Chapter 2
What is wave motion?

2.1 What is a wave?

As surprising as it may sound, there is no simple answer to this question. Better not ask what a wave is, but ask what can be said about a wave, explains J. Pierce [197]. The confusion is caused by the wave motion itself, which can be related to propagating disturbances or oscillations. Nevertheless, let us first present some definitions.

Truesdell and Noll [241] said:

> wave is a state moving into another state.

Short as this definition is, the question of what a state is remains unanswered. Surely we cannot give a political meaning to this term but should have physics in mind. More physical is the following definition [66]:

> wave is a disturbance which propagates from one point in a medium to other points without giving the medium as a whole any permanent displacement.

A disturbance means that a medium is deformed at a certain point and this disturbance is transmitted from one point to the next and so on. Consequently, a wave moving with a finite velocity should overcome the medium's resistance to deformation as well as the resistance to motion (i.e. inertia). This implies that waves can only occur in a medium in which energy can be stored in both kinetic and potential forms. In this way we may add one more definition:

> a wave is characterized by the transfer of energy from one point to another.

As mentioned before, we focus here on waves in solids. Let us introduce some more definitions:

> a solid is a substance that has a definite volume and shape and resists forces that tend to alter its volume or shape;
> a solid is a crystalline material in which the constituent atoms are arranged in a 3D lattice with certain symmetries.

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Clearly the first definition corresponds to the continuum theory [83] while the second definition corresponds to the theory of discrete media [158]. Based on the definitions given above, two basic types of wave motion are possible in solids. The motion of a disturbance (particle motion) can be in the direction of the wave motion, in which case the wave is called longitudinal. But the motion of a disturbance can also be transverse to the wave motion, and then the wave is called transverse. In terms of stress, a longitudinal wave transmits tensile and compressive stress while the transverse wave transmits shear stress. Notice that longitudinal waves are sometimes called dilatational, irrotational, or extension waves. Transverse waves are also called shear, rotational, distortion, or equivoluminal waves. Both of them are sometimes called body waves. If a solid has a free surface, surface or Rayleigh waves are possible. In this case, the motion of a disturbance is in the plane perpendicular to the free surface and parallel to the direction of propagation. The amplitude of the surface waves decreases with the depth measured from the free surface. In case of a solid-solid interface, the surface wave is called Stonely wave.

This short overview gives a first answer to the question of what a wave is. More on basic definitions and types of waves in solids can be found in monographs of Kolsky [134], Bland [28], Engelbrecht [66], and Maugin [158], to name just a few.

2.2 What is a wave equation?

The classical 1D wave equation in terms of a displacement u reads

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial^2 u}{\partial x^2}, \tag{2.1}$$

where x, t are space and time coordinates and c_0 is the velocity of the wave (a constant). Equation (2.1) is a hyperbolic equation, which means that the disturbance travels with a finite velocity c_0. Together with parabolic and elliptic equations, it forms the classical cornerstones of mathematical physics. One should note that the velocity c_0 is independent of the wavelength. In the 3D setting in coordinates x, y, z the wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right). \tag{2.2}$$

Equations (2.1) and (2.2) describe the propagation of a disturbance (excitation) generated by initial or boundary conditions. There is neither dissipation nor dispersion described by these equations (which is certainly not realistic). A remarkable property of Eq. (2.1) is that it has a closed solution (see, for example, [95]) for given initial conditions:

$$u(x,0) = F(x), \; u_t(x,0) = G(x). \tag{2.3}$$

Here and further, an index denotes a differentiation, so Eq. (2.1) can also be written as

$$u_{tt} = c_0^2 u_{xx}. \tag{2.4}$$

Indeed, after introducing new variables

$$\xi = x + c_0 t,\ \varsigma = x - c_0 t, \tag{2.5}$$

Eq. (2.1) yields

$$u_{\xi\varsigma} = 0, \tag{2.6}$$

which can be solved by direct integration. The solution is named after d'Alembert:

$$u(x,t) = \frac{1}{2}\left(F(x + c_0 t) + F(x - c_0 t)\right) + \frac{1}{2c_0}\int_{x-c_0 t}^{x+c_0 t} G(\alpha)d\alpha. \tag{2.7}$$

This solution explicitly shows waves propagating in two directions—to the right $(F(x - c_0 t))$ and to the left $(F(x + c_0 t))$. The conditions $x - c_0 t = 0$ and $x + c_0 t = 0$ define the fronts of these waves—see Fig. 2.1.

For zero initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0, \tag{2.8}$$

and a boundary condition

$$u(0,t) = R(t), \tag{2.9}$$

the solution for $t > x/c_0$ is

$$u(x,t) = R(t - x/c_0). \tag{2.10}$$

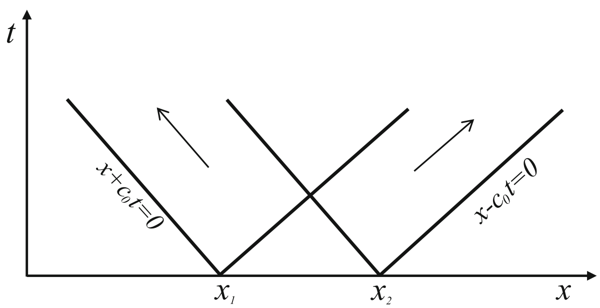

Fig. 2.1 Characteristic plane x, t for an excitation given in $x = [x_1, x_2]$. *Arrows* show the direction of propagation

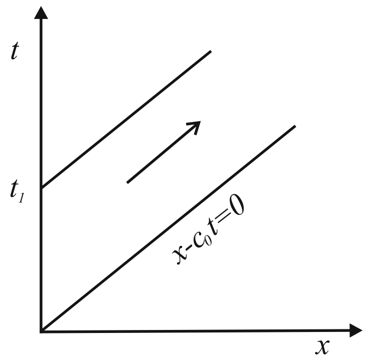

Fig. 2.2 Characteristic plane x, t for an excitation given in $t = [0, t_1]$

The corresponding x, t plane is shown in Fig. 2.2. More about problems with initial and boundary conditions, reflections from boundaries, etc. can be found in [4, 95].

From Fig. 2.1 it is easily seen that the wave equation (2.1) is a two-wave equation. These waves propagate separately except in the interaction area. This immediately stimulates the idea to construct a model for just one wave. Such an idea will be elaborated later in Chap. 5. In the classical case of (2.1) and the corresponding solutions (2.7) or (2.10), there will be no special advantages in one-wave equations. However, in more complicated cases they can provide substantial benefits when describing the wave process.

2.3 What is needed for wave motion?

As said in Sect. 2.1, waves propagate in media where kinetic and potential energy can be stored. In the simplest 1D case kinetic energy $\mathscr{K}$ and potential energy W can be determined by

$$\mathscr{K} = \frac{1}{2}\rho u_t^2, \quad W = \frac{1}{2}(\lambda + 2\mu)u_x^2, \tag{2.11}$$

where ρ is the density and λ, μ are Lamé parameters. The wave equation is then derived from the balance of momentum, resulting in

$$\rho u_{tt} = (\lambda + 2\mu)u_{xx}. \tag{2.12}$$

The left-hand side stems from the given kinetic energy resulting in acceleration, and the right-hand side from the given potential energy resulting in a force but in continua expressed by stress. It is easily seen that the velocity of the wave c_0 is determined by $c_0^2 = (\lambda + 2\mu)/\rho$. Note that in this simple case u_x is the deformation, and the stress σ is determined by $\sigma = (\lambda + 2\mu)u_x$ which is nothing but Hooke's

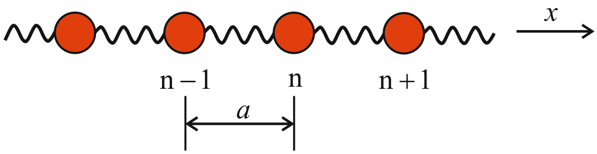

Fig. 2.3 1D chain of particles with equal masses m

law. Lamé parameters λ, μ actually determine the modulus of elasticity (Young's modulus). In engineering practice, this is denoted by $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$ [4].

Although the derivation of the wave equation in this case is enormously simplified, it definitely shows that density and elasticity of a medium (material) govern and determine the velocity of propagation.

The universality of the wave equation can be demonstrated by other simple cases. Let us take the waves in a one-dimensional chain of particles with equal masses, as shown in Fig. 2.3. The particles in such an infinite elastic chain are linked by elastic springs of stiffness k. The wave motion in this chain is governed by the equation of motion [158]:

$$m \frac{d^2 U_n}{dt^2} = k(U_{n+1} - 2U_n + U_{n-1}), \tag{2.13}$$

where U is the displacement. Equation (2.13) is nothing but Newton's 2nd law written for the n-th particle. If we now go to the continuum limit, then

$$U_n(t) = U(x_n, t), \quad U_{n\pm1}(t) = U(x_n \pm a, t). \tag{2.14}$$

We expand U into the Taylor series

$$U_{n\pm1} = U(x_n) \pm \frac{\partial U}{\partial x}\Big|_{x_n} a + \frac{1}{2}\frac{\partial^2 U}{\partial x^2}\Big|_{x_n} a^2 \pm \ldots. \tag{2.15}$$

With the expression (2.15), Eq. (2.13) then yields

$$U_{tt} - c_0^2 U_{xx} = 0, \tag{2.16}$$

where $c_0^2 = ka^2/m = E/\rho$. It is easily seen that

$$\rho = m/a^3, \quad E = k/a. \tag{2.17}$$

The model (2.13)—see Fig. 2.3—is the Born-von Kármán model [158].

Another simple example is a vibrating string which has attracted the attention of researchers over ages starting from Pythagoras (for a historical overview see [95]). Let us consider a string under tension T as shown in Fig. 2.4. The mass density per

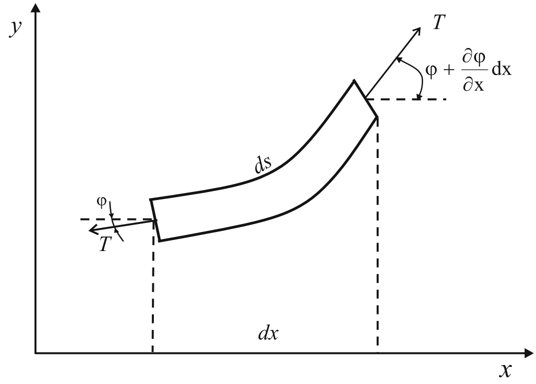

Fig. 2.4 Element of a string

unit length is ρ and the arc length ds of the string can be assumed $ds \sim dx$ because of small deflections. The equation of motion in the vertical direction is

$$-T\sin\varphi + T\sin\left(\varphi + \frac{\partial\varphi}{\partial x}dx\right) = \rho ds\frac{\partial^2\varphi}{\partial t^2}. \tag{2.18}$$

For small deflections we may use $\sin\varphi \sim \varphi$ and $\varphi \sim \partial y/\partial x$. Then Eq. (2.18) yields

$$T\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2}. \tag{2.19}$$

This equation is again a wave equation

$$\frac{\partial^2 y}{\partial t^2} - c_0^2\frac{\partial^2 y}{\partial x^2} = 0, \tag{2.20}$$

where $c_0^2 = T/\rho$.

These simple examples demonstrate that the backbone of wave motion—the wave equation—can be derived by using many different assumptions. Whether a medium is discrete or continuous, whether we have to deal with moving objects like a string, the outcome is the same. But real life is much more complicated than these simple cases, and in the next chapters of this book more questions will be asked.

2.4 How to measure waves?

Evidently the wave equation itself provides one important entity, namely the velocity c_0 which depends on the physical properties of the medium (material). Two more measures are dictated by the initial and boundary conditions, namely

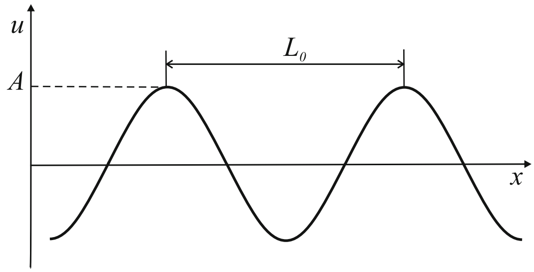

Fig. 2.5 Harmonic wave

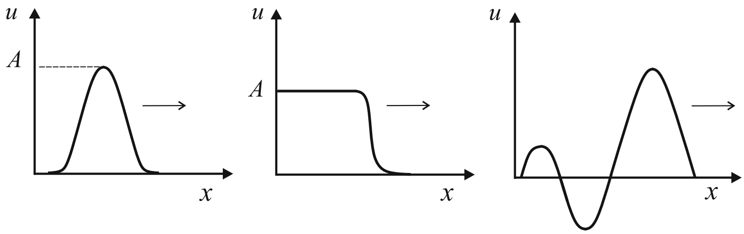

Fig. 2.6 Anharmonic waves

the wavelength L_0 and the amplitude A. Figure 2.5 shows these measures for a harmonic wave. Such a wave can be represented in the form

$$u = A\exp[i(kx - \omega t)]. \tag{2.21}$$

Or, if you consider only the real part of expression (2.21),

$$u = \mathrm{Re}(A)\cos(kx - \omega t), \tag{2.22}$$

where k is the wave number and ω is the angular frequency. In order for the waves as described by (2.21) or (2.22) to satisfy the wave equation, the condition $k = \omega/c_0$ must be satisfied. Using k and ω, we can determine the frequency $f = \omega/2\pi$ and the wavelength $L = 2\pi/k = 2\pi c_0/\omega = c_0/f$.

Harmonic waves play an important role in the analysis of waves, but the world is much more complicated and the waves can have an anharmonic shape. Some of such anharmonic profiles are shown in Fig. 2.6. Clearly one has to reconsider the notions of the wavelength and the frequency determined above. In addition, in many cases the fundamental wave equation (2.1) is not sufficient to describe the physical situation and must be modified. For example, the linearized Klein-Gordon equation reads

$$u_{tt} - c_0^2 u_{xx} + \beta^2 u = 0, \tag{2.23}$$

where $\beta = \text{const.}$ describes the motion of a string which is attached to a backing sheet [34]. If for the case of Eq. (2.1) we have

$$\omega = c_0 k, \tag{2.24}$$

then in the case of Eq. (2.23) we have

$$\omega = \pm(c_0^2 k^2 + \beta^2)^{1/2}. \tag{2.25}$$

Consequently, instead of expression (2.24) one should consider

$$\omega = \omega(k), \tag{2.26}$$

which is known as a dispersion relation. Combining the expressions (2.24) and (2.26) we determine the phase velocity to be

$$c_{ph} = \omega/k = \omega(k)/k. \tag{2.27}$$

If $c_{ph} = \text{const.}$ like in (2.24), the wave is nondispersive. If, however, $c_{ph} \neq \text{const.}$, the wave is dispersive. It means that waves with different wave numbers propagate with different velocities. In this context, we can also determine a so-called group velocity

$$c_{gr} = d\omega/dk. \tag{2.28}$$

The difference between c_{ph} and c_{gr} is shown in Fig. 2.7. In the dispersive case $c_{ph} \neq c_{gr}$, the energy transmitted by a wave propagates with group velocity c_{gr}. If $c_{gr} < c_{ph}$ then the dispersion is called normal; otherwise the dispersion is called anomalous. For more details, see Bland [28], Billingham and King [34], etc.

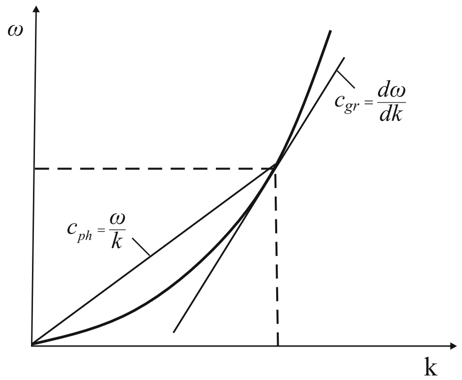

Fig. 2.7 Phase and group velocities

2.5 Why does the wave equation need to be modified?

The classical wave equation (2.1) is one of the fundamental equations in mathematical physics [230] and has had a large impact on the development of wave mechanics in many branches of physics. However, it describes the behaviour of a very simple physical situation with basic characteristics of wave motion: velocity, wave fronts, amplitude and wave length. In reality, one should also consider dispersion (as modelled by Eq. (2.23)), dissipation, nonlinearity, thermal effects, etc. This brings us to the main idea of this book:
how to modify the wave equation in order to model physical effects which influence the wave motion.

Actually, it means that we use the wave equation as a backbone, and the flesh is put around it reflecting rich physical phenomena involving interaction effects, velocity dependence on wavelength (dispersion), etc. However, the concept of finite velocity must always be incorporated. Returning to the flowchart of modelling (Fig. 1.1), however, the attention is focused on physical problems and their mathematical formulation together with solutions and interpretations of mathematical models. The model physical experiments, which are important without any doubt, are beyond the scope of this book. Questions on how to find the coefficients of the proposed models based on continuum theory are analytically treated by Janno and Engelbrecht [118].

As explained in the Introduction (Chap. 1), the inspirations for adding modifications are developed in a step-by-step chain of questions which occur naturally during the studies of wave motion. One certainly has to refer to many excellent monographs on waves, like those by Whitham [245], Bland [28], Lax [144] and others which form a basis for the present study.

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