

# Chapter 2

## Boolean and Superboolean Matrices

We introduce in this chapter the superboolean semiring  $\mathbb{SB}$  and the core of the theory of (boolean) matrices over  $\mathbb{SB}$ , with special emphasis on the concepts of independence of vectors and rank. These matrices are used to represent various kinds of algebraic and combinatorial objects, namely posets and simplicial complexes, especially matroids.

### 2.1 The Boolean and the Superboolean Semirings

A *commutative semiring* is an algebra  $(S, +, \cdot, 0, 1)$  of type  $(2, 2, 0, 0)$  satisfying the following properties:

- (CS1)  $(S, +, 0)$  and  $(S, \cdot, 1)$  are commutative monoids;
- (CS2)  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$  for all  $a, b, c \in S$ ;
- (CS3)  $a \cdot 0 = 0$  for every  $a \in S$ .

To avoid trivial cases, we assume that  $1 \neq 0$ . If the operations are implicit, we denote this semiring simply by  $S$ .

If we only require commutativity for the monoid  $(S, +, 0)$  and use both left-right versions of (CS2) and (CS3), we have the general concept of *semiring*.

Clearly, commutative semirings constitute a variety of algebras of type  $(2, 2, 0, 0)$ , and so universal algebra provides the concepts of congruence, homomorphism and subsemiring. In particular, an equivalence relation  $\sigma$  on  $S$  is said to be a *congruence* if

$$(a\sigma b \wedge a'\sigma b') \Rightarrow ((a + a')\sigma(b + b') \wedge (a \cdot a')\sigma(b \cdot b')).$$

In this case, we get induced operations on  $S/\sigma = \{a\sigma \mid a \in S\}$  through

$$a\sigma + b\sigma = (a + b)\sigma, \quad a\sigma \cdot b\sigma = (a \cdot b)\sigma.$$

If  $\sigma$  is not the universal relation, then  $1\sigma \neq 0\sigma$  and so  $(S/\sigma, +, \cdot, 0\sigma, 1\sigma)$  is also a commutative semiring, the *quotient* of  $S$  by  $\sigma$ .

The natural numbers  $\mathbb{N} = \{0, 1, 2, \dots\}$  under the usual addition and multiplication provide a most important example of a commutative semiring. For every  $m \in \mathbb{N}$ , we define a relation  $\sigma_m$  on  $\mathbb{N}$  through

$$a\sigma_m b \quad \text{if } a = b \text{ or } a, b \geq m.$$

Then  $\sigma_m$  is a congruence on  $\mathbb{N}$  and

$$n\sigma_m = \begin{cases} \{n\} & \text{if } n < m \\ \{m, m+1, \dots\} & \text{otherwise} \end{cases}$$

Hence  $\{0, \dots, m\}$  is a cross-section for  $\mathbb{N}/\sigma_m$ .

We can define the *boolean semiring* as the quotient

$$\mathbb{B} = \mathbb{N}/\sigma_1.$$

As usual, we view the elements of  $\mathbb{B}$  as the elements of the cross-section  $\{0, 1\}$ . Addition and multiplication are then described respectively by

$$\begin{array}{c|ccc} + & 0 & 1 & \\ \hline 0 & 0 & 1 & \\ 1 & 1 & 1 & \end{array} \qquad \begin{array}{c|ccc} \cdot & 0 & 1 & \\ \hline 0 & 0 & 0 & \\ 1 & 0 & 1 & \end{array}$$

Similarly, we can define the *superboolean semiring* as the quotient

$$\mathbb{SB} = \mathbb{N}/\sigma_2.$$

We can view the elements of  $\mathbb{SB}$  as the elements of the cross-section  $\{0, 1, 2\}$ . Addition and multiplication are then described respectively by

$$\begin{array}{c|cccc} + & 0 & 1 & 2 & \\ \hline 0 & 0 & 1 & 2 & \\ 1 & 1 & 2 & 2 & \\ 2 & 2 & 2 & 2 & \end{array} \qquad \begin{array}{c|cccc} \cdot & 0 & 1 & 2 & \\ \hline 0 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 2 & \\ 2 & 0 & 2 & 2 & \end{array}$$

Since  $1 + 1$  takes different values in both semirings, it follows that  $\mathbb{B}$  is not a subsemiring of  $\mathbb{SB}$ . However,  $\mathbb{B}$  is a homomorphic image of  $\mathbb{SB}$  (through the canonical mapping  $n\sigma_2 \mapsto n\sigma_1$ ).

For an alternative perspective of  $\mathbb{SB}$  as a supertropical semiring, the reader is referred to Sect. A.1 of the Appendix.

## 2.2 Superboolean Matrices

Given a semiring  $S$ , we denote by  $\mathcal{M}_{m \times n}(S)$  the set of all  $m \times n$  matrices with entries in  $S$ . We write also  $\mathcal{M}_n(S) = \mathcal{M}_{n \times n}(S)$ . Addition and multiplication are defined as usual.

Given  $M = (a_{ij}) \in \mathcal{M}_{m \times n}(S)$  and nonempty  $I \subseteq \{1, \dots, m\}$ ,  $J \subseteq \{1, \dots, n\}$ , we denote by  $M[I, J]$  the submatrix of  $M$  with entries  $a_{ij}$  ( $i \in I$ ,  $j \in J$ ). For all  $i \in \{1, \dots, m\}$  and  $j \in \{1, \dots, n\}$ , we write also

$$M[\bar{i}, \bar{j}] = M[\{1, \dots, m\} \setminus \{i\}, \{1, \dots, n\} \setminus \{j\}].$$

Finally, we denote by  $M[i, \_]$  the  $i$ th row vector of  $M$ , and by  $M[\_, j]$  the  $j$ th column vector of  $M$ .

The results we present in this section are valid for more general semirings (any *supertropical semifield*, actually, see [25, 31] and Sect. A.1 in the Appendix), but we shall discuss only the concrete case of  $\mathbb{SB}$ .

Let  $S_n$  denote the symmetric group on  $\{1, \dots, n\}$ . The *permanent* of a square matrix  $M = (m_{ij}) \in \mathcal{M}_n(\mathbb{SB})$  (a positive version of the determinant) is defined by

$$\text{Per } M = \sum_{\pi \in S_n} \prod_{i=1}^n m_{i, \pi(i)}.$$

Note that this formula coincides with the formula for the determinant of a square matrix over the two-element field  $\mathbb{Z}_2$  (but interpreting the operations in  $\mathbb{SB}$ ). The classical results on determinants involving only a rearrangement of the permutations extend naturally to  $\mathbb{SB}$ . Therefore we can state the two following propositions without proof:

**Proposition 2.2.1.** *Let  $M = (m_{ij}) \in \mathcal{M}_n(\mathbb{SB})$  and let  $p \in \{1, \dots, n\}$ . Then*

$$\text{per } M = \sum_{j=1}^n m_{pj} (\text{per } M[\bar{p}, \bar{j}]) = \sum_{i=1}^n m_{ip} (\text{per } M[\bar{i}, \bar{p}]).$$

**Proposition 2.2.2.** *The permanent of a square superboolean matrix remains unchanged by:*

- (i) *Permuting two columns;*
- (ii) *Permuting two rows;*
- (iii) *Transposition.*

Next we present definitions of independence and rank appropriate to the context of superboolean matrices, introduced by Izhakian in [25] (see also [28, 31]). For alternative notions in the context of semirings, see [15]. We need to introduce the *ghost ideal*

$$\mathcal{G} = \{0, 2\} \subseteq \mathbb{SB}$$

(see Sect. A.1 for more details on ghost ideals).

Let  $\mathbb{SB}^n$  denote the set of all vectors  $V = (v_1, \dots, v_n)$  with entries in  $\mathbb{SB}$ . Addition and the scalar product  $\mathbb{SB} \times \mathbb{SB}^n \rightarrow \mathbb{SB}^n$  are defined the obvious way.

We say that the vectors  $V^{(1)}, \dots, V^{(m)} \in \mathbb{SB}^n$  are *independent* if

$$\lambda_1 V^{(1)} + \dots + \lambda_m V^{(m)} \in \mathcal{G}^n \quad \text{implies} \quad \lambda_1, \dots, \lambda_m = 0$$

for all  $\lambda_1, \dots, \lambda_m \in \{0, 1\}$ . Otherwise, they are said to be *dependent*. The contrapositive yields that  $V^{(1)}, \dots, V^{(m)}$  are dependent if and only if there exists some nonempty  $I \subseteq \{1, \dots, m\}$  such that  $\sum_{i \in I} V^{(i)} \in \mathcal{G}^n$ .

The next lemma discusses independence when we extend the vectors by one further component:

**Lemma 2.2.3.** *Let  $X = \{V^{(1)}, \dots, V^{(m)}\} \subseteq \mathbb{SB}^n$  and  $Y = \{W^{(1)}, \dots, W^{(m)}\} \subseteq \mathbb{SB}^{n+1}$  be such that  $V^{(i)} = (v_1^{(i)}, \dots, v_n^{(i)})$  and  $W^{(i)} = (a^{(i)}, v_1^{(i)}, \dots, v_n^{(i)})$  for  $i \in \{1, \dots, m\}$ . Then:*

- (i) *If  $X$  is independent, so is  $Y$ ;*
- (ii) *If  $a^{(1)}, \dots, a^{(m)} \in \mathcal{G}$ , then  $X$  is independent if and only if  $Y$  is independent.*

*Proof.* (i) Assume that  $X$  is independent. Let  $\lambda_1, \dots, \lambda_m \in \{0, 1\}$  be such that  $\lambda_1 W^{(1)} + \dots + \lambda_m W^{(m)} \in \mathcal{G}^{n+1}$ . Then  $\lambda_1 V^{(1)} + \dots + \lambda_m V^{(m)} \in \mathcal{G}^n$  and so  $\lambda_1 = \dots = \lambda_m = 0$  since  $X$  is independent. Thus  $Y$  is independent.

- (ii) The direct implication follows from (i). Assume now that  $Y$  is independent. Let  $\lambda_1, \dots, \lambda_m \in \{0, 1\}$  be such that  $\lambda_1 V^{(1)} + \dots + \lambda_m V^{(m)} \in \mathcal{G}^n$ . Since  $a^{(1)}, \dots, a^{(m)} \in \mathcal{G}$ , we get  $\lambda_1 W^{(1)} + \dots + \lambda_m W^{(m)} \in \mathcal{G}^{n+1}$  and so  $\lambda_1 = \dots = \lambda_m = 0$  since  $Y$  is independent. Thus  $X$  is independent.  $\square$

We start now to address independence in the context of a matrix. Two matrices  $M$  and  $M'$  are said to be *congruent* and we write  $M \cong M'$  if we can transform one into the other by permuting rows and permuting columns independently. A row of a superboolean matrix is called a *marker* if it has one entry 1 and all the remaining entries are 0.

**Lemma 2.2.4 ([28, Cor. 3.4]).** *Let  $M \in \mathcal{M}_{m \times n}(\mathbb{SB})$  be such that the column vectors  $M[\_, j]$  ( $j \in \{1, \dots, n\}$ ) are independent. Then  $M$  has a marker.*

*Proof.* Let  $M = (a_{ij})$ . By independence, we must have

$$\sum_{j=1}^n M[\_, j] \notin \mathcal{G}^n.$$

Hence  $a_{i1} + \dots + a_{in} = 1$  for some  $i \in \{1, \dots, m\}$  and so the  $i$ th row of  $M$  is a marker.  $\square$

The following result discusses independence for the row/column vectors of a square superboolean matrix. The equivalence of the three first conditions is due to Izhakian [25] (see also [31]), the remaining equivalence to Izhakian and

Rhodes [28]. Recall that a square matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ ? & 1 & 0 & \dots & 0 \\ ? & ? & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ? & ? & ? & \dots & 1 \end{pmatrix}$$

is called *lower unitriangular*.

**Proposition 2.2.5** ([25, Th. 2.10], [28, Lemma 3.2]). *The following conditions are equivalent for every  $M \in \mathcal{M}_n(\mathbb{SB})$ :*

- (i) *The column vectors  $M[\_, j]$  ( $j \in \{1, \dots, n\}$ ) are independent;*
- (ii) *The row vectors  $M[i, \_]$  ( $i \in \{1, \dots, n\}$ ) are independent;*
- (iii)  *$\text{Per } M = 1$ ;*
- (iv)  *$M$  is congruent to some lower unitriangular matrix.*

*Proof.* (i)  $\Rightarrow$  (iv). We use induction on  $n$ . Since the implication holds trivially for  $n = 1$ , we assume that  $n > 1$  and the implication holds for  $n - 1$ . Assuming (i), it follows from Lemma 2.2.4 that  $M$  has a marker. By permuting the rows of  $M$  if needed, we may assume that the first row is a marker. By permuting columns if needed, we may assume that  $M[1, \_] = (1, 0, \dots, 0)$ . Let  $N = M[\bar{1}, \bar{1}]$ . Then the column vectors  $N[\_, j]$  ( $j \in \{1, \dots, n - 1\}$ ) are the column vectors  $M[\_, j]$  ( $j \in \{2, \dots, n\}$ ) with the first coordinate removed. Since this first coordinate is always 0, it follows from Lemma 2.2.3(ii) that the column vectors of  $N$  are independent. By the induction hypothesis,  $N$  is congruent to some lower unitriangular matrix  $N'$ , i.e. we can apply some sequence of row/column permutations to  $N$  to get  $N'$ . Now if we apply the same sequence of row/column permutations to the matrix  $M$  the first row of  $M$  remains unchanged, hence we obtain a lower unitriangular matrix as required.

(iv)  $\Rightarrow$  (iii). If  $M = (m_{ij})$  is lower unitriangular, then the unique  $\pi \in S_n$  such that  $\prod_{i=1}^n m_{i,i\pi}$  is nonzero is the identity permutation. Hence  $\text{Per } M = \prod_{i=1}^n m_{i,i} = 1$ . Finally, we apply Proposition 2.2.2.

(iii)  $\Rightarrow$  (i). We use induction on  $n$ . Since the implication holds trivially for  $n = 1$ , we assume that  $n > 1$  and the implication holds for  $n - 1$ . Note that, by Proposition 2.2.2, permuting rows or columns does not change the permanent. Clearly, the same happens with respect to the dependence of the column vectors.

Suppose first that  $M$  has no marker. Since  $M$  cannot have a row consisting only of zeroes in view of  $\text{Per } M = 1$ , we have at least two nonzero entries in each row of  $M$ . Since  $\text{Per } M = 1$ , we may also assume, (independently) permuting rows and columns if necessary, that  $M$  has no zero entries on the main diagonal.

We build a directed graph  $\Gamma = (V, E)$  with vertex set  $V = \{1, \dots, n\}$  and edges  $i \rightarrow j$  whenever  $m_{ij} \neq 0$ . By our assumption on the main diagonal, we have a loop at each vertex  $i$ . Moreover, each vertex  $i$  must have outdegree at least two, each of

the nonzero entries  $m_{ij}$  in the  $i$ th row producing an edge  $i \longrightarrow j$ . It follows that  $\Gamma$  must have a cycle

$$i_0 \longrightarrow i_1 \longrightarrow \dots \longrightarrow i_k = i_0$$

of length  $k \geq 2$  and so  $S_n$  contains two different permutations  $\pi$ , namely the identity and  $(i_0 i_1 \dots i_{k-1})$ , such that  $m_{i,i\pi} \neq 0$  for every  $i \in \{1, \dots, n\}$ . Hence  $\text{Per } M = 2$ , a contradiction.

Hence we may assume that  $M = (m_{ij})$  has a marker. Permuting rows and columns if necessary, we may indeed assume that  $M[1, \_ ] = (1, 0, \dots, 0)$ . Let  $N = M[\bar{1}, \bar{1}]$ . Then Proposition 2.2.1 yields  $\text{Per } N = \text{Per } M = 1$ . By the induction hypothesis, the column vectors of  $N$  are independent. Suppose that

$$\lambda_1 M[\_, 1] + \dots + \lambda_n M[\_, n] \in \mathcal{G}^n$$

for some  $\lambda_1, \dots, \lambda_n \in \{0, 1\}$ . Since  $M[1, \_ ]$  is a marker, we get  $\lambda_1 = 0$ . Since the column vectors of  $N$  are independent, we get  $\lambda_2 = \dots = \lambda_n = 0$  as well. Thus the column vectors of  $M$  are independent.

(ii)  $\Leftrightarrow$  (iii). Let  $M'$  denote the transpose matrix of  $M$ . By Proposition 2.2.2(iii), we have

$$\text{Per } M = 1 \quad \Leftrightarrow \quad \text{Per } M' = 1.$$

On the other hand, (ii) is equivalent to the column vectors  $M'[\_, j]$  ( $j \in \{1, \dots, n\}$ ) being independent. Now we use the equivalence (i)  $\Leftrightarrow$  (iii).  $\square$

A square matrix satisfying the above (equivalent) conditions is said to be *nonsingular*.

We consider now independence for any arbitrary nonempty subset of column vectors. Given (equipotent) nonempty  $I, J \subseteq \{1, \dots, n\}$ , we say that  $I$  is a *witness* for  $J$  in  $M$  if  $M[I, J]$  is nonsingular.

**Proposition 2.2.6 ([25, Th. 3.11]).** *The following conditions are equivalent for all  $M \in \mathcal{M}_{m \times n}(\mathbb{S}\mathbb{B})$  and  $J \subseteq \{1, \dots, n\}$  nonempty:*

- (i) *The column vectors  $M[\_, j]$  ( $j \in J$ ) are independent;*
- (ii)  *$J$  has a witness in  $M$ .*

*Proof.* (i)  $\Rightarrow$  (ii). We use induction on  $|J|$ . Since the implication holds trivially for  $|J| = 1$ , we assume that  $|J| > 1$  and the implication holds for smaller sets.

Applying Lemma 2.2.4 to the matrix  $M' = M[\{1, \dots, m\}, J]$ , it follows that  $M'$  has a marker. In view of Proposition 2.2.2, (independently) permuting rows and columns does not compromise the existence of a witness, hence we may assume that  $M'[1, \_ ] = (1, 0, \dots, 0)$  and  $j_1$  is the element of  $J$  corresponding to the first column of  $M'$ . Let  $N = M'[\bar{1}, \bar{1}]$ . Then the column vectors  $N[\_, j]$  ( $j \in \{1, \dots, |J| - 1\}$ ) are the column vectors  $M'[\_, j]$  ( $j \in \{2, \dots, |J|\}$ ) with the first coordinate removed. Since this first coordinate is always 0, it follows from

Lemma 2.2.3(ii) that the column vectors of  $N$  are independent. By the induction hypothesis,  $J \setminus \{j_1\}$  has some witness  $I$  in  $N$ . Write  $P = N[I, J \setminus \{j_1\}]$ . Then  $M[I \cup \{1\}, J]$  is of the form

$$\begin{pmatrix} 1 & 0 \\ \dots & P \end{pmatrix}$$

and so  $\text{Per } M[I \cup \{1\}, J] = \text{Per } P = 1$ . Hence  $I \cup \{1\}$  is a witness for  $J$  in  $M$ .

(ii)  $\Rightarrow$  (i). Assume that  $I$  is a witness for  $J$  in  $M$ . Let  $N = M[I, J]$ . By Proposition 2.2.5, the column vectors  $N[\_, j]$  ( $j \in \{1, \dots, |J|\}$ ) are independent. Thus the vectors  $M[\_, j]$  ( $j \in J$ ) are independent by Lemma 2.2.3(i).  $\square$

We can deduce a corollary on boolean matrices which will become useful in future chapters:

**Corollary 2.2.7.** *Let  $M \in \mathcal{M}_{m \times n}(\mathbb{B})$  and let  $M' \in \mathcal{M}_{(m+1) \times n}(\mathbb{B})$  be obtained by adding as an extra row the sum (in  $\mathbb{B}$ ) of two rows of  $M$ . Then the following conditions are equivalent for every  $J \subseteq \{1, \dots, n\}$ :*

- (i) *The column vectors  $M[\_, j]$  ( $j \in J$ ) are independent;*
- (ii) *The column vectors  $M'[\_, j]$  ( $j \in J$ ) are independent.*

*Proof.* (i)  $\Rightarrow$  (ii). By Lemma 2.2.3(i).

(ii)  $\Rightarrow$  (i). By Proposition 2.2.6,  $J$  has a witness  $I$  in  $M'$ . It is easy to see that if a marker  $u$  is the sum of some vectors in  $\mathbb{B}^k$ , then one of them is equal to  $u$ . Therefore, if the sum row occurs in  $M'[I, J]$ , we can always replace it by one of the summand rows and get a nonsingular matrix of the form  $M[K, J]$ .  $\square$

We are now ready to introduce the notion of rank of a superboolean matrix:

**Proposition 2.2.8 ([25, Th. 3.11]).** *The following numbers coincide for a given  $M \in \mathcal{M}_{m \times n}(\mathbb{SB})$ :*

- (i) *The maximum number of independent column vectors in  $M$ ;*
- (ii) *The maximum number of independent row vectors in  $M$ ;*
- (iii) *The maximum size of a subset  $J \subseteq \{1, \dots, n\}$  having a witness in  $M$ ;*
- (iv) *The maximum size of a nonsingular submatrix of  $M$ .*

*Proof.* Let  $M_{\iota_1}, \dots, M_{\iota_4}$  denote the integers defined by each of the conditions (i)–(iv) for  $M$ , respectively. The equality  $M_{\iota_3} = M_{\iota_4}$  follows from the definition of witness, and  $M_{\iota_1} = M_{\iota_3}$  follows from Proposition 2.2.6. Finally,  $M_{\iota_2} = M^{\iota_1} = M^{\iota_4}$ . Since  $M^{\iota_4} = M_{\iota_4}$  in view of Proposition 2.2.2(iii), we get  $M_{\iota_2} = M_{\iota_4}$ .  $\square$

The *rank* of a superboolean matrix  $M$ , denoted by  $\text{rk } M$ , is then the number given by any of the equivalent conditions of Proposition 2.2.8.

If  $M$  is a boolean matrix, we can still define  $\text{rk } M$  as its rank when viewed as a superboolean matrix. We note also that this notion of rank for boolean matrices does not coincide with the definition used by Berstel, Perrin and Reutenauer in [2, Section VI.3].

Boolean Representations of Simplicial Complexes and  
Matroids

Rhodes, J.; Silva, P.V.

2015, X, 173 p. 36 illus., Hardcover

ISBN: 978-3-319-15113-7