Chapter 2
Foundation: Passivity, Stability and Passivity-Based Motion Control

Abstract This chapter provides foundations not only for bilateral teleoperation but also for all of the subsequent chapters. Passivity, stability of dynamical systems, and several passivity-based motion control schemes are introduced.

In this chapter, we first introduce the notion of passivity, and then briefly review its close relation to stability. Subsequently, several representative passivity-based motion control laws are then presented. Remark that these contents form the foundation not only for Part I but also for the subsequent Parts II and III.

2.1 Passivity

This section defines passivity, which is a central concept of this book, and points out its preservation with respect to feedback and parallel interconnections. The contents of this section are based on the books [19, 39, 279, 297, 318].

2.1.1 Definition of Passivity

Let us first consider a system H illustrated in Fig. 2.1 with input $u(t) \in \mathbb{R}^p$ and output $y(t) \in \mathbb{R}^p$, $t \in \mathbb{R}_+$, $\mathbb{R}_+ := [0, \infty)$, where H is regarded as a mapping from an input signal space $\mathcal{U}$ to an output signal space $\mathcal{Y}$. Then, passivity of the map is defined as below.

Definition 2.1 The system $H : \mathcal{U} \to \mathcal{Y}$ with input $u \in \mathcal{U}$ and output $y \in \mathcal{Y}$ is said to be *passive* if there exists a constant $\beta \geq 0$ such that

$$\int_0^\tau y^T(t)u(t)dt \geq -\beta$$

for all input signals $u \in \mathcal{U}$ and for all $\tau \in \mathbb{R}_+$. In addition, H is said to be

T. Hatanaka et al., *Passivity-Based Control and Estimation in Networked Robotics*,
Communications and Control Engineering, DOI 10.1007/978-3-319-15171-7_2

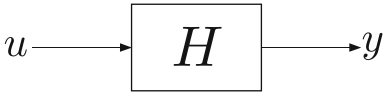


Fig. 2.1 A system from input u to output y

- *input strictly passive* if there exists a scalar $\delta_u > 0$ such that

$$\int_0^{\tau} y^T(t)u(t)dt \geq -\beta + \delta_u \int_0^{\tau} \|u(t)\|^2 dt,$$

- *output strictly passive* if there exists a scalar $\delta_y > 0$ such that

$$\int_0^{\tau} y^T(t)u(t)dt \geq -\beta + \delta_y \int_0^{\tau} \|y(t)\|^2 dt$$

for all input signals $u \in \mathcal{U}$ and for all $\tau \in \mathbb{R}_+$.[1] □

Remark that passivity is also defined for a static map $h : \mathbb{R}^p \to \mathbb{R}^p$. In the case, a static map $h : \mathbb{R}^p \to \mathbb{R}^p$ is passive if the inequality $(h(u))^T u \geq 0$ holds for all $u \in \mathbb{R}^p$. It is easily confirmed that the linear map $h(u) = ku$ with $k > 0$ is (both input- and output-) strictly passive.

Let us next introduce another system representation, namely the state space model

$$\dot{x} = f(x, u),\ x(0) = x_0 \in \mathbb{R}^n, \tag{2.1a}$$

$$y = h(x, u), \tag{2.1b}$$

with input vector $u(t) \in \mathbb{R}^p$, output vector $y(t) \in \mathbb{R}^p$, and the state vector $x(t) \in \mathbb{R}^n$. The input signal $u : \mathbb{R}_+ \to \mathbb{R}^p$ is assumed to be piecewise continuous in t and to be bounded for all $t \in \mathbb{R}_+$. Throughout this book, we are interested only in the case where the state trajectory satisfying (2.1a) is uniquely determined by the initial state x_0 and the input signal.[2] We also assume that the system (2.1) has an equilibrium at the origin, namely $f(0, 0) = 0$, and also that $h(0, 0) = 0$.

Passivity of the system (2.1) is defined as follows:

Definition 2.2 The system (2.1) from input u to output y is said to be *passive* if there exists a positive semidefinite function[3] $S : \mathbb{R}^n \to \mathbb{R}_+$, called *storage function*, such that

---

[1] Throughout this book, the notation $\|x\|$ for a vector x describes the vector 2-norm $\|x\| = \sqrt{x^T x}$ unless otherwise noted.

[2] See [159] for the details on the existence and uniqueness of the solution.

[3] A function $S(x) : \mathbb{R}^n \to \mathbb{R}$ is said to be *positive definite* if $S(0) = 0$ and $S(x) > 0\ \forall x \neq 0$. It is positive semidefinite if $S(0) = 0$ and $S(x) \geq 0\ \forall x \neq 0$ hold. Also, if $-S$ is positive definite (semidefinite), then S is said to be negative definite (semidefinite).

$$S(x(\tau)) - S(x_0) \leq \int_0^\tau y^T(t)u(t)dt \tag{2.2}$$

for all input signals $u : [0, \tau] \to \mathbb{R}^p$, initial states $x_0 \in \mathbb{R}^n$ and $\tau \in \mathbb{R}_+$. In addition, (2.1) is said to be

- *input strictly passive* if there exists a scalar $\delta_u > 0$ such that

$$S(x(\tau)) - S(x_0) \leq \int_0^\tau \left(y^T(t)u(t) - \delta_u \|u(t)\|^2\right) dt, \tag{2.3}$$

- *output strictly passive* if there exists a scalar $\delta_y > 0$ such that

$$S(x(\tau)) - S(x_0) \leq \int_0^\tau \left(y^T(t)u(t) - \delta_y \|y(t)\|^2\right) dt. \tag{2.4}$$

for all input signals $u : [0, \tau] \to \mathbb{R}^p$, initial states $x_0 \in \mathbb{R}^n$ and $\tau \in \mathbb{R}_+$. □

These two definitions are closely related to each other. Indeed, given an initial state x_0, the system (2.1) is known to define a causal[4] input–output mapping $H_{x_0} : \mathcal{U} \to \mathcal{Y}$ [318]. Now, since the term $S(x(\tau))$ in the left-hand side of (2.2) is nonnegative, we have

$$\int_0^\tau y^T(t)u(t)dt \geq -S(x_0).$$

Hence, taking $\beta = S(x_0) \geq 0$, we see that the map H_{x_0} is passive in the sense of Definition 2.1. The same statements hold for both of the input and output strict passivity.

In this book, we mainly focus on the system representation (2.1) rather than the input–output map, and hence mainly take the passivity of Definition 2.2 except for a portion of Part I. When the storage function $S(x)$ is continuously differentiable, the condition (2.2) can be transformed into another form. For such a function $S(x)$, the time derivative of $S(x(t))$ along the trajectories of (2.1) is given as

$$\dot{S} = \frac{d}{dt}S(x(t)) = \frac{\partial S}{\partial x}\dot{x} = \frac{\partial S}{\partial x} f(x, u),$$

where

$$\frac{\partial S}{\partial x} = \left[\frac{\partial S}{\partial x_1} \cdots \frac{\partial S}{\partial x_n}\right],$$

and $x_i,\ i = 1, \ldots, n$ is the ith element of x. Thus, if the inequality

[4] A map $H : \mathcal{U} \to \mathcal{Y}$ is said to be causal if the output $(H(u))(\tau)$ at any time $\tau \in \mathbb{R}_+$ is dependent only on the past and current profile of input $u(t),\ t \leq \tau$. See [318] for its formal definition.

$$\frac{\partial S}{\partial x} f(x, u) \leq y^T u = h^T(x, u)u \tag{2.5}$$

holds for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^p$, then the system (2.1) is passive, i.e., (2.2) holds. Indeed, it is confirmed that just integrating (2.5) from time 0 to τ yields (2.2). Similarly, the inequalities

$$\begin{aligned}
\frac{\partial S}{\partial x} f(x, u) &\leq y^T u - \delta_u \|u\|^2 \ \ \forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^p, \\
\frac{\partial S}{\partial x} f(x, u) &\leq y^T u - \delta_y \|y\|^2 \ \ \forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^p
\end{aligned} \tag{2.6}$$

for some $\delta_u > 0$ and $\delta_y > 0$ imply (2.3) and (2.4), respectively.

If the storage function S is the total energy of the system, the term $y^T(t)u(t)$ in (2.2) is regarded as the power supplied to the system at time t, and hence the integral $\int_0^\tau y^T(t)u(t)dt$ is as the energy supplied to the system within the time interval $[0, \tau]$. To see the meaning of the statement, let us introduce two typical examples of passive systems.

The first example is an electrical circuit, which gives origin to the word *passivity*. Let us consider the circuit in Fig. 2.2 consisting only of passive elements, a resistance, capacitor, and inductor. Kirchoff's law yields the differential equation

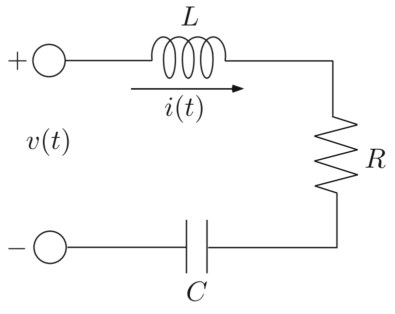
$$v = Ri + \frac{1}{C}\int_0^t i(\tau)d\tau + L\frac{di}{dt}, \tag{2.7}$$

describing the relation between the voltage v and current i. Using the electric charge Q, (2.7) is rewritten as

$$\dot{i} = -\frac{R}{L}i - \frac{1}{LC}Q + \frac{1}{L}v, \quad \dot{Q} = i.$$

Regard the voltage v and current i as the input u and output y of the system, respectively. Then, the system is described as

Fig. 2.2 RLC circuit



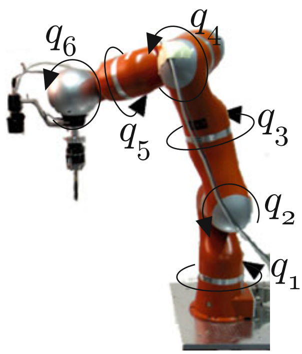


Fig. 2.3 Robot manipulator (Photo courtesy of Institute for Dynamic Systems and Control, ETH Zürich, Switzerland)

$$\dot{x} = -\begin{bmatrix} \frac{R}{L} & \frac{1}{LC} \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u, \quad x = \begin{bmatrix} i \\ Q \end{bmatrix}, \tag{2.8a}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x. \tag{2.8b}$$

Since the energy in the system is composed of the energy stored in the capacitor and the magnetic energy of the inductance, we let their summation be the storage function as

$$S(x) = \frac{1}{2C} Q^2 + \frac{L}{2} i^2 = \frac{1}{2} x^T \begin{bmatrix} L & 0 \\ 0 & \frac{1}{C} \end{bmatrix} x.$$

Then, the time derivative of S along the trajectories of (2.8) is given as

$$\begin{aligned} \dot{S} &= x^T \begin{bmatrix} L & 0 \\ 0 & \frac{1}{C} \end{bmatrix} \left(-\begin{bmatrix} \frac{R}{L} & \frac{1}{LC} \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u \right) \\ &= -x^T \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} x - \frac{1}{C} x^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + x^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ &= -Ry^2 + yu. \end{aligned} \tag{2.9}$$

The last equation holds because of the skew symmetry[5] of the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. We see from (2.6) and (2.9) that the circuit system is output strictly passive from input v to output i with $\delta_y = R$. Note that the product of the input $u = v$ and output $y = i$ is equal to the power in the sense of the circuit systems.

[5] A matrix $A \in \mathbb{R}^{n \times n}$ is said to be *skew symmetric* if $A + A^T = 0$.

We next consider an n-link robot manipulator. An example with $n = 6$ is shown in Fig. 2.3.[6] As seen in [297], the robot dynamics can be written as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau, \tag{2.10}$$

where $q \in \mathbb{R}^n$, $\dot{q} \in \mathbb{R}^n$ and $\ddot{q} \in \mathbb{R}^n$ represent the joint angles, velocities, and accelerations, respectively, and $\tau \in \mathbb{R}^n$ is the vector of the input torque. The matrix $M(q) \in \mathbb{R}^{n\times n}$ is the manipulator inertia matrix, $C(q, \dot{q}) \in \mathbb{R}^{n\times n}$ is the Coriolis matrix and $g(q) = \left(\frac{\partial P(q)}{\partial q}\right)^T \in \mathbb{R}^n$ with the potential energy $P(q)$ is the gravity vector. It is well known that (i) the inertia matrix $M(q)$ is positive definite,[7] and (ii) the matrix

$$\dot{M}(q) - 2C(q, \dot{q})$$

is skew-symmetric by defining $C(q, \dot{q})$ using the Christoffel symbols. See [297] for details on the properties.

Let us take the summation of the kinetic energy and potential energy as the storage function as

$$S(x) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q),\ x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}. \tag{2.11}$$

Then, the time derivative of S along the trajectories of (2.10) satisfies

$$\begin{aligned}
\dot{S} &= \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \left(\frac{\partial P(q)}{\partial q}\right)\dot{q} \\
&= \dot{q}^T \left(\tau - C(q, \dot{q})\dot{q} - g(q)\right) + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + g(q)^T\dot{q} \\
&= \dot{q}^T\tau - \frac{1}{2}\dot{q}^T \left(\dot{M}(q) - 2C(q, \dot{q})\right)\dot{q} = \dot{q}^T\tau.
\end{aligned} \tag{2.12}$$

The last equation holds because of the skew symmetry of $\dot{M}(q) - 2C(q, \dot{q})$. Equation (2.12) means passivity of the manipulator dynamics (2.10) from the input torque τ to the joint velocity $\dot{q}$. The inner product of the torque τ and velocity $\dot{q}$ is equal to the power in the sense of the mechanical systems.

Remark that, in both of the examples, skew symmetry of a matrix plays an important role in proving passivity in (2.9) and (2.12), respectively. Besides these examples, the matrix skew symmetry has been recognized as being closely related to the fundamental property of passivity [297]. In Sect. 5.2, we will present another example of the inseparable relation between passivity and skew symmetry.

[6] The robot actually has one more degree of freedom.

[7] A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *positive definite* if it is symmetric ($A = A^T$) and $x^T Ax > 0\ \forall x \neq 0$.

Before closing this subsection, we finally present one more example of passive systems, namely single integrator described as

$$\dot{x} = u$$
$$y = x$$

with $x, y, u \in \mathbb{R}^n$. Define the storage function $S(x) = \frac{1}{2}\|x\|^2$. Then, the time derivative of S along the system trajectories is given by

$$\dot{S} = x^T \dot{x} = y^T u, \tag{2.13}$$

which means passivity of the system.

2.1.2 Passivity Preservation for Interconnections

One of the most important properties of passivity is its preservation in terms of interconnections. Let us first consider the standard feedback connection illustrated in Fig. 2.4. Here, we suppose that both of subsystems H_1 and H_2 are represented by the state space models

$$H_1 : \begin{array}{l} \dot{x}_1 = f_1(x_1, u_1),\ x_1(0) = x_{10} \in \mathbb{R}^{n_1}, \\ y_1 = h_1(x_1, u_1) \end{array} \tag{2.14a}$$

$$H_2 : \begin{array}{l} \dot{x}_2 = f_2(x_2, u_2),\ x_2(0) = x_{20} \in \mathbb{R}^{n_2}, \\ y_2 = h_2(x_2, u_2) \end{array} \tag{2.14b}$$

with $u_1(t), u_2(t), y_1(t), y_2(t) \in \mathbb{R}^p$. Then, the feedback connection in Fig. 2.4 is described by the equations

$$u_1 = e_1 - y_2,\ u_2 = e_2 + y_1 \tag{2.15}$$

as well as (2.14). Define

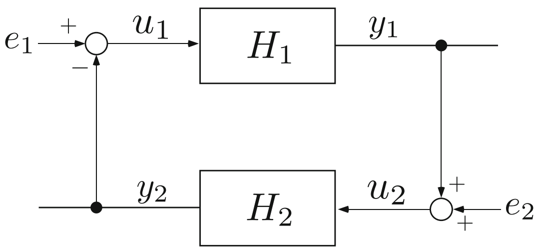


Fig. 2.4 Standard feedback connection

$$u := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \; e := \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \; y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The feedback connection (2.14), and (2.15) yields the system H_{ey} from input e to output y by eliminating u through substitution of (2.15) into (2.14). The block diagram of the system H_{ey} is illustrated in Fig. 2.5.

Now, the following theorem is known to be true, which proves passivity preservation in terms of the feedback interconnection in Fig. 2.4.

Theorem 2.3 *Consider the feedback connection in Fig. 2.4. Suppose that both of the subsystems H_1 and H_2 are passive. Then, the closed-loop system H_{ey} in Fig. 2.5 is also passive from input e to output y. In addition, when $e_2 \equiv 0$ holds,*[8] *the closed-loop system $H_{e_1y_1}$ with input e_1 and output y_1 in Fig. 2.6 is passive.* □

We also have the following theorem associated with strict passivity.

Theorem 2.4 *Consider the feedback connection in Fig. 2.4. Suppose that H_1 and H_2 are strictly output passive. Then, the closed-loop system H_{ey} is also strictly output passive from input e to output y. In addition, when $e_2 \equiv 0$, the closed-loop system $H_{e_1y_1}$ is strictly output passive if H_1 is passive and H_2 is strictly input passive, or H_1 is strictly output passive and H_2 is passive.* □

Let us next consider the parallel connection of two subsystems H_1 and H_2 with $u_1(t), u_2(t), y_1(t), y_2(t) \in \mathbb{R}^p$, as illustrated in Fig. 2.7. Then, the following theorem also holds, which proves preservation of passivity in terms of the parallel interconnection.

Theorem 2.5 *Consider the parallel connection in Fig. 2.7. Suppose that both of the subsystems H_1 and H_2 are passive. Then, the system in Fig. 2.7 is also passive from input $u = u_1 = u_2$ to output $y = y_1 + y_2$.* □

Finally, we consider the system H_1. Let us now transform the input u_1 and output y_1 as $u_1 = M(x_1)\bar{u}_1$ and $\bar{y}_1 = M^T(x_1)y_1$, respectively, using a matrix $M(x_1) \in \mathbb{R}^{p\times q}$. Then, it is straightforward to prove the following theorem for the system in Fig. 2.8 from the transformed input $\bar{u}$ to transformed output $\bar{y}$.

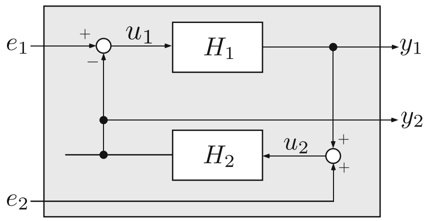


Fig. 2.5 Block diagram of H_{ey}

[8] The notation $e \equiv 0$ for any signal e means $e(t) = 0 \;\; \forall t \in \mathbb{R}_+$.

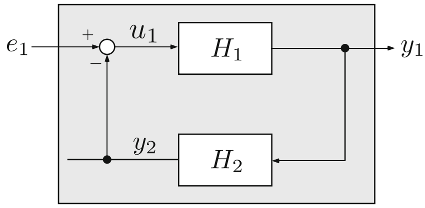


Fig. 2.6 Block diagram of $H_{e_1 y_1}$

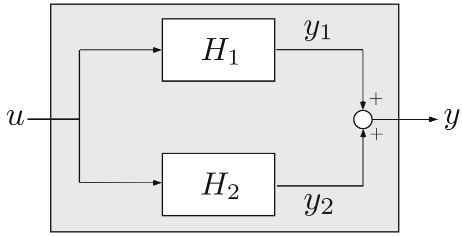


Fig. 2.7 Parallel connection of the systems H_1 and H_2

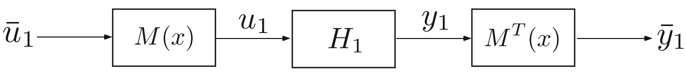


Fig. 2.8 Pre- and post-multiplications of a matrix and its transpose

Theorem 2.6 *Suppose that the subsystem H_1 is passive. Then, the system in Fig. 2.8 is also passive from the transformed input $\bar{u}_1$ to transformed output $\bar{y}_1$.* □

2.2 Stability of Dynamical Systems and Passivity

Another important feature of passivity is its close relation to stability of dynamical systems. The goal of this section is to introduce some stability concepts and clarify their relations to passivity. It is not our intention here to encompass the wide-range of stability theory but to extract, from the vast field, only the contents necessary for the subsequent chapters. If the readers are interested in the general theory, please refer to [159, 318].

2.2.1 $\mathcal{L}_2$ Stability

We start this section with introducing the notion of $\mathcal{L}_2$ stability, which is a kind of input–output stability. In the context of this book, it is sufficient to define the stability for systems described by the following state space model identical to (2.1) except for the dimension of $y(t)$.

$$\dot{x} = f(x, u),\ x(0) = x_0 \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^p \tag{2.16a}$$

$$y = h(x, u),\ \ y(t) \in \mathbb{R}^q \tag{2.16b}$$

However, we start with slightly formal descriptions using input–output maps.

Let us first denote by Ξ the set of all measurable real-valued functions of time $u : \mathbb{R}_+ \to \mathbb{R}^p$. Then, we define the set

$$\mathcal{L}_2 := \left\{ u \in \Xi \,\middle|\, \|u\|_{\mathcal{L}_2} < \infty \right\},$$

where

$$\|u\|_{\mathcal{L}_2} := \left(\int_0^\infty \|u(t)\|^2 dt \right)^{1/2}.$$

In order to deal with unbounded growing signals like $u(t) = t$, we also introduce the extended $\mathcal{L}_2$ space defined by

$$\mathcal{L}_{2e} := \left\{ u \in \Xi \,\middle|\, \|u_\tau\|_{\mathcal{L}_2} < \infty,\ \forall \tau \in \mathbb{R}_+ \right\},$$

where

$$u_\tau(t) := \begin{cases} u(t), & t \in [0, \tau] \\ 0, & t > \tau \end{cases}, \tag{2.17}$$

and it is called a truncation of u. It is clear from the definition that the set $\mathcal{L}_{2e}$ satisfies $\mathcal{L}_2 \subset \mathcal{L}_{2e}$.

We next consider a causal input–output mapping $H : \mathcal{L}_{2e}(\mathcal{U}) \to \mathcal{L}_{2e}(\mathcal{Y})$ from input signal space $\mathcal{L}_{2e}(\mathcal{U})$ to output signal space $\mathcal{L}_{2e}(\mathcal{Y})$. Then, $\mathcal{L}_2$ stability is defined as below.

Definition 2.7 A mapping $H : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is $\mathcal{L}_2$ stable if

$$u \in \mathcal{L}_2(\mathcal{U}) \Rightarrow H(u) \in \mathcal{L}_2(\mathcal{Y}).$$

□

We also introduce the notion of the $\mathcal{L}_2$-gain.

Definition 2.8 A mapping $H : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is said to have finite $\mathcal{L}_2$-gain if there exist nonnegative constants $\gamma \geq 0$ and $\beta \geq 0$ such that for all $\tau \in \mathbb{R}_+$

$$\|(Hu)_\tau\|_{\mathcal{L}_2} \leq \gamma \|u_\tau\|_{\mathcal{L}_2} + \beta \ \ \forall u \in \mathcal{L}_{2e}(\mathcal{U}) \tag{2.18}$$

holds. When H has finite $\mathcal{L}_2$-gain, the $\mathcal{L}_2$-gain of H is defined by

$$\gamma(H) := \inf\{\gamma |\ \exists \beta \text{ such that (2.18) holds.}\}.$$

□

Note that if H has finite $\mathcal{L}_2$-gain, then it is $\mathcal{L}_2$ stable [318].

Let us move on to the state space model (2.16). To extend the Definition 2.8 to the system (2.16), we need to impose an additional assumption that $y \in \mathcal{L}_{2e}$ is ensured for every initial state x_0 and every input u belonging to the class $\mathcal{L}_{2e}$. Under the assumption, once x_0 is fixed, a causal dynamic operator $H_{x_0} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is defined by the state space model (2.16). Accordingly, we define $\mathcal{L}_2$ stability for (2.16) as below.

Definition 2.9 The state space model (2.16) is said to be $\mathcal{L}_2$-stable if, for all x_0, the input–output map H_{x_0} is $\mathcal{L}_2$-stable in the sense of Definition 2.8. In addition, the model (2.16) is said to have finite $\mathcal{L}_2$-gain if there exists a finite γ such that, for every initial state x_0, there exists a finite constant $\beta(x_0)$ satisfying

$$\|(H_{x_0}u)_\tau\|_{\mathcal{L}_2} \leq \gamma\|u_\tau\|_{\mathcal{L}_2} + \beta(x_0)$$

for all $u \in \mathcal{L}_{2e}$ and $\tau \in \mathbb{R}_+$. □

Let us now clarify a relation between passivity in Definition 2.2 and $\mathcal{L}_2$ stability in Definition 2.9. For this purpose, the following theorem plays a key role.

Theorem 2.10 *If the system (2.16) with $p = q$ is output strictly passive, then it is $\mathcal{L}_2$ stable and has finite $\mathcal{L}_2$-gain.* □

More importantly, combining Theorem 2.10 with Theorem 2.4 yields the following remarkable theorem, known as *passivity theorem*, linking passivity with $\mathcal{L}_2$ stability of the closed-loop system.

Theorem 2.11 *Consider the feedback connection (2.14), and (2.15) in Fig. 2.4.*

- *If both of the subsystems H_1 and H_2 are output strictly passive, then the closed-loop system H_{ey} with input e and output y has finite $\mathcal{L}_2$-gain.*
- *When $e_2 \equiv 0$, the closed-loop system $H_{e_1y_1}$ with input e_1 and output y_1 has finite $\mathcal{L}_2$-gain if H_1 is passive and H_2 is input strictly passive, or H_1 is output strictly passive and H_2 is passive.* □

2.2.2 Lyapunov Stability

In this subsection, we briefly review the Lyapunov stability theory.

Let us consider the dynamical autonomous system

$$\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}^n \tag{2.19}$$

having the unique solution for each given x_0. We also assume without loss of generality that the origin $x = 0$ is an equilibrium of the system (2.19), i.e., $f(0) = 0$.

Definition 2.12 The equilibrium $x = 0$ of the system (2.19) is said to be *(Lyapunov) stable* if, for each $\varepsilon > 0$, there exists a positive constant $\delta = \delta(\varepsilon)$ such that

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \varepsilon \ \ \forall t \in \mathbb{R}_+.$$

In addition, $x = 0$ is said to be *asymptotically stable* if it is stable and there exists a positive constant δ such that

$$\|x_0\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0.$$

The point is *exponentially stable* if there exist positive constants δ, k, and λ such that

$$\|x(t)\| \leq k\|x_0\|e^{-\lambda t} \ \ \forall \|x_0\| \leq \delta.$$

□

We next introduce the widely known Lyapunov theorem as below.

Theorem 2.13 *Suppose that $x = 0$ is an equilibrium for (2.19) and $\mathcal{D} \subset \mathbb{R}^n$ is a domain containing the origin. Let $U : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function. Suppose that the function U is positive definite, i.e.,*

$$U(0) = 0 \, and \, U(x) > 0 \ \ \forall x \in \mathcal{D} \setminus \{0\}, \tag{2.20}$$

and satisfies

$$\dot{U}(x) \leq 0 \ \ \forall x \in \mathcal{D}$$

(such a function is called Lyapunov function). Then, the equilibrium $x = 0$ is stable. In addition, the condition

$$\dot{U} < 0 \ \ \forall x \in \mathcal{D} \setminus \{0\},$$

implies that $x = 0$ is asymptotically stable. Moreover, if $U(x)$ satisfies

$$\begin{aligned} &k_1\|x\|^a \leq U(x) \leq k_2\|x\|^a, \ k_1, k_2, a > 0 \\ &\dot{U}(x) \leq -k_3\|x\|^a, \ k_3 > 0 \end{aligned}$$

for all $x \in \mathcal{D}$, then $x = 0$ is exponentially stable. □

Let us next introduce LaSalle's invariance principle. A set $\mathcal{M}$ is said to be an *invariant set* for the system (2.19) if

$$x(0) \in \mathcal{M} \Rightarrow x(t) \in \mathcal{M} \ \ \forall t \in \mathbb{R},$$

and to be a *positively invariant set* for (2.19) if

$$x(0) \in \mathcal{M} \Rightarrow x(t) \in \mathcal{M} \ \ \forall t \in \mathbb{R}_+.$$

In addition, a solution x of (2.19) is said to approach a set $\mathcal{M}$ as the time t goes to infinity, if, for each $\varepsilon > 0$, there exists $T > 0$ such that

$$\inf_{y \in \mathcal{M}} \|x - y\| < \varepsilon \ \ \forall t > T.$$

By using the notions, we can state the LaSalle's invariance principle as below.

Theorem 2.14 *Let $\mathcal{P} \subset \mathcal{D}$ be a compact positively invariant set for the system (2.19) contained in a domain $\mathcal{D} \subset \mathbb{R}^n$. Let $U : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function satisfying $\dot{U}(x) \leq 0 \ \forall x \in \mathcal{P}$. Define the set $\mathcal{X}$ by $\mathcal{X} = \{x \in \mathcal{P} | \ \dot{U}(x) = 0\}$, and let $\mathcal{M}$ be the largest invariant set in $\mathcal{X}$. Then, every solution of (2.19) whose initial state is selected in $\mathcal{P}$ approaches the set $\mathcal{M}$ as t goes to infinity.* □

The following corollary directly links Theorem 2.14 with stability.

Corollary 2.15 *Let $x = 0$ be an equilibrium of (2.19), and $U : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable positive definite function such that $\dot{U}(x) \leq 0$ in $\mathcal{D}$, where $\mathcal{D} \subset \mathbb{R}^n$ is a domain containing the origin. Let $\mathcal{X} = \{x \in \mathcal{D} | \ \dot{U}(x) = 0\}$, and suppose that no solution other than $x \equiv 0$ can stay identically in $\mathcal{X}$. Then, the equilibrium $x = 0$ is asymptotically stable.* □

Let us next state relations between passivity and the above Lyapunov stability. Here, we consider the system in the form of

$$\dot{x} = f(x) + g(x)u, \ \ x(0) = x_0 \in \mathbb{R}^n, \tag{2.21a}$$

$$y = h(x), \tag{2.21b}$$

which is a special class of (2.1) and all the assumptions imposed on (2.21) are assumed to be true for (2.21) also. Remark that the system (2.21a) with $u \equiv 0$ is reduced to the form of the autonomous system (2.19).

Suppose that the system (2.21) is passive with respect to a continuously differentiable storage function S. Then, when $u \equiv 0$, the inequality

$$\dot{S}(x(t)) \leq 0$$

holds from (2.5). Thus, Theorem 2.13 immediately proves stability of the origin of (2.21) with $u \equiv 0$ if S is positive definite. However, the definition of passivity (Definition 2.2) requires S to be only positive *semidefinite*. Indeed, in the case of a positive semidefinite S, stability of the origin can be violated even if (2.1) is passive. We thus need to add an assumption to discuss Lyapunov stability for general passive systems.

Since $S(x) \geq 0$, the derivative $\dot{S}(x(t))$ is never negative when $S(x(t)) = 0$, regardless of $u(t)$. Now, (2.5) implies that $\dot{S}(x(t)) \leq y^T(t)u(t)$ holds at any time t and any $u(t)$. Thus, when $S(x(t)) = 0$, the term $y^T(t)u(t)$ must be nonnegative for all $u(t) \in \mathbb{R}^p$, which is equivalent to $y(t) = 0$. Hence, $h(x) = 0$ must hold if $S(x) = 0$. According to the observation, we introduce the following notion.

Definition 2.16 The system (2.21) is said to be *zero-state observable* if no solution of (2.21) with $u \equiv 0$, namely $\dot{x} = f(x)$, can stay identically in the set of states satisfying $h(x) = 0$, other than the solution $x \equiv 0$. □

Then, we have the following theorem.

Theorem 2.17 *Suppose that the system (2.21) is zero-state observable, and is passive with respect to the continuously differentiable positive semidefinite storage function $S(x) \geq 0$. Then the origin $x = 0$ is a stable equilibrium of the autonomous system (2.21a) with $u \equiv 0$.* □

Theorem 2.17 deals with only stability of the origin, and does not guarantee asymptotic stability. Indeed, passivity does not always guarantee asymptotic stability in the case of $u \equiv 0$ as exemplified by the integrator $\dot{x} = u,\ y = x$, which is passive as shown at the end of Sect. 2.1.1 and zero-state observable, but the solutions of $\dot{x} = 0$ do not go to zero. However, asymptotic stability is guaranteed for a restrictive class of passive systems, namely output strictly passive systems.

Theorem 2.18 *Suppose that the system (2.21) is zero-state observable and output strictly passive with respect to a continuously differentiable storage function $S(x)$. Then the origin $x = 0$ is an asymptotically stable equilibrium of the autonomous system (2.21a) with $u \equiv 0$.* □

The output strict passivity limits the applications of this theorem. However, combining it with Theorem 2.4 provides an important insight into stabilization of the origin via feedback control.

Let us now consider the feedback connection in Fig. 2.6, where the subsystem H_1 is assumed to be passive but is not always output strictly passive. Suppose that the subsystem H_2 in Fig. 2.6 is a static map

$$y_2 = ku_2, \quad k > 0. \tag{2.22}$$

Then, it is easy to confirm that the subsystem H_2 is an input strictly passive defined just after Definition 2.2. Then, Theorem 2.4 means that the closed-loop system $H_{e_1 y_1}$ is output strictly passive from input e_1 to output y_1. Consequently, Theorem 2.18 immediately proves asymptotic stability of the origin of H_1 in the absence of the external input ($e_1 \equiv 0$) under the observability condition. Formally, the above statements are summarized by the following corollary.

Corollary 2.19 *Consider a passive zero-state observable system (2.21). Then, the negative feedback $u(t) = -ky(t),\ k > 0$ asymptotically stabilizes the origin $x = 0$.* □

Though we restrict the feedback structure to a static linear feedback (2.22), the statement compatible with the corollary holds even if it is replaced by any input strictly passive system under observability conditions.

Some additional results used in this book are provided in Appendix A.

2.3 Passivity-Based Motion Control

In this section, we briefly review the basic results of passivity-based control of rigid robot manipulators whose dynamics is formulated as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau. \tag{2.23}$$

See (2.10) for the meaning of the notations. As stated in Sect. 2.1.1, the inertia matrix $M(q) \in \mathbb{R}^{n\times n}$ is positive definite and the matrix $\dot{M}(q) - 2C(q, \dot{q})$ is skew-symmetric.

The objective here is to design the torque τ so as to render the trajectory of the joint displacement q to a desired trajectory q_d. Here, we define the displacement error vector e_q by

$$e_q := q - q_d.$$

Takegaki and Arimoto [310] presented the following control law assuming that q_d is constant, i.e., $\dot{q}_d = 0$.

$$\tau = g(q) - K_p e_q + u \tag{2.24}$$

with a diagonal positive gain matrix K_p. Substituting (2.24) into (2.23) yields

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + K_p e_q = u. \tag{2.25}$$

Let us now define the storage function

$$S_{TA} := \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}e_q^T K_p e_q. \tag{2.26}$$

Then, the time derivative of S_{TA} along trajectories of (2.25) is given as

$$\begin{aligned}
\dot{S}_{TA} &= \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \dot{q}^T M(q)\ddot{q} + e_q^T K_p \dot{e}_q \\
&= \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \dot{q}^T(-C(q, \dot{q})\dot{q} - K_p e_q + u) + e_q^T K_p \dot{q} \\
&= \frac{1}{2}\dot{q}^T(\dot{M}(q) - 2C(q, \dot{q}))\dot{q} + \dot{q}^T u = \dot{q}^T u.
\end{aligned} \tag{2.27}$$

Equation (2.27) implies that the system (2.25) is passive from input u to output $\dot{q}$. Substituting $\dot{q} \equiv 0$ and $u \equiv 0$ into (2.25) yields $e_q \equiv 0$, which means zero-state observability of the system by taking $e_q = q - q_d$ as a state variable instead of q. Hence, closing the loop by a negative feedback

$$u = -K_q \dot{q} \tag{2.28}$$

with a diagonal positive gain matrix K_q guarantees asymptotic stability of the origin $e_q = 0,\ \dot{q} = 0$ from Corollary 2.19.

Comparing the two storage functions (2.11) and (2.26), we see that the potential energy function is shaped by the local feedback loop (2.24) so that its unique minimal value is taken at the desirable state $e_q = 0$. Such an energy shaping is called *passivation*, and the feedback control (2.28) is called *damping injection*. The two-stage control strategy, passivation, and damping injection, has been called *passivity-based control*.

Paden and Panja [243] extended the above result to a tracking problem such that $\dot{q}_d \neq 0$. They propose the passivation controller

$$\tau = M(q)\ddot{q}_d + C(q, \dot{q})\dot{q}_d + g(q) - K_p e_q + u. \tag{2.29}$$

Substituting (2.29) into (2.23) yields

$$M(q)\ddot{e}_q + C(q, \dot{q})\dot{e}_q + K_p e_q = u. \tag{2.30}$$

If we define the storage function

$$S_{PP} := \frac{1}{2}\dot{e}_q^T M(q)\dot{e}_q + \frac{1}{2}e_q^T K_p e_q,$$

its time derivative along the trajectories of (2.30) is given as

$$\begin{aligned}
\dot{S}_{PP} &= \frac{1}{2}\dot{e}_q^T \dot{M}(q)\dot{e}_q + \dot{e}_q^T M(q)\ddot{e}_q + e_q^T K_p \dot{e}_q \\
&= \frac{1}{2}\dot{e}_q^T \dot{M}(q)\dot{e}_q + \dot{e}_q^T(-C(q, \dot{q})\dot{e}_q - K_p e_q + u) + e_q^T K_p \dot{e}_q \\
&= \frac{1}{2}\dot{e}_q^T(\dot{M}(q) - 2C(q, \dot{q}))\dot{e}_q + \dot{e}_q^T u = \dot{e}_q^T u.
\end{aligned} \tag{2.31}$$

Equation (2.31) implies that the system (2.30) is passive from input u to output $\dot{e}_q$. Notice that the storage function is shaped so that it takes the minimal value at $e_q = \dot{e}_q = 0$. Substituting $\dot{e}_q \equiv 0$ and $u \equiv 0$ into (2.30) yields $e_q \equiv 0$, and hence the system is zero-state observable. Thus, the damping injection

$$u = -K_q \dot{e}_q$$

guarantees asymptotic stability of the origin $e_q = \dot{e}_q = 0$ from Corollary 2.19.

Meanwhile, Slotine and Li [285] presented the passivation controller

$$\tau = M(q)\dot{v} + C(q, \dot{q})v + g(q) + u, \tag{2.32}$$

where

$$v := \dot{q}_d - \Lambda e_q$$

with a diagonal positive gain matrix Λ. Let us now introduce the notation

$$r := \dot{q} - v = \dot{e}_q + \Lambda e_q.$$

Then, substituting (2.32) into (2.23) yields

$$M(q)\dot{r} + C(q, \dot{q})r = u. \tag{2.33}$$

It is immediate to see that (2.33) is passive from u to r with respect to the storage function

$$S_{SL} := \frac{1}{2}r^T M(q)r.$$

Thus, close the loop of u by

$$u = -K_r r = -K_r(\dot{e}_q + \Lambda e_q) \tag{2.34}$$

with a diagonal positive gain matrix K_r. Define the Lyapunov function candidate

$$U_{SL} := \frac{1}{2}r^T M(q)r + e_q^T \Lambda K_r e_q.$$

Then, its time derivative along the trajectories of (2.33) with (2.34) satisfies

$$\begin{aligned}
\dot{U}_{SL} &= \frac{1}{2}r^T \dot{M}(q)r + r^T M(q)\dot{r} + 2e_q^T \Lambda K_r \dot{e}_q \\
&= \frac{1}{2}r^T \dot{M}(q)r + r^T(-C(q, \dot{q})r + u) + 2e_q^T \Lambda K_r \dot{e}_q \\
&= \frac{1}{2}r^T(\dot{M}(q) - 2C(q, \dot{q}))r - r^T K_r r + 2e_q^T \Lambda K_r \dot{e}_q \\
&= -(\dot{e}_q + \Lambda e_q)^T K_r(\dot{e}_q + \Lambda e_q) + 2e_q^T \Lambda K_r \dot{e}_q \\
&= -\dot{e}_q^T K_r \dot{e}_q - e_q^T \Lambda K_r \Lambda e_q.
\end{aligned} \tag{2.35}$$

We see from (2.35) and Theorem 2.13 that the origin $e_q = \dot{e}_q = 0$ is asymptotically stable. It is not difficult to confirm that the stability is also exponential under the assumption of a constant norm bound on $M(q)$ [297].

We finally introduce the passivity-based adaptive control scheme [285, 297], assuming that the inertia matrix M, Coriolis matrix C, and gravity g contain time invariant uncertainties. Describing their estimates by $\hat{M}$, $\hat{C}$, and $\hat{g}$ respectively, the control input (2.32) and (2.34) are replaced by

$$\tau = \hat{M}(q)\dot{v} + \hat{C}(q, \dot{q})v + \hat{g}(q) - K_r r. \tag{2.36}$$

It is well known [297] that the left-hand side of (2.10) can be represented as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = Y(q, \dot{q}, \ddot{q})\psi, \tag{2.37}$$

where the notation Y, called the regressor, is a known function of the generalized coordinates and ψ is a constant uncertain parameter vector (See [297] for more details on the representation). Namely, the uncertainties in M, C, and g are summarized into those in ψ, and (2.36) is rewritten as

$$\tau = Y\hat{\psi} - K_r r. \tag{2.38}$$

Substituting (2.37) and (2.38) into (2.10) yields

$$M(q)\dot{r} + C(q, \dot{q})r + K_r r = Y\tilde{\psi},\ \tilde{\psi} := \hat{\psi} - \psi. \tag{2.39}$$

Here, we employ the following update rule of the estimate $\hat{\psi}$.

$$\dot{\hat{\psi}} = -\Gamma^{-1} Y^T r \tag{2.40}$$

Take the Lyapunov function

$$U_{AC} := \frac{1}{2} r^T M(q) r + e_q^T \Lambda K_r e_q + \frac{1}{2}\tilde{\psi}^T \Gamma \tilde{\psi} = U_{SL} + \frac{1}{2}\tilde{\psi}^T \Gamma \tilde{\psi}.$$

Then, from (2.35), the time derivative of U_{SL} along the trajectories of (2.39) is given by

$$\dot{U}_{SL} = -\dot{e}_q^T K_r \dot{e}_q - e_q^T \Lambda K_r \Lambda e_q + r^T Y \tilde{\psi}. \tag{2.41}$$

Meanwhile, the time derivative of $\frac{1}{2}\tilde{\psi}^T \Gamma \tilde{\psi}$ along the trajectories of (2.40) is

$$\frac{d}{dt}\left(\frac{1}{2}\tilde{\psi}^T \Gamma \tilde{\psi}\right) = \tilde{\psi}^T \Gamma \dot{\tilde{\psi}} = \tilde{\psi}^T \Gamma \dot{\hat{\psi}} = -\tilde{\psi}^T Y^T r, \tag{2.42}$$

where the second equation holds because ψ is constant. Combining (2.41) and (2.42) yields

$$\dot{U}_{AC} \leq -\dot{e}_q^T K_r \dot{e}_q - e_q^T \Lambda K_r \Lambda e_q, \tag{2.43}$$

which directly proves stability of the closed-loop system in the sense of Lyapunov and boundedness of the state r, e_q, and $\tilde{\psi}$. In addition, from the definition $r = \dot{e}_q + \Lambda e_q$ and boundedness of e_q, $\dot{e}_q$ is also bounded. Meanwhile, integrating (2.43) with respect to time means that e_q and $\dot{e}_q$ belong to $\mathcal{L}_2$. Namely, the Barbalat's Lemma (Lemma A.3) with $z(t) = e_q(t)$ proves $e_q \to 0$, which is equivalent to $q \to q_d$. In addition, if $\ddot{e}_q$ is bounded, $\dot{e}_q \to 0$, i.e., $\dot{q} \to \dot{q}_d$ is guaranteed from Lemma A.3.

http://www.springer.com/978-3-319-15170-0

Passivity-Based Control and Estimation in Networked Robotics
Hatanaka, T.; Chopra, N.; Fujita, M.; Spong, M.W.
2015, X, 349 p. 251 illus., 192 illus. in color., Hardcover
ISBN: 978-3-319-15170-0