

Chapter 2

Problem Formulation

In this chapter, we explain some important aspects of beamforming and differential arrays with a focus on the circular geometry. The problem of a DMA design is formulated while we progress in defining some useful concepts. We start with the definition of the steering vector for a plane wave with the conventional anechoic farfield model, which has an interesting structure. We give the general definition of the beampattern as well as its expression for differential arrays. We explain a fundamental property, which is the basis for the design of circular differential microphone arrays. We then derive the gain in signal-to-noise ratio (SNR), which is very useful in the evaluation of DMAs under different types of noise.

2.1 Signal Model

We consider a source signal (plane wave), in the farfield, that propagates in an anechoic acoustic environment at the speed of sound, i.e., $c = 340$ m/s, and impinges on a uniform circular array (UCA), of radius r , consisting of M omnidirectional microphones. The direction of the source signal to the array is parameterized by the azimuth angle θ . We assume that the center of the UCA coincides with the origin of the Cartesian coordinate system, azimuth angles are measured anti-clockwise from the x axis, i.e., at $\theta = 0$, and sensor 1 of the array is placed on the x axis, i.e., at $\theta = 0$ (see Fig. 2.1). When operating in the farfield, the time delay between microphone m and the center of the array is given by [1]

$$\tau_m = \frac{r}{c} \cos(\theta - \psi_m), \quad m = 1, 2, \dots, M, \quad (2.1)$$

where

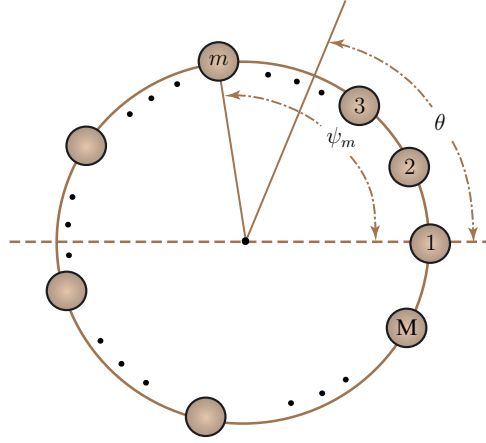


Fig. 2.1 Illustration of a uniform circular microphone array in the Cartesian coordinate system.

$$\psi_m = \frac{2\pi(m-1)}{M} \quad (2.2)$$

is the angular position of the m th array element. In this scenario, the steering vector of length M is

$$\begin{aligned} \mathbf{d}(\omega, \theta) &= [e^{j\omega\tau_1} \dots e^{j\omega\tau_M}]^T \\ &= [e^{j\omega r c^{-1} \cos(\theta - \psi_1)} \dots e^{j\omega r c^{-1} \cos(\theta - \psi_M)}]^T, \end{aligned} \quad (2.3)$$

where the superscript T is the transpose operator, $j = \sqrt{-1}$ is the imaginary unit, $\omega = 2\pi f$ is the angular frequency, and $f > 0$ is the temporal frequency. The acoustic wavelength is $\lambda = c/f$. For a UCA, the interelement spacing is

$$\begin{aligned} \delta &= 2r \sin\left(\frac{\pi}{M}\right) \\ &\approx \frac{2\pi r}{M}. \end{aligned} \quad (2.4)$$

In order to avoid spatial aliasing [2], which has the negative effect of creating grating lobes (i.e., copies of the main lobe, which usually points toward the desired signal), it is necessary that the interelement spacing is less than $\lambda/2$, i.e.,

$$M > \frac{4\pi r}{\lambda} = \frac{4\pi r f}{c}. \quad (2.5)$$

The condition (2.5) easily holds for small values of r (or δ) and in low frequencies but not in high frequencies.

Substituting the approximation of (2.4) into (2.3), the steering vector can be rewritten as

$$\mathbf{d}(f, \theta) \approx \left[e^{j f \delta c^{-1} \cos(\theta - \psi_1)} \dots e^{j M f \delta c^{-1} \cos(\theta - \psi_M)} \right]^T, \quad (2.6)$$

which is interesting to compare to the steering vector of a uniform linear array (ULA):

$$\mathbf{d}_L(\omega, \theta) = \left[1 e^{-j \omega \delta c^{-1} \cos \theta} \dots e^{-j (M-1) \omega \delta c^{-1} \cos \theta} \right]^T. \quad (2.7)$$

There are two fundamental differences between these two steering vectors. The first one is that (2.6) depends on $\cos(\theta - \psi_m)$ while (2.7) depends on $\cos \theta$. The other difference is that $\mathbf{d}(f, \theta) \neq \mathbf{d}(f, -\theta)$ while $\mathbf{d}_L(\omega, \theta) = \mathbf{d}_L(\omega, -\theta)$. The fact that $\mathbf{d}(\omega, \theta)$ has no apparent symmetry makes the design of circular differential microphone arrays (CDMAs) very different from the design of linear differential microphone arrays (LDMA). However, we still have an interesting symmetry for the entries of $\mathbf{d}(\omega, \theta)$.

Property 2.1. For the angle $\theta = 0$, let

$$\mathbf{d}(\omega, 0) = \left[D_1(\omega, 0) D_2(\omega, 0) \dots D_M(\omega, 0) \right]^T. \quad (2.8)$$

We have

$$D_{m+1}(\omega, 0) = D_{M-m+1}(\omega, 0), \quad m = 1, 2, \dots, M-1. \quad (2.9)$$

Proof. It is easy to check that the following relations hold

$$\cos(\psi_{m+1}) = \cos(\psi_{M-m+1}), \quad m = 1, 2, \dots, M-1. \quad (2.10)$$

Therefore, (2.9) is true.

For some particular values of θ , the steering vector of a UCA has another interesting property.

Property 2.2. For the angular positions, ψ_m , $m = 1, 2, \dots, M$, of the M sensors of a UCA, we have

$$\mathbf{d}(\omega, \psi_m) = \mathbf{P}_m \mathbf{d}(\omega, 0), \quad (2.11)$$

where $\mathbf{P}_m$ is an $M \times M$ permutation matrix whose form is

$$\mathbf{P}_m = \mathbf{P}_2^{m-1}, \quad m = 1, 2, \dots, M, \quad (2.12)$$

where $\mathbf{P}_1 = \mathbf{I}_M$, with $\mathbf{I}_M$ being the $M \times M$ identity matrix, and

$$\mathbf{P}_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}. \quad (2.13)$$

Proof. This property can be shown by using (2.9) and the relations:

$$\cos(\psi_i - \psi_j) = \cos(\psi_{|i-j|+1}), \quad i, j = 1, 2, \dots, M. \quad (2.14)$$

Since $\mathbf{P}_m$ is a permutation matrix, we have

$$\mathbf{P}_m^T \mathbf{P}_m = \mathbf{P}_m \mathbf{P}_m^T = \mathbf{I}_M. \quad (2.15)$$

As a result,

$$\mathbf{d}(\omega, 0) = \mathbf{P}_m^T \mathbf{d}(\omega, \psi_m). \quad (2.16)$$

To simplify the presentation of the equations in the rest of this text, we will also use the variable:

$$\varpi = \frac{\omega r}{c}. \quad (2.17)$$

Let us denote by θ_s the steering angle of the array. We consider designing fixed directional beamformers¹, like in DMAs, where the main lobe points at $\theta = \theta_s$ and the desired signal propagates from the same angle. We recall that for LDMAs, the optimal position is at $\theta = 0$ or π (endfire direction) and electronic steering (in the sense that the main lobe can be oriented to any possible direction without affecting the shape of the beampattern) is not really feasible. We will see that with CDMAs, we have much more flexibilities.

As pointed out in [3], there is a fundamental difference between differential arrays and filter-and-sum beamformers. In the latter category, the filters are optimized in such a way that the microphone signals are aligned in order to steer the main lobe in the direction of the desired signal, whereas in the former category the gains are optimized to steer a number of nulls in some specific directions.

The focus of this work is on the design, with small circular apertures, of beamformers whose beampatterns are very close to the ones obtained with “ideal” DMAs but in the direction θ_s . For that, a complex weight, $H_m^*(\omega, \theta_s)$, $m = 1, 2, \dots, M$, is applied at the output of each microphone, where the superscript $*$ denotes complex conjugation. The weighted outputs are then summed together to form the beamformer output. Putting all the

¹ The terms beamformer, beamforming, and beampattern may not be adequate in the context of DMAs but we will still use them for convenience.

gains together in a vector of length M , we get

$$\mathbf{h}(\omega, \theta_s) = [H_1(\omega, \theta_s) H_2(\omega, \theta_s) \cdots H_M(\omega, \theta_s)]^T. \quad (2.18)$$

Then, the objective is to design such a filter for any directivity pattern of any order when the array geometry is circular. The approach taken here is based on the fundamental observation that for all beampatterns of interest, some constraints must be fulfilled at all frequencies given that the number of microphones is equal to M . In other words, we select a certain number of fundamental constraints from a well-defined beampattern of a DMA to design $\mathbf{h}(\omega, \theta_s)$. The case $M = 2$ has no interest in this investigation since it is equivalent to a ULA, which has been extensively studied in the literature, and corresponds to a first-order DMA [4], [5].

In the next two sections, we discuss some fundamental measures. We are only interested in narrowband measures. The broadband measures can be easily deduced from their respective narrowband counterparts [6].

2.2 Beampattern

Each beamformer has a pattern of directional sensitivity, i.e., it has different sensitivities from sounds arriving from different directions. The beampattern or directivity pattern describes the sensitivity of the beamformer to a plane wave (source signal) impinging on the UCA from the direction θ . Mathematically, it is defined as

$$\begin{aligned} \mathcal{B}[\mathbf{h}(\omega, \theta_s), \theta] &= \mathbf{h}^H(\omega, \theta_s) \mathbf{d}(\omega, \theta) \\ &= \sum_{m=1}^M H_m^*(\omega, \theta_s) e^{j\varpi \cos(\theta - \psi_m)}, \end{aligned} \quad (2.19)$$

where the superscript H is the conjugate-transpose operator.

The frequency-independent beampattern of an N th-order DMA is well known. Its definition is usually given for a steering angle of 0 [4]. For any steering angle, θ_s , this beampattern is defined as

$$\mathcal{B}_N(\theta - \theta_s) = \sum_{n=0}^N a_{N,n} \cos^n(\theta - \theta_s), \quad (2.20)$$

where $a_{N,n}$, $n = 0, 1, \dots, N$, are real coefficients. The different values of these coefficients determine the different directivity patterns of the N th-order DMA. In the direction of the desired signal, i.e., for $\theta = \theta_s$, the beampattern must be equal to 1, i.e., $\mathcal{B}_N(0) = 1$. Therefore, we have

$$\sum_{n=0}^N a_{N,n} = 1. \quad (2.21)$$

As a result, we always choose the first coefficient as

$$a_{N,0} = 1 - \sum_{n=1}^N a_{N,n}. \quad (2.22)$$

The most interesting patterns have at least one null in some direction. We also have the important property:

$$\mathcal{B}_N(\theta - \theta_s) = \mathcal{B}_N(-\theta + \theta_s). \quad (2.23)$$

The directivity pattern of any DMA is an even function. Therefore, on a polar plot², $\mathcal{B}_N(\theta - \theta_s)$ is symmetric about the axis $\theta_s; \theta_s + \pi$ and any DMA design can be restricted to this range. It follows from (2.20) that an N th-order DMA has at most N (distinct) nulls in this range.

The directivity factor (see also Section 2.3) of an N th-order DMA, defined as the ratio between the directivity pattern at the direction $\theta = \theta_s$ and the averaged directivity pattern over the whole space, is³ [4], [7], [8]

$$\begin{aligned} \mathcal{G}_N &= \frac{\mathcal{B}_N^2(0)}{\frac{1}{\pi} \int_{\theta_s}^{\theta_s+\pi} \mathcal{B}_N^2(\theta - \theta_s) d\theta} \\ &= \frac{\pi}{\int_{\theta_s}^{\theta_s+\pi} \left[\sum_{n=0}^N a_{N,n} \cos^n(\theta - \theta_s) \right]^2 d\theta} \end{aligned} \quad (2.24)$$

and what we call the directivity index is

$$\mathcal{D}_N = 10 \log_{10} \mathcal{G}_N. \quad (2.25)$$

We find that the first-order, second-order, and third-order directivity factors are

² Polar patterns are a very convenient way to describe the directional sensitivity of the DMAs.

³ This situation corresponds to the cylindrically isotropic noise field.

$$\mathcal{G}_1 = \frac{1}{a_{1,0}^2 + \frac{1}{2}a_{1,1}^2}, \quad (2.26)$$

$$\mathcal{G}_2 = \frac{1}{a_{2,0}^2 + \frac{1}{2}a_{2,1}^2 + \frac{3}{8}a_{2,2}^2 + a_{2,0}a_{2,2}}, \quad (2.27)$$

$$\mathcal{G}_3 = \frac{1}{a_{3,0}^2 + \frac{1}{2}a_{3,1}^2 + \frac{3}{8}a_{3,2}^2 + a_{3,0}a_{3,2} + \frac{5}{16}a_{3,3}^2 + \frac{3}{4}a_{3,1}a_{3,3}}. \quad (2.28)$$

The hypercardioid is the pattern obtained from the maximization of the directivity factor⁴.

The front-to-back ratio is defined as the ratio of the power of the output of the array to signals propagating from the front-half plane (rotated by θ_s) to the output power for signals arriving from the rear-half plane (rotated by θ_s) [9]. This ratio, for the cylindrically isotropic noise field, is mathematically defined as [4], [9]

$$\mathcal{F}_N = \frac{\int_{\theta_s}^{\theta_s+\pi/2} \mathcal{B}_N^2(\theta - \theta_s) d\theta}{\int_{\theta_s+\pi/2}^{\pi} \mathcal{B}_N^2(\theta - \theta_s) d\theta}. \quad (2.29)$$

The supercardioid is the pattern obtained from the maximization of the front-to-back ratio⁵ [9].

First-order directivity patterns have the form:

$$\mathcal{B}_1(\theta - \theta_s) = (1 - a_{1,1}) + a_{1,1} \cos(\theta - \theta_s) \quad (2.30)$$

and the most important ones are as follows.

- Dipole: $a_{1,1} = 1$, null at $\cos(\theta - \theta_s) = 0$, and $\mathcal{D}_1 = 3$ dB.
- Cardioid: $a_{1,1} = \frac{1}{2}$, null at $\cos(\theta - \theta_s) = -1$, and $\mathcal{D}_1 = 4.3$ dB.
- Hypercardioid: $a_{1,1} = \frac{2}{3}$, null at $\cos(\theta - \theta_s) = -1/2$, and $\mathcal{D}_1 = 4.8$ dB.
- Supercardioid: $a_{1,1} = 2 - \sqrt{2}$, null at $\cos(\theta - \theta_s) = (1 - \sqrt{2})/(2 - \sqrt{2})$, and $\mathcal{D}_1 = 4.6$ dB.

Figure 2.2 shows these different polar patterns for $\theta_s = 0$. What is exactly shown are the values of the magnitude squared beampattern in dB, i.e., $10 \log_{10} \mathcal{B}_1^2(\theta)$.

Second-order beampatterns are described by the equation:

⁴ Another type of hypercardioid can be obtained by maximizing the directivity factor in the presence of a spherically isotropic noise field. There is not much difference, however, between the two patterns.

⁵ Another type of supercardioid can be obtained by maximizing the front-to-back ratio in the presence of a spherically isotropic noise field. There is not much difference, however, between the two patterns.

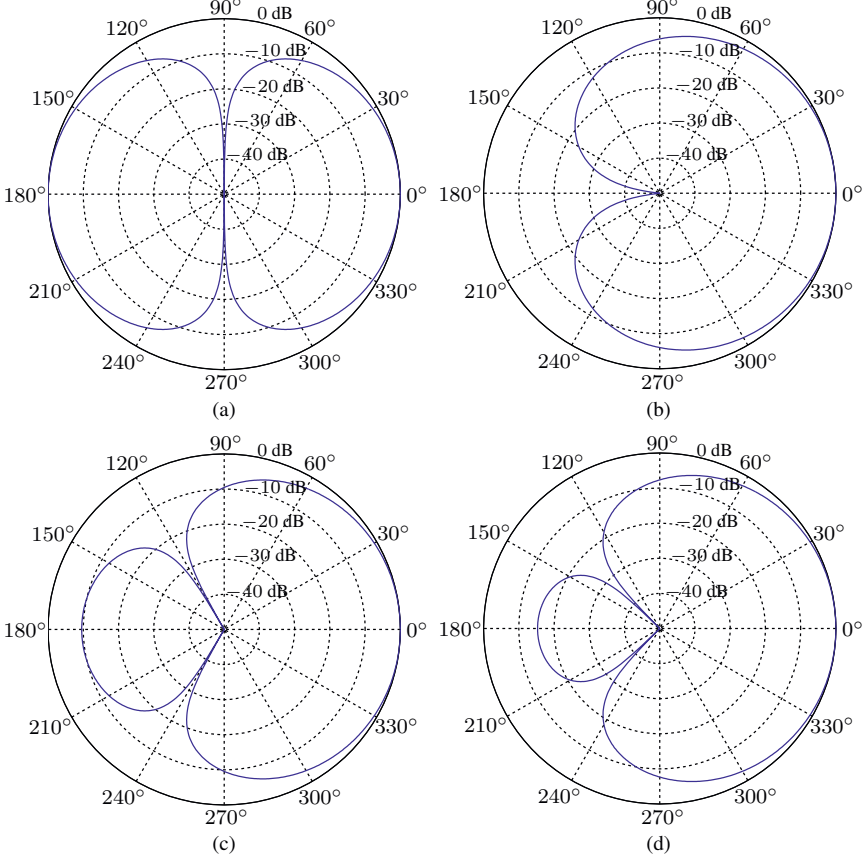


Fig. 2.2 First-order directivity patterns: (a) dipole, (b) cardioid, (c) hypercardioid, and (d) supercardioid.

$$\mathcal{B}_2(\theta - \theta_s) = (1 - a_{2,1} - a_{2,2}) + a_{2,1} \cos(\theta - \theta_s) + a_{2,2} \cos^2(\theta - \theta_s). \quad (2.31)$$

The second-order dipole has a null at $\cos(\theta - \theta_s) = 0$ and a one (maximum) at $\cos(\theta - \theta_s) = -1$. Replacing these values in (2.31), we find that $a_{2,1} = 0$ and $a_{2,2} = 1$. By analogy to the first-order and second-order dipoles, we define the N th-order dipole as

$$\mathcal{B}_{D,N}(\theta - \theta_s) = \cos^N(\theta - \theta_s), \quad (2.32)$$

implying that $a_{N,N} = 1$ and $a_{N,N-1} = a_{N,N-2} = \dots = a_{N,0} = 0$. The N th-order dipole has only one (distinct) null (in the range $\theta_s; \theta_s + \pi$) at $\theta = \theta_s + \pi/2$. The directivity indices of the second-order and third-order dipoles are, respectively, $\mathcal{D}_2 = 4.3$ dB and $\mathcal{D}_3 = 5.1$ dB.

The most well-known second-order cardioid has two nulls; one at $\cos(\theta - \theta_s) = -1$ and the other one at $\cos(\theta - \theta_s) = 0$. From these values, we easily deduce from (2.31) that $a_{2,1} = a_{2,2} = \frac{1}{2}$. By analogy to the first-order and second-order cardioids, we define the N th-order cardioid as

$$\mathcal{B}_{C,N}(\theta - \theta_s) = \left[\frac{1}{2} + \frac{1}{2} \cos(\theta - \theta_s) \right] \cos^{N-1}(\theta - \theta_s), \quad (2.33)$$

implying that $a_{N,N} = a_{N,N-1} = \frac{1}{2}$ and $a_{N,N-2} = a_{N,N-3} = \dots = a_{N,0} = 0$. This N th-order cardioid has only two distinct nulls (in the range $\theta_s; \theta_s + \pi$): one at $\theta = \theta_s + \pi/2$ and the other one at $\theta = \theta_s + \pi$. The directivity indices of the second-order and third-order cardioids are, respectively, $\mathcal{D}_2 = 6.6$ dB and $\mathcal{D}_3 = 7.6$ dB.

The N th-order hypercardioid and supercardioid are characterized by the fact that they have N distinct nulls in the interval $\theta_s < \theta < \theta_s + \pi$. Hence, their general beampattern is

$$\mathcal{B}_{HS,N}(\theta - \theta_s) = \prod_{n=1}^N [\zeta_{N,n} + (1 - \zeta_{N,n}) \cos(\theta - \theta_s)]. \quad (2.34)$$

Third-order beampatterns have the form

$$\begin{aligned} \mathcal{B}_3(\theta - \theta_s) = & (1 - a_{3,1} - a_{3,2} - a_{3,3}) + a_{3,1} \cos(\theta - \theta_s) \\ & + a_{3,2} \cos^2(\theta - \theta_s) + a_{3,3} \cos^3(\theta - \theta_s). \end{aligned} \quad (2.35)$$

We give the values of $a_{N,n}$ and $\mathcal{D}_N$ for some examples of hypercardioid and supercardioid [4], [8]:

- second-order hypercardioid, $a_{2,1} = \frac{2}{5}$, $a_{2,2} = \frac{4}{5}$, $\mathcal{D}_2 = 7$ dB;
- second-order supercardioid, $a_{2,1} \approx 0.484$, $a_{2,2} \approx 0.413$, $\mathcal{D}_2 = 6.3$ dB;
- third-order hypercardioid, $a_{3,1} = -\frac{4}{7}$, $a_{3,2} = \frac{4}{7}$, $a_{3,3} = \frac{8}{7}$, $\mathcal{D}_3 = 8.4$ dB; and
- third-order supercardioid, $a_{3,1} \approx 0.217$, $a_{3,2} \approx 0.475$, $a_{3,3} \approx 0.286$, $\mathcal{D}_3 = 7.2$ dB.

Figures 2.3 and 2.4 depict the different second-order and third-order directivity patterns discussed above for $\theta_s = 0$.

Now, let us get back to the general definition of the UCA beampattern given in (2.19). To make the analysis of a CDMA similar to an LDMA, we assume that $\theta_s = 0$. Then, we can simplify the notation by writing $\mathbf{h}(\omega, \theta_s) = \mathbf{h}(\omega)$. We will discuss the more general case of any steering angle later on. In order to be able to design a CDMA of any order, its beampattern must be an even function like in an LDMA, i.e., we must have

$$\mathcal{B}[\mathbf{h}(\omega), \theta] = \mathcal{B}[\mathbf{h}(\omega), -\theta]. \quad (2.36)$$

In the following, we will use the relations:

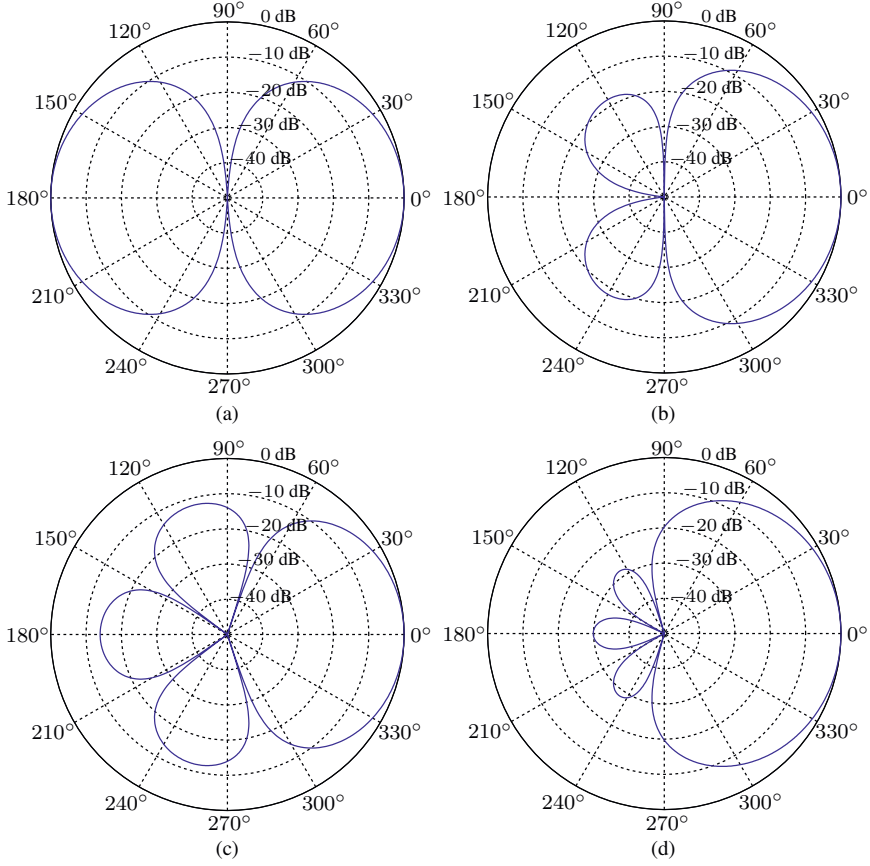


Fig. 2.3 Second-order directivity patterns: (a) dipole, (b) cardioid, (c) hypercardioid, and (d) supercardioid.

$$\cos(\theta + \psi_m) = \cos(\theta - \psi_{M-m+2}), \quad m = 1, 2, \dots, M. \quad (2.37)$$

For $M = 2$, we have

$$\begin{aligned} \mathcal{B}[\mathbf{h}(\omega), -\theta] &= H_1^*(\omega) e^{j\varpi \cos \theta} + H_2^*(\omega) e^{j\varpi \cos(\theta + \psi_2)} \\ &= H_1^*(\omega) e^{j\varpi \cos \theta} + H_2^*(\omega) e^{j\varpi \cos(\theta - \psi_2)} \\ &= \mathcal{B}[\mathbf{h}(\omega), \theta]. \end{aligned} \quad (2.38)$$

For any positive integer M , it is obvious that

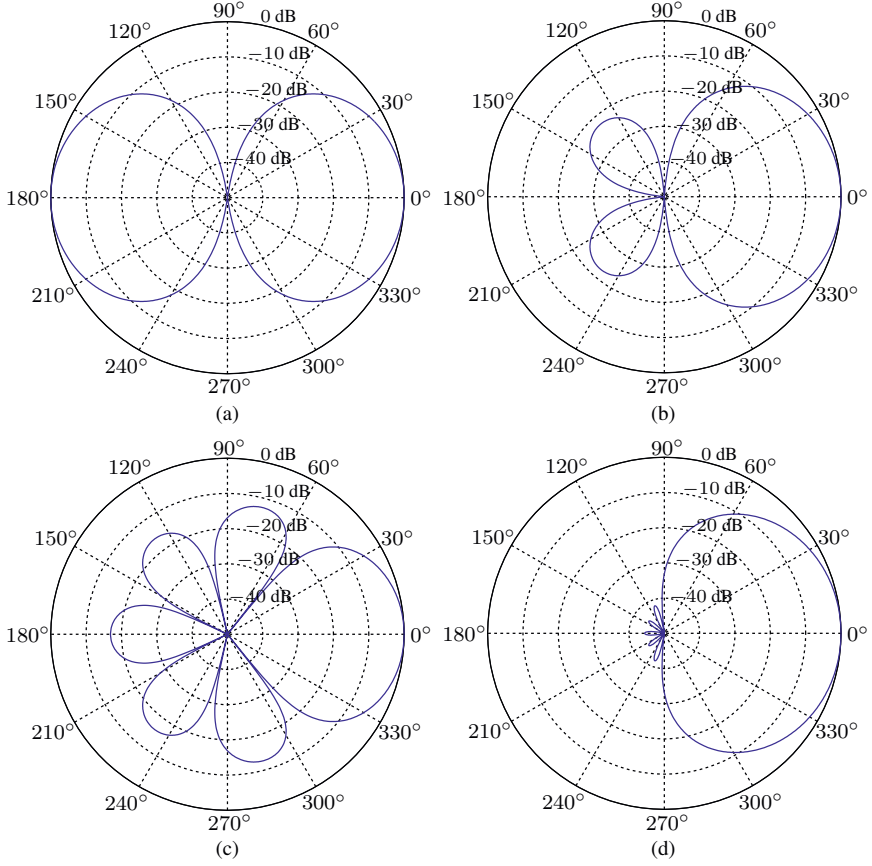


Fig. 2.4 Third-order directivity patterns: (a) dipole, (b) cardioid, (c) hypercardioid, and (d) supercardioid.

$$\begin{aligned}
 \mathcal{B}[\mathbf{h}(\omega), -\theta] &= \sum_{m=1}^M H_m^*(\omega) e^{j\varpi \cos(-\theta - \psi_m)} \\
 &= \sum_{m=1}^M H_m^*(\omega) e^{j\varpi \cos(\theta + \psi_m)}.
 \end{aligned} \tag{2.39}$$

Substituting (2.37) into (2.39), we get

$$\mathcal{B}[\mathbf{h}(\omega), -\theta] = \sum_{m=1}^M H_m^*(\omega) e^{j\varpi \cos(\theta - \psi_{M-m+2})}. \tag{2.40}$$

Therefore, (2.36) is verified if and only if

$$H_{m+1}(\omega) = H_{M-m+1}(\omega), \quad m = 1, 2, \dots, M-1. \quad (2.41)$$

We observe that in the filter $\mathbf{h}(\omega)$ of length M , only its first $\lfloor \frac{M}{2} \rfloor + 1$ coefficients need to be optimized, where $\lfloor x \rfloor$ is the integer part of x . This also means that only $\lfloor \frac{M}{2} \rfloor + 1$ independent constraints are possible. As a result, with a UCA of M microphones, we can design any differential array up to the order $\lfloor \frac{M}{2} \rfloor$ at the steering angle $\theta_s = 0$.

It is clear that we can design filters in such a way that the directivity patterns are all identical (up to a rotation) for

$$\theta_s = \psi_m = \frac{2\pi(m-1)}{M}, \quad m = 1, 2, \dots, M, \quad (2.42)$$

because of the perfect alignment of the desired source signal (at $\theta_s = \psi_m$), the position of the m th microphone, and the center of the array. Also, from this axis, the beampattern is perfectly symmetric. It is possible to electronically steer the array in other directions (than $\theta_s = \psi_m$), but the shape of the resulting patterns may not resemble anymore to the well-known ones. We can perfectly steer at $\theta_s = \psi_m$ because of Property 2.2.

We give the following fundamental property, which is the basis for the design of CDMAs.

Property 2.3. With M microphones and the designed filter, $\mathbf{h}(\omega)$, with the symmetry constraint [eq. (2.41)], we can build a CDMA up to the order $\lfloor \frac{M}{2} \rfloor$, which can perfectly steer in M different directions $\psi_m = \frac{2\pi(m-1)}{M}$, $m = 1, 2, \dots, M$. We recall that ψ_m is the position of the m th microphone. Steering in other directions is also possible but there is no guaranty that the original directivity pattern will not be affected.

In practice, we only need to design the filter for $\theta_s = 0$ with the conditions on the coefficients of $\mathbf{h}(\omega)$ given in (2.41). The filter corresponding to $\theta_s = \psi_m$ is easily obtained by simply permuting the coefficients of the filter designed for $\theta_s = 0$.

2.3 Gain in Signal-to-Noise Ratio (SNR)

We recall that the desired signal comes from the angle θ_s . In this case, the m th microphone signal is given by

$$Y_m(\omega) = e^{j\omega \cos(\theta_s - \psi_m)} X(\omega) + V_m(\omega), \quad m = 1, 2, \dots, M, \quad (2.43)$$

where $X(\omega)$ is the desired signal and $V_m(\omega)$ is the additive noise at the m th microphone. In a vector form, (2.43) becomes

$$\begin{aligned}\mathbf{y}(\omega) &= [Y_1(\omega) Y_2(\omega) \cdots Y_M(\omega)]^T \\ &= \mathbf{d}(\omega, \theta_s) X(\omega) + \mathbf{v}(\omega),\end{aligned}\quad (2.44)$$

where $\mathbf{d}(\omega, \theta_s)$ is the steering vector at $\theta = \theta_s$ and the noise signal vector, $\mathbf{v}(\omega)$, is defined similarly to $\mathbf{y}(\omega)$.

The beamformer output is simply

$$\begin{aligned}Z(\omega) &= \sum_{m=1}^M H_m^*(\omega, \theta_s) Y_m(\omega) \\ &= \mathbf{h}^H(\omega, \theta_s) \mathbf{y}(\omega) \\ &= \mathbf{h}^H(\omega, \theta_s) \mathbf{d}(\omega, \theta_s) X(\omega) + \mathbf{h}^H(\omega, \theta_s) \mathbf{v}(\omega),\end{aligned}\quad (2.45)$$

where $Z(\omega)$ is an estimate of the desired signal, $X(\omega)$.

If we consider the first microphone as the reference, we can define the input signal-to-noise ratio (SNR) with respect to this reference as

$$\text{iSNR}(\omega) = \frac{\phi_X(\omega)}{\phi_{V_1}(\omega)}, \quad (2.46)$$

where $\phi_X(\omega) = E[|X(\omega)|^2]$ and $\phi_{V_1}(\omega) = E[|V_1(\omega)|^2]$ are the variances of $X(\omega)$ and $V_1(\omega)$, respectively, with $E[\cdot]$ denoting mathematical expectation.

The output SNR is defined as

$$\begin{aligned}\text{oSNR}[\mathbf{h}(\omega, \theta_s)] &= \phi_X(\omega) \frac{|\mathbf{h}^H(\omega, \theta_s) \mathbf{d}(\omega, \theta_s)|^2}{\mathbf{h}^H(\omega, \theta_s) \mathbf{\Phi}_{\mathbf{v}}(\omega) \mathbf{h}(\omega, \theta_s)} \\ &= \frac{\phi_X(\omega)}{\phi_{V_1}(\omega)} \times \frac{|\mathbf{h}^H(\omega, \theta_s) \mathbf{d}(\omega, \theta_s)|^2}{\mathbf{h}^H(\omega, \theta_s) \mathbf{\Gamma}_{\mathbf{v}}(\omega) \mathbf{h}(\omega, \theta_s)},\end{aligned}\quad (2.47)$$

where

$$\mathbf{\Phi}_{\mathbf{v}}(\omega) = E[\mathbf{v}(\omega) \mathbf{v}^H(\omega)] \quad (2.48)$$

and

$$\mathbf{\Gamma}_{\mathbf{v}}(\omega) = \frac{\mathbf{\Phi}_{\mathbf{v}}(\omega)}{\phi_{V_1}(\omega)} \quad (2.49)$$

are the correlation and pseudo-coherence matrices of $\mathbf{v}(\omega)$, respectively.

The definition of the gain in SNR is easily derived from the previous definitions, i.e.,

$$\begin{aligned}\mathcal{G}[\mathbf{h}(\omega, \theta_s)] &= \frac{\text{oSNR}[\mathbf{h}(\omega, \theta_s)]}{\text{iSNR}(\omega)} \\ &= \frac{|\mathbf{h}^H(\omega, \theta_s) \mathbf{d}(\omega, \theta_s)|^2}{\mathbf{h}^H(\omega, \theta_s) \mathbf{\Gamma}_{\mathbf{v}}(\omega) \mathbf{h}(\omega, \theta_s)}.\end{aligned}\quad (2.50)$$

Assume that the matrix $\mathbf{\Gamma}_{\mathbf{v}}(\omega)$ is nonsingular. In this case, for any two vectors $\mathbf{h}(\omega, \theta_s)$ and $\mathbf{d}(\omega, \theta_s)$, we have

$$\begin{aligned}|\mathbf{h}^H(\omega, \theta_s) \mathbf{d}(\omega, \theta_s)|^2 &\leq [\mathbf{h}^H(\omega, \theta_s) \mathbf{\Gamma}_{\mathbf{v}}(\omega) \mathbf{h}(\omega, \theta_s)] \\ &\quad \times [\mathbf{d}^H(\omega, \theta_s) \mathbf{\Gamma}_{\mathbf{v}}^{-1}(\omega) \mathbf{d}(\omega, \theta_s)],\end{aligned}\quad (2.51)$$

with equality if and only if $\mathbf{h}(\omega, \theta_s) \propto \mathbf{\Gamma}_{\mathbf{v}}^{-1}(\omega) \mathbf{d}(\omega, \theta_s)$. Using the inequality (2.51) in (2.50), we deduce an upper bound for the gain:

$$\begin{aligned}\mathcal{G}[\mathbf{h}(\omega, \theta_s)] &\leq \mathbf{d}^H(\omega, \theta_s) \mathbf{\Gamma}_{\mathbf{v}}^{-1}(\omega) \mathbf{d}(\omega, \theta_s) \\ &\leq \text{tr}[\mathbf{\Gamma}_{\mathbf{v}}^{-1}(\omega)] \text{tr}[\mathbf{d}(\omega, \theta_s) \mathbf{d}^H(\omega, \theta_s)] \\ &\leq M \text{tr}[\mathbf{\Gamma}_{\mathbf{v}}^{-1}(\omega)],\end{aligned}\quad (2.52)$$

where $\text{tr}[\cdot]$ is the trace of a square matrix. We observe how the gain is upper bounded [as long as $\mathbf{\Gamma}_{\mathbf{v}}(\omega)$ is nonsingular] and depends on the number of microphones as well as on the nature of the noise.

In our context, the distortionless constraint is desired, i.e.,

$$\mathbf{h}^H(\omega, \theta_s) \mathbf{d}(\omega, \theta_s) = 1. \quad (2.53)$$

As a consequence, it is easy to see that the filter:

$$\mathbf{h}_{\max}(\omega, \theta_s) = \frac{\mathbf{\Gamma}_{\mathbf{v}}^{-1}(\omega) \mathbf{d}(\omega, \theta_s)}{\mathbf{d}^H(\omega, \theta_s) \mathbf{\Gamma}_{\mathbf{v}}^{-1}(\omega) \mathbf{d}(\omega, \theta_s)} \quad (2.54)$$

maximizes the gain, which is given by

$$\mathcal{G}[\mathbf{h}_{\max}(\omega, \theta_s)] = \mathbf{d}^H(\omega, \theta_s) \mathbf{\Gamma}_{\mathbf{v}}^{-1}(\omega) \mathbf{d}(\omega, \theta_s). \quad (2.55)$$

We are interested in three types of noise.

- The temporally and spatially white noise with the same variance at all microphones⁶. In this case, $\mathbf{\Gamma}_{\mathbf{v}}(\omega) = \mathbf{I}_M$. Therefore, the white noise gain is

$$\begin{aligned}\mathcal{G}_{\text{wn}}[\mathbf{h}(\omega, \theta_s)] &= \frac{|\mathbf{h}^H(\omega, \theta_s) \mathbf{d}(\omega, \theta_s)|^2}{\mathbf{h}^H(\omega, \theta_s) \mathbf{h}(\omega, \theta_s)} \\ &= \frac{1}{\mathbf{h}^H(\omega, \theta_s) \mathbf{h}(\omega, \theta_s)},\end{aligned}\quad (2.56)$$

⁶ This noise models the sensor noise.

where in the second line of (2.56), the distortionless constraint is assumed. With the delay-and-sum beamformer:

$$\mathbf{h}_{\text{DS}}(\omega, \theta_s) = \frac{\mathbf{d}(\omega, \theta_s)}{M}, \quad (2.57)$$

we find the maximum possible white noise gain, which is

$$\mathcal{G}_{\text{wn,max}}(\omega) = M. \quad (2.58)$$

In general, the white noise gain of an N th-order CDMA is

$$\mathcal{G}_{\text{wn},N}[\mathbf{h}(\omega, \theta_s)] = \frac{1}{\mathbf{h}^H(\omega, \theta_s) \mathbf{h}(\omega, \theta_s)} \leq M. \quad (2.59)$$

We will see how the white noise may be amplified by CDMA's, especially in low frequencies.

- The diffuse noise⁷:

$$\begin{aligned} [\mathbf{\Gamma}_{\mathbf{v}}(\omega)]_{ij} &= [\mathbf{\Gamma}_{\text{dn}}(\omega)]_{ij} \\ &= \text{sinc}\left(\frac{\omega \delta_{ij}}{c}\right), \end{aligned} \quad (2.60)$$

where

$$\text{sinc}(x) = \frac{\sin x}{x} \quad (2.61)$$

and

$$\delta_{ij} = 2r \left| \sin \left[\frac{\pi(i-j)}{M} \right] \right| \quad (2.62)$$

is the distance between microphones i and j . In this scenario, the gain in SNR, $\mathcal{G}_{\text{dn}}[\mathbf{h}(\omega, \theta_s)]$, is called the directivity factor and the directivity index is simply defined as [2], [4]

$$\mathcal{D}[\mathbf{h}(\omega, \theta_s)] = 10 \log_{10} \mathcal{G}_{\text{dn}}[\mathbf{h}(\omega, \theta_s)]. \quad (2.63)$$

With diffuse noise, the filter $\mathbf{h}(\omega, \theta_s)$ is often found by maximizing the directivity factor. As a result, the optimal filter is given in (2.54) by simply replacing $\mathbf{\Gamma}_{\mathbf{v}}(\omega)$ with $\mathbf{\Gamma}_{\text{dn}}(\omega)$. For a ULA, this filter corresponds to the hypercardioid of order $M-1$, but not for a UCA. We will get back to this point in Chapter 6.

- The noise comes from a point source at the angle θ_n . In this case, the pseudo-coherence matrix is

⁷ This situation corresponds to the spherically isotropic noise field.

$$\mathbf{\Gamma}_{\mathbf{v}}(\omega) = \mathbf{d}(\omega, \theta_n) \mathbf{d}^H(\omega, \theta_n), \quad (2.64)$$

where $\mathbf{d}(\omega, \theta_n)$ is the steering vector of the noise source. We observe from (2.64) that the pseudo-coherence matrix is singular. In fact, this is the only possibility where the gain in SNR, $\mathcal{G}_{\text{ns}}[\mathbf{h}(\omega, \theta_s)]$, is not upper bounded and can go to infinity. We deduce that this gain is

$$\begin{aligned} \mathcal{G}_{\text{ns}}[\mathbf{h}(\omega, \theta_s)] &= \frac{|\mathbf{h}^H(\omega, \theta_s) \mathbf{d}(\omega, \theta_s)|^2}{|\mathbf{h}^H(\omega, \theta_s) \mathbf{d}(\omega, \theta_n)|^2} \\ &= \frac{1}{|\mathbf{h}^H(\omega, \theta_s) \mathbf{d}(\omega, \theta_n)|^2}. \end{aligned} \quad (2.65)$$

When the noise and desired signals come from the same direction, i.e., when $\theta_n = \theta_s$, then there is no possible gain, i.e., $\mathcal{G}_{\text{ns}}[\mathbf{h}(\omega, \theta_s)] = 1$, $\forall \mathbf{h}(\omega, \theta_s)$. We also deduce the gain of an N th-order DMA:

$$\mathcal{G}_{\text{ns},N}(\theta_n - \theta_s) = \frac{1}{|\mathcal{B}_N(\theta_n - \theta_s)|^2}. \quad (2.66)$$

The gain $\mathcal{G}_{\text{ns}}[\mathbf{h}(\omega, \theta_s)]$ or $\mathcal{G}_{\text{ns},N}(\theta_n - \theta_s)$ depends only on the beam pattern; so the plot of this gain as a function of θ_n is equivalent to the polar pattern.

We now give an important property concerning $\mathbf{\Gamma}_{\text{dn}}(\omega)$.

Property 2.4. The diffuse noise pseudo-coherence matrix is a circulant matrix.

Proof. We recall that, by its own definition, $\mathbf{\Gamma}_{\text{dn}}(\omega)$ is symmetric and Toeplitz. A Toeplitz matrix is circulant if and only if the elements of $\mathbf{\Gamma}_{\text{dn}}(\omega)$ at the lines 1, 2, ..., $M-1$ and last column are equal to the elements at the lines 2, 3, ..., M and first column, i.e.,

$$[\mathbf{\Gamma}_{\text{dn}}(\omega)]_{mM} = [\mathbf{\Gamma}_{\text{dn}}(\omega)]_{(m+1)1}, \quad m = 1, 2, \dots, M-1. \quad (2.67)$$

This is true if we can show that

$$\delta_{mM} = \delta_{(m+1)1}, \quad m = 1, 2, \dots, M-1. \quad (2.68)$$

This is, indeed, straightforward to verify by using the trigonometric identity:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta. \quad (2.69)$$

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