematical physics. In this theory, the macroscopic behavior of a system composed by a large number of particles (atoms/molecules) is described by means of probability distributions on the phase space of the microscopic configurations of the system. More precisely, the basic postulate is that the equilibrium values of macroscopic observables are obtained as averages (respect to these probabilities) of appropriate functions of the microscopic configurations. If the system is confined in a bounded region, these probabilities are given by the so called *Gibbs ensembles* (or *finite volume Gibbs states*). These probability distributions are stationary with respect to the time evolution of the underlying mechanical system, but this is only a necessary condition for a dynamical justification for their use to calculate equilibrium quantities. This is a central question of statistical mechanics, which is discussed in any classic textbook or review on the subject, see, e.g., [11, 14].

Three different type of ensembles are introduced, the *microcanonical*, the *canonical*, and the *grand canonical* ensemble. The microcanonical ensemble describes the thermal equilibrium of an isolated system with a large number N of degree of freedom, and it is defined by the uniform probability distribution on the iso-energetic surface in the phase space of the system (this choice is also known as the principle of equal a priori probabilities). The other ensembles are derived from the microcanonical one to describe the equilibrium of not isolated systems. To be more concrete, we consider a classical system composed by N identical particles with Hamiltonian,

$$H_N(x) = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \sum_{1 \le j < i \le N} \Phi(\mathbf{r}_i - \mathbf{r}_j) ,$$

where $x = (\mathbf{r}_1, \dots, \mathbf{r}_N, \mathbf{p}_1, \dots, \mathbf{p}_N)$ denotes the point in the phase space of positions and momenta of the particles, and Φ is a two-body potential. If the system is in thermal contact with a reservoir at temperature T, then its equilibrium is described by the canonical ensemble, defined as the probability distribution with density in phase space proportional to $\exp(-H_N(x)/kT)$, where k is the Boltzmann constant. Finally, if the system can also exchange particles with the reservoir, then the equilibrium is described by the grand canonical ensemble, which for later purposes we describe in more detail. Denoting by $\Gamma_{N,V}$ the phase space for N particles confined in the region V, the grand canonical ensemble is defined by the rule,

$$\langle G \rangle_V = \frac{1 + \sum_{N=1}^{\infty} (1/N!) \int_{\Gamma_{N,V}} \mathrm{d}x \, \exp(-\beta H_N(x) + \lambda N) \, G_N(x)}{Z_V(\beta, \lambda)} , \tag{1.5}$$

where $\beta = (kT)^{-1}$ and λ, named *chemical potential*, is a positive parameter; $G_N(x)$ is the function representing the observable G in a system of N particles, and the normalization constant

$$Z_V(\beta, \lambda) = 1 + \sum_{N=1}^{\infty} \frac{1}{N!} \int_{\Gamma_{N,V}} \mathrm{d}x \, \exp(-\beta H_N(x) + \lambda N) \tag{1.6}$$

is called the (grand canonical) *partition function*.

According to the theory, the partition function contains all the information on the thermodynamic properties of the system. More precisely, these are obtained by identifying $p_V(\beta, \lambda) = (\beta |V|)^{-1} \log Z_V(\beta, \lambda)$ with the thermodynamic pressure, as a function of inverse temperature and chemical potential. Since the pressure should be an intensive function (independent of V), this identification is not satisfactory. The reason of this apparent discrepancy depends on the macroscopic size of a real system, which is much larger than the microscopic scale. Therefore, to have a good thermodynamic behavior it is sufficient the existence of the limit $p(\beta, \lambda) = \lim_{V \to \mathbb{R}^d} p_V(\beta, \lambda)$, where $V \to \mathbb{R}^d$ means that V invades the whole space in a reasonable way (e.g., if $\{V\}$ is a sequence of cubes). Accordingly, it is such limit $p(\beta, \lambda)$ that has to be identified with the thermodynamic pressure.

The existence of the limiting pressure can be proved under quite soft conditions on the pair potential Φ. For example, this is the case if the pair potential is finite range and *stable*, where Φ stable means that there exists a constant $B \geq 0$ such that for all N the potential energy of N particles is bounded below by $-BN$, i.e.,

$$\sum_{1 \leq j < i \leq N} \Phi(\mathbf{r}_i - \mathbf{r}_j) \geq -BN \qquad \forall\, N \in \mathbb{N}\,.$$

In other part of physics this property is called *saturation* of the force. This means that N particles cannot produce an energy larger than N (this fact forbids negative interaction in the origin). We remark that even for V bounded, the convergence of the series in (1.6) and then the existence of $p_V(\beta, \lambda)$ needs some assumption on the interaction (obviously, stability is sufficient).

This limiting procedure is known as *thermodynamic limit* and can be repeated in the case of the microcanonical and canonical ensembles, by providing similar formulae for the entropy and free energy respectively, as functions of the appropriate thermodynamic variables. The problem of showing that all these ensembles give the same values for the thermodynamic functions is called the problem of thermodynamic equivalence of ensembles. A solution to this problem is given in Ruelle's book [12].

As well as asking whether different ensembles lead to the same thermodynamic potentials, we could wonder whether they lead to the same local properties. This means to consider *local observables* (i.e., functions on phase space which depend only on the positions and velocities of the particles in bounded regions) and ask whether, in the thermodynamic limit, different ensembles lead to the same averages for each function of this type. Local equivalence of ensembles is a stronger property than thermodynamic equivalence of ensembles. For example, for values of the parameters corresponding to a phase transition, due to the possibility of long range correlations, the limiting averages of some local variables could even be ill defined.

Instead of considering an infinite sequence of finite systems, as in the theory of the thermodynamic limit, we can deal with just one system, which is infinite from the beginning. This simplification bypasses the finite size effects and allows a clear formulation of questions relating to phase transitions and correlation functions. The price to pay is that it requires a much more sophisticated mathematics with respect to the finite-system approach.

In particular, for a classical infinite system, the phase space is infinite-dimensional and the Gibbs ensembles cannot be described by means of phase-space densities. The method used alternatively is to characterize the probability measures on the infinite-dimensional phase space by specifying the expectations of all the local observables. More precisely, the phase space Γ is the collection of sequences $\mathbf{X} = \{(\mathbf{r}_i, \mathbf{p}_i)\}_{i \in \mathbb{N}}$ which are locally finite (i.e., the number of particles inside any bounded region is finite), equipped with the topology of local convergence. The equilibrium states, often called (infinite volume) *Gibbs states*, are then defined as those Borel probability measures on Γ satisfying the so called *DLR equilibrium condition*, formulated by Dobrushin [6] and independently by Lanford and Ruelle [10]. For finite range forces, the *DLR* equations are the mathematical expression of the condition that if the particles outside a finite region are held fixed, then the particles inside the region have a Gibbs grand canonical distribution in the external field produced by the particles outside. In more detail, the probability ν satisfies the *DLR* condition if for any local function G and any finite region Λ which contains the support of G we have,

$$\int \nu(\mathrm{d}\mathbf{X})G(\mathbf{X}) = \int \nu(\mathrm{d}\mathbf{X})\langle G\rangle_{\Lambda,\mathbf{X}_{\Lambda^c}} \ ,$$

where $\mathbf{X}_{\Lambda^c}$ is the restriction of $\mathbf{X}$ to Λ^c and $\langle\cdot\rangle_{\Lambda,\mathbf{X}_{\Lambda^c}}$ is the Gibbs grand canonical distribution with boundary condition $\mathbf{X}_{\Lambda^c}$, which is defined as in (1.5) with $H_N(x)$ replaced by $H_N(x|\mathbf{X}_{\Lambda^c}) = H_N(x)+W_\Lambda(x|\mathbf{X}_{\Lambda^c})$, where $W_\Lambda(x|\mathbf{X}_{\Lambda^c})$ is the potential energy due to the pair interaction Φ between the N particles of the configuration $x \in \Gamma_{\Lambda,N}$ and the particles outside Λ of the locally finite configuration $\mathbf{X}$.

For stable and finite-range forces this condition is equivalent to the condition that the state be a limit of grand canonical states with suitable boundary conditions. This justifies the physical interpretation of the infinite volume Gibbs states as the different thermodynamic phases. In particular, the nonuniqueness of solution (for given values of β and λ) of the *DLR* equations corresponds to the occurrence of a phase transition. We address the interested reader to the books of Georgii [8] and Ruelle [12] for an exhaustive exposition of the theory of Gibbs states.

Existence and good thermodynamic behavior of the Gibbs states depend on the nature of the pair potential Φ. For finite range forces, a sufficient condition is that Φ be *superstable* [13], i.e., there are constants $B_1 > 0$ and $B_2 \geq 0$ such that for any finite configuration of particles $\{\mathbf{r}_1, \ldots, \mathbf{r}_N\}$, $N \in \mathbb{N}$, and for any bounded region $\Lambda \subset \mathbb{R}^d$,

$$\sum_{i<j} \chi(\mathbf{r}_i \in \Lambda)\chi(\mathbf{r}_j \in \Lambda)\Phi(\mathbf{r}_i - \mathbf{r}_j) \geq \frac{B_1 N(\Lambda)^2}{|\Lambda|} - B_2 N(\Lambda) \ , \tag{1.7}$$

where $\chi(A)$ denotes the characteristic function of the set A, $N(\Lambda)$ is the number of particles inside Λ, and $|\Lambda|$ is the volume of Λ. Note that the stability property previously introduced simply means that (1.7) holds with $B_1 = 0$.

Superstability is a technical assumption, slightly stronger that stability. It can be proved [12] that Φ is superstable if it can be written as the sum of a *stable* interaction plus a nonnegative interaction which is positive and continuous in the origin. Actually, the only interesting potential which is stable but not superstable is given by $\Phi = 0$.

We address the reader to the fundamental paper by Ruelle [13] for a detailed study of the statistical mechanics of systems with superstable interactions. For our purposes, we just recall here a crucial support property of the equilibrium states in systems with this type of interaction. Such property asserts that the energy and number of particles in any bounded region of the space cannot fluctuate too much, in the following sense.

We introduce, for any $\mu \in \mathbb{R}^d$ and $R > 0$,

$$Q(\mathbf{X}; \mu, R) := \sum_i \chi(|\mathbf{r}_i - \mu| < R) \left\{ \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} \sum_{j:j \neq i} \Phi(\mathbf{r}_i - \mathbf{r}_j) + \bar{B} \right\} ,$$

which controls the energy and number of particles in the open ball of center μ and radius R. The positive constant $\bar{B}$ is chosen large enough to have $Q \geq 0$; it is easy to verify that such choice is possible as Φ is finite range and satisfies (1.7). Let now ν be any Gibbs state, i.e., any solution to the *DLR* equation for given β and μ. Then, using Ruelle's superstability estimates [13], a direct calculation shows that, for appropriate $\alpha, c > 0$,

$$\int \nu(\mathrm{d}\mathbf{X}) \exp(\alpha Q(\mathbf{X}; \mu, R)) \leq \exp(cR^d) \qquad \forall\, \mu \in \mathbb{R}^d \quad \forall\, R > 0 .$$

By the exponential Chebyshev's inequality it follows that $\nu(Q(\mathbf{X}; \mu, R) > \alpha^{-1}\xi R^d) \leq \exp[(c - \xi)R^d]$ for any $\xi > 0$. From this last estimate it is not difficult to show that, setting

$$Q(\mathbf{X}) := \sup_{\mu} \sup_{R: R > \log(\mathrm{e} + |\mu|)} \frac{Q(\mathbf{X}; \mu, R)}{R^d} ,$$

one has $\nu(Q(\mathbf{X}) > K) \leq a \exp(-bK)$ for suitable $a, b > 0$ and any $K \geq 0$; we omit the details, see also [7]. In particular, by the Borell–Cantelli lemma,

$$\nu(Q(\mathbf{X}) < \infty) = 1 .$$

Otherwise stated, in the case of short range and superstable interactions, the support of any equilibrium state is contained in the set of locally finite configurations which have local energy and number of particle fluctuations only of logarithmic order.

1.1.2 Choice of the Initial Data

In the nonequilibrium case (as in our setting) the situation is much more complicated, but the admissible interactions remain of the superstable type. Indeed, superstability implies that it is very expensive (in term of energy) to have many particles in bounded regions of the space, so that local conservation of energy may prevent from collapses in finite time.

Concerning the choice of initial conditions, we want a set of full measure with respect to at least the Gibbs states. For instance, a set of microscopic states in which the velocities of all the particles are uniformly bounded is exceptional and to know its time evolution is not so important. While, from the discussion in the previous subsection, we know that in order to consider configurations which are typical for the thermodynamic states, we need to allow initial data with logarithmic divergences in the velocities and local densities.

The bare existence of the dynamics is not enough for our purposes. In fact, we also need nontrivial informations on the long time behavior of the system. Nowadays, this kind of knowledge has been obtained only in one spatial dimension. Of course, a one dimensional world is a little strange, but it is the first step to face the problem. Moreover, by "one dimension" we do not solely mean particles moving along a straight line, but also particles moving in a region which is infinitely extended along one direction only.

Throughout this chapter we shall assume that the particles of the background mutually interact via a positive potential with a finite range. Reasonably, we could make the assumption that the interaction be superstable, including in this way also negative interactions. The generalization of positive interactions with a finite range by superstable potentials with a long range term have been done many times. For instance, concerning the existence of the dynamics in three dimension for bounded interaction, this has been done in [5], which generalizes the results of the pioneering paper [4]. The calculations are very cumbersome. We hope that in our context also this generalization may be performed, but the explicit study is long and nontrivial and it has not been done so far.

We conclude with a notation warning valid in the rest of this chapter: in the sequel, if not further specified, we shall denote by C a generic positive constant whose numerical value may change from line to line and it may possibly depend only on the interactions Φ and Ψ.

1.2 Infinite Dynamics in One Dimension: Existence and Long Time Behavior

In this section we present the key tools which allow to prove the existence of the infinite dynamics in one dimension, with a good control on its long time behavior. For explanatory reasons, we consider here the simplest case of a gas of particles of unit mass moving along a straight line and disregard the presence of the heavy particle. We assume that the particles interact among themselves by means of a non-negative, symmetric, short-range, two-body potential Φ of the form

$$\Phi(x) = \Phi_1(x) + a|x|^{-b} \, , \tag{1.8}$$

where $a \geq 0$, $b > 0$, and Φ_1 is twice differentiable and symmetric. If Φ is finite at the origin we assume $\Phi(0) > 0$, which guarantees Φ to be superstable [12]. Without loss of generality we assume that Φ has range not greater than one, i.e.,

$$\Phi(x) = 0 \qquad \text{if} \quad |x| > 1 \, . \tag{1.9}$$

We denote by $(r_i, v_i) \in \mathbb{R}^2$ the position and velocity of the particles. The state $X = \{(r_i, v_i)\}_{i\in\mathbb{N}}$ is assumed to have a locally finite density and energy. In particular, for any $\mu \in \mathbb{R}$ and $R > 0$, it is well defined the quantity

$$Q(X; \mu, R) := \sum_i \chi_i(\mu, R) \left\{ \frac{v_i^2}{2} + \frac{1}{2} \sum_{j:j\neq i} \Phi(r_i - r_j) + 1 \right\}, \tag{1.10}$$

where $\chi_i(\mu, R) = \chi(|r_i - \mu| \leq R)$. According to the discussion of the previous section, in order to consider configurations which are typical for the thermodynamic states, we allow initial data with logarithmic divergences in the velocities and local densities. In the present context, by defining

$$Q(X) := \sup_{\mu} \sup_{R:R>\log(\mathrm{e}+|\mu|)} \frac{Q(X; \mu, R)}{2R}, \tag{1.11}$$

the set

$$\mathscr{X} := \{X : Q(X) < \infty\} \tag{1.12}$$

has a full measure with respect to any Gibbs state.

Given $X \in \mathscr{X}$ and $n \in \mathbb{N}$ let $I_n := \{i \in \mathbb{N}: |r_i| \leq n\}$. The n-partial dynamics $t \mapsto X^{(n)}(t) = \{(r_i^{(n)}(t), v_i^{(n)}(t))\}_{i\in I_n}$ is defined as the solution to the Cauchy problem

$$\begin{cases} \ddot{r}_i^{(n)}(t) = -\displaystyle\sum_{j\in I_n:j\neq i} \Phi'(r_i^{(n)}(t) - r_j^{(n)}(t)), & i \in I_n, \\ X^{(n)}(0) = \{(r_i, v_i)\}_{i\in I_n}. \end{cases} \tag{1.13}$$

Theorem 1.1 *For $X \in \mathscr{X}$ the following limits exist,*

$$\lim_{n\to\infty} r_i^{(n)}(t) = r_i(t), \quad \lim_{n\to\infty} v_i^{(n)}(t) = v_i(t), \quad i \in \mathbb{N}. \tag{1.14}$$

Moreover, the flow $t \mapsto X(t) = \{(r_i(t), v_i(t))\}_{i\in\mathbb{N}}$ is the unique (global) solution to

$$\begin{cases} \ddot{r}_i(t) = -\displaystyle\sum_{j\neq i} \Phi'(r_i(t) - r_j(t)), & i \in \mathbb{N}, \\ X(0) = X. \end{cases} \tag{1.15}$$

such that $X(t) \in \mathscr{X}$. Finally,

$$|v_i(t)| \leq C \left[\sqrt{Q(X)\log(\mathrm{e} + |r_i| + Q(X))} + Q(X)t \right] \quad \forall\, i \in \mathbb{N} \quad \forall\, t \geq 0, \tag{1.16}$$

and, for any $\mu \in \mathbb{R}$ *and* $R > \log(\mathrm{e} + |\mu|)$,

$$Q(X(t);\mu,R) \le CQ(X)\left[R + \log(\mathrm{e} + Q(X)) + (1 + Q(X))t^2\right] \quad \forall\, t \ge 0\,. \tag{1.17}$$

A basic tool in the proof of this theorem is an estimate on the growth in time of the local density and energy, which is the content of the following lemma. The idea behind the proof is to use the local conservation of energy and number of particles, combined with the superstability of the potential, to control the variation in time of the local density and energy with these same quantities, thus obtaining a differential inequality which gives the desired estimate. The proof turns out to be a little bit technical as, to control the variation of energy by the energy itself, we need to work with a mollified version of $Q(X;\mu,R)$.

Lemma 1.2 *There exists a constant* $K_0 > 0$ *such that, for any* $X \in \mathscr{X}$ *and* $n \in \mathbb{N}$,

$$\sup_{\mu} Q(X^{(n)}(t);\mu,R_n(t)) \le K_0 Q(X) R_n(t) \qquad \forall\, t \ge 0\,, \tag{1.18}$$

where

$$R_n(t) := \log(\mathrm{e} + n) + \int_0^t \mathrm{d}s\; V_n(s) \tag{1.19}$$

and

$$V_n(t) := \max_{i \in I_n} \sup_{s \in [0,t]} |v_i^{(n)}(s)|\,. \tag{1.20}$$

Proof We introduce the following mollified version of $Q(X;\mu,R)$,

$$W(X;\mu,R) := \sum_i f_i^{\mu,R} \left\{ \frac{v_i^2}{2} + \frac{1}{2}\sum_{j:j\ne i} \Phi(r_i - r_j) + 1 \right\}\,, \tag{1.21}$$

where

$$f_i^{\mu,R} = f\left(\frac{|r_i - \mu|}{R}\right) \tag{1.22}$$

and $f \in C^\infty(\mathbb{R}_+)$ is not increasing and satisfies: $f(x) = 1$ for $x \in [0,1]$, $f(x) = 0$ for $x \ge 2$, and $|f'(x)| \le 2$. Clearly,

$$Q(X;\mu,R) \le W(X;\mu,R) \le Q(X;\mu,2R)\,. \tag{1.23}$$

For $0 \le s \le t$, we define

$$R_n(t,s) := \log(\mathrm{e}+n) + \int_0^t \mathrm{d}\tau \; V_n(\tau) + \int_s^t \mathrm{d}\tau \; V_n(\tau) \tag{1.24}$$

(note that $R_n(t,t) = R_n(t)$ and $R_n(t,0) \le 2R_n(t)$) and compute

$$\partial_s W(X^{(n)}(s);\mu, R_n(t,s)) = \sum_i \left[\kappa_i(t,s)\varepsilon_i(s) + f_i^{\mu,R_n(t,s)}\dot{\varepsilon}_i(s)\right] , \tag{1.25}$$

where, denoting by $r_i^{\mu}(s)$ the sign of $r_i(s) - \mu$,

$$\kappa_i(t,s) = f'\left(\frac{|r_i(s)-\mu|}{R_n(t,s)}\right)\left[\frac{r_i^{\mu}(s)v_i(s)}{R_n(t,s)} - \frac{\partial_s R_n(t,s)}{R_n(t,s)^2}|r_i(s)-\mu|\right] ,$$
$$\varepsilon_i(s) = \frac{v_i(s)^2}{2} + \frac{1}{2}\sum_{j:j\ne i}\Phi(r_i(s)-r_j(s)) + 1,$$

and, to simplify notation, we have omitted the explicit dependence on n of r_i, v_i, κ_i, and ε_i.

Since $f'(|y|) \le 0$, $f'(|y|) = 0$ if $|y| \le 1$, $\partial_s R_n(t,s) = -V_n(s)$, and $|v_i(s)| \le V_n(s)$, then $\kappa_i(t,s) \le 0$. On the other hand, from the equations of motion,

$$\dot{\varepsilon}_i(s) = -\sum_{j:j\ne i}\Phi'(r_i(s)-r_j(s))\frac{v_i(s)+v_j(s)}{2} .$$

Then, by (1.25) and using Φ' is odd,

$$\partial_s W(X^{(n)}(s);\mu, R_n(t,s)) \le -\sum_{i\ne j}\left(f_i^{\mu,R_n(t,s)} - f_j^{\mu,R_n(t,s)}\right)\Phi'(r_i(s)-r_j(s))\frac{v_i(s)}{2} . \tag{1.26}$$

From (1.8) we have $|q|\,|\Phi'(q)| \le C[1+\Phi(q)]$ for any $q \ne 0$. Then, by the inequality

$$\left|f_i^{\mu,R} - f_j^{\mu,R}\right| \le 2\,\frac{|r_i - r_j|}{R}\left[\chi_i(\mu,2R) + \chi_j(\mu,2R)\right] ,$$

and since $R_n(t,s) > 1$, the modulus of the double sum in the right-hand side of (1.26) can be bounded from above by

$$-C\frac{\partial_s R_n(t,s)}{R_n(t,s)}\sum_{i\ne j}[1+\Phi(r_i(s)-r_j(s))]\chi_i(\mu,4R_n(t,s))\chi_j(\mu,4R_n(t,s))\chi_{i,j}(s) , \tag{1.27}$$

where we shortened $\chi_{i,j}(s) = \chi(|r_i(s) - r_j(s)| \le 1)$. Since Φ is superstable, by arguing as in the proof of [4, Eq. (2.15)], the double sum in the right-hand side of (1.27) can be bounded by $CW(X^{(n)}(s); \mu, 4R_n(t,s))$; moreover, setting

$$W(X; R) := \sup_{\mu} W(X; \mu, R) \;, \tag{1.28}$$

it can be proved that

$$W(X; \mu, 2R) \le CW(X; R) \tag{1.29}$$

(see, e.g., [3, 4]), and hence, by (1.26),

$$\partial_s W(X^{(n)}(s); \mu, R_n(t,s)) \le -C \frac{\partial_s R_n(t,s)}{R_n(t,s)} W(X^{(n)}(s); R_n(t,s)) \;,$$

from which, by integrating and taking the supremum on μ,

$$\begin{aligned} W(X^{(n)}(s); R_n(t,s)) &\le W(X^{(n)}(0); R_n(t,0)) \\ &\qquad - C \int_0^s \mathrm{d}\tau \, \frac{\partial_\tau R_n(t,\tau)}{R_n(t,\tau)} \, W(X^{(n)}(\tau); R_n(t,\tau)) \;, \end{aligned} \tag{1.30}$$

whence

$$W(X^{(n)}(s); R_n(t,s)) \le W(X^{(n)}(0); R_n(t,0)) \left(\frac{R_n(t,0)}{R_n(t,s)} \right)^C \;.$$

Setting $s = t$ and using that $R_n(t,0) \le 2R_n(t,t) = 2R_n(t)$,

$$W(X^{(n)}(t); R_n(t)) \le CW(X^{(n)}(0); R_n(t)) \;.$$

Then, from (1.23), (1.28), and definition (1.11), we conclude that

$$\begin{aligned} Q(X^{(n)}(t); \mu, R_n(t)) &\le CW(X^{(n)}(0); R_n(t)) \\ &\le C \sup_{\mu} Q(X^{(n)}(0); \mu, 2R_n(t)) \\ &\le 4CQ(X) R_n(t) \;, \end{aligned}$$

which proves (1.18). □

We have thus proved that the growth of the local energy is controlled by the maximal displacement of the particles. Since the potential is positive, the former gives an upper bound on the square of the maximal velocity of the particles. But the maximal velocity (multiplied by the length of the time interval) is an upper bound for the maximal displacement. As a result, one obtain a bound on the maximal velocity, which depends linearly on time and on the square root of the initial energy particle and density. The precise statement is the content of the following lemma.

Lemma 1.3 *There exists a constant* $K_1 > 0$ *such that, for any* $X \in \mathscr{X}$, $n \in \mathbb{N}$, *and* $i \in I_n$,

$$|v_i^{(n)}(t)| \le K_1 \left[\sqrt{Q(X)\log(\mathrm{e}+n)} + Q(X)t \right] \qquad \forall\, t \ge 0\,. \tag{1.31}$$

Proof Let $N_n(\mu, t)$ be the number of the particles $i \in I_n$ such that $|r_i^{(n)}(t) - \mu| \le R_n(t)$ and $|v_i^{(n)}(t)| > b_n(t)$, with

$$b_n(t) = K\sqrt{Q(X)\log(\mathrm{e}+n)} + \frac{V_n(t)}{2}\,, \tag{1.32}$$

where $K > 0$ will be fixed later and V_n is defined in (1.20). Clearly:

$$Q(X^{(n)}(t); \mu, R_n(t)) > \frac{b_n(t)^2}{2} N_n(\mu, t)$$

so that, by inequality (1.18) and using the definitions (1.19) and (1.20),

$$N_n(\mu, t) < 8K_0 Q(X) \frac{\log(\mathrm{e}+n) + tV_n(t)}{\left(2K\sqrt{Q(X)\log(\mathrm{e}+n)} + V_n(t)\right)^2}\,,$$

from which, after neglecting some positive terms,

$$N_n(\mu, t) < \frac{2K_0}{K^2} + \frac{8K_0 Q(X)t}{2K\sqrt{Q(X)} + V_n(t)}\,. \tag{1.33}$$

We now choose $K = 2\sqrt{2K_0}$; by (1.33), if $V_n(t) \ge 32K_0 Q(X)t$ then $N_n(\mu, t) < 1/2$, i.e., $N_n(\mu, t) = 0$. The above argument is independent of μ, so that $V_n(t) \ge 32K_0 Q(X)t$ actually implies $|v_i^{(n)}(t)| \le b_n(t)$ for all $i \in I_n$. Since $b_n(t)$ is not decreasing, we have in fact $V_n(t) \le b_n(t)$ when $V_n(t) \ge 32K_0 Q(X)t$. Recalling the definition (1.32), we conclude that

$$V_n(t) \le 2K\sqrt{Q(X)\log(\mathrm{e}+n)} + 64K_0 Q(X)t \qquad \forall\, t \ge 0\,. \tag{1.34}$$

By (1.34) the inequality (1.31) follows for $K_1 = \max\{2K, 64K_0\}$. ☐

Proof of Theorem 1.1 The estimate (1.31) gives an upper bound for the velocities of particles evolving according to the n-partial dynamics, which is independent of n.

This is the key ingredient for the proof of the existence and locality of the infinite dynamics, which is now achieved via a standard iterative procedure.

Let

$$\delta_i(n,t) := |r_i^{(n)}(t) - r_i^{(n-1)}(t)| + |v_i^{(n)}(t) - v_i^{(n-1)}(t)| \, . \tag{1.35}$$

From the equations of motion in integral form we have,

$$\begin{cases} v_i^{(n)}(t) = v_i - \int_0^t \mathrm{d}s \sum_{j \in I_n : j \neq i} \Phi'(r_i^{(n)}(s) - r_j^{(n)}(s)) \, , \\ r_i^{(n)}(t) = r_i + v_i t - \int_0^t \mathrm{d}s \; (t-s) \sum_{j \in I_n : j \neq i} \Phi'(r_i^{(n)}(s) - r_j^{(n)}(s)) \, . \end{cases} \tag{1.36}$$

From (1.35) and (1.36) it follows that, for any $i \in I_{n-1}$,

$$|\delta_i(n,t)| \leq (1+t) \int_0^t \mathrm{d}s \; G_i^{(n)}(s) \, , \tag{1.37}$$

where

$$G_i^{(n)}(s) := \left| \sum_{j \in I_n : \, j \neq i} \Phi'(r_i^{(n)}(s) - r_j^{(n)}(s)) - \sum_{j \in I_{n-1} : \, j \neq i} \Phi'(r_i^{(n-1)}(s) - r_j^{(n-1)}(s)) \right| .$$

By (1.9) and (1.31), each particle $i \in I_n$ may interact during the time $[0,t]$ only with the particles j such that $|r_j - r_i| \leq p_n(t)$, with

$$p_n(t) := 1 + 2K_1 t \left[\sqrt{Q(X) \log(\mathrm{e}+n)} + Q(X)t \right] . \tag{1.38}$$

We now fix $k \in \mathbb{N}$ and define

$$n(k) := \min\{m \in \mathbb{N} : n > 1 + k + p_n(t) \quad \forall \, n \geq m\} \, . \tag{1.39}$$

For $n \geq n(k)$ each particle $i \in I_k$ does not interact, during the time $[0,t]$, with the particles $j \in I_n \setminus I_{n-1}$. Moreover, since Φ is of the form (1.8),

$$|\Phi'(\xi) - \Phi'(\zeta)| \leq C \left[\Phi(\xi)^\eta + \Phi(\zeta)^\eta + \chi(|\xi| \leq 1) + \chi(|\zeta| \leq 1)\right] |\xi - \zeta| \, ,$$

where $\eta = (b+1)/b$. Therefore, for any $n \geq n(k)$, $s \geq 0$, and $i \in I_k$,

$$\begin{aligned} G_i^{(n)}(s) \leq C \sum_{j : j \neq i}{}^{*} & \left[\Phi(r_i^{(n)}(s) - r_j^{(n)}(s))^\eta + \Phi(r_i^{(n-1)}(s) - r_j^{(n-1)}(s))^\eta + 1 \right] \\ & \times \left[\delta_i(n,s) + \delta_j(n,s) \right] \, , \end{aligned}$$

where $\sum_{j:j\neq i}^{*}$ denotes the sums restricted to all the particles $j \in I_{n-1}$ closer than 1 to $r_i^{(n)}(s)$ or $r_i^{(n-1)}(s)$. Using the definition (1.10) and introducing

$$u_k(n,t) := \sup_{i\in I_k} \delta_i(n,t), \tag{1.40}$$

by (1.37) and the above bounds, for any $t \geq 0$,

$$u_k(n,t) \leq C(1+t)\int_0^t \mathrm{d}s \; \sup_{\mu} \left[Q(X^{(n)}(s);\mu,1) + Q(X^{(n-1)}(s);\mu,1)\right]^{\eta} u_{k_1}(n,s) \;, \tag{1.41}$$

where $k_1 = \mathrm{Int}\,[k + p_n(t)] + 1$ (Int $[\cdot]$ denotes the integer part of $\cdot$). On the other hand, from (1.18) and (1.31), for any $s \in [0,t]$,

$$Q(X^{(n)}(s);\mu,1) \leq K_0 Q(X)\left\{\log(\mathrm{e}+n) + K_1 t\left[\sqrt{Q(X)\log(\mathrm{e}+n)} + Q(X)t\right]\right\} . \tag{1.42}$$

Therefore, by (1.41), we obtain the following integral inequality, valid for any $t \geq 0$,

$$u_k(n,t) \leq g_n(t)\int_0^t \mathrm{d}s \; u_{k_1}(n,s), \tag{1.43}$$

where

$$g_n(t) := K_2(1+t)^{2\eta+1}\left\{Q(X)\left[\log(\mathrm{e}+n) + p_n(t)\right]\right\}^{2\eta} . \tag{1.44}$$

with $K_2 > 0$ large enough. Setting $k_q = \mathrm{Int}\,[k_{q-1} + p_n(t)] + 1$, $q \in \mathbb{N}$, and $k_0 = k$, we can iterate the inequality (1.43) ℓ times, with

$$\ell := \mathrm{Int}\left[\frac{n-k-1}{1+p_n(t)}\right] \tag{1.45}$$

(which ensures $n > n(k_{\ell-1})$). Since $u_k(n,t) \leq a_n(t)$ with

$$a_n(t) := 2K_1(1+t)\left[\sqrt{Q(X)\log(\mathrm{e}+n)} + Q(X)t\right] , \tag{1.46}$$

we finally get the following bound,

$$u_k(n,t) \leq a_n(t)\frac{[g_n(t)t]^{\ell}}{\ell!} . \tag{1.47}$$

Recalling the definitions (1.35), (1.40), the existence of the infinite dynamics via the limits (1.14) now follows from the absolute convergence, uniform on compact time intervals, of the series $\sum_n u_k(n,t)$, which is a straightforward consequence of (1.47). Uniqueness can be proved with similar reasonings and it is omitted.

To prove the bound (1.16), we choose $k = \text{Int}\,[|r_i|] + 1$ and let

$$n^* := \text{Int}\left[\alpha(k^2 + Q(X)^{4\eta+2})\mathrm{e}^t\right],$$

with $\alpha > 1$ to be fixed later. From (1.31), $v_i^{(n^*)}(t)$ satisfies a bound like (1.16). On the other hand, by (1.40),

$$|v_i(t) - v_i^{(n^*)}(t)| \le \sum_{n'=n^*}^{\infty} u_k(n',t) \ . \tag{1.48}$$

From definition (1.39) it is easy to check that there exists α_0 such that if $\alpha \ge \alpha_0$ then $n^* \ge n(k)$ for all $k \ge 1$. We can then use (1.47) to bound each term in the sum on the right-hand side of (1.48). Moreover, recalling definitions (1.38), (1.44), (1.46), and (1.45), there exists $K_3 > 1$ such that, for all $n' \ge n^*$,

$$\begin{aligned} t &\le \log(\mathrm{e}+n')\ , & p_{n'}(t) &\le K_3[1+Q(X)]\log^2(\mathrm{e}+n')\ ,\\ g_{n'}(t) &\le K_3[1+Q(X)]^{4\eta}\log^{6\eta+1}(\mathrm{e}+n')\ , & a_{n'}(t) &\le K_3[1+Q(X)]\log^2(\mathrm{e}+n')\ ,\\ \ell &\ge \frac{n'-k-1}{2K_3[1+Q(X)]\log^2(\mathrm{e}+n')}\ . \end{aligned} \tag{1.49}$$

Inserting the bounds above in (1.47) and using Stirling formula we get,

$$u_k(n',t) \le K_3[1+Q(X)]\exp\left[-\ell\,\log\frac{n'-k-1}{2\mathrm{e}K_3^3[1+Q(X)]^{4\eta+2}\log^{6(\eta+1)}(\mathrm{e}+n')}\right]. \tag{1.50}$$

Since $n^* \ge \alpha[k^2 + Q(X)^{4\eta+2}]$, there is $\alpha_1 \ge \alpha_0$ such that the log in the square brackets on the right-hand side of (1.50) is not smaller than 1 for all $Q(X)$, $k \in \mathbb{N}$, $\alpha \ge \alpha_1$, and $n' \ge n^*$. Hence, from (1.48), the last bound in (1.49), and (1.50) we obtain, for all $\alpha \ge \alpha_1$,

$$|v_i(t) - v_i^{(n^*)}(t)| \le K_3[1+Q(X)]\sum_{n'\ge n^*}\exp\left[-\frac{n'-k-1}{2K_3[1+Q(X)]\log^2(\mathrm{e}+n')}\right]. \tag{1.51}$$

Since $n^* \ge \alpha[k^2 + Q(X)^{4\eta+2}]$, by choosing $\alpha \ge \alpha_1$ large enough, the right-hand side is bounded uniformly with respect to $Q(X)$ and $k \in \mathbb{N}$. The bound (1.16) is thus proved.

We are left with the proof of (1.17). By (1.23) it is enough to prove (1.17) with $Q(X(t);\mu,R)$ replaced by $W(X(t);\mu,R)$. Given $\alpha_2 \ge 1$ let

$$n_0 = \text{Int}\left[\alpha_2(\mathrm{e} + Q(X)^{4\eta+2})\mathrm{e}^{2(R+t)}\right] + 1\ .$$

Since $\log(\mathrm{e}+n_0) > R$, by (1.18), (1.31), and (1.23),

$$\begin{aligned} W(X^{(n_0)}(t);\mu,R) &\le Q(X^{(n_0)}(t);\mu,2R_{n_0}(t)) \le C\sup_\nu Q(X^{(n_0)}(t);\nu,R_{n_0}(t)) \\ &\le CQ(X)R_{n_0}(t) \le CQ(X)\left[\log(\mathrm{e}+n_0)+Q(X)t^2\right] \\ &\le CQ(X)\left[R+\log(\mathrm{e}+Q(X))+(1+Q(X))t^2\right]\,, \end{aligned}$$

where in the second inequality we used the positivity of the potential, see [3, Eq. (A.12)]. On the other hand,

$$\begin{aligned} W(X(t);\mu,R) &\le W(X^{(n_0)}(t);\mu,R) \\ &\quad + \sum_{n>n_0} |W(X^{(n)}(t);\mu,R) - W(X^{(n-1)}(t);\mu,R)|. \end{aligned} \tag{1.52}$$

Let us estimate the sum on the right-hand side of (1.52). We have,

$$\begin{aligned} &|W(X^{(n)}(t);\mu,R) - W(X^{(n-1)}(t);\mu,R)| \\ &\le \sum_i f\left(\frac{|r_i^{(n)}(t)-\mu|}{R}\right)\left|\varepsilon_i^{(n)}-\varepsilon_i^{(n-1)}\right| \\ &\quad + \sum_i \left| f\left(\frac{|r_i^{(n)}(t)-\mu|}{R}\right) - f\left(\frac{|r_i^{(n-1)}(t)-\mu|}{R}\right)\right| \varepsilon_i^{(n-1)}\,, \end{aligned} \tag{1.53}$$

where

$$\varepsilon_i^{(n)} = \frac{|v_i^{(n)}(t)|^2}{2} + \frac{1}{2}\sum_{j\in I_n: j\neq i} \Phi(r_i^{(n)}(t)-r_j^{(n)}(t)) + 1\,.$$

By (1.16), which is obviously valid also for the n-partial dynamics, if $|r_i^{(n)}(t)-\mu| \le 2R$ then all the particles $j \in I_n$ such that $|r_i^{(n)}(t)-r_j^{(n)}(t)| \le 1$ or $|r_i^{(n-1)}(t) - r_j^{(n-1)}(t)| \le 1$ are initially contained in the interval with center μ and radius $R(t)$, where $R(t) = C[R + Q(X)(1+t^2)]$. In particular, by choosing α_2 large enough, for any $n \ge n_0$ each particle i such that $|r_i^{(n)}(t)-\mu| \le 2R$ does not interact with the particles $j \in I_n \setminus I_{n-1}$, so that

$$\left|\varepsilon_i^{(n)} - \varepsilon_i^{(n-1)}\right| \le C\left[\frac{|v_i^{(n)}(t)| + |v_i^{(n-1)}(t)|}{2}\delta_i(n,t) + \sum_{j:j\neq i}{}^{*} \left[\delta_i(n,t)+\delta_j(n,t)\right]\right],$$

(recall the definition (1.35)) where $\sum^*_{j:j\neq i}$ denotes the sum restricted to all the particles $j \in I_{n-1}$ such that $|r_i^{(n)}(t) - r_j^{(n)}(t)| \le 1$ or $|r_i^{(n-1)}(t) - r_j^{(n-1)}(t)| \le 1$.

The number of these particles is thus bounded by $N(X;\mu, R(t)) \leq 2Q(X)R(t)$, where we used (1.60), (1.61), and that $R > \log(e + |\mu|)$. Then, setting $\varDelta_n(t) := \max\{\delta_i(n,t) : |r_i - \mu| \leq R(t)\}$ and using (1.16), if i is such that $|r_i - \mu| \leq R(t)$, for any $n > n_0$,

$$\begin{aligned} \left|\varepsilon_i^{(n)} - \varepsilon_i^{(n-1)}\right| &\leq C\left[\sqrt{Q(X)\log(e + |\mu| + R(t))} + Q(X)(t + R(t))\right] \varDelta_n(t) \\ &\leq C[1 + Q(X)^2]\log^2(e + n)\,\varDelta_n(t)\ , \end{aligned} \tag{1.54}$$

On the other hand,

$$\begin{aligned} &\left| f\left(\frac{|r_i^{(n)}(t) - \mu|}{R}\right) - f\left(\frac{|r_i^{(n-1)}(t) - \mu|}{R}\right)\right| \\ &\quad\leq 2\frac{|r_i^{(n)}(t) - r_i^{(n-1)}(t)|}{R}\chi\big(|r_i^{(n-1)}(t) - \mu| \leq \delta_i(n,t) + 2R\big) \\ &\quad\leq C\chi\big(|r_i^{(n-1)}(t) - \mu| \leq \varDelta_n(t) + 2R\big)\,\varDelta_n(t)\ . \end{aligned} \tag{1.55}$$

By the same argument leading to (1.50), (1.51), and the by definition of n_0, if α_0 is large enough,

$$\varDelta_n(t) \leq C[1 + Q(X)]\exp\left[-\frac{n}{C[1 + Q(X)]\log^2(e + n)}\right] \qquad \forall\, n > n_0\ . \tag{1.56}$$

In particular $\varDelta_n(t) \leq C$. Then, inserting the bounds (1.54) and (1.55) in (1.53),

$$\begin{aligned} &|W(X^{(n)}(t);\mu, R) - W(X^{(n-1)}(t);\mu, R)| \\ &\quad\leq C[1 + Q(X)^2]\log^2(e + n)N(X^{(n)}(t);\mu, 2R)\,\varDelta_n(t) \\ &\qquad + W(X^{(n-1)}(t);\mu, \varDelta_n(t) + 2R)\,\varDelta_n(t) \\ &\quad\leq C[1 + Q(X)^2]\log^2(e + n)W(X^{(n)}(t);\mu, 2R_n(t))\,\varDelta_n(t) \\ &\qquad + W(X^{(n-1)}(t);\mu, C + 2R_{n-1}(t))\,\varDelta_n(t) \\ &\quad\leq C[1 + Q(X)^2]\log^2(e + n)\,[R_{n-1}(t) + R_n(t)]\ \varDelta_n(t)\ , \end{aligned}$$

where in the last inequality we used the positivity of the potential, (1.23), and (1.18). Again by (1.16) we have that $R_n(t) \leq C[1 + Q(X)^2]\log^2(e + n)$ for $n \geq n_0$. By (1.56) we then conclude that the sum on the right-hand side of (1.52) is bounded by a constant. □

1.3 Runaway Effects for Bounded Body/Medium Interactions in One Dimension

This section is the core of the chapter, where we rigorously prove the runaway effect for not singular interaction, by analyzing the two specific models shortly described in Sect. 1.1.

1.3.1 The Quasi-One-Dimensional Model

In this model the gas is confined in an infinitely extended tube of $\mathbb{R}^3$. More precisely, let $\mathbf{n}$ be a fixed unit vector in $\mathbb{R}^3$. For any $\mathbf{q} \in \mathbb{R}^3$, we denote by $\mathbf{q}^\perp = \mathbf{q} - (\mathbf{q}\cdot\mathbf{n})\mathbf{n}$ its orthogonal projection. Given $L > 0$, let $\Omega := \{\mathbf{q} \in \mathbb{R}^3 : |\mathbf{q}^\perp| < L\}$ be the infinite tube of radius L and symmetry axis $\mathbf{n}$. The heavy particle of mass M is subjected to the force $\mathbf{F} = F\mathbf{n}$, $F > 0$, and it is coupled with the infinite system of particles of unit mass by means of a non-negative, symmetric, twice differentiable, short-range, two-body potential Ψ. The particles interact among themselves by means of a non-negative, symmetric, short-range, two-body potential Φ of the form

$$\Phi(\mathbf{q}) = \Phi_1(\mathbf{q}) + a|\mathbf{q}|^{-b} \ , \tag{1.57}$$

where $a \geq 0$, $b > 0$, and Φ_1 is twice differentiable and symmetric. As in the previous section, if Φ is finite at the origin we assume $\Phi(0) > 0$. Without loss of generality we assume that both Ψ and Φ have range not greater than one, i.e.,

$$\Phi(\mathbf{q}) = 0, \quad \Psi(\mathbf{q}) = 0 \qquad \text{if} \quad |\mathbf{q}| > 1 \ . \tag{1.58}$$

We force the system to stay confined inside the tube Ω, by requiring that all the particles are subjected to a one-body potential of the form

$$\Theta(\mathbf{q}) = \frac{\theta_h(|\mathbf{q}^\perp|)}{(L - |\mathbf{q}^\perp|)^\gamma} \ , \qquad \mathbf{q} \in \Omega \ , \tag{1.59}$$

where $\gamma > 0$, $h \in (0, L)$, and $\theta_h(s)$, $s \in \mathbb{R}^+$, is a non-negative, twice differentiable function, identically zero for $s \leq h$ and strictly positive at $s = L$.

The state of the system is determined by the position and velocity $\mathbf{x} = (\mathbf{r}, \mathbf{v})$ of the heavy particle and those $\mathbf{x}_i = (\mathbf{r}_i, \mathbf{v}_i)$, $i \in \mathbb{N}$, of the other particles. We denote by $\mathbf{X} = \{\mathbf{x}_i\}_{i\in\mathbb{N}}$ the state of the infinite extended system, which is assumed to have a locally finite density and energy. In particular it is well defined, for any $\mu \in \mathbb{R}$ and $R > 0$,

$$Q(\mathbf{X}; \mu, R) := \sum_i \chi_i(\mu, R) \left\{ \frac{\mathbf{v}_i^2}{2} + \Theta(\mathbf{r}_i) + \frac{1}{2} \sum_{j:j\neq i} \Phi(\mathbf{r}_i - \mathbf{r}_j) + 1 \right\} \ , \tag{1.60}$$

where $\chi_i(\mu, R) = \chi(|\mathbf{r}_i \cdot \mathbf{n} - \mu| \leq R)$. Analogously to the one dimensional case of the previous section, by defining

$$Q(\mathbf{X}) := \sup_{\mu} \sup_{R:R>\log(\mathrm{e}+|\mu|)} \frac{Q(\mathbf{X}; \mu, R)}{2R} , \tag{1.61}$$

the set $\mathscr{X} := \{\mathbf{X} : Q(\mathbf{X}) < \infty\}$ has a full measure with respect to any Gibbs state. The time evolution $t \mapsto (\mathbf{x}(t), \mathbf{X}(t))$ is defined by the solutions of the Newton equations,

$$\begin{cases} \ddot{\mathbf{r}}(t) = \mathbf{G}(\mathbf{x}(t), \mathbf{X}(t)) , \\ \ddot{\mathbf{r}}_i(t) = \mathbf{G}_i(\mathbf{x}(t), \mathbf{X}(t)) , \quad i \in \mathbb{N} , \\ \mathbf{x}(0) = (0, 0), \quad \mathbf{X}(0) = \mathbf{X} , \end{cases} \tag{1.62}$$

where

$$\mathbf{G}(\mathbf{x}, \mathbf{X}) := -M^{-1}\left[\sum_j \nabla\Psi(\mathbf{r} - \mathbf{r}_j) + \nabla\Theta(\mathbf{r})\right] + M^{-1}\mathbf{F} , \tag{1.63}$$

$$\mathbf{G}_i(\mathbf{x}, \mathbf{X}) := -\sum_{j:j\neq i} \nabla\Phi(\mathbf{r}_i - \mathbf{r}_j) - \nabla\Psi(\mathbf{r}_i - \mathbf{r}) - \nabla\Theta(\mathbf{r}_i) , \quad i \in \mathbb{N} , \tag{1.64}$$

and, without loss of generality, we assumed that the heavy particle is initially located at $\mathbf{r} = 0$ with velocity $\mathbf{v} = 0$.

The Cauchy problem for this system of infinite equations is well posed when the initial condition $\mathbf{X}$ is chosen in the set $\mathscr{X}$, and the solution can be constructed as a limit of the n-partial dynamics, here defined as follows. Given $\mathbf{X} \in \mathscr{X}$ and $n \in \mathbb{N}$, let $I_n := \{i \in \mathbb{N} : \mathbf{r}_i \in \Omega(0, n)\}$, where $\Omega(\mu, R) := \{\mathbf{r} \in \Omega : |\mathbf{r}_i \cdot \mathbf{n} - \mu| \leq R\}$. The n-partial dynamics $t \mapsto (\mathbf{x}^{(n)}(t), \mathbf{X}^{(n)}(t))$, $\mathbf{X}^{(n)}(t) = \{\mathbf{x}_i^{(n)}(t)\}_{i\in I_n}$, is the solution of the differential system,

$$\begin{cases} \ddot{\mathbf{r}}^{(n)}(t) = \mathbf{G}(\mathbf{x}^{(n)}(t), \mathbf{X}^{(n)}(t)) , \\ \ddot{\mathbf{r}}_i^{(n)}(t) = \mathbf{G}_i(\mathbf{x}^{(n)}(t), \mathbf{X}^{(n)}(t)) , \quad i \in I_n , \\ \mathbf{x}^{(n)}(0) = (0, 0), \quad \mathbf{X}^{(n)}(0) = \{\mathbf{x}_i\}_{i\in I_n} . \end{cases} \tag{1.65}$$

For notational convenience we introduce the vector,

$$\mathbf{E} = E\mathbf{n} := M^{-1}\mathbf{F} , \tag{1.66}$$

which will be used in the sequel.

Theorem 1.4 *For each* $\mathbf{X} \in \mathscr{X}$ *there exists a unique flow* $t \mapsto (\mathbf{x}(t), \mathbf{X}(t))$*,* $\mathbf{X}(t) = \{\mathbf{x}_i(t)\}_{i\in\mathbb{N}} \in \mathscr{X}$ *satisfying* (1.62)*. Moreover, for any* $t \geq 0$*,*

$$\lim_{n\to\infty} \mathbf{x}^{(n)}(t) = \mathbf{x}(t) \ , \qquad \lim_{n\to\infty} \mathbf{x}_i^{(n)}(t) = \mathbf{x}_i(t) \quad \forall\, i \in \mathbb{N} \ . \tag{1.67}$$

Finally, setting

$$Q_E(\mathbf{X}) = Q(\mathbf{X}) + E \ , \tag{1.68}$$

for any $\mathbf{X} \in \mathscr{X}$*,* $i \in \mathbb{N}$*, and* $t \geq 0$*,*

$$|\mathbf{v}(t)| \leq C \left[\sqrt{Q_E(\mathbf{X}) \log(\mathrm{e} + Q_E(\mathbf{X}))} + Q_E(\mathbf{X}) t \right] \ , \tag{1.69}$$

$$|\mathbf{v}_i(t)| \leq C \left[\sqrt{Q_E(\mathbf{X}) \log(\mathrm{e} + |x_i| + Q_E(\mathbf{X}))} + Q_E(\mathbf{X}) t \right] \ , \tag{1.70}$$

and, for any $\mu \in \mathbb{R}$*,* $R > \log(\mathrm{e} + |\mu|)$*, and* $t \geq 0$*,*

$$Q(\mathbf{X}(t); \mu, R) \leq C Q_E(\mathbf{X}) \left[R + \log(\mathrm{e} + Q_E(\mathbf{X})) + (1 + Q_E(\mathbf{X})) t^2 \right] \ . \tag{1.71}$$

The strategy used to prove Theorem 1.1 can be easily adapted here and we omit the details. We just remark that the analogous of Lemma 1.2 has to be proved for the local energy and density of the whole system. More precisely, after denoting by $\mathbf{x}_0 = (\mathbf{r}_0, \mathbf{v}_0)$ the position and velocity of the heavy particle and setting $\hat{\mathbf{X}} = (\mathbf{x}_0, \mathbf{X})$, we define

$$\hat{Q}(\hat{\mathbf{X}}; \mu, R) := \sum_i \chi_i(\mu, R) \left\{ \frac{(M-1)\delta_{i,0} + 1}{2} \mathbf{v}_i^2 + \Theta(\mathbf{r}_i) + \frac{1}{2} \sum_{j: j \neq i} \hat{\Phi}_{i,j} + 1 \right\} \ ,$$

where $\hat{\Phi}_{i,j} = \Phi(\mathbf{r}_i - \mathbf{r}_j)$ if $i, j \geq 1$ and $\hat{\Phi}_{i,j} = \Psi(\mathbf{r}_i - \mathbf{r}_j)$ if $i = 0$ or $j = 0$. Then, calling $\hat{\mathbf{X}}^{(n)}(t) = (\mathbf{x}_0^{(n)}(t), \mathbf{X}^{(n)}(t))$,

$$\sup_\mu \hat{Q}(\hat{\mathbf{X}}^{(n)}(t); \mu, \hat{R}_n(t)) \leq C Q_E(\mathbf{X}) \hat{R}_n(t) \quad \forall\, t \geq 0 \ , \tag{1.72}$$

where

$$\hat{R}_n(t) := \log(\mathrm{e} + n) + \int_0^t \mathrm{d}s\ \hat{V}_n(s), \qquad \hat{V}_n(t) := \max_{i \in I_n \cup \{0\}} \sup_{s \in [0,t]} |\mathbf{v}_i^{(n)}(s) \cdot \mathbf{n}| \ .$$

We also remark that the analogous of (1.26) is now,

$$\partial_s \hat{W}(\hat{\mathbf{X}}^{(n)}(s); \mu, \hat{R}_n(t,s)) \le -f_0^{\mu, R_n(t,s)} \mathbf{E} \cdot \mathbf{v}_0(s) - \sum_{i \neq j} \left(f_i^{\mu, \hat{R}_n(t,s)} - f_j^{\mu, \hat{R}_n(t,s)} \right) \nabla \hat{\Phi}_{i,j} \cdot \frac{\mathbf{v}_i(s)}{2} ,$$

where $\hat{W}(\hat{\mathbf{X}}; \mu, R)$ is a mollified version of $\hat{Q}(\hat{\mathbf{X}}; \mu, R)$ similar to (1.21) and $\hat{R}_n(t,s)$ is defined as $R_n(t,s)$ in (1.24) with $V_n(\cdot)$ replaced by $\hat{V}_n(\cdot)$. Noticing that

$$\int_0^s \mathrm{d}\tau \; f_0^{\mu, R_n(t,s)} |\mathbf{E} \cdot \mathbf{v}_0(\tau)| \le E \hat{R}_n(t) ,$$

the same reasoning leading to the integral inequality (1.30) gives,

$$\hat{W}(\hat{\mathbf{X}}^{(n)}(s); \hat{R}_n(t,s)) \le \hat{W}(\hat{\mathbf{X}}^{(n)}(0); \hat{R}_n(t,0)) + E \hat{R}_n(t) - C \int_0^s \mathrm{d}\tau \; \frac{\partial_\tau \hat{R}_n(t,\tau)}{\hat{R}_n(t,\tau)} W(\hat{\mathbf{X}}^{(n)}(\tau); \hat{R}_n(t,\tau)) ,$$

which can be integrated, obtaining,

$$\sup_\mu \hat{W}(\hat{\mathbf{X}}^{(n)}(t); \mu, \hat{R}_n(t)) \le C \left[E \hat{R}_n(t) + \sup_\mu \hat{W}(\hat{\mathbf{X}}^{(n)}(0); \mu, \hat{R}_n(t)) \right] ,$$

from which one easily concludes the proof of (1.72). We finally notice that the presence of the confining potential Θ does not cause problems in the iterative procedure since, as for Φ, the variation of its gradient can be controlled with a suitable power of the potential itself.

Here we state the main result on this model.

Theorem 1.5 *There exist positive constants C_0 and C_1 such that for any $\mathbf{X} \in \mathscr{X}$ the following holds. Let $t \mapsto (\mathbf{x}(t), \mathbf{X}(t))$ be the unique solution of Eqs.* (1.62) *and recall* (1.66)*. If $[\log(\mathrm{e} + E)]^{-1} E > C_0 Q(\mathbf{X})$ then, for any $t \ge 0$,*

$$|\mathbf{v}(t) - \mathbf{E}t| \le C_0 Q(\mathbf{X}) \left(\frac{\log(\mathrm{e} + E)}{\sqrt{E}} + t \right) \tag{1.73}$$

and, for any $i \in \mathbb{N}$,

$$|\mathbf{v}_i(t)| \le C_1 \left[\sqrt{Q(\mathbf{X}) \log(\mathrm{e} + |\mathbf{r}_i \cdot \mathbf{n}| + E)} + Q(\mathbf{X}) t \right] . \tag{1.74}$$

The meaning of this theorem appears evident from the following corollary.

Corollary 1.6 *For each Gibbs state $\langle\cdot\rangle$ of the background system there exists a threshold $\bar{E} > 0$ such that, for any $E > \bar{E}$,*

$$\lim_{t\to\infty} \langle \mathbf{v}(t) \cdot \mathbf{n} \rangle = \infty \, .$$

The above corollary is an immediate consequence of the bound (1.69) and Theorem 1.5 since for any Gibbs state $\langle\cdot\rangle$ there exist A, $B > 0$ such that $\langle \chi(Q(\mathbf{X}) > \kappa) \rangle \leq A \exp[-B\kappa]$ for any $\kappa > 0$, see, e.g., [7]. It is clear that the same result holds not only for Gibbs states but for any reasonable equilibrium or non-equilibrium thermodynamic state.

The rigorous proof of Theorem 1.5 is given in [1]. Here we shall only give a sketch of it. But let us first briefly discuss the main ideas. Disregarding for the moment the interaction of the heavy particle with the background, we observe that, after a time of order $1/\sqrt{E}$, the heavy particle reaches a velocity of order $\sqrt{E}$. On the other hand, during this time, its displacement is bounded by a constant so that the heavy particle can interact only with a finite number (not depending on E) of particles. Since the interaction is assumed to be bounded, we may expect that, even taking into account this interaction, if E is very large then the velocity of the heavy particle is still of order $\sqrt{E}$ at a time of order $1/\sqrt{E}$. After this time another mechanism takes place: since also in this quasi-one-dimensional model the velocities of the background particles may increase at most linearly in time, the heavy particle is now much faster than all the particles it meets. Hence, it interacts with each of them for a very short time and, since the interaction is bounded, this implies that also the momentum transferred during the scattering process is very small. Clearly, as time goes by, the number of particles which may interact with the heavy one increases. But for E large enough we may suppose the heavy particle to accelerate so rapidly that, in a unit time, the momentum transferred by the other particles (which is of order [number of collisions] $\cdot$ [time of collision]) remains bounded by a constant smaller than E: if this happens the heavy particle will increase its velocity indefinitely.

If the external field E is not large, the above mechanism does not work, and the heavy particle can exchange a large part of its energy with the background. However, the velocity of both the heavy and background particles may increase at most linearly in time, as shown in Theorem 1.4, see (1.69) and (1.70).

To prove rigorously the above picture, we consider the maximal time for which the horizontal velocity of the heavy particle remains close enough to Et, and the absolute velocity of the particles which may interact with the former is much smaller than Et. By choosing E large enough, this maximal time is positive. Then, by analyzing the dynamics up to this time, we obtain sharper estimates implying, by a continuity argument, that this time is actually infinite. After that, the inequalities (1.73) and (1.74) will be a byproduct of the above bounds.

Before explaining the main steps of the above strategy, we remark that it is not possible to work directly with the infinite dynamics, since a control of the explicit dependence on E and $Q(\mathbf{X})$ in the limiting procedure is needed. Therefore, we analyze the n-partial dynamics, by obtaining bounds which are uniform in n.

Step 1 (definitions). Let

$$U_n(t) := \max_i G\Big(\inf_{s\in[0,t]} |\mathbf{r}_i^{(n)}(s)\cdot\mathbf{n}| - \sqrt{E}t - \frac{6Et^2}{5}\Big) \sup_{s\in[0,t]} |\mathbf{v}_i^{(n)}(s)| \,, \tag{1.75}$$

where $G \in C(\mathbb{R})$ is not increasing and satisfying: $G(x) = 1$ for $x \le 1$, $G(x) = 0$ for $x \ge 2$. We next define,

$$T_n := \sup\left\{t \ge 0\colon \max\{U_n(s); |\mathbf{v}^{(n)}(s)\cdot\mathbf{n} - Es|\} \le \sqrt{E} + \frac{Es}{5} \quad \forall\, s \in [0,t]\right\} \,, \tag{1.76}$$

setting $T_n = 0$ if the above set is empty. By (1.58), for any $t \in [0, T_n)$, the ith particle can interact with the heavy one during the time $[0,t]$ only if $i \in A_n(t)$, where

$$A_n(t) := \left\{i \in I_n \ : \ \inf_{s\in[0,t]} |\mathbf{r}_i^{(n)}(s)\cdot\mathbf{n}| \le 1 + \sqrt{E}t + \frac{6Et^2}{5}\right\} \,. \tag{1.77}$$

Observe also that $U_n(\cdot)$ is a continuous and not decreasing function such that

$$\max_{i\in A_n(t)} \sup_{s\in[0,t]} |\mathbf{v}_i^{(n)}(s)| \le U_n(t) \le \max_{i\in \bar{A}_n(t)} \sup_{s\in[0,t]} |\mathbf{v}_i^{(n)}(s)| \,, \tag{1.78}$$

where

$$\bar{A}_n(t) := \left\{i \in I_n \ : \ \inf_{s\in[0,t]} |\mathbf{r}_i^{(n)}(s)\cdot\mathbf{n}| \le 2 + \sqrt{E}t + \frac{6Et^2}{5}\right\} \,. \tag{1.79}$$

Since, by definition (1.61),

$$|\mathbf{v}_i| \le 2\sqrt{Q(\mathbf{X})\log(\mathrm{e} + |\mathbf{r}_i\cdot\mathbf{n}|)} \qquad \forall\, i \in \mathbb{N} \,, \tag{1.80}$$

it follows that, setting $C^* := 16\log(\mathrm{e}+2)$,

$$U_n(0) \le \frac{\sqrt{E}}{2} \qquad \forall\, n \in \mathbb{N} \quad \forall\, E \ge C^*Q(\mathbf{X}) \,.$$

Recalling that $\mathbf{v}(0) = 0$, by continuity we conclude that if $E \ge C^*Q(\mathbf{X})$ then $T_n > 0$ for all $n \in \mathbb{N}$.

Step 2 (energy estimate). We study the n-partial dynamics for $E \ge C^*Q(\mathbf{X})$ and $t \in [0, T_n)$. We essentially show that up to time T_n the heavy particle does not exchange too much energy with the background. This is the content of the following estimate on the growth in time of the local density and energy (1.60) of the background. In fact, with respect to (1.72), here the estimate is uniform for large E. More precisely, for any $\mathbf{X} \in \mathscr{X}$, $E \geq C^* Q(\mathbf{X})$, and $n \in \mathbb{N}$,

$$\sup_{\mu} Q(\mathbf{X}^{(n)}(t); \mu, R_n(t)) \leq CQ(\mathbf{X}) R_n(t) \qquad \forall\, t \in [0, T_n) \;, \tag{1.81}$$

where

$$R_n(t) := \log(\mathrm{e}+n) + \int_0^t \mathrm{d}s\; V_n(s) \;, \qquad V_n(t) := \max_{i \in I_n} \sup_{s \in [0,t]} |\mathbf{v}_i^{(n)}(s) \cdot \mathbf{n}| \;. \tag{1.82}$$

The proof of (1.81) is essentially the same as that of Lemma 1.2. The main difference is that the analogous of (1.26) for the mollified version W of Q now reads,

$$\begin{aligned} \partial_s W(\mathbf{X}^{(n)}(s); \mu, R_n(t,s)) \leq & -\sum_i f_i^{\mu, R_n(t,s)} \nabla \Psi(\mathbf{r}_i^{(n)}(s) - \mathbf{r}^{(n)}(s)) \cdot \mathbf{v}_i^{(n)}(s) \\ & - \frac{1}{2} \sum_{i \neq j} \left(f_i^{\mu, R_n(t,s)} - f_j^{\mu, R_n(t,s)} \right) \nabla \Phi(\mathbf{r}_i^{(n)}(s) - \mathbf{r}_j^{(n)}(s)) \cdot \mathbf{v}_i^{(n)}(s) \;. \end{aligned}$$

Therefore, the same reasoning leading to the integral inequality (1.30) gives in this case,

$$\begin{aligned} W(\mathbf{X}^{(n)}(s); R_n(t,s)) \leq\; & W(\mathbf{X}^{(n)}(0); R_n(t,0)) \\ & + \sup_{\mu} \int_0^s \mathrm{d}\tau \sum_i f_i^{\mu, R_n(t,s)} |\nabla \Psi(\mathbf{r}_i^{(n)}(\tau) - \mathbf{r}^{(n)}(\tau)) \cdot \mathbf{v}_i^{(n)}(\tau)| \\ & - C \int_0^s \mathrm{d}\tau\, \frac{\partial_\tau R_n(t,\tau)}{R_n(t,\tau)}\, W(\mathbf{X}^{(n)}(\tau); R_n(t,\tau)) \;. \end{aligned} \tag{1.83}$$

We observe that in the sum on the right-hand side of (1.83) only the particles which are initially in $\Omega(\mu, 4R_n(t,0))$ can contribute; the number of these particles is bounded by $W(\mathbf{X}^{(n)}(0); 4R_n(t,0))$. Moreover, letting

$$t_E := \frac{20}{\sqrt{E}} \;, \tag{1.84}$$

by the definition of T_n, if $T_n > t_E$ then

$$\left[\mathbf{v}^{(n)}(\tau) - \mathbf{v}_i^{(n)}(\tau)\right] \cdot \mathbf{n} \geq \frac{E\tau}{2} \qquad \forall\, i \in A_n(\tau) \quad \forall\, \tau \in (t_E, T_n) \;, \tag{1.85}$$

and

$$
\begin{aligned}
&\int_0^s \mathrm{d}\tau \ \sum_i f_i^{\mu,R_n(t,\tau)} \left|\nabla\Psi(\mathbf{r}_i^{(n)}(\tau)-\mathbf{r}^{(n)}(\tau))\cdot\mathbf{v}_i^{(n)}(\tau)\right| \\
&\quad \le W(\mathbf{X}^{(n)}(0);\mu,4R_n(t,0))\|\nabla\Psi\|_\infty \Bigg\{ \left(\sqrt{E}+\frac{Et_E}{5}\right) t_E \\
&\qquad + \chi(T_n>t_E)\max_i \int_{t_E}^{T_n} \mathrm{d}\tau \ \chi(|\mathbf{r}_i^{(n)}(\tau)-\mathbf{r}^{(n)}(\tau)|\le 1)|\mathbf{v}_i^{(n)}(\tau)| \Bigg\} \ .
\end{aligned}
\tag{1.86}
$$

We now observe that if $T_n > t_E$ then the bound (1.85) implies that for each $i \in \mathbb{N}$ there exists an interval $[s_{i1}, s_{i2}] \subseteq [t_E, T_n)$ such that

$$
\chi\big(\tau\in[t_E,T_n)\big)\chi(|\mathbf{r}_i^{(n)}(\tau)-\mathbf{r}^{(n)}(\tau)|\le 1) \le \chi([s_{i1},s_{i2}]), \quad |s_{i2}-s_{i1}| \le \frac{2}{Es_{i1}} \ .
$$

Since $|\mathbf{v}_i^{(n)}(\tau)| \le \sqrt{E} + E\tau/5$, it follows that

$$
\begin{aligned}
&\chi(T_n>t_E)\int_{t_E}^{T_n} \mathrm{d}\tau \ \chi(|\mathbf{r}_i^{(n)}(\tau)-\mathbf{r}^{(n)}(\tau)|\le 1)|\mathbf{v}_i^{(n)}(\tau)| \\
&\quad \le \frac{2}{Es_{i1}}\left(\sqrt{E}+\frac{Es_{i1}}{5}+\frac{2}{5s_{i1}}\right) \ ,
\end{aligned}
$$

and the right-hand side of the above inequality is bounded by a constant since $Es_{i1} \ge Et_E$. Hence, by (1.86), we conclude that, for any $\mu \in \mathbb{R}$,

$$
\int_0^s \mathrm{d}\tau \ \sum_i f_i^{\mu,R_n(t,\tau)} \left|\nabla\Psi(\mathbf{r}_i^{(n)}(\tau)-\mathbf{r}^{(n)}(\tau))\cdot\mathbf{v}_i^{(n)}(\tau)\right| \le CW(\mathbf{X}^{(n)}(0);\mu,4R_n(t,0)) \ .
\tag{1.87}
$$

Inserting (1.87) in (1.83) we obtain a differential inequality which can be solved, finally getting

$$
W(\mathbf{X}^{(n)}(t);R_n(t)) \le CW(\mathbf{X}^{(n)}(0);R_n(t)) \ .
$$

from which (1.81) follows by arguing as in the proof of Lemma 1.2.

Step 3 (velocity bounds). As a corollary of the above estimate the velocities of the particles background increase at most linearly in time with a rate uniformly bounded with respect to E. More precisely, by arguing as in the proof of Lemma 1.3, we now deduce from (1.81) that for any $\mathbf{X} \in \mathscr{X}$ and $E \ge C^* Q(\mathbf{X})$, $n \in \mathbb{N}$, and $i \in I_n$,

$$
|\mathbf{v}_i^{(n)}(t)| \le C\left[\sqrt{Q(\mathbf{X})\log(\mathrm{e}+n)} + Q(\mathbf{X})t\right] \qquad \forall\, t \in [0,T_n) \ .
\tag{1.88}
$$

By (1.88) we can apply an iterative procedure as in the proof of Theorem 1.1. As a result we get the following estimate, whose proof is omitted: there exist $C_1 > 0$ such that, for any $\mathbf{X} \in \mathscr{X}$, $E \geq C^* Q(\mathbf{X})$, $i \in \mathbb{N}$, and $n \geq |\mathbf{r}_i \cdot \mathbf{n}|$,

$$|\mathbf{v}_i^{(n)}(t)| \leq C_1 \left[\sqrt{Q(\mathbf{X}) \log(\mathrm{e} + |\mathbf{r}_i \cdot \mathbf{n}| + E)} + Q(\mathbf{X})t \right] \quad \forall\, t \in [0, \bar{T}_n)\ , \tag{1.89}$$

where

$$\bar{T}_n := \min_{n' \leq n} T_{n'}\ . \tag{1.90}$$

Step 4 (bootstrap argument) By exploiting the dynamics and using (1.88) it is now possible to show that if E is large enough then $T_n = \infty$ for any $n \in \mathbb{N}$ and sharper estimates do hold. This is the content of the following proposition.

Proposition 1.7 *There exists $C_2 \geq C^*$ such that, for each given $\mathbf{X} \in \mathscr{X}$, if* $[\log(\mathrm{e}+E)]^{-1} E \geq C_2 Q(\mathbf{X})$ *then $T_n = \infty$ for any $n \in \mathbb{N}$.*

Proof We shall prove that there exists $C_2 \geq C^*$ such that, for any $\mathbf{X} \in \mathscr{X}$, $[\log(\mathrm{e}+E)]^{-1} E \geq C_2 Q(\mathbf{X})$, $n \in \mathbb{N}$, and $k \leq n$,

$$U_k(t) \leq \frac{1}{2}\left(\sqrt{E} + \frac{Et}{5}\right) \quad \forall\, t \in [0, \bar{T}_n)\ , \tag{1.91}$$

$$|\mathbf{v}^{(k)}(t) \cdot \mathbf{n} - Et| \leq \frac{1}{2}\left(\sqrt{E} + \frac{Et}{5}\right) \quad \forall\, t \in [0, \bar{T}_n)\ . \tag{1.92}$$

By continuity this implies that $T_k > \bar{T}_n$ for all $k \leq n$ (which contradicts the definition of $\bar{T}_n$) unless $\bar{T}_n = \infty$, and the proposition is proved.

In the sequel we shall assume $E \geq \max\{C^*; 4C_1\} Q(\mathbf{X})$. Let $n \in \mathbb{N}$ and $k \leq n$. By (1.89) and recalling the definitions of T_k and $\bar{A}_k(t)$, see (1.76) and (1.79), for $t \in [0, \bar{T}_n)$ the initial position $\mathbf{r}_i$ of each particle $i \in \bar{A}_k(t)$ has to verify the inequality,

$$|\mathbf{r}_i \cdot \mathbf{n}| - C_1 t \sqrt{Q(\mathbf{X}) \log(\mathrm{e} + |\mathbf{r}_i \cdot \mathbf{n}| + E)} - C_1 Q(\mathbf{X}) t^2 \leq 2 + \sqrt{E}t + \frac{6E}{5} t^2\ .$$

Since $E \geq \max\{C^*; 4C_1\} Q(\mathbf{X})$ and $C^* > 1$, the above inequality implies

$$|\mathbf{r}_i \cdot \mathbf{n}| - C_1 \sqrt{E}t \sqrt{\log(\mathrm{e} + |\mathbf{r}_i \cdot \mathbf{n}| + E)} \leq 2 + \sqrt{E}t + 2Et^2\ , \tag{1.93}$$

from which it follows that there exists $\rho > 1$ such that, for any $k \leq n$ and $t \in [0, \bar{T}_n)$,

$$i \in \bar{A}_k(t) \implies \mathbf{r}_i \in \Omega(0, R_E(t)) \qquad \text{with} \quad R_E(t) := \rho \left[\log(\mathrm{e} + E) + Et^2 \right]\ , \tag{1.94}$$

and hence, by (1.60) and (1.61), recalling also that $A_k(t) \subseteq \bar{A}_k(t)$,

$$|A_k(t)| \le |\bar{A}_k(t)| \le 2Q(\mathbf{X})R_E(t)\ , \tag{1.95}$$

where for any finite set B we denote by $|B|$ its cardinality.

By (1.78), (1.94), and (1.89),

$$U_k(t) \le C\left[\sqrt{Q(\mathbf{X})\log(\mathrm{e}+E+Et^2)} + Q(\mathbf{X})t\right] \qquad \forall\, t \in [0,\bar{T}_n)\ . \tag{1.96}$$

Consider now the difference $|\mathbf{v}^{(k)}(t)\cdot\mathbf{n} - Et|$. By the equations of motion, since $\nabla\Theta\cdot\mathbf{n} = 0$, we have

$$|\mathbf{v}^{(k)}(t)\cdot\mathbf{n} - Et| \le \frac{\|\nabla\Psi\|_\infty}{M}\sum_j \int_0^t \mathrm{d}s\ \chi(|\mathbf{r}^{(k)}(s) - \mathbf{r}_j^{(k)}(s)| \le 1)\ . \tag{1.97}$$

By (1.84) and (1.85) for $n = k$, we estimate, for any $t \in [0,\bar{T}_n)$,

$$|\mathbf{v}^{(k)}(t)\cdot\mathbf{n} - Et| \le \frac{\|\nabla\Psi\|_\infty}{M}\left\{|A_k(\min\{t;t_E\})|\,t_E + F_k(t)\right\}\ , \tag{1.98}$$

where

$$F_k(t) = \chi(\bar{T}_n > t > t_E)\sum_j \int_{t_E}^t \mathrm{d}s\ \chi(|\mathbf{r}^{(k)}(s) - \mathbf{r}_j^{(k)}(s)| \le 1)\ . \tag{1.99}$$

By (1.95) and the definition (1.84),

$$|A_k(\min\{t;t_E\})|\,t_E \le CQ(\mathbf{X})\frac{\log(\mathrm{e}+E)}{\sqrt{E}}\ . \tag{1.100}$$

To bound the term $F_k(t)$ we use (1.85) with $k = n$. Given $K > 0$ to be fixed later, let $q_0 \in \mathbb{N}$ be such that $2^{K+q_0} < R_E(t) \le 2^{K+q_0+1}$ ($R_E(t)$ as in (1.94)) and define

$$\begin{aligned} \bar{N} &= \{j \in \mathbb{N} : |\mathbf{r}_j\cdot\mathbf{n}| \le 2^K\}\ , \\ N_q &= \{j \in \mathbb{N} : 2^{K+q} < |\mathbf{r}_j\cdot\mathbf{n}| \le 2^{K+q+1}\}\ , \qquad q = 0,\ldots,q_0\ . \end{aligned}$$

By (1.85), for any $t \in [0,\bar{T}_n)$,

$$F_k(t) \le \frac{2|\bar{N}|}{Et_E} + \sum_{q=0}^{q_0}\frac{2|N_q|}{Et_{k,q}}\chi(\bar{T}_n > t_E)\ , \tag{1.101}$$

where, for $\bar{T}_n > t_E$,

$$t_{k,q} := \min_{j \in N_q} \inf\{s \in [t_E, \bar{T}_n) \ : \ |\mathbf{r}^{(k)}(s) - \mathbf{r}_j^{(k)}(s)| \le 1\} \ ,$$

setting $t_{k,q} = \infty$ if the above set is empty for all $j \in N_q$. We choose K such that $2^K = \bar{c} \log(\mathrm{e} + E)$ with $\bar{c}$ to be fixed below and so large that $2^{K+x-1} > \log(\mathrm{e} + 2^{K+x+1})$ for all $x \ge 0$ and $E \ge 0$. Then, since $2^{K+q+1} - 2^{K+q} = 2^{K+q}$, by (1.60) and (1.61) we have $|N_q| \le 2Q(\mathbf{X})2^{K+q}$. On the other hand, by inequality (1.93) which is valid for all $i \in A_k(t)$, the time $t_{k,q}$ has to satisfy the condition:

$$2^{K+q} - C_1\sqrt{E}t_{k,q}\sqrt{\log(\mathrm{e} + 2^{K+q+1} + E)} \le 2 + \sqrt{E}t_{k,q} + 2Et_{k,q}^2 \ .$$

It follows that if $\bar{c}$ is chosen sufficiently large then $t_{k,q} \ge C\sqrt{2^{K+q}/E}$. Finally, again by (1.60) and (1.61), we have $|\bar{N}| \le 2Q(\mathbf{X})2^K$. Inserting all the previous bounds in (1.101) we finally obtain,

$$\begin{aligned} F_k(t) &\le \frac{CQ(\mathbf{X})}{\sqrt{E}} \left(2^K + \sum_{q=0}^{q_0} 2^{(K+q)/2} \right) \le \frac{CQ(\mathbf{X})}{\sqrt{E}} \left(2^K + 4\sqrt{R_E(t)} \right) \\ &\le CQ(\mathbf{X}) \left(\frac{\log(\mathrm{e} + E)}{\sqrt{E}} + t \right) \ , \end{aligned} \tag{1.102}$$

where we used $2^{(K+q_0)/2} \le \sqrt{R_E(t)}$. By (1.98), (1.100), and (1.102), we conclude that, for any $E \ge \max\{C^*; 4C_1\}Q(\mathbf{X})$ and $k \le n$,

$$|\mathbf{v}^{(k)}(t) \cdot \mathbf{n} - Et| \le CQ(\mathbf{X}) \left(\frac{\log(\mathrm{e} + E)}{\sqrt{E}} + t \right) \qquad \forall\, t \in [0, \bar{T}_n) \ . \tag{1.103}$$

By (1.96), (1.103), and by choosing $C_2 \ge \max\{C^*; 4C_1\}$ large enough, the inequalities (1.91) and (1.92) are verified for all E such that $[\log(\mathrm{e} + E)]^{-1}E \ge C_2 Q(\mathbf{X})$. The proposition is proved. □

Step 5 (conclusion). We can now conclude the proof of the estimates (1.73) and (1.74). The latter holds with C_1 as in (1.89), and $C_0 \ge C_2$ large enough. In fact, for $[\log(\mathrm{e}+E)]^{-1}E \ge C_2 Q(\mathbf{X})$, since $\bar{T}_n = \infty$, the bounds (1.89) and (1.103) hold for all $t \ge 0$. The former implies the inequality (1.74) for the corresponding infinite dynamics. The inequality (1.73) follows from (1.103) and an analogous upper bound for $|\mathbf{v}^{(n)}(t)^\perp|$, which we next prove. Let

$$\mathscr{E}_n^\perp(t) := \frac{M}{2}|\mathbf{v}^{(n)}(t)^\perp|^2 + \Theta(\mathbf{r}^{(n)}(t)) \ .$$

From the equations of motion, since $\nabla\Theta \cdot \mathbf{n} = 0$,

$$\dot{\mathscr{E}}_n^{\perp}(t) = -\sum_j \nabla\Psi(\mathbf{r}^{(n)}(t) - \mathbf{r}_j^{(n)}(t)) \cdot \mathbf{v}^{(n)}(t)^{\perp} ,$$

and hence, since Θ is non-negative,

$$\left|\dot{\mathscr{E}}_n^{\perp}(t)\right| \le \|\nabla\Psi\|_\infty \sqrt{\frac{2\mathscr{E}_n^{\perp}(t)}{M}} \sum_j \chi(|\mathbf{r}^{(n)}(t) - \mathbf{r}_j^{(n)}(t)| \le 1) .$$

Then, setting $\bar{\mathscr{E}}_n^{\perp}(t) = \sup_{s\in[0,t]} \mathscr{E}_n^{\perp}(s)$ and using $\mathscr{E}_n^{\perp}(0) = 0$, we obtain:

$$|\mathbf{v}^{(n)}(t)^{\perp}| \le \sqrt{\frac{2\bar{\mathscr{E}}_n^{\perp}(t)}{M}} \le \frac{2\|\nabla\Psi\|_\infty}{M} \int_0^t \mathrm{d}s \sum_j \chi(|\mathbf{r}^{(n)}(s) - \mathbf{r}_j^{(n)}(s)| \le 1) .$$

We have already found an upper bound for the right-hand side of the above inequality, see the analysis done starting from (1.97) to prove (1.103) (but now with $\bar{T}_n = \infty$). We conclude that, for $[\log(\mathrm{e} + E)]^{-1} E \ge C_2 Q(\mathbf{X})$,

$$|\mathbf{v}^{(n)}(t)^{\perp}| \le CQ(\mathbf{X}) \left(\frac{\log(\mathrm{e} + E)}{\sqrt{E}} + t \right) \qquad \forall\, t \ge 0 . \tag{1.104}$$

By choosing $C_0 > C_2$ large enough, the inequality (1.73) follows from (1.103) and (1.104) for any E such that $[\log(\mathrm{e} + E)]^{-1} E \ge C_0 Q(\mathbf{X})$.

1.3.2 The One-Dimensional Model: Violation of Ohm's Law

We try now to remove the assumption $E > \bar{E}$ of Corollary 1.6. We are able to prove it rigorously only in a particular one-dimensional model (particles interacting with a nonnegative, finite range, smooth pair potential), but we believe that the result holds also in higher dimension. Actually, we need to show that the growth of the velocity of a background particle is sub-linear in time. Our approach depends on the (strict) one dimensionality of the system: in one dimension the conservations of total impulse and total energy impose that after a binary collision the outgoing velocities are exactly the same of the ingoing ones. Of course, the reality is not so simple because there are multiple collisions, nevertheless in [2] we show that if a particle is much faster than the others its velocity remains almost unchanged during the scattering process. As a consequence we obtain the required bounds on the growth of the velocities of the background particles, which allow to consider the case of small external force. However, we believe that this is a useful technical tool but not an essential one (also in higher dimension a fast particle does not change too much its velocity in a binary collision).

We then study the system (1.1) in dimension $d = 1$, i.e.,

$$\begin{cases} \ddot{r}(t) = E - M^{-1} \sum_j \Psi'(r(t) - r_j(t)) \ , \\ \\ \ddot{r}_i(t) = -\Psi'(r_i(t) - r(t)) - \sum_{j \neq i} \Phi'(r_i(t) - r_j(t)) \ , \quad i \in \mathbb{N} \ , \\ \\ (r(0), v(0)) = (0, 0) \qquad X(0) = X \end{cases} \tag{1.105}$$

where we drop the bold font, set $m = 1$, and $E = F/M$ as in the previous section. The initial condition $X = \{(r_i, v_i)\}_{i \in \mathbb{N}}$ is chosen in $\mathscr{X}$ as defined in (1.12). We assume that both Φ and Ψ are nonnegative, symmetric, finite range, and smooth pair potentials. We also require $\Phi(0) > 0$ so that Φ is superstable. In [2] the following theorem is proved.

Theorem 1.8 *For any Gibbs state $\langle \cdot \rangle$ of the background system and any intensity of E,*

$$\liminf_{t \to \infty} \frac{\langle v(t) \rangle}{t} > 0 \ .$$

We recall that the Ohm's law states a proportionality between the external force and the mean velocity (linear response). The previous theorem means that for a bounded particle/background interaction the Ohm's law is not valid. We need singular interactions. Heuristic arguments in [1] suggest how large must be the divergence, but we will discuss this point in the next section with more details.

The proof of Theorem 1.8 is nontrivial and rather technical. In the rest of the section we briefly sketch the strategy, by only giving the main ideas without entering into details.

As already claimed, the crucial point is a new estimate on the growth in time of the background particle velocity. Since the argument leading to this new estimate is not really affected by the presence of the heavy particle, we discuss it in the case when the latter is absent. Therefore, we consider the Cauchy problem (1.15) and assume that the potential satisfies (1.8) with $a = 0$ (i.e., it is nonnegative, finite range, smooth, and superstable) and (1.9).

Theorem 1.9 *There exist $K \geq 1$ and $a_0 \in (0, 1)$ such that for any $a \in (0, a_0]$ and $X \in \mathscr{X}$ the following holds. Let $t \mapsto X(t) = \{(r_i(t), v_i(t)\}_{i \in \mathbb{N}}$ be the solution to Eqs. (1.15). Then, for any $i \in \mathbb{N}$ and $t \geq 0$,*

$$|v_i(t)| \leq at \qquad \forall\, t \geq T_a(X, r_i) \ , \tag{1.106}$$

where

$$T_a(X, r_i) = a^{-K[1+Q(X)^3]/a^2} \sqrt{\log(\mathrm{e} + |r_i|)} \ . \tag{1.107}$$

The theorem is proved by contradiction: we assume that at time T large enough the velocity absolute value of a particle is greater than aT and we then show that this particle does not change very much its velocity during the backward motion, so that the latter is initially larger than $aT/2$. For T large enough this fact contradicts the assumptions on the initial data.

The proof is based on a nontrivial perturbative analysis of the collision processes. To explain the general idea, let us consider the following particular and very simple situation: given $a \in (0,1)$, at a large time T there is only one fast particle, say the i^{th} particle, with velocity larger than aT, while the other particles have velocities with absolute value smaller than $aT/4$ during the whole time interval $[0,T]$. More precisely, we assume this situation occurs with

$$T > K_1 \frac{1+Q(X)}{a^2}\sqrt{\log(\mathrm{e}+|r_i|)}\;, \tag{1.108}$$

where $K_1 > 1$ is to be fixed later. Let

$$T_* := \inf\left\{t \in (0,T_*) \colon v_i(t) = \frac{aT}{2}\right\}\;,$$

setting $T_* = 0$ if the above set is empty, and define, for $t \in [T_*, T]$,

$$p_i(t) := v_i(t) + \sum_{j\neq i} \frac{\Phi(r_i(t)-r_j(t))}{v_i(t)-v_j(t)}\;. \tag{1.109}$$

From the equations of motion and recalling that Φ is symmetric we have,

$$\begin{aligned}\dot{p}_i(t) = &-\sum_{j\neq i}\sum_{s\neq i} \frac{\Phi(r_i(t)-r_j(t))\Phi'(r_i(t)-r_s(t))}{(v_i(t)-v_j(t))^2}\\ &+\sum_{j\neq i}\sum_{s\neq j} \frac{\Phi(r_i(t)-r_j(t))\Phi'(r_j(t)-r_s(t))}{(v_i(t)-v_j(t))^2}\;.\end{aligned}$$

Since $v_i(t) - v_j(t) \geq aT/4$ for $t \in [T_*, T]$ and Φ, Φ' are bounded with support in $[-1,1]$, it follows that

$$\begin{aligned}|p_i(T)-p_i(T_*)| &\leq \frac{C}{(aT)^2}\int_{T_*}^{T} \mathrm{d}s\; N_i(s)^2 \leq \frac{C}{(aT)^2}\left[\sup_{\tau\in[0,T]} N_i(\tau)\right]\int_{T_*}^{T}\mathrm{d}s\; N_i(s)\\ &= \frac{C}{(aT)^2}\left[\sup_{\tau\in[0,T]} N_i(\tau)\right]\sum_{j\neq i}\int_{T_*}^{T}\mathrm{d}s\;\chi(|r_j(s)-r_i(s)|\leq 2)\;,\end{aligned} \tag{1.110}$$

where $N_i(s)$ clearly denotes the number of particles which are contained at time s in the interval $[r_i(s) - 2, r_i(s) + 2]$. We now observe that since the potential Φ is superstable with range bounded by 1, denoting by $N(X; \mu, R)$ the number of particles of the configuration X contained in the interval $[\mu - R, \mu + R]$,

$$N(X; \mu, R)^2 \leq CQ(X; \mu, R) \qquad \forall\, X \in \mathscr{X} \quad \forall\, \mu \in \mathbb{R} \quad \forall\, R \geq 1 \,. \tag{1.111}$$

Therefore, by (1.17), and (1.108), for any $\tau \in [0, T]$,

$$\begin{aligned} N_i(\tau)^2 &\leq CQ(X(\tau); r_i(\tau), 2) \\ &\leq CQ(X)[\log(\mathrm{e} + |r_i|) + \log(\mathrm{e} + Q(X)) + (1 + Q(X))\tau^2] \\ &\leq CQ(X)[1 + Q(X)]T^2 \,. \end{aligned}$$

Since $v_i(t) - v_j(t) \geq aT/4$, the time integrals on the right-hand side of (1.110) are not greater than $16/(aT)$. Consequently,

$$|p_i(T) - p_i(T_*)| \leq \frac{C\sqrt{Q(X)[1 + Q(X)]}}{a^3T^2}\bar{N}_i(T) \,,$$

where $\bar{N}_i(T)$ is the total number of particles j such that $|r_j(s) - r_i(s)| \leq 2$ for some $s \in [0, T]$. Now, by (1.16) and (1.108), for such particles we have,

$$|r_j - r_i| \leq 2 + 2C\left[\sqrt{Q(X)\log(\mathrm{e} + |r_i| + Q(X))} + Q(X)T\right] \leq C[1 + Q(X)]T^2 \,,$$

whence, by (1.111) and (1.108),

$$\begin{aligned} \bar{N}_i(T)^2 &\leq N(X; r_i, C[1 + Q(X)]T^2)^2 \leq CQ(X; r_i, C[1 + Q(X)]T^2) \\ &\leq CQ(X)[\log(\mathrm{e} + |r_i|) + C[1 + Q(X)]T^2] \\ &\leq CQ(X)[1 + Q(X)]T^2 \,. \end{aligned}$$

Therefore,

$$|p_i(T) - p_i(T_*)| \leq \frac{CQ(X)[1 + Q(X)]}{a^3T} \,.$$

On the other hand, for any $t \in [T_*, T]$,

$$|v(t) - p(t)| \leq \frac{C}{aT}N_i(t) \leq \frac{C\sqrt{Q(X)[1 + Q(X)]}}{a} \,,$$

whence

$$|v(T) - v(T_*)| \le C \left[\frac{Q(X)[1+Q(X)]}{a^3 T} + \frac{\sqrt{Q(X)[1+Q(X)]}}{a} \right].$$

By (1.108), if K_1 is large enough the right-hand side in the above display is smaller than $aT/2$ so that $T_* = 0$. Since $|v_i| \le \sqrt{2Q(X)\log(\mathrm{e} + |r_i|)}$ we get a contradiction, therefore $v_i(T) \le aT$ if T satisfies (1.108).

Notice that in the above argument we used that the average force acting on the fast particle is very small for two reasons: firstly, the fast particle interacts with a slow one for a very short time (inversely proportional to the velocity gap); secondly, there is a compensation effect in the action of the forces during a collision of two particle (this fact is proved introducing the quantity $p_i(t)$, a sort of "adiabatic invariant"). The fast particle thus returns to the initial time essentially with the same velocity aT and this fact gives an absurd for large T.

Of course, the assumption that the fast particle is alone and the background does not increase its velocity is too drastic. Concerning the first question, we now make an essential observation. Consider the set $\mathscr{P} = \{j \in \mathbb{N}\colon |r_j - r_i| \le L_T\}$ with $L_T = C_*[1 + Q(X)]T^2$. By (1.16) (we still assume (1.108)), for C_* sufficiently large $\mathscr{P}$ contains all the particles which can interact with the ith one during the time interval $[0, T]$. Moreover $|v_j(t)| \le C[1 + Q(X)]T^2$ for any $j \in \mathscr{P}$. Now, Eq. (1.17) implies that, for each time $t \in [0, T]$, the total number of fast particles in $\mathscr{P}$ does not depend on T. Indeed, setting $\mathscr{I}_t = \{j \in \mathscr{P}\colon |v_j(t)| > aT/4\}$, we have

$$|\mathscr{I}_t| \le \frac{32}{(aT)^2} Q(X(t); r_i, L_T + C[1 + Q(X)]T^2) \le \frac{\bar{C}\, Q(X)[1 + Q(X)]}{a^2},$$

for a suitable $\bar{C} > 0$.

For T large enough this fact imposes that there exists a velocity gap between the fast and slow particles. Then we can find an ε small enough such that in the interval $[(1-\varepsilon)T, T]$ the background does not increase very much its velocity and the fast particles remain such. We emphasize that the control on the background is nontrivial, but we do not discuss it here and address the interested reader to [2]. So the effect of the background on the fast particles is small. We must now control the mutual interactions among the fast particles: it is possible to show that each fast particle after some collisions either remains alone (and so it does not change its velocity) or it remains in a small cluster (in momentum space), whose center of mass is almost unchanged. In conclusion, the velocity of each fast particle in the interval $[(1-\varepsilon)T, T]$ is almost unchanged. Repeating ε^{-1} times this estimate, we arrive to an absurd.

The dependence on the small parameter a of $T_a(X, r_i)$ in (1.107) is very bad: we have to wait a super-exponentially large (with respect to a^{-1}) time to catch the asymptotic estimate on the ith particle velocity. On the other hand, this choice is a useful mathematical device to control the effect of the mutual interaction among the fast particles. In fact, it guarantees an a priori bound on the maximal number of refinements into small clusters which are needed to follow the evolution of the fast particles during the time interval $[(1-\varepsilon)T, T]$.

Let us go back to the whole system (1.105). The following theorems are proved in [2].

Theorem 1.10 *Given $Q > 0$ and $L > 0$ let*

$$\mathscr{B}_{Q,L} := \{X \in \mathscr{X} : Q(X) \le Q, \quad |r_i| \ge L \quad \forall\, i \in \mathbb{N}\} . \tag{1.112}$$

Then, for each $E > 0$ and $Q > 0$ there exists $L_0 > 0$ such that, for any $L \ge L_0$ and $X \in \mathscr{B}_{Q,L}$,

$$\lim_{t\to\infty} \frac{v(t)}{t} = E . \tag{1.113}$$

Theorem 1.11 *For any $E \ge 0$ and $X \in \mathscr{X}$,*

$$\liminf_{t\to\infty} \frac{v(t)}{t} \ge 0 . \tag{1.114}$$

Theorem 1.8 is an easy consequence of these theorems. In fact, by well known properties of the *DLR* states, see, e.g., [7], the subset $\mathscr{B}_{Q,L}$ has positive measure w.r.t the state $\langle\cdot\rangle$ for Q large enough (depending on $\langle\cdot\rangle$) and for any $L \ge 0$. Then, by fixing Q large enough and L as in Theorem 1.10,

$$\begin{aligned}\liminf_{t\to\infty} \frac{\langle v(t)\rangle}{t} &\ge \left\langle \liminf_{t\to\infty} \frac{v(t)}{t} \right\rangle = \left\langle \chi(\mathscr{B}_{Q,L}) \lim_{t\to\infty} \frac{v(t)}{t} \right\rangle \\ &\quad + \left\langle \chi(\mathscr{B}^c_{Q,L}) \liminf_{t\to\infty} \frac{v(t)}{t} \right\rangle \ge \left\langle \chi(\mathscr{B}_{Q,L}) \right\rangle E > 0 ,\end{aligned}$$

where in the first inequality we used Fatou's lemma. Obviously, the same result holds not only for Gibbs states but for any reasonable equilibrium or non-equilibrium thermodynamic state.

Remark 1.12 By exploiting the proof of Theorem 1.10 in [2] it is easy to check that we actually prove also the following statement. Assume the heavy particle is initially in the position $r = 0$ with a positive velocity v_0. Then for each $X \in \mathscr{X}$ and $E > 0$ there exists a threshold $\bar{v}_0$ such that Eq. (1.113) holds for any $v_0 \ge \bar{v}_0$. This Hamiltonian model is thus an example of *runaway* particle, see, e.g., [9]. We also remark that (1.113) improves the results of the previous section not only because is valid for any intensity of E, but also in view of the fact that predicts an asymptotic uniformly accelerated motion for the heavy particle (with acceleration equal to E). We recall that this result for the Cauchy problem (1.105) when E is large enough with respect to $Q(X)$ was already proved in [1].

The proof of Theorem 1.10 can be summarized as follows. We start by noticing that the analogous of Theorem 1.4 clearly holds in the present case. Therefore, if $t \mapsto \{(r(t), v(t)); X(t)\}$, $X(t) = \{(r_i(t), v_i(t))\}_{i\in\mathbb{N}}$, denotes the solution to Eqs. (1.105) and $Q_E(X) := Q(X) + E$, for any $X \in \mathscr{X}$, $i \in \mathbb{N}$, and $t \geq 0$,

$$|v(t)| \leq C \left[\sqrt{Q_E(X) \log(\mathrm{e} + Q_E(X))} + Q_E(X)t \right] , \tag{1.115}$$

$$|v_i(t)| \leq C \left[\sqrt{Q_E(X) \log(\mathrm{e} + |r_i| + Q_E(X))} + Q_E(X)t \right] , \tag{1.116}$$

and, for any $\mu \in \mathbb{R}$, $R > \log(\mathrm{e} + |\mu|)$, and $t \geq 0$,

$$Q(X(t); \mu, R) \leq CQ_E(X) \left[R + \log(\mathrm{e} + Q_E(X)) + (1 + Q_E(X))t^2 \right] . \tag{1.117}$$

We fix $E, Q > 0$ and define $Q_E := Q + E$ so that $Q_E(X) \leq Q_E$ for any $X \in \mathscr{B}_{Q,L}$. By (1.115) and (1.116), it is easy to deduce that there exists a constant $\bar{C} \geq 1$ such that, for any $X \in \mathscr{B}_{Q,L}$ and $i \in \mathbb{N}$,

$$\inf_{s\in[0,t]} |r(s) - r_i(s)| \leq 2 \quad \Longrightarrow \quad |r_i| \leq Y_t := \bar{C} \left[\log(\mathrm{e} + Q_E) + Q_E t^2 \right] , \tag{1.118}$$

from which it follows that if $|r_i| \geq 2\bar{C} \log(\mathrm{e} + Q_E)$ then

$$\inf_{s\in[0,t]} |r(s) - r_i(s)| \leq 2 \quad \Longrightarrow \quad t \geq \sqrt{\frac{|r_i|}{2\bar{C}Q_E}} . \tag{1.119}$$

The parameter $L_0 = L_0(E, Q)$ in the statement of the theorem is then chosen in the following way. Let

$$0 < a_0 \leq \frac{\min\{1; E\}}{8} , \qquad K \geq 1 ,$$

be two parameters to be fixed later. Then $L_0 \geq 2\bar{C} \log(\mathrm{e} + Q_E)$ is chosen large enough that, for any $a \in (0, a_0]$, $K \geq 1$, and $L \geq L_0$,

$$T \geq a^{-K(1+Q_E^3)/a^2} \sqrt{\log\left[\mathrm{e} + Y_T + 1\right]} \qquad \forall\, T \geq T_L := \sqrt{\frac{L}{2\bar{C}Q_E}} . \tag{1.120}$$

Define now,

$$U(t) := \max_i G(|r_i| - Y_t) \sup_{s\in[0,t]} |v_i(s)| , \tag{1.121}$$

with Y_t as in (1.118) and $G \in C(\mathbb{R})$ a not increasing function satisfying: $G(x) = 1$ for $x \le 0$, $G(x) = 0$ for $x \ge 1$. By (1.118), the continuous and not decreasing function $U(\cdot)$ is an upper bound for the maximal velocity of any particle which may interact with the heavy one during the time $[0, t]$. We next define,

$$T^* := \sup \left\{ t > 0 \colon \max\{U(s); |v(s) - Es|\} \le \frac{Es}{4} \quad \forall\, s \in [0, t] \right\} , \tag{1.122}$$

setting $T^* = 0$ if the above set is empty. By (1.119), the definition of T_L in (1.120), and recalling that $|r_i| \ge L \ge 2\bar{C} \log(\mathrm{e} + Q_E)$, we have $U(t) = 0$ and $v(t) = Et$ for $t \le T_L$, whence $T^* > T_L$. We next prove that if a_0 is small enough and K is large enough then

$$\max\{U(t); |v(t) - Et|\} \le \frac{Et}{8} \qquad \forall\, t \in [T_L, T^*) , \tag{1.123}$$

which implies, by continuity, $T^* = \infty$. Moreover, since we actually prove that $|v(t) - Et| = \mathscr{O}(\log t)$ for $t < T^*$, the limit (1.113) follows.

To bound $U(t)$ we can apply Theorem 1.9 in the present context. To this end, we first notice that by (1.120) and the definition (1.107), if $T \ge T_L$ then $T \ge T_a(X, r_i)$ for any i such that $|r_i| \le Y_T + 1$. Moreover, since for $t < T^*$ the heavy particle is much faster than the particles it meets, the interaction with this particle does not affect too much the velocity of each background particle up to this time. More precisely, for any $0 \le \tau_1 \le \tau_2 < T^*$ and $i \in \mathbb{N}$,

$$\left| \int_{\tau_1}^{\tau_2} \mathrm{d}s \; \nabla\Psi(r_i(s) - r(s)) \right| \le \frac{2\|\nabla\Psi\|_\infty}{ET_L} . \tag{1.124}$$

Note in fact that the i-th particle may interact with the heavy one only after the time T_L, and hence for a time not bigger than $2/(E\tau_1) \le 2/(ET_L)$. From the previous estimate the strategy used for proving (1.106) applies in this case almost unchanged, getting,

$$|v_i(T)| \le aT \qquad \forall\, T \in [T_L, T^*) \quad \forall\, i \;:\; |r_i| \le Y_T + 1 , \tag{1.125}$$

which in particular implies, by the definition of $U(\cdot)$,

$$U(t) \le \max\{|v_i(t)| \;:\; |r_i| \le Y_t + 1\} \le \frac{Et}{8} \tag{1.126}$$

(recall we assume $a_0 \le E/8$).

We are left with an upper bound for $|v(t) - Et|$ when $T_L < t < T^*$ (recall in fact that $|v(t) - Et| = 0$ for $t \le T_L$). Define,

$$p(t) := v(t) - Et + \sum_i \frac{\Psi(r(t) - r_i(t))}{M(v(t) - v_i(t))} .$$

By the equations of motion,

$$\dot{p}(t) = \sum_{i,j} \frac{\Psi(r(t) - r_i(t))\nabla\Psi(r(t) - r_j(t))}{M^2[v(t) - v_i(t)]^2} - \sum_{i \neq j} \frac{\Psi(r(t) - r_i(t))\nabla\Phi(r_i(t) - r_j(t))}{M[v(t) - v_i(t)]^2}$$
$$- \sum_i \frac{\Psi(r(t) - r_i(t))[E + \nabla\Psi(r_i(t) - r(t))]}{M[v(t) - v_i(t)]^2} ,$$

and therefore, for $T_L \le t < T^*$,

$$|p(t) - p(T_L)| \le C \int_{T_L}^t \mathrm{d}s \, \frac{N(s)}{Es} \left[\frac{1}{s} + \frac{1 + N(s)}{Es} \right] , \tag{1.127}$$

where $N(s) = N(X(s); r(s), 2)$. By (1.111) and (1.71),

$$N(s)^2 \le CQ(X(s); r(s), 2) \le CQ_E \left[\log(\mathrm{e} + Q_E) + (1 + Q_E)s^2 \right] , \tag{1.128}$$

where we used that $|r(s)| \le Cs[\sqrt{Q_E \log(\mathrm{e} + Q_E)} + Q_E s]$, which follows by (1.69). The term $[1 + N(s)]/(Es)$ in (1.127) can be bounded using (1.128); by the definitions of L and T_L we thus obtain, for any $t \ge T_L$,

$$\begin{aligned} |p(t) - p(T_L)| &\le C \frac{1 + Q_E}{E} \int_{T_L}^t \mathrm{d}s \, \frac{N(s)}{Es} \\ &= C \frac{1 + Q_E}{E} \sum_j \int_{T_L}^t \mathrm{d}s \, \frac{1}{Es} \chi(|r(s) - r_j(s)| \le 2) . \end{aligned}$$

An upper bound for the right-hand side of the above inequality can be obtained as it follows. We first observe that by (1.118) and the definition (1.112) only the particles which are initially in $[-Y_t, -L] \cup [L, Y_t]$ may contribute to the above integral. We next define $N_q = \{j \in \mathbb{N} \, : \, 2^q < |r_j| \le 2^{q+1}\}$. Since $t < T^*$,

$$\sum_j \int_{T_L}^t \mathrm{d}s \, \frac{1}{Es} \chi(|r_j(s) - r(s)| \le 2) \le \sum_{q=q_L}^{q_t+1} \frac{4|N_q|}{E^2 t_q^2} , \tag{1.129}$$

where q_L [resp. q_t] is the integer such that $2^{q_L} < L \le 2^{q_L+1}$ [resp. $2^{q_t} < Y_t \le 2^{q_t+1}$] and

$$t_q := \min_{j \in N_q} \inf\{s \in [T_L, T^*) \, : \, |r(s) - r_j(s)| \le 2\} ,$$

setting $t_q = \infty$ if the above set is empty for all $j \in N_q$. We may assume L so large (i.e., a_0 small enough) that $2^{q-1} > \log(\mathrm{e} + 2^{q+1})$ for any $q \ge q_L$. Then, since

$2^{q+1} - 2^q = 2^q$, by (1.60) and (1.61) we have $|N_q| \le 2Q(X)2^q \le Q_E 2^{q+1}$. On the other hand, from (1.119) it follows that $t_q > \sqrt{2^{q-1}/(\bar{C} Q_E)}$. Inserting the previous bounds in (1.129) we obtain, for any $t \in [T_L, T^*)$,

$$\sum_j \int_{T_L}^t \mathrm{d}s \, \frac{1}{Es} \chi(|r_j(s) - r(s)| \le 2) \le C \frac{Q_E^2}{E^2} \log \frac{Y_t}{L} \le C \frac{Q_E^2}{E^2} \log t \ ,$$

so that

$$|p(t) - p(T_L)| \le C \frac{1 + Q_E^3}{E^3} \log t \qquad \forall\, t \in [T_L, T^*) \ . \tag{1.130}$$

Since $v(T_L) - ET_L = 0$ we have,

$$|v(t) - Et| \le |v(t) - Et - p(t)| + |p(t) - p(T_L)| + |v(T_L) - ET_L - p(T_L)| \ .$$

By the definition of $p(t)$, the first [resp. third] term on the right-hand side is smaller than a constant multiple of $N(t)/(Et)$ [resp. $N(T_L)/(ET_L)$], which we have already shown to be bounded by $C(1 + Q_E)/E$ for any $t \ge T_L$. Finally, the second term is bounded in (1.130). In conclusion,

$$|v(t) - Et| \le C \frac{1 + Q_E}{E} \left(1 + \frac{Q_E^2}{E^2} \log t \right) \qquad \forall\, t \in [T_L, T^*) \ , \tag{1.131}$$

which in particular implies, if a_0 is small enough, $|v(t) - Et| \le Et/8$ for all $t \in [T_L, T^*)$. By (1.126) Eq. (1.123) is thus proved, whence $T^* = \infty$. The limit (1.113) then follows from (1.131).

Concerning Theorem 1.11, we have to prove that there exist $a_0 \in (0, 1)$ and $T_a = T_a(X, E) > 0$ such that, for any $a \in (0, a_0]$,

$$v(t) \ge -at \qquad \forall\, t \ge T_a \ . \tag{1.132}$$

The proof is much more involved and cannot be achieved by the same strategy used in the proof of Theorem 1.9. Let us firstly review the proof of the case without external force. We divide the time interval $[0, T]$ in many subintervals $[T_{k-1}, T_k]$ and the fast particles into many disjoint clusters. By using the equations of motion and some tricks we prove that the velocity of the center of mass of each cluster remains almost constant. Of course a cluster may increase its size due to the internal forces, thus approaching an adjacent one. But the first time τ_1 (in the backward evolution) when this happens, we decompose the set of fast particles into smaller clusters which remain disjoint until a time τ_2 and so on. The important point is that the number of clusters increases at each step, so that the number of steps is not bigger than the cardinality of the set of the fast particles. This procedure holds in each time interval and we go back to time zero with some fast particles, thus getting a contradiction because of the initial data we have chosen.

The above strategy fails in the present context. In fact, due to the presence of the external force, the above scheme does not hold for the cluster containing the heavy particle: the velocity of its center of mass decreases during the backward evolution. Therefore this cluster could approach the adjacent one without modifying its size. For this reason, a refinement into smaller clusters does not anymore guarantee that the number of clusters increases. We then need a nontrivial modification of that part of the proof, that we do not discuss here and address the interested readers to [2].

1.3.3 Higher Dimensions and Open Problems

It is reasonable that the violation of Ohm's law for bounded interactions holds also in the case of particles in the tube, but we are not able to prove it rigorously. In fact, in one dimension a very fast particle interacts once and for a short time with a slow particle. In the tube it is not so: a particle could be very fast because of a high transversal velocity, thus remaining near the origin and interacting many times with a slow one. As time goes by, it increases its transversal velocity and then changes its direction and moves along the **n**-axis. Of course this behavior is very unlikely, but it is hard to be excluded.

We can introduce a strange model to overcome this effect. The background particles freely move in the whole tridimensional space $\mathbb{R}^3$, but they are attracted by the **n**-axis via an external force of potential

$$\Theta(\mathbf{q}) = \theta(|\mathbf{q}^{\perp}|)\ |\mathbf{q}^{\perp}|^{\alpha}\ , \tag{1.133}$$

where $\mathbf{q}^{\perp} = \mathbf{q} - (\mathbf{q}\cdot\mathbf{n})\mathbf{n}$, $\alpha \in (0,1)$, and $\theta(s)$, $s \in \mathbb{R}^{+}$, is a non-negative, twice differentiable function, identically zero for $s \leq 1/2$ and equal to one for $s \geq 1$. This potential plays the role of the confining one-body potential in (1.59). This model has not been investigated.

We have no rigorous results for singular interaction particle/background, except the hard core case, that will be discussed in the next sections in the framework of the mean field approximation. The difficulty arises from the fact that a single collision could affect the motion of the heavy particle. Obviously, if the heavy particle is very fast only quasi-central collisions are important and they become very few. This fact suggests some heuristic considerations on the divergence necessary to forbid large velocities [1], but a rigorous analysis seems too hard. See also Sect. 2.4, where such a heuristic analysis is done in the context of mean field models.

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http://www.springer.com/978-3-319-14758-1

Mathematical Models of Viscous Friction
Buttà, P.; Cavallaro, G.; Marchioro, C.
2015, XIV, 134 p. 5 illus., Softcover
ISBN: 978-3-319-14758-1