

Chapter 2

Dynamical Systems

Abstract We give the definition of dynamical system and a classification of such systems: finite-dimensional and infinite-dimensional systems; continuous-time and discrete-time systems; continuous and discontinuous systems; autonomous and non-autonomous systems; and composite systems. Classes of finite-dimensional dynamical systems that we address include systems determined by ordinary differential equations, ordinary differential inequalities, ordinary difference equations, and ordinary difference inequalities. *General* classes of infinite-dimensional dynamical systems that we address include systems determined by differential equations and inclusions defined on Banach spaces and systems determined by linear and nonlinear semigroups. *Specific* classes of infinite-dimensional dynamical systems that we address include systems determined by functional differential equations, Volterra integrodifferential equations, and certain classes of partial differential equations. For all cases, we present specific examples.

Our main objective in the present chapter is to define a dynamical system and to present several important classes of dynamical systems. The chapter is organized into twelve sections.

In the first section we establish some of the notation that we require in this chapter, as well as in the subsequent chapters. Next, in the second section we present precise definitions for dynamical system and related concepts. We introduce finite-dimensional dynamical systems determined by ordinary differential equations in the third section, by differential inequalities in the fourth section, and by ordinary difference equations and inequalities in the fifth section. In the sixth section, we address infinite-dimensional dynamical systems determined by differential equations and inclusions defined on Banach spaces, and in the seventh and eighth sections we consider special cases of infinite-dimensional dynamical systems determined by functional differential equations and Volterra integrodifferential equations, respectively. In the ninth section we discuss dynamical systems determined by semigroups defined on Banach and Hilbert spaces, and in the tenth section we treat dynamical systems determined by specific classes of partial differential equations. Finally, we address composite dynamical systems in the eleventh section and discontinuous dynamical systems in the twelfth section.

The specific classes of dynamical systems that we consider in this chapter are very important. However, there are of course many other important classes of dynamical systems, not even alluded to in the present chapter. We address one such class of systems in Chapter 5, determined by discrete-event systems.

Much of the material presented in Sections 2.3–2.10 constitutes background material and concerns the well posedness (existence, uniqueness, continuation, and continuity with respect to initial conditions of solutions) of a great variety of equations (resp., systems). Even if practical, it still would distract from our objectives on hand if we were to present proofs for these results. Instead, we give detailed references where to find such proofs, and in some cases, we give hints (in the problem section) on how to prove some of these results. The above is in contrast with our presentations in the remainder of this book where we prove all results (except some, concerning additional background material).

2.1 Notation

Let Y, Z be arbitrary sets. Then $Y \cup Z, Y \cap Z, Y - Z$, and $Y \times Z$ denote the union, intersection, difference, and Cartesian product of Y and Z , respectively. If Y is a subset of Z , we write $Y \subset Z$ and if x is an element of Y , we write $x \in Y$. We denote a mapping f of Y into Z by $f: Y \rightarrow Z$ and we denote the set of all mappings from Y into Z by $\{Y \rightarrow Z\}$. Let \emptyset denote the empty set.

Let \mathbb{R} denote the set of real numbers, let $\mathbb{R}^+ = [0, \infty)$, let \mathbb{N} denote the set of nonnegative integers (i.e., $\mathbb{N} = \{0, 1, 2, \dots\}$), and let \mathbb{C} denote the set of complex numbers. Let $J \subset \mathbb{R}$ denote an interval (i.e., $J = [a, b), (a, b], [a, b]$, or $(a, b), b > a$, with $J = (-\infty, \infty) = \mathbb{R}$ allowed). If Y_1, \dots, Y_n are n arbitrary sets, their Cartesian product is denoted by $Y_1 \times \dots \times Y_n$, and if in particular $Y = Y_1 = \dots = Y_n$ we write Y^n .

Let \mathbb{R}^n denote real n -space. If $x \in \mathbb{R}^n$, $x^T = (x_1, \dots, x_n)$ denotes the transpose of x . Also, if $x, y \in \mathbb{R}^n$, then $x \leq y$ signifies $x_i \leq y_i$, $x < y$ signifies $x_i < y_i$, and $x > 0$ signifies $x_i > 0$ for all $i = 1, \dots, n$. We let $|\cdot|$ denote the Euclidean norm; that is, for $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, $|x| = (x^T x)^{1/2} = (\sum_{i=1}^n x_i^2)^{1/2}$.

Let $A = [a_{ij}]_{n \times n}$ denote a real $n \times n$ matrix (i.e., $A \in \mathbb{R}^{n \times n}$) and let A^T denote the transpose of A . The matrix norm $|\cdot|$, induced by the Euclidean vector norm (defined on \mathbb{R}^n), is defined by

$$|A| = \inf \{ \alpha \in \mathbb{R}^+ : \alpha |x| \geq |Ax|, x \in \mathbb{R}^n \} = [\lambda_M(A^T A)]^{1/2}$$

where $\lambda_M(A^T A)$ denotes the largest eigenvalue of $A^T A$ (recall that the eigenvalues of symmetric matrices are real). In the interests of clarity, we also use the notation $\|\cdot\|$ to distinguish the norm of a matrix (e.g., $\|A\|$) from the norm of a vector (e.g., $|x|$).

We let $L_p[G, U]$, $1 \leq p \leq \infty$, denote the usual Lebesgue space of all Lebesgue measurable functions with domain G and range U . The norm in $L_p[G, U]$ is usually denoted $\|\cdot\|_p$, or $\|\cdot\|_{L_p}$ if more explicit notation is required.

We let (X, d) be a metric space, where X denotes the underlying set and d denotes the metric. When the choice of the particular metric used is clear from context, we speak of a metric space X , rather than (X, d) .

If Y and Z are metric spaces and if $f: Y \rightarrow Z$, and if f is continuous, we write $f \in C[Y, Z]$; that is, $C[Y, Z]$ denotes the set of all continuous mappings from Y to Z . We denote the inverse of a mapping f , if it exists, by f^{-1} .

A function $\psi \in C[[0, r_1], \mathbb{R}^+]$ (resp., $\psi \in C[\mathbb{R}^+, \mathbb{R}^+]$) is said to belong to *class* \mathcal{K} (i.e., $\psi \in \mathcal{K}$) if $\psi(0) = 0$ and if ψ is strictly increasing on $[0, r_1]$ (resp., on \mathbb{R}^+). If $\psi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$, if $\psi \in \mathcal{K}$, and if $\lim_{r \rightarrow \infty} \psi(r) = \infty$, then ψ is said to belong to *class* \mathcal{K}_∞ (i.e., $\psi \in \mathcal{K}_\infty$).

For a function $f: \mathbb{R} \rightarrow \mathbb{R}$, we denote the upper right-hand, upper left-hand, lower right-hand, and lower left-hand Dini derivatives by D^+f , D^-f , D_+f , and D_-f , respectively. When we have a fixed Dini derivative of f in mind, we simply write Df , in place of the preceding notation.

2.2 Dynamical Systems

In characterizing the notion of dynamical system, we require the concepts of motion and family of motions.

Definition 2.2.1. Let (X, d) be a metric space, let $A \subset X$, and let $T \subset \mathbb{R}$. For any fixed $a \in A$, $t_0 \in T$, a mapping $p(\cdot, a, t_0): T_{a, t_0} \rightarrow X$ is called a *motion* if $p(t_0, a, t_0) = a$ where $T_{a, t_0} = [t_0, t_1) \cap T$, $t_1 > t_0$, and t_1 is finite or infinite. \square

Definition 2.2.2. A subset S of the set

$$\bigcup_{(a, t_0) \in A \times T} \{T_{a, t_0} \rightarrow X\}$$

is called a *family of motions* if for every $p(\cdot, a, t_0) \in S$, we have $p(t_0, a, t_0) = a$. \square

Definition 2.2.3. The four-tuple $\{T, X, A, S\}$ is called a *dynamical system*. \square

In Definitions 2.2.1 and 2.2.2 we find it useful to think of X as *state space*, T as *time set*, t_0 as *initial time*, a as the *initial condition* of the motion $p(\cdot, a, t_0)$, and A as the *set of initial conditions*. Note that in our definition of motion, we allow in general more than one motion to initiate from a given pair of *initial data*, (a, t_0) .

When in Definition 2.2.3, $T = J \subset \mathbb{R}^+$ (with $J = \mathbb{R}^+$ allowed), we speak of a *continuous-time dynamical system* and when $T = J \cap \mathbb{N}$ (with $J \cap \mathbb{N} = \mathbb{N}$ allowed) we speak of a *discrete-time dynamical system*. Also, when in Definition 2.2.3, X is a finite-dimensional vector space, we speak of a *finite-dimensional dynamical system*, and otherwise, of an *infinite-dimensional dynamical system*. Furthermore,

if in a continuous-time dynamical system all motions (i.e., all elements of S) are continuous with respect to time t , we speak of a *continuous dynamical system*. If at least one motion of a continuous-time dynamical system is not continuous with respect to t , we speak of a *discontinuous dynamical system*.

When in Definition 2.2.3, T , X , and A are known from context, we frequently speak of a *dynamical system* S , or even of a *system* S , rather than a dynamical system $\{T, X, A, S\}$.

Definition 2.2.4. A dynamical system $\{T, X_1, A_1, S_1\}$ is called a *dynamical subsystem*, or simply, a *subsystem* of a dynamical system $\{T, X, A, S\}$ if $X_1 \subset X$, $A_1 \subset A$, and $S_1 \subset S$. \square

Definition 2.2.5. A motion $p = p(\cdot, a, t_0)$ in a dynamical system $\{T, X, A, S\}$ is said to be *bounded* if there exist an $x_0 \in X$ and a $\beta > 0$ such that $d(p(t, a, t_0), x_0) < \beta$ for all $t \in T_{a, t_0}$. \square

Definition 2.2.6. A motion $p^* = p^*(\cdot, a, t_0)$ defined on $[t_0, c) \cap T$ is called a *continuation* of another motion $p = p(\cdot, a, t_0)$ defined on $[t_0, b) \cap T$ if $p = p^*$ on $[t_0, b) \cap T$, $c > b$, and $[b, c) \cap T \neq \emptyset$. We say that p is *noncontinuable* if no continuation of p exists. Also, $p = p(\cdot, a, t_0)$ is said to be *continuable forward for all time* if there exists a continuation $p^* = p^*(\cdot, a, t_0)$ of p that is defined on $[t_0, \infty) \cap T$, where it is assumed that for any $\alpha > 0$, $[\alpha, \infty) \cap T \neq \emptyset$. \square

In the remainder of this chapter, we present several important classes of dynamical systems. Most of this material serves as required background for the remainder of this book.

2.3 Ordinary Differential Equations

In this section we summarize some essential facts from the qualitative theory of ordinary differential equations that we require as background material and we show that the solutions of differential equations determine continuous, finite-dimensional dynamical systems.

2.3.1 Initial value problems

Let $D \subset \mathbb{R}^{n+1}$ be a domain (an open connected set), let $x = (x_1, \dots, x_n)^T$ denote elements of \mathbb{R}^n , and let elements of D be denoted by (t, x) . When x is a vector-valued function of t , let

$$\dot{x} = \frac{dx}{dt} = \left(\frac{dx_1}{dt}, \dots, \frac{dx_n}{dt} \right)^T = (\dot{x}_1, \dots, \dot{x}_n)^T.$$

For a given function $f_i: D \rightarrow \mathbb{R}$, $i = 1, \dots, n$, let $f = (f_1, \dots, f_n)^T$. Consider *systems of first-order ordinary differential equations* given by

$$\dot{x}_i = f_i(t, x_1, \dots, x_n), \quad i = 1, \dots, n. \quad (E_i)$$

Equation (E_i) can be written more compactly as

$$\dot{x} = f(t, x). \quad (E)$$

A *solution* of (E) is an n vector-valued differentiable function φ defined on a real interval $J = (a, b)$ (we express this by $f \in C^1[J, \mathbb{R}^n]$) such that $(t, \varphi(t)) \in D$ for all $t \in J$ and such that

$$\dot{\varphi}(t) = f(t, \varphi(t))$$

for all $t \in J$. We also allow the cases when $J = [a, b)$, $J = (a, b]$, or $J = [a, b]$. When $J = [a, b]$, then $\dot{\varphi}(a)$ is interpreted as the right-side derivative and $\dot{\varphi}(b)$ is interpreted as the left-side derivative.

For $(t_0, x_0) \in D$, the *initial value problem* associated with (E) is given by

$$\dot{x} = f(t, x), \quad x(t_0) = x_0. \quad (I_E)$$

An n vector-valued function φ is a *solution* of (I_E) if φ is a solution of (E) which is defined on $[t_0, b)$ and if $\varphi(t_0) = x_0$. To denote the dependence of the solutions of (I_E) on the initial data (t_0, x_0) , we frequently write $\varphi(t, t_0, x_0)$. However, when the initial data are clear from context, we often write $\varphi(t)$ in place of $\varphi(t, t_0, x_0)$.

When $f \in C[D, \mathbb{R}^n]$, φ is a solution of (I_E) if and only if φ satisfies the integral equation

$$\varphi(t) = x_0 + \int_{t_0}^t f(s, \varphi(s)) ds \quad (\tilde{E})$$

for $t \in [t_0, b)$. In (\tilde{E}) , we have used the notation

$$\int_{t_0}^t f(s, \varphi(s)) ds = \left[\int_{t_0}^t f_1(s, \varphi(s)) ds, \dots, \int_{t_0}^t f_n(s, \varphi(s)) ds \right]^T.$$

2.3.2 Existence, uniqueness, and continuation of solutions

The following examples demonstrate that we need to impose restrictions on the right-hand side of (E) to ensure the existence, uniqueness, and continuation of solutions of the initial value problem (I_E) .

Example 2.3.1. For the scalar initial value problem

$$\dot{x} = g(x), \quad x(0) = 0 \quad (2.1)$$

where $x \in \mathbb{R}$ and

$$g(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

there is no differentiable function φ that satisfies (2.1). Therefore, this initial value problem has *no solution* (in the sense defined above). \square

Example 2.3.2. The initial value problem

$$\dot{x} = x^{1/3}, \quad x(t_0) = 0$$

where $x \in \mathbb{R}$, has at least two solutions given by

$$\varphi_1(t) = \left[\frac{2}{3}(t - t_0) \right]^{3/2}$$

and $\varphi_2(t) = 0$ for $t \geq t_0$. \square

Example 2.3.3. The scalar initial value problem

$$\dot{x} = ax, \quad x(t_0) = x_0$$

where $x \in \mathbb{R}$, has a unique solution given by $\varphi(t) = e^{a(t-t_0)}x(t_0)$ for $t \geq t_0$. \square

The following result, called the *Peano–Cauchy Existence Theorem*, provides a set of sufficient conditions for the existence of solutions of the initial value problem (I_E) .

Theorem 2.3.1. *Let $f \in C[D, \mathbb{R}^n]$. Then for any $(t_0, x_0) \in D$, the initial value problem (I_E) has a solution defined on $[t_0, t_0 + c)$ for some $c > 0$.* \square

The next result provides a set of sufficient conditions for the uniqueness of solutions of the initial value problem (I_E) .

Theorem 2.3.2. *Let $f \in C[D, \mathbb{R}^n]$. Assume that for every compact set $K \subset D$, f satisfies the Lipschitz condition*

$$|f(t, x) - f(t, y)| \leq L_K |x - y| \quad (2.2)$$

for all $(t, x), (t, y) \in K$ where L_K is a constant depending only on K . Then (I_E) has at most one solution on any interval $[t_0, t_0 + c)$, $c > 0$. \square

In the problem section we provide details for the proofs of Theorems 2.3.1 and 2.3.2. Alternatively, the reader may wish to refer, for example, to Miller and Michel [37] for proofs of these results.

Next, let φ be a solution of (E) on an interval J . By a *continuation* or *extension* of φ we mean an extension φ_0 of φ to a larger interval J_0 in such a way that the extension solves (E) on J_0 . Then φ is said to be *continued* or *extended* to the larger interval J_0 . When no such continuation is possible, then φ is said to be *noncontinuable*.

Example 2.3.4. The differential equation

$$\dot{x} = x^2$$

has a solution $\varphi(t) = 1/(1-t)$ defined on $J = (-1, 1)$. This solution is continuable to the left to $-\infty$ and is not continuable to the right. \square

Example 2.3.5. The differential equation

$$\dot{x} = x^{1/3} \quad (2.3)$$

where $x \in \mathbb{R}$, has a solution $\psi(t) \equiv 0$ on $J = (-\infty, 0)$. This solution is continuable to the right in more than one way. For example, both $\psi_1(t) \equiv 0$ and $\psi_2(t) = (2t/3)^{3/2}$ are solutions of (2.3) for $t \geq 0$. \square

Before stating the next result, we require the following concept.

Definition 2.3.1. A solution φ of (E) defined on the interval (a, b) is said to be *bounded* if there exists a $\beta > 0$ such that $|\varphi(t)| < \beta$ for all $t \in (a, b)$, where β may depend on φ . \square

In the next result we provide a set of sufficient conditions for the continuability of solutions of (E) .

Theorem 2.3.3. *Let $f \in C[J \times \mathbb{R}^n, \mathbb{R}^n]$ where $J = (a, b)$ is a finite or an infinite interval. Assume that every solution of (E) is bounded. Then every solution of (E) can be continued to the entire interval $J = (a, b)$. \square*

In the problem section we give details for the proof of the above result. Alternatively, the reader may want to refer, for example, to Miller and Michel [37] for the proof of this result.

In Chapter 6 we establish sufficient conditions that ensure the boundedness of the solutions of (E) , using the Lyapunov stability theory (refer to Example 6.2.9).

2.3.3 *Dynamical systems determined by ordinary differential equations*

On \mathbb{R}^n we define the metric d , using the Euclidean norm $|\cdot|$, by

$$d(x, y) = |x - y| = \left[\sum_{i=1}^n (x_i - y_i)^2 \right]^{1/2}$$

for all $x, y \in \mathbb{R}^n$. Let $A \subset \mathbb{R}^n$ be an open set, let $J \subset \mathbb{R}$ be a finite or an infinite open interval, and let $D = J \times A$. Assume that for (E) and (I_E) $f \in C[D, \mathbb{R}^n]$. In view of Theorem 2.3.1, (I_E) has at least one solution on $[t_0, t_0 + c)$ for some $c > 0$. Let S_{t_0, x_0} denote the set of all the solutions of (I_E) and let $S_E = \cup_{(t_0, x_0) \in D} S_{t_0, x_0}$. Then S_E constitutes the set of all the solutions of (E) that are defined on any half closed (resp., half open) interval $[a, b) \subset J$.

Let $T = J$ and $A \subset X = \mathbb{R}^n$. Then $\{T, X, A, S_E\}$ is a dynamical system in the sense of Definition 2.2.3. When $D = J \times A$ is understood from context, we refer to this dynamical system simply as S_E and we call S_E the *dynamical system determined by (E)* .

We note in particular if $D = \mathbb{R}^+ \times \mathbb{R}^n$ and if for (E) , $f \in C[D, \mathbb{R}^n]$, and if every motion in S_E is bounded, then in view of Theorem 2.3.3, every motion of S_E is continuable forward for all time (see Definition 2.2.6).

We conclude this subsection with the following important example.

Example 2.3.6. Let $A \in C[\mathbb{R}^+, \mathbb{R}^{n \times n}]$ and consider the *linear homogeneous ordinary differential equation*

$$\dot{x} = A(t)x. \quad (LH)$$

The existence and uniqueness of solutions of the initial value problems determined by (LH) are ensured by Theorems 2.3.1 and 2.3.2. In Chapter 6 (see Example 6.2.9 and Corollary 6.2.1) we show that all the motions of the dynamical systems S_{LH} determined by (LH) are continuable forward for all time (resp., all the solutions of (LH) can be continued to ∞). \square

2.3.4 Two specific examples

In the following we consider two important special cases which we revisit several times.

Example 2.3.7. Conservative dynamical systems, encountered in classical mechanics, contain no energy-dissipating elements and are characterized by means of the *Hamiltonian function* $H(p, q)$, where $q^T = (q_1, \dots, q_n)$ denotes n *generalized position coordinates* and $p^T = (p_1, \dots, p_n)$ denotes n *generalized momentum coordinates*. We assume that $H(p, q)$ is of the form

$$H(p, q) = T(q, \dot{q}) + W(q) \quad (2.4)$$

where T denotes the *kinetic energy*, W denotes the *potential energy* of the system, and $\dot{q} = dq/dt$. These energy terms are determined from the path-independent line integrals

$$T(q, \dot{q}) = \int_0^{\dot{q}} p(q, \xi)^T d\xi = \int_0^{\dot{q}} \sum_{i=1}^n p_i(q, \xi) d\xi_i \quad (2.5)$$

$$W(q) = \int_0^q f(\eta)^T d\eta = \int_0^q \sum_{i=1}^n f_i(\eta) d\eta_i \quad (2.6)$$

where $f_i, i = 1, \dots, n$, denote *generalized potential forces*.

Necessary and sufficient conditions for the path independence of the integral (2.5) are given by

$$\frac{\partial p_i}{\partial \dot{q}_j}(q, \dot{q}) = \frac{\partial p_j}{\partial \dot{q}_i}(q, \dot{q}), \quad i, j = 1, \dots, n. \quad (2.7)$$

A similar statement can be made for (2.6).

Conservative dynamical systems are now given by the system of $2n$ differential equations

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i}(p, q), & i = 1, \dots, n \\ \dot{p}_i = -\frac{\partial H}{\partial q_i}(p, q), & i = 1, \dots, n. \end{cases} \quad (2.8)$$

If we compute the derivative of $H(p, q)$ with respect to time t , evaluated along the solutions of (2.8) (given by $q_i(t), p_i(t), i = 1, \dots, n$), we obtain

$$\frac{dH}{dt}(p(t), q(t)) = \sum_{i=1}^n \frac{\partial H}{\partial p_i}(p, q) \dot{p}_i + \sum_{i=1}^n \frac{\partial H}{\partial q_i}(p, q) \dot{q}_i$$

$$\begin{aligned}
&= - \sum_{i=1}^n \frac{\partial H}{\partial p_i}(p, q) \frac{\partial H}{\partial q_i}(p, q) + \sum_{i=1}^n \frac{\partial H}{\partial q_i}(p, q) \frac{\partial H}{\partial p_i}(p, q) \\
&\equiv 0.
\end{aligned}$$

Thus, in a conservative dynamical system (2.8), the Hamiltonian (i.e., the total energy in the system) is constant along the solutions of (2.8).

Along with initial data $q_i(t_0)$, $p_i(t_0)$, $i = 1, \dots, n$, the equations (2.8) determine an initial value problem. If the right-hand side of (2.8) is Lipschitz continuous, then according to Theorems 2.3.1 and 2.3.2, this initial value problem has unique solutions for all initial data that can be continued forward for all time. The set of the solutions of (2.8) generated by varying the initial data $(t_0, q(t_0), p(t_0))$ over $\mathbb{R} \times \mathbb{R}^{2n}$ determines a dynamical system in the sense of Definition 2.2.3. \square

Example 2.3.8 (Lagrange's Equation). If the preceding dynamical system is modified to contain elements that dissipate energy, such as viscous friction elements in mechanical systems and resistors in electric circuits, we employ *Lagrange's equation* in describing such systems. For a system of n degrees of freedom, this equation is given by

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i}(q, \dot{q}) \right) - \frac{\partial L}{\partial q_i}(q, \dot{q}) + \frac{\partial D}{\partial \dot{q}_i}(\dot{q}) = F_i, \quad i = 1, \dots, n \quad (2.9)$$

where $q^T = (q_1, \dots, q_n)$ denotes the *generalized position vector*. The function $L(q, \dot{q})$ is called the *Lagrangian* and is defined as

$$L(q, \dot{q}) = T(q, \dot{q}) - W(q);$$

that is, it is the difference between the kinetic energy T (see (2.5)) and the potential energy W (see (2.6)).

The function $D(\dot{q})$ denotes *Rayleigh's dissipation function* which is assumed to be of the form

$$D(\dot{q}) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \beta_{ij} \dot{q}_i \dot{q}_j$$

where $Q = [\beta_{ij}]$ is a symmetric, positive semidefinite matrix. The dissipation function D represents one-half the rate at which energy is dissipated as heat (produced by friction in mechanical systems and resistance in electric circuits).

The term F_i , $i = 1, \dots, n$, in (2.9) denotes applied force and includes all external forces associated with the i th coordinate. The force F_i is defined to be positive when it acts to increase the value of q_i .

System (2.9) consists of n second-order ordinary differential equations that can be changed into a system of $2n$ first-order ordinary differential equations by letting $x_1 = q_1, x_2 = \dot{q}_1, \dots, x_{2n-1} = q_n, x_{2n} = \dot{q}_n$. This system of equations, along with

given initial data $x_i(t_0)$, $i = 1, \dots, 2n$, constitutes an initial value problem. If the functions L and D are sufficiently smooth, as in the preceding example, then for every set of initial data, the initial value problem has unique solutions that can be continued forward for all time. Furthermore, similarly as in the preceding example, this initial value problem determines a dynamical system. \square

2.4 Ordinary Differential Inequalities

Let $J \subset \mathbb{R}$ be a finite or an infinite interval and let D denote a fixed Dini derivative. (For example, if $\varphi \in C[J, \mathbb{R}^n]$, then $D\varphi$ denotes one of the four different Dini derivatives $D^+\varphi$, $D_+\varphi$, $D^-\varphi$, $D_-\varphi$.) Let $g \in C[J \times (\mathbb{R}^+)^n, \mathbb{R}^n]$ where $g(t, 0) \geq 0$ for all $t \in J$. We consider differential inequalities given by

$$Dx \leq g(t, x). \quad (EI)$$

We say that $\varphi \in C[[t_0, t_1), (\mathbb{R}^+)^n]$ is a *solution* of (EI) if $(D\varphi)(t) \leq g(t, \varphi(t))$ for all $t \in [t_0, t_1) \subset J$. Associated with (EI) we consider the *initial value problem*

$$Dx \leq g(t, x), \quad x(t_0) = x_0 \quad (IEI)$$

where $t_0 \in J$ and $x_0 \in \mathbb{R}_+^n \cup \{0\}$ and where $\mathbb{R}_+ = (0, \infty)$. $\varphi \in C[[t_0, t_1), (\mathbb{R}^+)^n]$ is said to be a *solution* of (IEI) if φ is a solution of (EI) and if $\varphi(t_0) = x_0$ (recall that $\mathbb{R}^+ = [0, \infty)$).

For $x_0 \in \mathbb{R}_+^n$, the existence of solutions of (IEI) follows from the existence of the initial value problem

$$\dot{x} = g(t, x), \quad x(t_0) = x_0$$

where $t_0 \in J$ and $x_0 \in \mathbb{R}_+^n$. Note that when $x_0 = 0$, then $\varphi(t) \equiv 0$ is a solution of (IEI) .

Let $T = J$, $A = \mathbb{R}_+^n \cup \{0\} \subset X = (\mathbb{R}^+)^n$, and let X be equipped with the Euclidean metric. Let S_{t_0, x_0} denote the set of all solutions of (IEI) , and let

$$S_{EI} = \bigcup_{(t_0, x_0) \in J \times A} S_{t_0, x_0}.$$

Then S_{EI} is the set of all the solutions of (EI) with their initial values belonging to A . It now follows that $\{T, X, A, S_{EI}\}$ is a dynamical system. We refer to this system simply as *system* S_{EI} . We have occasion to use this system in subsequent chapters as a *comparison system*.

2.5 Difference Equations and Inequalities

The present section consists of two parts.

2.5.1 Difference equations

We now consider systems of *first-order difference equations* of the form

$$x(k+1) = f(k, x(k)) \quad (D)$$

where $k \in \mathbb{N}$, $x(k) \in \mathbb{R}^n$, and $f: \mathbb{N} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Associated with (D) we have the *initial value problem*

$$x(k+1) = f(k, x(k)), \quad x(k_0) = x_0 \quad (I_D)$$

where $k_0 \in \mathbb{N}$, $x_0 \in \mathbb{R}^n$, and $k \in \mathbb{N}_{k_0} \triangleq [k_0, \infty) \cap \mathbb{N}$. We say that an n vector-valued function φ defined on \mathbb{N}_{k_0} is a *solution* of (I_D) if $\varphi(k+1) = f(k, \varphi(k))$ and $\varphi(k_0) = x_0$ for all $k \in \mathbb{N}_{k_0}$. Any solution of (I_D) is also said to be a solution of (D).

Because f in (D) is a function, there are no difficulties that need to be addressed concerning the existence, uniqueness, and continuation of solutions of (I_D) . Indeed, these issues follow readily from induction and the fact that the solutions of (I_D) are defined on \mathbb{N}_{k_0} .

Let $\varphi(\cdot, k_0, x_0): \mathbb{N}_{k_0} \rightarrow \mathbb{R}^n$ denote the unique solution of (I_D) for $x(k_0) = x_0$ and let $S_D = \cup_{(k_0, x_0) \in \mathbb{N} \times \mathbb{R}^n} \{\varphi(\cdot, k_0, x_0)\}$. Then S_D is the set of all possible solutions of (D) defined on \mathbb{N}_{k_0} for all $k_0 \in \mathbb{N}$.

Let $T = \mathbb{N}$ and $X = A = \mathbb{R}^n$ and let X be equipped with the Euclidean metric. Then $\{T, X, A, S_D\}$ is a discrete-time, finite-dimensional dynamical system (see Definition 2.2.3). Moreover, every motion of this dynamical system, which for short we denote by S_D , is continuable forward for all time.

Example 2.5.1. Important examples of dynamical systems determined by difference equations include *second-order sections* of *digital filters* in *direct form*, depicted in the block diagram of Figure 2.1.

In such filters, the type of overflow nonlinearity that is used depends on the type of arithmetic used. Frequently used overflow nonlinearities include the saturation function defined by

$$\text{sat}(\theta) = \begin{cases} 1, & \theta \geq 1 \\ \theta, & -1 < \theta < 1 \\ -1, & \theta \leq -1. \end{cases} \quad (2.10)$$

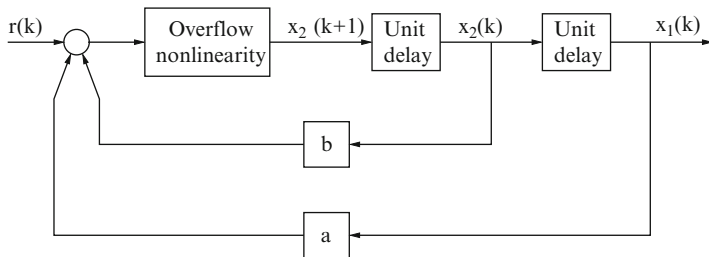


Fig. 2.1 Digital filter in direct form.

Letting r denote the external input to the filter, the equations that describe the filter are now given by

$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = \text{sat}[ax_1(k) + bx_2(k) + r(k)]. \end{cases} \quad (2.11)$$

With $r(k)$ given for $k \in \mathbb{N}$, (2.11) possesses a unique solution $\varphi(k, k_0, x_0)$ for every set of initial data $(k_0, x_0) \in \mathbb{N} \times \mathbb{R}^n$ that exists for all $k \geq k_0$, where $x_0 = [x_1(k_0), x_2(k_0)]^T$. The set of all solutions of (2.11) generated by varying (k_0, x_0) over $\mathbb{N} \times \mathbb{R}^n$, determines a dynamical system. \square

2.5.2 Difference inequalities

We conclude the present section with a brief discussion of *systems of difference inequalities* given by

$$x(k+1) \leq g(k, x(k)) \quad (DI)$$

where $k \in \mathbb{N}$ and $g: \mathbb{N} \times (\mathbb{R}^+)^n \rightarrow (\mathbb{R}^+)^n$ with $g(k, 0) \geq 0$ for all $k \in \mathbb{N}$. A function $\varphi: \mathbb{N}_{k_0} \rightarrow (\mathbb{R}^+)^n$ is a *solution* of (DI) if

$$\varphi(k+1) \leq g(k, \varphi(k))$$

for all $k \in \mathbb{N}_{k_0}$. In this case $\varphi(k_0)$ is an *initial value*. For *any* initial value $x_0 \in (\mathbb{R}^+)^n$, solutions of (DI) exist. For example, the solution of the initial value problem

$$x(k+1) = g(k, x(k)), \quad x(k_0) = x_0$$

is such a solution of (DI) (refer to Subsection 2.5.1 above).

Let $T = \mathbb{N}$, $A = X = (\mathbb{R}^+)^n$ and let S_{DI} denote the set of all solutions of (DI) defined on \mathbb{N}_{k_0} for any $k_0 \in \mathbb{N}$. Then $\{T, X, A, S_{DI}\}$ is a finite-dimensional, discrete-time dynamical system. We have occasion to make use of this system as a *comparison system* in subsequent chapters.

2.6 Differential Equations and Inclusions Defined on Banach Spaces

The present section consists of two parts.

2.6.1 Differential equations defined on Banach spaces

In order to put the presentations of the subsequent sections of this chapter into a clearer context, we briefly consider differential equations in Banach spaces. A general form of a system of first-order differential equations in a Banach space X is given by

$$\dot{x}(t) = F(t, x(t)) \quad (GE)$$

where $F: \mathbb{R}^+ \times C \rightarrow X$, $C \subset X$. Associated with (GE) we have the *initial value problem* given by

$$\dot{x}(t) = F(t, x(t)), \quad x(t_0) = x_0 \quad (I_{GE})$$

where $t_0 \in \mathbb{R}^+$, $t \geq t_0 \geq 0$, and $x_0 \in C \subset X$. Under appropriate assumptions, which ensure the existence of solutions of (GE) , the initial value problem (I_{GE}) determines a continuous-time, infinite-dimensional dynamical system, denoted by S_{GE} , which consists of all the solutions $x(t, t_0, x_0)$ of (I_{GE}) with $x(t_0, x_0, t_0) = x_0$ for all $t_0 \in \mathbb{R}^+$ and $x_0 \in C$.

For the conditions of existence, uniqueness, and continuation of solutions of the initial value problem (I_{GE}) , the reader may want to refer, for example, to Lakshmikantham and Leela [26] and Lasota and Yorke [27]. For example, if F is continuously differentiable, or at least locally Lipschitz continuous, then the theory of existence, uniqueness, and continuation of solutions of (I_{GE}) is essentially the same as for the finite-dimensional case we addressed in Section 2.3 when discussing ordinary differential equations (see, e.g., Dieudonné [11, Chapter 10, Section 4]). This is further demonstrated in Sections 2.7 and 2.8, where we concern ourselves with special classes of dynamical systems defined on Banach spaces, described by functional differential equations and Volterra integrodifferential equations, respectively. In general, however, issues concerning the well posedness of

initial value problems (I_{GE}) can be quite complicated. For example, as shown in Godunov [15], if F in (GE) is only continuous, then (I_{GE}) may not have a solution. Throughout this book, we assume that (I_{GE}) and the associated dynamical systems are well posed.

Important classes of infinite-dimensional continuous-time dynamical systems are determined by partial differential equations. Such systems are addressed in Section 2.10. In the analysis of initial and boundary value problems determined by partial differential equations, semigroups play an important role. Semigroups, which are important in their own right in determining a great variety of dynamical systems, are treated in Section 2.9. We show how such systems may frequently be viewed as special cases of (GE) and (I_{GE}) .

2.6.2 Differential inclusions defined on Banach spaces

In many applications (e.g., in certain classes of partial differential equations), the function F in (GE) may be discontinuous or even multivalued. This generality gives rise to *differential inclusions in Banach spaces*. One such form of systems of differential inclusions is briefly discussed in the following.

Let Ω be an open subset of a Banach space X , let 2^X denote the set of all subsets of X , let \emptyset be the empty set, and let $F: \mathbb{R}^+ \times \Omega \rightarrow 2^X - \emptyset$ be a *set-valued mapping*. We consider systems of differential inclusions given by ([1, 34])

$$\dot{x}(t) \in F(t, x) \quad (GI)$$

where $t \in \mathbb{R}^+$, $x \in \Omega$, and $\dot{x}(t) = dx(t)/dt$. Associated with (GI) , we have the *initial value problem*

$$\dot{x}(t) \in F(t, x), \quad x(t_0) = x_0 \quad (I_{GI})$$

where $t_0 \in \mathbb{R}^+$ and $x_0 \in \Omega$.

A differentiable function φ defined on an interval $[t_0, t_1)$ (t_1 may be infinite) is said to be a *solution of (I_{GI})* if $\varphi(t_0) = x_0$ and if $\dot{\varphi}(t) \in F(t, \varphi(t))$ for all $t \in [t_0, t_1)$. We call any solution of (I_{GI}) a *solution of (GI)* .

Now let

$$S_{GI} = \{\varphi(\cdot, t_0, x_0): \varphi(\cdot, t_0, x_0) \text{ is a solution of } (I_{GI}) \text{ defined on } [t_0, t_1), t_1 > t_0, t_0 \in \mathbb{R}^+, x_0 \in \Omega\}.$$

Then S_{GI} is a dynamical system that we call the *dynamical system determined by (GI)* .

In the following, we consider some specific cases.

Example 2.6.1. Let Ω be an open subset of \mathbb{R}^n and let $f^m, f^M \in C[\mathbb{R}^+ \times \Omega, \mathbb{R}^n]$ where $f^m(t, x) \leq f^M(t, x)$ for all $(t, x) \in \mathbb{R}^+ \times \Omega$ where inequality of vectors is to be interpreted componentwise. Now consider systems of differential inequalities given by

$$f^m(t, x) \leq \dot{x} \leq f^M(t, x) \quad (IE)$$

where $\dot{x} = dx/dt$.

A function $\varphi \in C^1[[t_0, t_1], \Omega]$, where $t_0 \in \mathbb{R}^+$ and where t_1 may be finite or infinite, is said to be a solution of (IE) if for all $t \in [t_0, t_1)$,

$$f^m(t, \varphi(t)) \leq \dot{\varphi}(t) \leq f^M(t, \varphi(t)).$$

We refer to the set of all the solutions of (IE), denoted by S_{IE} , as the *dynamical system determined by (IE)*.

The existence of the solutions of (IE) is guaranteed by the existence of the solutions of systems of ordinary differential equations. Thus, for any $f \in C[\Omega \times \mathbb{R}^+, \mathbb{R}^n]$ satisfying

$$f^m(t, x) \leq f(t, x) \leq f^M(t, x) \quad (IE)$$

for all $(t, x) \in \mathbb{R}^+ \times \Omega$, any solution of the equation

$$\dot{x} = f(t, x) \quad (E)$$

must also be a solution of (IE).

It is clear that S_{IE} is a specific example of a dynamical system determined by differential inclusions. \square

Example 2.6.2. Consider systems described by the set of equations

$$\dot{x} = Ax + Bu \quad (2.12a)$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, \dot{x} = dx/dt$, and

$$u(t) = [g_1(c_1^T x(t - \tau)), \dots, g_m(c_m^T x(t - \tau))]^T \quad (2.12b)$$

where $\tau > 0, C = [c_1, \dots, c_m] \in \mathbb{R}^{m \times n}$, and $g_i \in C[\mathbb{R}, \mathbb{R}], i = 1, \dots, m$, satisfy the *sector conditions*

$$\delta_i \sigma^2 \leq g_i(\sigma) \sigma \leq \Delta_i \sigma^2 \quad (2.12c)$$

where $\Delta_i \geq \delta_i \geq 0, i = 1, \dots, m$.

System (2.12) defines a feedback control system consisting of a linear plant and nonlinear controllers that take transportation delays into account. The sector conditions (2.12c) allow for deterministic uncertainties associated with the control actuators.

We refer to the set of all the solutions of system (2.12a)–(2.12c), denoted by $S_{(2.12)}$, as the dynamical system determined by (2.12). It is clear that $S_{(2.12)}$ is a specific example of a dynamical system determined by differential inclusions. \square

We conclude by noting that the system (2.12a)–(2.12c) is a *differential-difference equation*. Such equations are special cases of *functional differential equations*, which we address next.

2.7 Functional Differential Equations

Let C_r denote the set $C[[-r, 0], \mathbb{R}^n]$ with norm defined by

$$\|\varphi\| = \max \{|\varphi(t)| : -r \leq t \leq 0\}. \quad (2.13)$$

For a given function $x(\cdot)$ defined on $[-r, c)$, $c > 0$, let x_t be the function determined by $x_t(s) = x(t+s)$ for $-r \leq s \leq 0$ and $t \in [0, c)$. A *retarded functional differential equation* with delay r is defined as

$$\dot{x}(t) = F(t, x_t) \quad (F)$$

where $F: \Omega \rightarrow \mathbb{R}^n$ and Ω is an open set in $\mathbb{R} \times C_r$. A differentiable function $p \in C[[t_0 - r, t_0 + c), \mathbb{R}^n]$, $c > 0$, is a solution of equation (F) if $(t, p_t) \in \Omega$ for $t \in [t_0, t_0 + c)$ and $\dot{p} = F(t, p_t)$ for $t \in [t_0, t_0 + c)$.

At first glance it may appear that the functional differential equation (F) is not a special case of the general differential equation (GE) defined on a Banach space X (refer to Subsection 2.6.1), because for the former, the range of the function F is in \mathbb{R}^n (and not in C_r), and for the latter, the range of the function F is in $C \subset X$. However, it turns out that the functional differential equation (F) can be transformed into an equivalent equation which is a special case of (GE). To see this, we note that

$$\begin{aligned} \dot{x}_t(s) &= \dot{x}(t+s), \quad -r \leq s \leq 0 \\ &= \lim_{h \rightarrow 0^+} \frac{1}{h} [x(t+h+s) - x(t+s)], \quad -r \leq s \leq 0 \\ &= \lim_{h \rightarrow 0^+} \frac{1}{h} [x_{t+h}(s) - x_t(s)] \\ &\triangleq \frac{d}{dt} x_t(s). \end{aligned}$$

Defining F_t by

$$F_t(t, x_t)(s) = F(t + s, x_{t+s}), \quad -r \leq s \leq 0,$$

it follows that the functional differential equation (F) can equivalently be expressed by the equation

$$\dot{x}_t = \frac{d}{dt}x_t = F_t(t, x_t) \quad (\tilde{F})$$

which is a special case of equation (GE) because the range of F_t is in C_r .

Example 2.7.1. Linear retarded functional differential equations have the form

$$\dot{x} = L(x_t) \quad (LF)$$

where L is a linear operator defined on C_r given by the Riemann–Stieltjes integral

$$L(\varphi) = \int_{-r}^0 [dB(s)]\varphi(s) \quad (2.14)$$

where $B(s) = [b_{ij}(s)]$ is an $n \times n$ matrix whose entries are functions of bounded variation on $[-r, 0]$ (see, e.g., Yoshizawa [47]).

A special case of (LF) are linear differential-difference equations given by

$$\dot{x}(t) = A_1x(t) + B_1x(t - r) \quad (2.15)$$

where A_1 and B_1 are constant matrices (see, e.g., Bellman and Cooke [4]). □

Example 2.7.2. As a special case of the above example, we consider the scalar equation

$$\dot{x}(t) = \int_{-r}^0 x(t + s)d\eta(s) \quad (2.16)$$

where η is a function of bounded variation on $[-r, 0]$ and the integral in (2.16) denotes a Riemann–Stieltjes integral. Defining $L: C_r \rightarrow \mathbb{R}$ by

$$L(\varphi) = \int_{-r}^0 \varphi(s)d\eta(s),$$

we can rewrite (2.16) as

$$\dot{x}(t) = L(x_t).$$

If in particular, we consider the *scalar differential-difference equation*

$$\dot{x}(t) = ax(t) + bx(t-r), \quad (2.17)$$

where a, b are real constants and $t \in [0, c)$, and if we let

$$\eta(s) = \begin{cases} 0, & s = -r \\ b, & -r < s < 0 \\ a + b, & s = r \end{cases}$$

then we obtain in the present case

$$L(\varphi) = \int_{-r}^0 \varphi(s) d\eta(s) = a\varphi(0) + b\varphi(-r). \quad \square$$

We now associate with (F) the *initial value problem*

$$\dot{x}(t) = F(t, x_t), \quad x_{t_0} = \psi \quad (I_F)$$

where $(t_0, \psi) \in \Omega \subset \mathbb{R} \times C_r$. A function $p \in C[[t_0 - r, t_0 + c), \mathbb{R}^n]$, $c > 0$, is a *solution* of (I_F) if p is a solution of (F) and if $p_{t_0} = \psi$ (i.e., $p_{t_0}(s) = p(t_0 + s) = \psi(s)$ for $-r \leq s < 0$).

If in (F) the function F is continuous, then $p \in C[[t_0 - r, t_0 + c), \mathbb{R}^n]$, $c > 0$, is a solution of (I_F) if and only if

$$\begin{cases} p(t) = \psi(t - t_0), & t_0 - r \leq t \leq t_0 \\ p(t) = \psi(0) + \int_{t_0}^t F(s, p_s) ds, & t > t_0. \end{cases} \quad (2.18)$$

Alternatively, if we define an operator T on the function space $C[[t_0 - r, t_0 + c), \mathbb{R}^n]$ by

$$\begin{cases} (T)(t) = \psi(t - t_0), & t_0 - r \leq t \leq t_0 \\ (T)(t) = \psi(0) + \int_{t_0}^t F(x, p_s) ds, & t > t_0 \end{cases} \quad (2.19)$$

then p is a solution of (I_F) if and only if p is a *fixed point* of the operator T , that is, if and only if $Tp = p$. Note that when p satisfies (2.18), then the continuity of p implies the differentiability of p on $[t_0, c)$.

Similarly as in the case of ordinary differential equations (see Theorem 2.3.1), the following result provides a set of sufficient conditions for the existence of solutions of the initial value problem (I_F) .

Theorem 2.7.1. *Let Ω be an open set in $\mathbb{R} \times C_r$ and let $F \in C[\Omega, \mathbb{R}^n]$. Then for any $(t_0, \psi) \in \Omega$, (I_F) has a solution defined on $[t_0 - r, t_0 + c)$ for some $c > 0$. \square*

In the problem section we provide details for the proof of Theorem 2.7.1.

Similarly as in the case of ordinary differential equations (see Theorem 2.3.2), the next result provides a set of sufficient conditions for the uniqueness of solutions of the initial value problem (I_F) .

Theorem 2.7.2. *Let Ω be an open set in $\mathbb{R} \times C_r$ and assume that on every compact set $K \subset \Omega$, F satisfies the Lipschitz condition*

$$|F(t, x) - F(t, y)| \leq L_K \|x - y\| \quad (2.20)$$

for all $(t, x), (t, y) \in K$, where L_K is a constant that depends only on K , $|\cdot|$ is a norm on \mathbb{R}^n , and $\|\cdot\|$ is the norm defined on C_r in (2.13). Then (I_F) has at most one solution on the interval $[t_0 - r, t_0 + c)$ for any $c > 0$. \square

In the problem section we provide details for the proof of Theorem 2.7.2. Also, in Chapter 4, we prove a more general uniqueness result, applicable to differential equations defined on Banach spaces, in the context of the Lyapunov theory. Theorem 2.7.2 is a special case of that result (refer to Example 4.4.1).

Now let $p \in C[[t_0 - r, b), \mathbb{R}^n]$ be a solution of (F) where $b > t_0$. We say that p_0 is a *continuation* of p if there exists a $b_0 > b$ such that $p_0 \in C[[t_0 - r, b_0), \mathbb{R}^n]$ is a solution of (F) with the property that $p_0(t) = p(t)$ for $t \in [t_0 - r, b)$. A solution p of (F) is said to be *noncontinuable* if no such continuation exists.

Before giving a continuation result for (F) , we recall that a mapping $F: X_1 \rightarrow X_2$, where X_1 and X_2 are metric spaces, is said to be *completely continuous* if F is continuous and if the closure of $F(B) = \{F(x): x \in B\}$ is compact for every bounded closed set $B \subset X_1$.

Theorem 2.7.3. *Let $\Omega = [t_0 - r, a) \times C_r$ where $a > t_0$ is finite or infinite. Assume that $F: \Omega \rightarrow \mathbb{R}^n$ is completely continuous and that every solution of (F) is bounded. Then every solution of (F) can be extended to the entire interval $[t_0 - r, a)$. \square*

In the problem section we provide details for the proof of Theorem 2.7.3. In Chapter 3 we present results that ensure the boundedness of the solutions of (F) , using Lyapunov stability theory.

Now let $A \subset C_r$ be an open set, let $J \subset \mathbb{R}$ be a finite or an infinite interval, and let $\Omega = J \times A$. Assume that $F \in C[\Omega, \mathbb{R}^n]$. Then (I_F) has at least one solution defined on $[t_0 - r, t_0 + c)$ (see Theorem 2.7.1). Let $S_{t_0, \psi}$ denote the set of all the solutions of (I_F) and let $S_F = \cup_{(t_0, \psi) \in \Omega} S_{t_0, \psi}$. Then S_F is the set of the solutions of (F) that are defined on any half closed (resp., half open) interval $[a, b) \subset J$.

Next, let $T = J$ and let $A \subset X = C_r$ with the metric determined by the norm $\|\cdot\|$ given in (2.13). Then $\{T, X, A, S_F\}$ is a dynamical system in the sense of Definition 2.2.3. When T , X , and A are known from the context, we refer to this dynamical system simply as S_F and we speak of the *dynamical system determined by (F)* .

Finally, we note that if in particular $\Omega = \mathbb{R}^+ \times C_r$ and $F: \Omega \rightarrow \mathbb{R}^n$ is completely continuous and if every motion of S_F is bounded, then in view of Theorem 2.7.3, every motion of S_F is continuable forward for all time.

When F in equation (F) is a function of t , x_t , and \dot{x}_t (rather than t and x_t), then the resulting equation is called a *neutral functional differential equation*. As in the case of retarded functional differential equations, such equations determine dynamical systems. We do not pursue systems of this type in this book.

2.8 Volterra Integrodifferential Equations

Volterra integrodifferential equations may be viewed as retarded functional differential equations with infinite delay; that is,

$$\dot{x}(t) = F(t, x_t) \quad (V)$$

where the interval $[-r, 0]$ is replaced by the interval $(-\infty, 0]$. This necessitates the use of a *fading memory space* X which consists of all measurable functions $\varphi: (-\infty, 0] \rightarrow \mathbb{R}^n$ with the property that φ is continuous on $-r \leq t \leq 0$ and that for every $\varphi \in X$, the function $\|\cdot\|$ defined by

$$\|\varphi\| = \sup \{|\varphi(t)|: -r \leq t \leq 0\} + \int_{-\infty}^{-r} p(t)|\varphi(t)|dt \quad (2.21)$$

is finite, where $p: (-\infty, -r) \rightarrow \mathbb{R}$ is a positive, continuously differentiable function such that $\dot{p}(t) \geq 0$ on $(-\infty, -r)$. It can easily be verified that this function is a norm on X .

More generally other choices of norms for X include

$$\|\varphi\| = \sup \{|\varphi(t)|: -r \leq t \leq 0\} + \left[\int_{-\infty}^{-r} p(t)|\varphi(t)|^q dt \right]^{1/q} \quad (2.22)$$

where $q \in [1, \infty)$. If in particular $q = 2$ and $r = 0$, then the norm (2.22) is induced by the inner product

$$\|\varphi\|^2 = \langle \varphi, \varphi \rangle = \langle \varphi(0), \varphi(0) \rangle + \int_{-\infty}^0 p(t) \langle \varphi(t), \varphi(t) \rangle dt. \quad (2.23)$$

It can readily be shown that when X is equipped with (2.22), then $(X, \|\cdot\|)$ is a Banach space and when X is equipped with the inner product (2.23), then $(X, \langle \cdot, \cdot \rangle)$ is a Hilbert space.

Associated with (V) is the *initial value problem*

$$\dot{x}(t) = F(t, x_t), \quad x_{t_0} = \psi \quad (I_V)$$

where $(t_0, \psi) \in \mathbb{R}^+ \times X$. A function $\varphi \in C[(t_0, t_0 + c), \mathbb{R}^n]$, $c > 0$, is a *solution*

of (I_V) if φ is a solution of (V) (i.e., $\dot{\varphi}(t) = F(t, \varphi_t)$ for $t \in [t_0, t_0 + c)$), and if $\varphi_{t_0} = \psi$ (i.e., $\varphi_{t_0}(s) = \varphi(t_0 + s) = \psi(s)$ for $-\infty < s \leq 0$).

We do not present results here concerning the existence, uniqueness, and continuation of solutions of (I_V) . Instead, we refer the reader to Hale [20] for such results.

Let $T = \mathbb{R}^+$ and $A \subset X$, let $S_{t_0, \psi}$ denote the set of all the solutions of (I_V) and let $S_V = \cup_{(t_0, \psi)} S_{t_0, \psi}$. Then S_V denotes the set of all the solutions of (V) that are defined on any interval $[a, b) \subset \mathbb{R}^+$ and $\{T, X, A, S_V\}$ is a dynamical system. When the context is clear, we simply speak of the *dynamical system* S_V .

An important class of Volterra integrodifferential equations are *linear Volterra integrodifferential equations* of the form

$$\dot{x}(t) = Ax_t(0) + \int_{-\infty}^0 K(s)x_t(s)ds \quad (LV)$$

which can equivalently be expressed as

$$\dot{x}(t) = Ax(t) + \int_{-\infty}^t K(s-t)x(s)ds \quad (2.24)$$

for $t \geq 0$, where $A \in \mathbb{R}^{n \times n}$ and $K = [k_{ij}]$ is a matrix-valued function with elements $k_{ij} \in L_1((-\infty, 0), \mathbb{R})$, $1 \leq i, j \leq n$.

Now let

$$X_p = \{\psi: (-\infty, 0] \rightarrow \mathbb{R}^n \text{ and } \psi: (-\infty, 0) \rightarrow \mathbb{R}^n \text{ belong to } L_p((-\infty, 0), \mathbb{R}^n)\}$$

and let X_p be equipped with a norm given by

$$\|\psi\| = |\psi(0)| + \left[\int_{-\infty}^0 |\psi(t)|^p dt \right]^{1/p} \quad (2.25)$$

where $p \in [1, \infty)$, and let

$$Y_p = \{\psi \in X_p: \dot{\psi} \in L_p((-\infty, 0), \mathbb{R}^n) \text{ and} \\ \psi(t) = \psi(0) + \int_0^t \dot{\psi}(s)ds \text{ for all } t \geq 0\}.$$

Associated with (LV) we have the *initial value problem*

$$\begin{cases} \dot{x}(t) = Ax_t(0) + \int_{-\infty}^0 K(s)x_t(s)ds, & t \geq 0 \\ x(t) = \psi(t), & t \leq 0 \end{cases} \quad (I_{LV})$$

where $\psi \in Y_p$.

In Barbu and Grossman [3], the following result is established for (I_{LV}) .

Theorem 2.8.1. *For any $\psi \in Y_p$, the initial value problem (I_{LV}) has a unique solution $x(t, \psi)$ that is defined on $(-\infty, \infty)$. \square*

For any $t_0 \in \mathbb{R}$, let $y(t, \psi, t_0) = x(t - t_0, \psi)$ where $x(t, \psi)$ denotes the unique solution of (I_{LV}) . Let

$$S_{LV} = \{y = y(t, \psi, t_0) : t_0 \in \mathbb{R}, \psi \in Y_p\},$$

let $T = \mathbb{R}$, and let $A = Y_p \subset X_p = X$. Then $\{T, X, A, S_{LV}\}$ is a dynamical system, which for short, we simply refer to as *dynamical system S_{LV}* , or as the *dynamical system determined by (LV)* .

In the following example we consider a simple model of the dynamics of a multicore nuclear reactor. We revisit this model in Chapter 9.

Example 2.8.1 ([31] Point kinetics model of a multicore nuclear reactor). We consider the point kinetics model of a multicore nuclear reactor with l cores described by the equations

$$\left\{ \begin{array}{l} \Lambda_i \dot{p}_i(t) = [\rho_i(t) - \varepsilon_i - \beta_i] p_i(t) + \rho_i(t) + \sum_{k=1}^6 \beta_{ki} c_{ki}(t) \\ \quad + \sum_{j=1}^l \varepsilon_{ji} (P_{j0}/P_{i0}) \int_{-\infty}^t h_{ji}(t-s) p_j(s) ds \\ \dot{c}_{ki}(t) = \lambda_{ki} [p_i(t) - c_{ki}(t)], \quad i = 1, \dots, l, \quad k = 1, \dots, 6 \end{array} \right. \quad (2.26)$$

where $p_i: \mathbb{R} \rightarrow \mathbb{R}$ and $c_{ki}: \mathbb{R} \rightarrow \mathbb{R}$ represent the power in the i th core and the concentration of the k th precursor in the i th core, respectively. The constants Λ_i , ε_i , β_{ki} , ε_{ji} , P_{i0} , and λ_{ki} are all positive and

$$\beta_i = \sum_{k=1}^6 \beta_{ki}.$$

The functions $h_{ji} \in L_1(\mathbb{R}^+, \mathbb{R})$. They determine the coupling between cores due to neutron migration from the j th to the i th core. The function ρ_i represents the reactivity of the i th core which we assume to have the form

$$\rho_i(t) = \int_{-\infty}^t w_i(t-s) p_i(s) ds \quad (2.27)$$

where $w_i \in L_1[\mathbb{R}^+, \mathbb{R}]$. The functions $p_i(t)$ and $c_{ki}(t)$ are assumed to be known, bounded, continuous functions defined on $-\infty < t < 0$.

In the present context, a physically realistic assumption is that $c_{ki}(t)e^{\lambda_{ki}t} \rightarrow 0$ as $t \rightarrow -\infty$. Under this assumption, we can solve for c_{ki} in terms of p_i to obtain

$$c_{ki}(t) = \int_{-\infty}^t \lambda_{ki} e^{-\lambda_{ki}(t-s)} p_i(s) ds. \quad (2.28)$$

Using (2.27) and (2.28) to eliminate ρ_i and c_{ki} from (2.26), we obtain l Volterra integrodifferential equations for $p_i(t)$, $i = 1, \dots, l$. To express these equations in a more compact form, we let

$$F_i(t) = \Lambda_i^{-1} \left[w_i(t) + \sum_{k=1}^6 \beta_{ki} \lambda_{ki} e^{-\lambda_{ki} t} + \varepsilon_{ii} h_{ii} \right],$$

$$K_i = \Lambda_i^{-1} [\varepsilon_i + \beta_i],$$

$$n_i(t) = \Lambda_i^{-1} w_i(t), \quad \text{and}$$

$$G_{ij} = \frac{\varepsilon_{ij} P_{j0} h_{ji}(t)}{\Lambda_i P_{i0}}.$$

With $p_i(t)$ defined on $-\infty < t < \infty$, we have

$$\begin{aligned} \dot{p}_i(t) = & -K_i p_i(t) + \int_{-\infty}^t F_i(t-s) p_i(s) ds + p_i(t) \int_{-\infty}^t n_i(t-s) p_i(s) ds \\ & + \sum_{j=1, j \neq i}^l \int_{-\infty}^t G_{ij}(t-s) p_j(s) ds, \quad i = 1, \dots, l \end{aligned}$$

for $t \geq 0$ and $p_i(t) = \varphi_i(t)$ defined on $-\infty < t \leq 0$ where $\varphi_i \in Z_i$, the fading memory space of all absolutely continuous functions ψ_i defined on $(-\infty, 0]$ such that

$$\|\psi_i\|^2 = |\psi_i(0)|^2 + \int_{-\infty}^0 |\psi_i(s)|^2 e^{L_i s} ds < \infty,$$

where $L_i > 0$ is a constant. We address the choice of L_i in Chapter 9, when studying the stability properties of (2.26). The set of all solutions of system (2.26), generated by varying φ_i over Z_i , $i = 1, \dots, l$, determines a dynamical system. \square

2.9 Semigroups

We now address linear and nonlinear semigroups that generate large classes of dynamical systems. Before addressing the subject on hand we need to introduce some additional notation.

2.9.1 Notation

Let X and Z denote Banach spaces and let $\|\cdot\|$ denote norms on such spaces. Also, Hilbert spaces are denoted X, Z , or H with inner product $\langle \cdot, \cdot \rangle$. In this case, the norm of $x \in H$ is given by $\|x\| = \langle x, x \rangle^{1/2}$.

Let A be a linear operator defined on a domain $D(A) \subset X$ with range in Z . We call A *closed* if its graph, $Gr(A) = \{(x, Ax) \in X \times Z : x \in D(A)\}$ is a closed subset of $X \times Z$ and we call A *bounded* if it maps bounded sets in X into bounded sets in Z , or equivalently, if it is continuous.

Subsequently, $I: X \rightarrow X$ denotes the identity transformation. Given a closed linear operator $A: D(A) \rightarrow X, D(A) \subset X$, we define the *resolvent set* of A , $\rho(A)$, as the set of all points λ in the complex plane such that the linear transformation $(A - \lambda I)$ has a bounded inverse, $(A - \lambda I)^{-1}: X \rightarrow X$. The complement of $\rho(A)$, denoted $\sigma(A)$, is called the *spectral set* or the *spectrum* of A .

Finally, given a bounded linear operator $A: D(A) \rightarrow Z, D(A) \subset X$, its norm is defined by

$$\|A\| = \sup \{\|Ax\| : \|x\| = 1\}.$$

2.9.2 C_0 -semigroups

Consider a process whose evolution in time can be described by a linear differential equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \in D(A) \tag{I_L}$$

for $t \in \mathbb{R}^+$. Here $A: D(A) \rightarrow X$ is assumed to be a linear operator with domain $D(A)$ dense in X . Moreover, A is always assumed to be closed or else to have an extension \bar{A} that is closed. By a *strong solution* $x(t)$ of (I_L) we mean a function $x: \mathbb{R}^+ \rightarrow D(A)$ such that $\dot{x}(t)$ exists and is continuous on $\mathbb{R}^+ \rightarrow X$ and such that (I_L) is true. The *abstract initial value problem* (I_L) is said to be *well posed* if for each $x_0 \in D(A)$, there is one and only one strong solution $x(t, x_0)$ of (I_L) defined on $0 \leq t < \infty$ and if in addition $x(t, x_0)$ *depends continuously* on (t, x_0) in the sense that given any $N > 0$ there is an $M > 0$ such that $\|x(t, x_0)\| \leq M$ when $0 \leq t \leq N$ and $\|x_0\| \leq N$.

If (I_L) is well posed, then there is an operator T defined by $T(t)x_0 = x(t, x_0)$ which is (for each fixed t) a bounded linear mapping from $D(A)$ to X . We call $T(t)x_0 = x(t, x_0), t \geq 0$, a *trajectory* of (I_L) for x_0 . Because $T(t)$ is bounded, it has a continuous extension from $D(A)$ to the larger domain X . The trajectories $x(t, x_0) = T(t)x_0$ for $x_0 \in X$ but $x_0 \notin D(A)$ are called *generalized solutions* of (I_L) . The resulting family of operators $\{T(t): t \in \mathbb{R}^+\}$ is called a C_0 -semigroup or a *linear semigroup*.

Independent of the above discussion, we now define C_0 -semigroup.

Definition 2.9.1 ([21, 23, 39]). A one-parameter family of bounded linear operators $T(t): X \rightarrow X, t \in \mathbb{R}^+$, is said to be a C_0 -semigroup, or a *linear semigroup*, if

- (i) $T(0) = I$ (I is the identity operator on X);
- (ii) $T(t + s) = T(t)T(s)$ for any $t, s \in \mathbb{R}^+$; and
- (iii) $\lim_{t \rightarrow 0^+} T(t)x = x$ for all $x \in X$. □

Evidently, every C_0 -semigroup is generated by some abstract differential equation of the form (I_L) .

Definition 2.9.2. Given any C_0 -semigroup $T(t)$, its *infinitesimal generator* is the operator defined by

$$Ax = \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t}$$

where $D(A)$ consists of all $x \in X$ for which this limit exists. □

Theorem 2.9.1 ([39]). For a C_0 -semigroup $T(t)$, there exist an $\omega \geq 0$ and an $M \geq 1$ such that

$$\|T(t)\| \leq Me^{\omega t}. \quad \square$$

The next result provides necessary and sufficient conditions for a given linear operator A to be the infinitesimal generator of some C_0 -semigroup.

Theorem 2.9.2 ([21, 39] **Hille–Yoshida–Phillips Theorem**). A linear operator A is the infinitesimal generator of a C_0 -semigroup $T(t)$ satisfying $\|T(t)\| \leq Me^{\omega t}$, if and only if

- (i) A is closed and $D(A)$ is dense in X ; and
- (ii) the resolvent set $\rho(A)$ of A contains (ω, ∞) and

$$\|(A - \lambda I)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n} \quad \text{for all } \lambda > \omega,$$

$n = 1, 2, \dots$, where I denotes the identity operator on X . □

A C_0 -semigroup of contractions is a C_0 -semigroup $T(t)$ satisfying $\|T(t)\| \leq 1$ (i.e., in Theorem 2.9.1, $M = 1$ and $\omega = 0$). Such semigroups are of particular interest in Hilbert spaces.

Definition 2.9.3. A linear operator $A: D(A) \rightarrow H, D(A) \subset H$, on a Hilbert space H is said to be *dissipative* if $\operatorname{Re}\langle Ax, x \rangle \leq 0$ for all $x \in D(A)$. □

For C_0 -semigroups of contractions we have the following result.

Theorem 2.9.3. If A is the infinitesimal generator of a C_0 -semigroup of contractions on a Hilbert space H , then A is dissipative and the range of $(A - \lambda I)$ is all of H for any $\lambda > 0$. Conversely, if A is dissipative and if the range of $(A - \lambda I)$ is H for at least one constant $\lambda_0 > 0$, then A is closed and A is the infinitesimal generator of a C_0 -semigroup of contractions. □

The above result is useful in the study of parabolic partial differential equations (Section 2.10).

For linear semigroups with generator A one can deduce many important qualitative properties by determining the spectrum of A . Some of these are summarized in the following results (refer to Slemrod [42]).

Theorem 2.9.4. *Given any two real numbers α and β with $\alpha < \beta$ there exists a C_0 -semigroup $T(t)$ on a Hilbert space H such that $\operatorname{Re} \lambda \leq \alpha$ for all $\lambda \in \sigma(A)$ and in addition $\|T(t)\| = e^{\beta t}$ for all $t \geq 0$. \square*

The next result applies to the following class of semigroups.

Definition 2.9.4. A C_0 -semigroup $T(t)$ is called *differentiable* for $t > r$ if for each $x \in X$, $T(t)x$ is continuously differentiable on $r < t < \infty$. \square

For example, a system of linear time-invariant functional differential equations with delay $[-r, 0]$ (as discussed in the last subsection of this section) determines a semigroup that is differentiable for $t > r$. Also, systems of parabolic partial differential equations (as discussed in the next section) normally generate semigroups that are differentiable for $t > 0$. In the finite-dimensional case (when $X = \mathbb{R}^n$), for linear semigroups the generator A must be an $n \times n$ matrix whose spectrum is the set of eigenvalues $\{\lambda\}$ of A . Such semigroups are differentiable as well for $t > 0$.

Following Slemrod [42] we have the following result.

Theorem 2.9.5. *If $T(t)$ is a C_0 -semigroup that is differentiable for $t > r$, if A is its generator, and if $\operatorname{Re} \lambda \leq -\alpha_0$ for all $\lambda \in \sigma(A)$, then given any positive $\alpha < \alpha_0$, there is a constant $K(\alpha) > 0$ such that $\|T(t)\| \leq K(\alpha)e^{-\alpha t}$ for all $t > r$. \square*

We conclude by defining the *dynamical system determined by a C_0 -semigroup $T(t)$* as

$$S_{C_0} = \{p = p(\cdot, x_0, t_0): p(t, x, t_0) \triangleq T(t - t_0)x, t_0 \in \mathbb{R}^+, t \geq t_0, x \in X\}.$$

We consider some specific examples of dynamical systems determined by C_0 -semigroups in the last subsection of this section.

2.9.3 Nonlinear semigroups

A nonlinear semigroup is a generalization of the notion of C_0 -semigroup. In arriving at this generalization, the linear initial value problem (I_L) is replaced by the nonlinear initial value problem

$$\dot{x}(t) = A(x(t)), \quad x(0) = x_0 \quad (I_N)$$

where $A: D(A) \rightarrow X$ is a nonlinear mapping. As mentioned already in Section 2.6 (in connection with initial value problem (I_{GE})) if A is continuously differentiable (or at least locally Lipschitz continuous), then the theory of existence, uniqueness,

and continuation of solutions of (I_N) is the same as in the finite-dimensional case (see Dieudonné [11, Chapter 10, Section 4]). If A is only continuous, then (I_N) need not to have any solution at all (see Dieudonné [11, p. 287, Problem 5]). In general, one wishes to have a theory that includes nonlinear partial differential equations. This mandates that A be allowed to be only defined on a dense set $D(A)$ and to be discontinuous. For such functions A , the accretive property (defined later) generalizes the Lipschitz property.

Definition 2.9.5 ([5, 8, 9, 15, 25, 27]). Assume that C is a subset of a Banach space X . A family of one-parameter (nonlinear) operators $T(t): C \rightarrow C$, $t \in \mathbb{R}^+$, is said to be a *nonlinear semigroup* defined on C if

- (i) $T(0)x = x$ for $x \in C$;
- (ii) $T(t+s)x = T(t)T(s)x$ for $t, s \in \mathbb{R}^+$, $x \in C$; and
- (iii) $T(t)x$ is continuous in (t, x) on $\mathbb{R}^+ \times C$. □

A nonlinear semigroup $T(t)$ is called a *quasi-contractive* semigroup if there is a number $w \in \mathbb{R}$ such that

$$\|T(t)x - T(t)y\| \leq e^{wt} \|x - y\| \quad (2.29)$$

for all $t \in \mathbb{R}^+$ and for all $x, y \in C$. If in (2.29) $w \leq 0$, then $T(t)$ is called a *contraction semigroup*. Note that $C = X$ is allowed as a special case.

The mapping A in (I_N) is sometimes multivalued (i.e., a relation) and in general must be extended to be multivalued if it is to generate a quasi-contractive semigroup. Thus, we assume that $A(x)$, $x \in X$, is a subset of X and we identify A with its graph,

$$Gr(A) = \{(x, y): x \in X \text{ and } y \in A(x)\} \subset X \times X.$$

In this case the *domain* of A , written as $D(A)$, is the set of all $x \in X$ for which $A(x) \neq \emptyset$, the *range* of A is the set

$$Ra(A) = \cup \{A(x): x \in D(A)\},$$

and the *inverse* of A at any point y is defined as the set

$$A^{-1}(y) = \{x \in X: y \in A(x)\}.$$

Let λ be a real or complex scalar. Then λA is defined by

$$(\lambda A)(x) = \{\lambda y: y \in A(x)\}$$

and $A + B$ is defined by

$$(A + B)(x) = A(x) + B(x) = \{y + z: y \in A(x), z \in B(x)\}.$$

Definition 2.9.6. A multivalued operator A is said to generate a nonlinear semigroup $T(t)$ on C if

$$T(t)x = \lim_{n \rightarrow \infty} \left(I - \frac{t}{n} A \right)^{-n} (x)$$

for all $x \in C$.

The *infinitesimal generator* A_s of a nonlinear semigroup $T(t)$ is defined by

$$A_s(x) = \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t}, \quad x \in D(A_s)$$

for all x such that this limit exists. The operator A and the infinitesimal generator A_s are generally different operators. \square

Definition 2.9.7. A multivalued operator A on X is said to be *w-accretive* if

$$\|(x_1 - \lambda y_1) - (x_2 - \lambda y_2)\| \geq (1 - \lambda w) \|x_1 - x_2\| \quad (2.30)$$

for all $\lambda \geq 0$ and for all $x_i \in D(A)$ and $y_i \in A(x_i)$, $i = 1, 2$. \square

If, in particular, X is a Hilbert space, then (2.30) reduces to

$$\langle (wx_1 - y_1) - (wx_2 - y_2), x_1 - x_2 \rangle \geq 0. \quad (2.31)$$

The above property for the nonlinear case is analogous to $(A - wI)$ being dissipative in the linear symmetric case.

Theorem 2.9.6. Assume that A is *w-accretive* and that for each $\lambda \in (0, \lambda_0)$,

$$Ra(I - \lambda A) \supset C = \overline{D(A)}$$

where $\overline{D(A)}$ denotes the closure of $D(A)$ and $\lambda_0 > 0$ is a constant. Then A generates a quasi-contractive semigroup $T(t)$ on C with

$$\|T(t)x - T(t)y\| \leq e^{wt} \|x - y\|$$

for all $t \in \mathbb{R}^+$ and for all $x, y \in C$. \square

In general, the trajectories $T(t)x$ determined by the semigroup in Theorem 2.9.6 are generalized solutions of (I_N) that need not be differentiable. Indeed, an example is discussed in Crandall and Liggett [9, Section 4], where $w = 0$, $\overline{D(A)} = X$, A generates a quasi-contraction $T(t)$ but the infinitesimal generator A_s has an empty domain. This means that not even one trajectory $T(t)x$ is differentiable at even one time t . If the graph of A is closed, then A is always an extension of the infinitesimal generator A_s . So whenever $x(t) = T(t)x$ has a derivative, then $\dot{x}(t)$ must be in $A(x(t))$.

The situation is more reasonable in the setting of a Hilbert space H . If A is w -accretive and closed (i.e., its graph is a closed subset of $H \times H$), then for any $x \in D(A)$ the set $A(x)$ is closed and convex. Thus, there is an element $A^0(x) \in A(x)$ such that $A^0(x)$ is the element of $A(x)$ closest to the origin. Given a trajectory $x(t) = T(t)x$, the right derivative

$$D^+x(t) = \lim_{h \rightarrow 0^+} \frac{x(t+h) - x(t)}{h}$$

must exist at all points $t \in \mathbb{R}^+$ and be continuous except possibly at a countably infinite set of points. The derivative $\dot{x}(t)$ exists and is equal to $D^+x(t)$ at all points where $D^+x(t)$ is continuous. Furthermore,

$$D^+x(t) = A^0(x(t))$$

for all $t \geq 0$. These results can be generalized to any space X that is uniformly convex. (Refer to Dunford and Schwarz [12, p. 74], for the definition of a uniformly convex space. In particular, any L_p space, $1 < p < \infty$, is a uniformly convex space.)

Definition 2.9.8. A trajectory $x(t) = T(t)x_0$ is called a *strong solution* of (I_N) if $x(t)$ is absolutely continuous on any bounded subset of \mathbb{R}^+ (so that $\dot{x}(t)$ exists almost everywhere) if $x(t) \in D(A)$ and if $\dot{x}(t) \in A(x(t))$ almost everywhere on \mathbb{R}^+ . \square

We also have

Definition 2.9.9. The initial value problem (I_N) is called *well posed* on C if there is a semigroup $T(t)$ such that for any $x_0 \in D(A)$, $T(t)x_0$ is a strong solution of (I_N) , and if $\overline{D(A)} = C$. \square

We summarize the above discussion in the following theorem (see [9, p. 267]).

Theorem 2.9.7. If X is a Hilbert space or a uniformly convex Banach space, A is w -accretive and closed, and

$$Ra(I - \lambda A) \supset clco \ D(A)$$

for all sufficiently small positive λ , where $clco$ denotes the closure of the convex hull, then the initial value problem (I_N) is well posed on $C = \overline{D(A)}$ and $\dot{x}(t) = A^0(x(t))$ almost everywhere on \mathbb{R}^+ . \square

We conclude by defining the dynamical system determined by a nonlinear semigroup $T(t)$ as

$$S_N = \{p = p(\cdot, x, t_0): p(t, x, t_0) \triangleq T(t - t_0)x, t_0 \in \mathbb{R}^+, t \geq t_0, x \in C\}.$$

We consider in the next subsection several specific examples of semigroups.

2.9.4 Examples of semigroups

We now consider several classes of important semigroups that arise in applications and we provide some related background material which we find useful in subsequent chapters.

Example 2.9.1 (Ordinary differential equations). Consider initial value problems described by a system of *autonomous* first-order ordinary differential equations given by

$$\dot{x} = g(x), \quad x(0) = x_0 \quad (2.32)$$

where $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and where it is assumed that g satisfies the Lipschitz condition

$$|g(x) - g(y)| \leq L|x - y| \quad (2.33)$$

for all $x, y \in \mathbb{R}^n$. In this case g is w -accretive with $w = L$ and (2.33) implies that g is continuous on \mathbb{R}^n . This continuity implies that the graph of g is closed. By Theorem 2.9.7 there exist a semigroup $T(t)$ and a subset $D \subset \mathbb{R}^n$ such that D is dense in \mathbb{R}^n and for any $x_0 \in D$, any solution $x(t) = T(t)x_0$ of (2.32) is absolutely continuous on any finite interval in \mathbb{R}^+ . In the present case $D = \mathbb{R}^n$ and $T(t)$ is a quasi-contractive semigroup with

$$|T(t)x - T(t)y| \leq e^{Lt}|x - y| \quad (2.34)$$

for all $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$.

Now assume that in (2.32) $g(x) = Ax$ where $A \in \mathbb{R}^{n \times n}$; that is,

$$\dot{x} = Ax, \quad x(0) = x_0. \quad (2.35)$$

In the present case (2.35) determines a differentiable C_0 -semigroup with generator A . The spectrum of A , $\sigma(A)$, coincides with the set of all eigenvalues of A , $\{\lambda\}$. Now according to Theorem 2.9.5, if $\operatorname{Re} \lambda \leq -\alpha_0$ for all $\lambda \in \sigma(A)$, where $\alpha_0 > 0$ is a constant, then given any positive $\alpha < \alpha_0$, there is a constant $K(\alpha) > 0$ such that

$$\|T(t)\| \leq K(\alpha)e^{-\alpha t}, \quad t \in \mathbb{R}^+. \quad (2.36)$$

□

Example 2.9.2 (Functional differential equations). Consider initial value problems described by a system of *autonomous* first-order functional differential equations

$$\begin{cases} \dot{x}(t) = F(x_t), & t > 0 \\ x(t) = \psi(t), & -r \leq t \leq 0 \end{cases} \quad (2.37)$$

where $F: C_r \rightarrow \mathbb{R}^n$. (For the notation used in this example, refer to Section 2.7.) Assume that F satisfies the Lipschitz condition

$$|F(\xi) - F(\eta)| \leq K \|\xi - \eta\| \quad (2.38)$$

for all $\xi, \eta \in C_r$. Under these conditions, the initial value problem (2.37) has a unique solution for every initial condition ψ , denoted by $p(t, \psi)$ which is defined for all $t \in \mathbb{R}^+$ (refer to Section 2.7). In this case $T(t)\psi = p_t(\cdot, \psi)$, or equivalently, $(T(t)\psi)(s) = p(t + s, \psi)$ defines a quasi-contractive semigroup on C_r . Define $A: D(A) \rightarrow C_r$ by

$$A\psi = \dot{\psi}, \quad D(A) = \{\psi \in C_r: \dot{\psi} \in C_r \text{ and } \dot{\psi}(0) = F(\psi)\}. \quad (2.39)$$

Then $D(A)$ is dense in C_r , A is the generator and also the infinitesimal generator of $T(t)$, and $T(t)$ is differentiable for $t > r$.

If in (2.37) $F = L$ is the linear mapping from C_r to \mathbb{R}^n defined in (2.14), we have

$$\dot{x} = L(x_t) \quad (2.40)$$

where

$$L(\varphi) = \int_{-r}^0 [dB(s)]\varphi(s). \quad (2.41)$$

In this case the semigroup $T(t)$ is a C_0 -semigroup. The spectrum of its generator consists of all solutions of the equation

$$\det \left(\int_{-r}^0 e^{\lambda s} [dB(s)] - \lambda I \right) = 0. \quad (2.42)$$

If all solutions of (2.42) satisfy the relation $\operatorname{Re} \lambda \leq -\gamma_0$ for some $\gamma_0 > 0$, then given any positive $\gamma < \gamma_0$, there is a constant $K(\gamma) > 0$ such that

$$\|T(t)\| \leq K(\gamma)e^{-\gamma t}, \quad t \in \mathbb{R}^+ \quad (2.43)$$

(refer to Theorem 2.9.5). □

Example 2.9.3 (Volterra integrodifferential equations). We consider the class of Volterra integrodifferential equations given in Section 2.8,

$$\begin{cases} \dot{x}(t) = Ax(t) + \int_{-\infty}^t K(s-t)x(s)ds, & t \geq 0 \\ x(t) = \varphi(t), & -\infty < t \leq 0 \end{cases} \quad (2.44)$$

where $A \in \mathbb{R}^{n \times n}$ and $K \in L_1[(-\infty, 0), \mathbb{R}^{n \times n}]$; that is, K is an $n \times n$ matrix-valued function whose entries $k_{ij} \in L_1[(-\infty, 0), \mathbb{R}]$. Let X_p , $1 \leq p < \infty$, be defined as in Section 2.8. Then

$$X_p \simeq L_p[(-\infty, 0), \mathbb{R}^n] \times \mathbb{R}^n \quad (2.45)$$

where \simeq denotes an isomorphic relation. To see this, note that for any $\varphi \in X_p$, $\varphi|_{(-\infty, 0)} \in L_p[(-\infty, 0), \mathbb{R}^n]$, $\varphi(0) \in \mathbb{R}^n$. Conversely, for any $\psi \in L_p[(-\infty, 0), \mathbb{R}^n]$ and $Z \in \mathbb{R}^n$, there is a unique $\varphi \in X_p$ such that $\varphi|_{(-\infty, 0)} = \psi$, and $\varphi(0) = Z$. In this case, if we denote $\varphi = (Z, \psi)$, the norm defined by (2.25) can now be written as

$$\|\varphi\| = \|(Z, \psi)\| = |Z| + \left[\int_{-\infty}^0 |\psi(s)|^p ds \right]^{1/p}, \quad 1 \leq p < \infty. \quad (2.46)$$

We now define an operator \tilde{A} by

$$\tilde{A}(Z, \psi) = \left(AZ + \int_{-\infty}^0 K(s)\psi(s)ds, \dot{\psi} \right) \quad (2.47)$$

on the domain

$$D(\tilde{A}) = \{(Z, \psi): \dot{\psi} \in L_p[(-\infty, 0), \mathbb{R}^n] \text{ and } \psi(t) = Z + \int_0^t \dot{\psi}(s)ds \text{ for all } t \leq 0\}. \quad (2.48)$$

Then \tilde{A} is an infinitesimal generator of a C_0 -semigroup $T(t)$ on X_p . Furthermore, when $(Z, \psi) \in D(\tilde{A})$, the equation

$$(x(t), x_t) = T(t)(Z, \psi) \quad (2.49)$$

determines a function $x(t)$ which is the unique solution of (2.44) (refer to Barbu and Grossman [3]).

If $\operatorname{Re} \lambda > 0$, then $\lambda \in \sigma(\tilde{A})$ if and only if

$$\det \left(A + \int_{-\infty}^0 e^{\lambda s} K(s)ds - \lambda I \right) = 0. \quad (2.50)$$

On the other hand, if $\operatorname{Re} \lambda \leq 0$, then λ is always in $\sigma(\tilde{A})$. □

There are many other important classes of semigroups, including those that are determined by partial differential equations. We address some of these in the next section.

2.10 Partial Differential Equations

In our discussion of partial differential equations we require additional nomenclature.

2.10.1 Notation

A *vector index* or *exponent* is a vector $\alpha^T = (\alpha_1, \dots, \alpha_n)$ whose components are nonnegative integers, $|\alpha| = \sum_{j=1}^n \alpha_j$, and for any $x \in \mathbb{R}^n$,

$$x^\alpha = (x_1, x_2, \dots, x_n)^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Let $D_k = i(\partial/\partial x_k)$ for $k = 1, \dots, n$, where $i = (-1)^{1/2}$ and let $D = (D_1, D_2, \dots, D_n)$ so that

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}. \quad (2.51)$$

In the sequel we let Ω be a domain in \mathbb{R}^n (i.e., Ω is a connected open set) with boundary $\partial\Omega$ and closure $\overline{\Omega}$. We assume that $\partial\Omega$ is of *class* C^k for suitable $k \geq 1$. By this we mean that for each $x \in \partial\Omega$, there is a ball B with center at x such that $\partial\Omega \cap B$ can be represented in the form

$$x_i = \varphi(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

for some $i, i = 1, \dots, n$, with φ continuously differentiable up to order k . This smoothness is easily seen to be true for the type of regions that normally occur in applications.

Also, let $C^l[\Omega, \mathbb{C}]$ denote the set of all complex-valued functions defined on Ω whose derivatives up to order l are continuous. For $u \in C^l[\Omega, \mathbb{C}]$, $l \in \mathbb{N}$, we define the norm

$$\|u\|_l = \left(\int_{\Omega} \sum_{|\alpha| \leq l} |D^\alpha u|^2 \right)^{1/2}. \quad (2.52)$$

Let

$$\tilde{C}^l[\Omega, \mathbb{C}] = \{u \in C^l[\Omega, \mathbb{C}]: \|u\|_l < \infty\}$$

and let

$$C_0^l[\Omega, \mathbb{C}] = \{u \in C^l[\Omega, \mathbb{C}]: u = 0 \text{ in a neighborhood of } \partial\Omega\}.$$

We define $H^l[\Omega, \mathbb{C}]$ and $H_0^l[\Omega, \mathbb{C}]$ to be the completions in the norm $\|\cdot\|_l$ of the spaces $\tilde{C}^l[\Omega, \mathbb{C}]$ and $C_0^l[\Omega, \mathbb{C}]$, respectively. In a similar manner, we can define the spaces $H^l[\Omega, \mathbb{R}]$ and $H_0^l[\Omega, \mathbb{R}]$. The spaces defined above are sometimes called *Sobolev spaces*. Their construction builds “zero boundary conditions” into, for example, $H_0^l[\Omega, \mathbb{R}]$.

Finally, we define $C^\infty[\Omega, \mathbb{C}] = \cap_{l \in \mathbb{N}} C^l[\Omega, \mathbb{C}]$ and we say that $u \in C^\infty[\bar{\Omega}, \mathbb{C}]$ if $D^\alpha u$ can be extended to be a continuous function on $\bar{\Omega}$ for any $\alpha \in \mathbb{N}^n$. We define $C^\infty[\Omega, \mathbb{R}]$ and $C^\infty[\bar{\Omega}, \mathbb{R}]$ in a similar manner. Occasionally, we say that u is a real-valued *smooth function on Ω* (on $\bar{\Omega}$) if $u \in C^\infty[\Omega, \mathbb{R}]$ (if $u \in C^\infty[\bar{\Omega}, \mathbb{R}]$). Complex-valued smooth functions on Ω (on $\bar{\Omega}$) are defined similarly.

2.10.2 Linear equations with constant coefficients

Given $r \times r$ complex constant square matrices A_α , $\alpha \in \mathbb{N}^n$, let

$$A(D) = \sum_{|\alpha| \leq m} A_\alpha D^\alpha,$$

and consider the initial value problem

$$\frac{\partial u}{\partial t}(t, x) = A(D)u(t, x), \quad u(0, x) = \psi(x) \quad (I_P)$$

where $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, $\psi \in L_2[\mathbb{R}^n, \mathbb{C}^r]$ are given, and $u: \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{C}^r$ is to be determined.

Proceeding intuitively for the moment, we apply L_2 -Fourier transforms to (I_P) to obtain

$$\frac{\partial \tilde{u}(t, \omega)}{\partial t} = A(\omega) \tilde{u}(t, \omega), \quad \tilde{u}(0, \omega) = \tilde{\psi}(\omega)$$

where $A(\omega) = \sum_{|\alpha| \leq m} A_\alpha \omega^\alpha$ for all $\omega \in \mathbb{R}^n$. In order to have a solution such that $u(t, x)$ and $(\partial u / \partial t)(t, x)$ are in L_2 over $x \in \mathbb{R}^n$, it is necessary that $A(\omega) \tilde{u}(t, \omega)$ be in L_2 over $\omega \in \mathbb{R}^n$. This places some restrictions on $A(\omega)$. For the proof of the next result, refer to Krein [23, p. 163].

Theorem 2.10.1. *The mapping $T(t)\psi = u(t, \cdot)$ defined by the solutions $u(t, x)$ of (I_P) determines a C_0 -semigroup on $X = L_2[\mathbb{R}^n, \mathbb{C}]$ if and only if there exists a nonsingular matrix $S(\omega)$ and a constant $K > 0$ such that for all $\omega \in \mathbb{R}^n$, the following conditions are satisfied.*

- (i) $|S(\omega)| \leq K$ and $|S(\omega)^{-1}| \leq K$.
- (ii) $S(\omega)A(\omega)S(\omega)^{-1} = [C_{ij}(\omega)]$ is upper triangular.
- (iii) $\operatorname{Re} C_{rr}(\omega) \leq \cdots \leq \operatorname{Re} C_{11}(\omega) \leq K$.
- (iv) $|C_{ik}(\omega)| \leq K(1 + |\operatorname{Re} C_{ii}(\omega)|)$ for $k = i + 1, \dots, r$.

□

Parabolic equations (i.e., equations for which $A(D)$ is strongly elliptic, defined later) satisfy these conditions whereas hyperbolic equations do not. We demonstrate this in the next examples.

Example 2.10.1. Consider a special case of (I_P) with $r = 1, m = n = 2$, given by

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu \\ u(0, x) = \psi(x). \end{cases} \quad (2.53)$$

For $\omega = (\omega_1, \omega_2)^T \in \mathbb{R}^2$ we have

$$A(\omega) = -\omega_1^2 - \omega_2^2 + ia\omega_1 + ib\omega_2 + c = C_{11}(\omega).$$

Clearly, $\operatorname{Re} A(\omega) = -\omega_1^2 - \omega_2^2 + c \leq c$ for all $\omega \in \mathbb{R}^2$. Therefore, all the hypotheses of Theorem 2.10.1 are satisfied and thus, (2.53) determines a C_0 -semigroup on $X = L_2[\mathbb{R}^2, \mathbb{C}]$. \square

Example 2.10.2. Consider the initial value problem determined by the wave equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \\ u(0, x) = \psi(x). \end{cases} \quad (2.54)$$

The above equation can equivalently be expressed by

$$\frac{\partial u_1}{\partial t} = u_2, \quad \frac{\partial u_2}{\partial t} = \frac{\partial^2 u_1}{\partial x^2}$$

with $u_1 = u$ and $u_2 = \partial u / \partial t$. Equation (2.54) is a specific case of (I_P) with $r = 2, m = 2, n = 1$, and

$$A(\omega) = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}.$$

The eigenvalues of $A(\omega)$ are given by $C_{11}(\omega) = i\omega$ and $C_{22}(\omega) = -i\omega$. In order that the hypotheses of Theorem 2.10.1 be satisfied, there must exist an $S(\omega)$ such that $S(\omega)A(\omega)S(\omega)^{-1} = C(\omega)$ where $C(\omega)$ is upper triangular with diagonal elements $C_{11}(\omega)$ and $C_{22}(\omega)$. Then

$$A(\omega)S(\omega)^{-1} = S(\omega)^{-1} \begin{bmatrix} i\omega & C_{12}(\omega) \\ 0 & -i\omega \end{bmatrix}.$$

Let

$$S(\omega)^{-1} = \begin{bmatrix} x_1(\omega) & y_1(\omega) \\ x_2(\omega) & y_2(\omega) \end{bmatrix}.$$

A straightforward calculation yields

$$S(\omega)^{-1} = \begin{bmatrix} x_1(\omega) & y_1(\omega) \\ i\omega x_1(\omega) & C_{12}(\omega)x_1(\omega) - i\omega y_1(\omega) \end{bmatrix}$$

and

$$S(\omega) = \frac{1}{[C_{12}(\omega)x_1(\omega)^2 - 2i\omega x_1(\omega)y_1(\omega)]} \begin{bmatrix} C_{12}(\omega)x_1(\omega) - i\omega y_1(\omega) - y_1(\omega) \\ -i\omega x_1(\omega) & x_1(\omega) \end{bmatrix}.$$

Because $\operatorname{Re} C_{11}(\omega) = 0$, condition (iv) in Theorem 2.10.1 implies that $|C_{12}(\omega)| \leq K$ and condition (i) of this theorem implies that all elements of $S(\omega)$ and $S(\omega)^{-1}$ are bounded by K . Thus,

$$|C_{12}(\omega)x_1(\omega) - i\omega y_1(\omega)| \leq K$$

and

$$|\omega| |C_{12}(\omega)x_1(\omega) - 2i\omega y_1(\omega)|^{-1} \leq K$$

can be combined to yield

$$\begin{aligned} |\omega|/K &\leq |C_{12}(\omega)x_1(\omega) - i\omega y_1(\omega)| + |i\omega y_1(\omega)| \\ &\leq K + |i\omega y_1(\omega)| \\ &\leq 2K + |C_{12}(\omega)x_1(\omega)|. \end{aligned}$$

Using $|C_{12}(\omega)| \leq K$ and $|x_1(\omega)| \leq K$ for all $\omega \in \mathbb{R}$, we obtain

$$|\omega|/K \leq 2K + K^2$$

for all $\omega \in \mathbb{R}$. But this is impossible. Thus, no matrix $S(\omega)$ as asserted above exists. Therefore, the solutions of (2.54) do not generate a C_0 -semigroup. \square

2.10.3 Linear parabolic equations with smooth coefficients

In the following $\Omega \subset \mathbb{R}^n$ is assumed to be a bounded domain with smooth boundary $\partial\Omega$. We consider the differential operator of order $2m$ given by

$$A(t, x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(t, x) D^\alpha \quad (2.55)$$

where $\alpha \in \mathbb{N}^n$, D^α is defined in (2.51) and the coefficients $a_\alpha(t, x)$ are complex-valued functions defined on $[0, T_0) \times \overline{\Omega}$ where $T_0 > 0$ is allowed to be infinite. The *principal part* of $A(t, x, D)$ is the operator given by

$$A'(t, x, D) = \sum_{|\alpha|=2m} a_\alpha(t, x) D^\alpha \quad (2.56)$$

and $A(t, x, D)$ is said to be *strongly elliptic* if there exists a constant $c > 0$ such that

$$\operatorname{Re} A'(t, x, \xi) \geq c |\xi|^{2m}$$

for all $t \in [0, T_0)$, $x \in \Omega$, and $\xi \in \mathbb{R}^n$.

In the following, we consider *linear, parabolic partial differential equations* with initial conditions and boundary conditions given by

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) + A(t, x, D)u(t, x) = f(t, x) & \text{on } (0, T_0) \times \Omega \\ D^\alpha u(t, x) = 0, \quad |\alpha| < m & \text{on } (0, T_0) \times \partial\Omega \\ u(0, x) = u_0(x) & \text{on } \Omega \end{cases} \quad (I_{PP})$$

where f and u_0 are complex-valued functions defined on $(0, T_0) \times \Omega$ and Ω , respectively.

Using the theory of Sobolev spaces, generalized functions (distributions), and differentiation in the distribution sense, the following result concerning the well posedness of (I_{PP}) (involving generalized solutions for (I_{PP})) has been established (see, e.g., Pazy [39] and Friedman [14]).

Theorem 2.10.2. *For (I_{PP}) , assume the following.*

- (i) $A(t, x, D)$ is *strongly elliptic*.
- (ii) $f, a_\alpha \in C^\infty[[0, T_0] \times \overline{\Omega}, \mathbb{C}]$ for all $|\alpha| \leq 2m$.
- (iii) $u_0 \in C^\infty[\overline{\Omega}, \mathbb{C}]$.
- (iv) $\lim_{x \rightarrow \partial\Omega} D^\alpha u_0(x) = 0$ for all $|\alpha| < m$.

Then there exists a unique solution $u \in C^\infty[[0, T_0] \times \overline{\Omega}, \mathbb{C}]$. □

If the operator $A(t, x, D)$ and the functions f and u_0 are real-valued, then Theorem 2.10.2 is still true with the solution u of (I_{PP}) being real-valued.

Now let $T = [0, T_0]$ and $X = A = C^\infty[\overline{\Omega}, \mathbb{C}]$ and let S_{t_0, u_0} denote the set of the (unique) solutions of (I_{PP}) , where in (I_{PP}) , $u(0, x) = u_0(x)$ on Ω is replaced by $u(t_0, x) = u_0(x)$ on Ω with $t_0 \in [0, T_0]$. Let $S_{PP} = \cup_{(t_0, u_0) \in [0, T_0] \times A} S_{t_0, u_0}$. Then $\{T, X, A, S_{PP}\}$ is a dynamical system. When T, X , and A are known from context, we refer to this system simply as *dynamical system S_{PP}* .

Because $A(t, x, D)$ is in general time-varying, (I_{PP}) will in general not generate a semigroup. However, in the special case when $A(t, x, D) \equiv A(x, D)$, the following result has been established (refer, e.g., to Pazy [39]).

Theorem 2.10.3. *In (I_{PP}) , let*

$$A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha$$

be strongly elliptic on Ω and let $Au \triangleq A(x, D)u$ be defined on

$$D(A) = H^{2m}[\Omega, \mathbb{C}] \cap H_0^m[\Omega, \mathbb{C}].$$

Then A is the infinitesimal generator of a C_0 -semigroup on $L_2[\Omega, \mathbb{C}]$. \square

We conclude by pointing out that dynamical systems (as well as nonlinear semigroups) are determined by nonlinear partial differential equations as well. We do not pursue this topic in this book.

2.11 Composite Dynamical Systems

Problems that arise in science and technology are frequently described by a mixture of equations. For example, in control theory, feedback systems usually consist of an interconnection of several blocks, such as the plant, the sensors, the actuators, and the controller. Depending on the application, these components are characterized by different types of equations. For example, in the case of distributed parameter systems, the plant may be described by a partial differential equation, a functional differential equation, or by a Volterra integrodifferential equation, and the remaining blocks may be characterized by ordinary differential equations or ordinary difference equations. In particular, the description of digital controllers involves ordinary difference equations.

The above is an example of a large class of *composite systems*. Depending on the context, such systems are also referred to in the literature as *interconnected systems* and *decentralized systems* (e.g., [31]). When the motions of some of the system components evolve along different notions of time (continuous time \mathbb{R}^+ and discrete time \mathbb{N}) such systems are usually referred to as *hybrid systems* (e.g., [45, 46]).

In the present section, we confine our attention to interconnected (resp., composite) dynamical systems whose motion components all evolve along the *same* notion of time. In the next section, where we address discontinuous dynamical systems, and specific examples of hybrid dynamical systems, we relax this requirement. A metric space (X, d) is said to be *nontrivial* if X is neither empty nor a singleton, it is said to be *decomposable* if there are nontrivial metric spaces (X_1, d_1) and (X_2, d_2) such that $X = X_1 \times X_2$, and it is said to be *undecomposable* if it is not decomposable.

Now let (X, d) , (X_i, d_i) , $i = 1, \dots, l$, be metric spaces. We assume that $X = X_1 \times \dots \times X_l$ and that there are constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 d(x, y) \leq \sum_{i=1}^l d_i(x_i, y_i) \leq c_2 d(x, y)$$

for all $x, y \in X$, where $x = [x_1, \dots, x_l]^T$, $y = [y_1, \dots, y_l]^T$, $x_i \in X_i$, and $y_i \in X_i$, $i = 1, \dots, l$. We can define the metric d on X in a variety of ways, including, for example,

$$d(x, y) = \sum_{i=1}^l d_i(x_i, y_i).$$

Definition 2.11.1 ([34]). A dynamical system $\{T, X, A, S\}$ is called a *composite dynamical system* if the metric space (X, d) can be decomposed as $X = X_1 \times \dots \times X_l$, $l \geq 2$, where X_1, \dots, X_l are nontrivial and undecomposable metric spaces with metrics d_1, \dots, d_l , respectively, and if there exist two metric spaces X_i and X_j , $i, j = 1, \dots, l$, $i \neq j$, that are not isometric. \square

The following example may be viewed as a distributed control (in contrast to a boundary control) of a plant that is governed by the heat equation and a controller that is governed by a system of first-order ordinary differential equations. The variables for the controller and the plant are represented by $z_1 = z_1(t)$ and $z_2 = z_2(t, x)$, respectively.

Example 2.11.1 ([31, 40]). We consider the composite system described by the equations

$$\begin{cases} \dot{z}_1(t) = Az_1(t) + \int_{\Omega} f(x)z_2(t, x)dx, & t \in \mathbb{R}^+ \\ \frac{\partial z_2}{\partial t}(t, x) = \alpha \Delta z_2(t, x) + g(x)c^T z_1(t), & (t, x) \in \mathbb{R}^+ \times \Omega \\ z_2(t, x) = 0 & (t, x) \in \mathbb{R}^+ \times \partial\Omega \end{cases} \quad (2.57)$$

where $z_1 \in \mathbb{R}^m$, $z_2 \in \mathbb{R}$, $A \in \mathbb{R}^{m \times m}$, $c \in \mathbb{R}^m$, f and $g \in L_2[\Omega, \mathbb{R}]$, $\alpha > 0$, Ω is a bounded domain in \mathbb{R}^n with a smooth boundary $\partial\Omega$, and Δ denotes the Laplacian (i.e., $\Delta = \sum_{i=1}^n \partial^2 / \partial x_i^2$). The system of equations (2.57) may be viewed as a differential equation in the Banach space $X \triangleq \mathbb{R}^m \times H_0[\Omega, \mathbb{R}]$ where $H_0[\Omega, \mathbb{R}]$ is the completion of $C_0[\Omega, \mathbb{R}]$ with respect to the L_2 -norm and $H_0[\Omega, \mathbb{R}] \subset L_2[\Omega, \mathbb{R}]$ (refer to Section 2.10). For every initial condition $z_0 = [z_{10}, z_{20}] \in \mathbb{R}^m \times H_0[\Omega, \mathbb{R}]$, there exists a unique solution $z(t, z_0)$ which depends continuously on z_0 . For a proof of the well posedness of system (2.57), refer to [31].

The set of all solutions of (2.57) clearly determines a composite dynamical system. \square

2.12 Discontinuous Dynamical Systems

All of the various types of dynamical systems that we have considered thus far include either discrete-time dynamical systems or continuous continuous-time dynamical systems (which we simply call continuous dynamical systems). In the present section we address discontinuous dynamical systems (continuous-time dynamical systems with motions that need not be continuous), which we abbreviate as DDS. Although the classes of DDS which we consider are very general, we have to put some restrictions on the types of discontinuities that we allow. To motivate the discussion of this section and to fix some of the ideas involved, we first consider an important specific example.

In Figure 2.2 we depict in block diagram form a configuration that is applicable to many classes of DDS, including hybrid systems and switched systems. There is a block that contains continuous-time dynamics, a block that contains phenomena which evolve at discrete points in time (discrete-time dynamics) or at discrete events, and a block that contains interface elements for the above system components. The block that contains the continuous-time dynamics is usually characterized by one or several types of the equations or inequalities defined on \mathbb{R}^+ enumerated in the previous sections (Sections 2.3, 2.4, and 2.6–2.10) whereas the block on the right in Figure 2.2 is usually characterized by difference equations or difference inequalities of the type addressed in Section 2.5 or it may contain other types of discrete characterizations involving, for example, Petri nets, logic commands, various types of discrete-event systems, and the like. The block labeled Interface Elements may vary from the very simple to the very complicated. At the simplest level, this block involves samplers and sample and hold elements. The sampling process may involve only one uniform rate, or it may be nonuniform (variable rate sampling), or there may be several different (uniform or nonuniform) sampling rates occurring simultaneously (multirate sampling).

Example 2.12.1 ([29, 46]). Perhaps the simplest specific example of the above class of systems are *sampled-data control systems* described by the equations

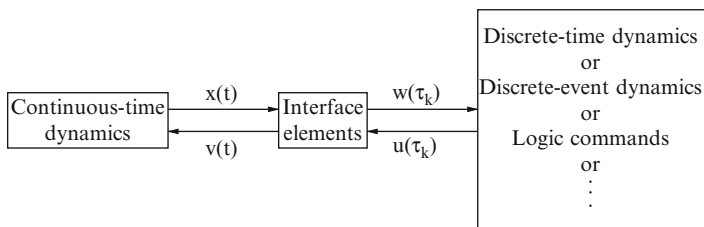


Fig. 2.2 DDS configuration.

$$\begin{cases} \dot{x}(t) = A_k x(t) + B_k v(t), & \tau_k \leq t < \tau_{k+1} \\ u(\tau_{k+1}) = C_k u(\tau_k) + D_k w(\tau_k), \\ v(t) = u(\tau_k), & \tau_k \leq t < \tau_{k+1} \\ w(\tau_k) = x(\tau_{k+1}^-), \end{cases} \quad (2.58)$$

where $k \in \mathbb{N}$, $t \in \mathbb{R}^+$, $x(t) \in \mathbb{R}^n$, $u(\tau_k) \in \mathbb{R}^m$, $\{\tau_k\}$ denotes sampling instants, A_k, B_k, C_k, D_k are real matrices of appropriate dimensions, $v(\cdot)$ and $w(\cdot)$ are interface variables, and $x(\tau^-) = \lim_{\theta \rightarrow 0^+} x(\tau - \theta)$.

Now define $\tilde{x}(t) = x(t)$, $t \geq \tau_0$ and $\tilde{u}(t) = v(t) = u(\tau_k)$, $\tau_k \leq t < \tau_{k+1}$, $k \in \mathbb{N}$. Then $\tilde{x}(t) = x(t^-)$ at $t = \tau_k$ and $\tilde{u}(t^-) = u(\tau_k)$ at $t = \tau_{k+1}$ for all $k \in \mathbb{N}$. Let $y(t)^T = [\tilde{x}(t)^T, \tilde{u}(t)^T]$. Letting

$$F_k = \begin{bmatrix} A_k & B_k \\ 0 & 0 \end{bmatrix}, \quad H_k = \begin{bmatrix} I & 0 \\ D_k & C_k \end{bmatrix}$$

where I denotes the $n \times n$ identity matrix, the system (2.58) can be described by the discontinuous ordinary differential equation

$$\begin{cases} \dot{y}(t) = F_k y(t), & \tau_k \leq t < \tau_{k+1} \\ y(t) = H_k y(t^-), & t = \tau_{k+1}, \quad k \in \mathbb{N}. \end{cases} \quad (2.59)$$

Next, for $k \in \mathbb{N}$, let $y_k(t, y_k, \tau_k)$, $t \geq \tau_k$, denote the unique solution of the initial value problem

$$\begin{cases} \dot{y}(t) = F_k y(t), \\ y(\tau_k) = y_k. \end{cases} \quad (2.60)$$

Then clearly, for every $y_0 \in \mathbb{R}^{n+m}$, the unique solution of the DDS (2.59) is given by

$$y(t, y_0, \tau_0) = y_k(t, y_k, \tau_k), \quad \tau_k \leq t < \tau_{k+1}, \quad k \in \mathbb{N}.$$

Thus, the solutions of (2.59) are made up of an infinite sequence of solution segments determined by the solutions of (2.60), $k \in \mathbb{N}$, and these solutions may be discontinuous at the points of discontinuity given by $\{\tau_k\}$, $k = 1, 2, \dots$. Finally, it is clear that the solutions of (2.59) determine a DDS. \square

In Chapter 3 we develop a stability theory for general DDS, $\{\mathbb{R}^+, X, A, S\}$, defined on metric spaces, and in subsequent chapters, we specialize this theory for specific classes of finite-dimensional and infinite-dimensional dynamical systems determined by various equations and semigroups of the type described in the present chapter. In order to establish meaningful and reasonable results, it is necessary to impose some restrictions on the discontinuities of the motions $p \in S$, which of course should conform to assumptions that one needs to make in the modeling

process of the DDS. Unless explicitly stated otherwise, we assume throughout this book that for a given discontinuous motion $p \in S$, the set of discontinuities is *unbounded* and *discrete* and is of the form

$$E_{1p} = \{\tau_1^p, \tau_2^p, \dots : \tau_1^p < \tau_2^p < \dots\}.$$

In the above expression, E_{1p} signifies the fact that in general, different motions may possess different sets of times at which discontinuities may occur. Because in most cases, the particular set E_{1p} in question is clear from context, we usually suppress the p -notation and simply write

$$E_1 = \{\tau_1, \tau_2, \dots : \tau_1 < \tau_2 < \dots\}.$$

Furthermore, we assume that E_1 has no finite accumulation points.

In the remainder of this section we consider several important specific classes of DDS.

2.12.1 Ordinary differential equations

The sampled-data control system (2.58) which equivalently is represented by the discontinuous differential equation (2.59) is a special case of discontinuous ordinary differential equations of the form

$$\begin{cases} \dot{x}(t) = f_k(t, x(t)), & \tau_k \leq t < \tau_{k+1}, \\ x(t) = g_k(x(t^-)), & t = \tau_{k+1}, k \in \mathbb{N} \end{cases} \quad (SE)$$

where for each $k \in \mathbb{N}$, $f_k \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$, $g_k: \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $x(\tau^-)$ is given in Example 2.12.1.

Associated with (SE), we consider the family of initial value problems given by

$$\begin{cases} \dot{x}(t) = f_k(t, x(t)) \\ x(\tau_k) = x_k, \end{cases} \quad (SE_k)$$

$k \in \mathbb{N}$. We assume that for (τ_k, x_k) , (SE_k) possesses a unique solution $x^{(k)}(t, x_k, \tau_k)$ which exists for all $t \in [\tau_k, \infty)$ (refer to Section 2.3 for conditions that ensure this). Then for every $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$, $t_0 = \tau_0$, (SE) has a unique solution $x(t, x_0, t_0)$ that exists for all $t \in [t_0, \infty)$. This solution is made up of a sequence of continuous solution segments $x^{(k)}(t, x_k, \tau_k)$, defined over the intervals $[\tau_k, \tau_{k+1})$ with initial conditions (τ_k, x_k) , $k \in \mathbb{N}$, where $x_{k+1} = x(\tau_{k+1}) = g_k(x(\tau_{k+1}^-))$, $k \in \mathbb{N}$ and the initial conditions $(\tau_0 = t_0, x_0)$ are given. At the points $\{\tau_{k+1}\}$, $k \in \mathbb{N}$, the solutions of (SE) have *possible* jumps (determined by $g_k(\cdot)$).

The set of all the solutions of (SE), S_{SE} , determines a DDS, $\{\mathbb{R}^+, X, A, S_{SE}\}$, where $X = A = \mathbb{R}^n$.

2.12.2 Functional differential equations ([43])

For the notation that we use in the present subsection, the reader should refer to Section 2.7.

We first consider a family of initial value problems described by *continuous* retarded functional differential equations (RFDEs) of the form

$$\begin{cases} \dot{x}(t) = F_k(t, x_t), \\ x_{\tau_k} = \varphi_k, \end{cases} \quad (SF_k)$$

$k \in \mathbb{N}$. For each $k \in \mathbb{N}$ we assume that $F_k \in C[\mathbb{R}^+ \times C_r, \mathbb{R}^n]$ and that (SF_k) is well posed so that for every $(\tau_k, \varphi_k) \in \mathbb{R}^+ \times C_r$, (SF_k) possesses a unique *continuous* solution $x^{(k)}(t, \varphi_k, \tau_k)$ that exists for all $t \in [\tau_k, \infty)$. (For conditions that ensure this, refer to Section 2.7.)

We now consider *discontinuous* RFDEs of the form

$$\begin{cases} \dot{x}(t) = F_k(t, x_t), & \tau_k \leq t < \tau_{k+1} \\ x_{\tau_{k+1}} = G_k(x_{\tau_{k+1}}^-), & k \in \mathbb{N} \end{cases} \quad (SF)$$

where for each $k \in \mathbb{N}$, F_k is assumed to possess the identical properties given in (SF_k) and $G_k: C_r \rightarrow C_r$. Thus, at $t = \tau_{k+1}$, the mapping $G_k(\cdot)$ assigns to every state $x_{\tau_{k+1}}^-$ ($x_{\tau_{k+1}}^-(\theta) = x(\tau_{k+1}^- + \theta)$, $-r \leq \theta \leq 0$) unambiguously a state $x_{\tau_{k+1}}$ ($x_{\tau_{k+1}}(\theta) = x(\tau_{k+1} + \theta)$, $-r \leq \theta \leq 0$).

Under the above assumptions for (SF) and (SF_k) , it is now clear that for every $(t_0, \varphi_0) \in \mathbb{R}^+ \times C_r$, $t_0 = \tau_0$, (SF) has a unique solution $x(t, \varphi_0, t_0)$ that exists for all $t \in [t_0, \infty)$. This solution is made up of a sequence of continuous solution segments $x^{(k)}(t, \varphi_k, \tau_k)$ defined over the intervals $[\tau_k, \tau_{k+1})$, $k \in \mathbb{N}$, with initial conditions (τ_k, φ_k) , where $\varphi_k = x_{\tau_k}$, $k = 1, 2, \dots$ and where $(\tau_0 = t_0, \varphi_0)$ are given. At the points $\{\tau_{k+1}\}$, $k \in \mathbb{N}$, the solutions of (SF) have *possible* jumps (determined by $G_k(\cdot)$).

It is clear that (SF) determines an *infinite-dimensional* DDS, $\{T, X, A, S\}$, where $T = \mathbb{R}^+$, $X = A = C_r$, the metric on X is determined by the norm $\|\cdot\|$ defined on C_r (i.e., $d(\varphi, \eta) = \|\varphi - \eta\|$ for all $\varphi, \eta \in C_r$), and S denotes the set of all the solutions of (SF) corresponding to all possible initial conditions $(t_0, \varphi_0) \in \mathbb{R}^+ \times C_r$. In the interests of brevity, we refer to this DDS as “system (SF) ” or as “ (SF) ”.

2.12.3 Differential equations in Banach spaces ([32])

We first consider a family of *initial value Cauchy problems in Banach space* X of the form

$$\begin{cases} \dot{x}(t) = F_k(t, x(t)), & t \geq \tau_k, \\ x(\tau_k) = x_k \end{cases} \quad (SG_k)$$

for $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, we assume that $F_k: \mathbb{R}^+ \times X \rightarrow X$ and that $\dot{x} = dx/dt$. We assume that for every $(\tau_k, x_k) \in \mathbb{R}^+ \times X$, (SG_k) possesses a unique solution $x^{(k)}(t, x_k, \tau_k)$ that exists for all $t \in [\tau_k, \infty)$. We express this by saying that (SG_k) is *well posed*.

We now consider *discontinuous initial value problems in Banach space* X given by

$$\begin{cases} \dot{x}(t) = F_k(t, x(t)), & \tau_k \leq t < \tau_{k+1} \\ x(\tau_{k+1}) = g_k(x(\tau_{k+1}^-)), & k \in \mathbb{N} \end{cases} \quad (SG)$$

where for each $k \in \mathbb{N}$, F_k is assumed to possess the identical properties given in (SG_k) and where $g_k: X \rightarrow X$. Under these assumptions, it is clear that for every $(t_0, x_0) \in \mathbb{R}^+ \times X$, $t_0 = \tau_0$, (SG) has a unique solution $x(t, x_0, t_0)$ that exists for all $t \in [t_0, \infty)$. This solution is made up of a sequence of solution segments $x^{(k)}(t, x_k, \tau_k)$, defined over the intervals $[\tau_k, \tau_{k+1})$, $k \in \mathbb{N}$, with initial conditions (τ_k, x_k) , where $x_k = x(\tau_k)$, $k = 1, 2, \dots$, and where $(\tau_0 = t_0, x_0)$ are given. At the points $\{\tau_{k+1}\}$, $k \in \mathbb{N}$, the solutions of (SG) have *possible* jumps (determined by $g_k(\cdot)$).

Consistent with the characterization of a *discontinuous dynamical system* given in Section 2.2, it is clear from the above that system (SG) determines a DDS, $\{T, X, A, S\}$, where $T = \mathbb{R}^+$, $A = X$, the metric on X is determined by the norm $\|\cdot\|$ defined on X (i.e., $d(x, y) = \|x - y\|$ for all $x, y \in X$), and S denotes the set of all the solutions of (SG) corresponding to all possible initial conditions $(t_0, x_0) \in \mathbb{R}^+ \times X$. In the interests of brevity, we refer to this DDS simply as “system (SG) ”, or simply as “ (SG) ”.

Remark 2.12.1. When in (SE) , $f_i \neq f_j$ for some $i, j \in \mathbb{N}$ and $g_k(x) = x$ for all $k \in \mathbb{N}$, then every motion $x(\cdot, x_0, t_0)$ of (SE) is *continuous* for all $t \geq t_0$ while the corresponding time derivative, $\dot{x}(\cdot, x_0, t_0)$, is discontinuous at appropriate points in time. In this case, $\{\mathbb{R}^+, X, A, S_{SE}\}$ is a *continuous* dynamical system. In the present context we view continuous dynamical systems as being special cases of discontinuous dynamical systems (DDS). Similar statements can be made for infinite-dimensional dynamical systems determined by functional differential equations (SF) and differential equations in Banach spaces (SG) .

2.12.4 Semigroups ([33])

We require a given collection of linear or nonlinear semigroups $\mathcal{T} = \{T_i(t)\}$ defined on a Banach space X , or on a set $C \subset X$, respectively; and a given collection of linear and continuous operators $\mathcal{H} = \{H_j\}(H_j: X \rightarrow X)$, or of nonlinear and continuous operators $(H_j: C \rightarrow C)$; and a given discrete and unbounded set $E = \{t_0 = \tau_0, \tau_1, \tau_2, \dots; \tau_0 < \tau_1 < \tau_2 \dots\} \subset \mathbb{R}^+$ with no finite accumulation points. The number of elements in \mathcal{T} and \mathcal{H} may be *finite* or *infinite*.

We now consider dynamical systems whose *motions* $y(\cdot, y_0, t_0)$ with *initial time* $t_0 = \tau_0 \in \mathbb{R}^+$ and *initial state* $y(t_0) = y_0 \in X$ (resp., $y_0 \in C \subset X$) are given by

$$\begin{cases} y(t, y_0, t_0) = T_k(t - \tau_k)(y(\tau_k)), & \tau_k \leq t < \tau_{k+1}, \\ y(t) = H_k(y(t^-)), & t = \tau_{k+1}, \quad k \in \mathbb{N}. \end{cases} \quad (SH)$$

We define the DDS *determined by semigroups* as

$$\begin{aligned} S = \{y = y(\cdot, x, t_0): & y(t, x, t_0) = T_k(t - \tau_k)(y(\tau_k)), \quad \tau_k \leq t < \tau_{k+1}, \\ & y(t) = H_k(y(t^-)), \quad t = \tau_{k+1}, \quad k \in \mathbb{N}, \quad t_0 = \tau_0 \in \mathbb{R}^+, \\ & y(\tau_0) = x \in X, \text{ resp., } x \in C \subset X\}. \end{aligned} \quad (2.61)$$

Note that every motion $y(\cdot, x, t_0)$ is unique, with $y(t_0, x, t_0) = x$, exists for all $t \geq t_0$, and is continuous with respect to t on $[t_0, \infty) - \{\tau_1, \tau_2, \dots\}$, and that at $t = \tau_k$, $k = 1, 2, \dots$, $y(\cdot, x, t_0)$ may be discontinuous. We call the set $E_1 = \{\tau_1, \tau_2, \dots\}$ the set of discontinuities for the motion $y(\cdot, x, t_0)$.

When in (2.61), \mathcal{T} consists of C_0 -semigroups, we speak of a DDS *determined by linear semigroups* and we denote this system by S_{DC_0} . Similarly, when in (2.61), \mathcal{T} consists of nonlinear semigroups, we speak of a DDS *determined by nonlinear semigroups* and we denote this system by S_{DN} . When the types of the elements in \mathcal{T} are not specified, we simply speak of a DDS *determined by semigroups* and we denote this system, as in (2.61), by S .

Finally, if in the case of S_{DC_0} , the elements in \mathcal{H} are linear, we use in (SH) the notation $T_k(t - \tau_k)(y(\tau_k)) = T_k(t - \tau_k)y(\tau_k)$ and $H_k(y(t^-)) = H_k y(t^-)$.

Next, a few observations may be in order:

- (a) For different initial conditions (x, t_0) , resulting in different motions $y(\cdot, x, t_0)$, we allow the set of discontinuities $E_1 = \{\tau_1, \tau_2, \dots\}$, the set of semigroups $\{T_k\} \subset \mathcal{T}$, and the set of functions $\{H_k\} \subset \mathcal{H}$ to differ, and accordingly, the notation $E_1^{x, t_0} = \{\tau_1^{x, t_0}, \tau_2^{x, t_0}, \dots\}, \{T_k^{x, t_0}\}$ and $\{H_k^{x, t_0}\}$ might be more appropriate. However, because in all cases, all meaning is clear from context, we do not use such superscripts.
- (b) S_{DC_0} and S_{DN} are very general classes of DDS and include large classes of finite-dimensional dynamical systems determined by ordinary differential equations and inequalities and large classes of infinite-dimensional dynamical systems determined by differential-difference equations, functional differential equations, Volterra integrodifferential equations, certain classes of partial differential equations, and the like.
- (c) The dynamical system models S_{DC_0} and S_{DN} are very flexible and include as special cases many of the DDS considered in the literature (e.g., [2, 10, 28–30, 46]), as well as general autonomous continuous dynamical systems: (i) If $T_k(t) = T$ for all k (\mathcal{T} has only one element) and if $H_k = I$ for all k , where I denotes the identity transformation, then S_{DC_0} reduces to an *autonomous*,

linear, continuous dynamical system and S_{DN} reduces to an *autonomous, nonlinear, continuous dynamical system*. (ii) In the case of *dynamical systems subjected to impulse effects* (see, e.g., [2]), one would choose $T_k(t) = T(t)$ for all k whereas the impulse effects are captured by an infinite family of functions $\mathcal{H} = \{H_k\}$. (iii) In the case of *switched systems*, frequently only a *finite number* of systems that are being switched are required and so in this case one would choose a finite family of semigroups, $\mathcal{T} = \{T_i(t)\}$ (see, e.g., [10, 46]); and so forth. (iv) Perhaps it needs pointing out that even though systems S_{DN} and S_{DC_0} are *determined* by families of semigroups (and nonlinearities), by themselves they are *not* semigroups, inasmuch as in general, they are time-varying and do not satisfy the hypotheses (i)–(iii) in Definitions 2.9.1 and 2.9.5.

We conclude with a specific example involving partial differential equations.

Example 2.12.2 ([33] DDS determined by the heat equation). We let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and we let $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ denote the *Laplacian*. Also, we let $X = H^2[\Omega, \mathbb{R}] \cap H_0^1[\Omega, \mathbb{R}]$ where $H_0^1[\Omega, \mathbb{R}]$ and $H^2[\Omega, \mathbb{R}]$ are *Sobolev spaces* (refer to Section 2.10). For any $\varphi \in X$, we define the H^1 -norm by

$$\|\varphi\|_{H^1}^2 = \int_{\Omega} (|\nabla\varphi|^2 + |\varphi|^2) dx \quad (2.62)$$

where $\nabla\varphi^T = (\partial\varphi/\partial x_1, \dots, \partial\varphi/\partial x_n)$.

We now consider DDS determined by the equations

$$\begin{cases} \frac{\partial u}{\partial t} = a_k \Delta u, & (t, x) \in [\tau_k, \tau_{k+1}) \times \Omega \\ u(t, \cdot) = g_k(u(t^-, \cdot)) \triangleq \varphi_{k+1}(\cdot), & t = \tau_{k+1} \\ u(t_0, x) = \varphi_0(x), & x \in \Omega \\ u(t, x) = 0, & (t, x) \in [t_0, \infty) \times \partial\Omega, \quad k \in \mathbb{N} \end{cases} \quad (2.63)$$

where $\varphi_0 \in X$, $a_k > 0$, $k \in \mathbb{N}$ are constants, $\{g_k\}$ is a given family of mappings with $g_k \in C[X, X]$, $k \in \mathbb{N}$, and $E = \{t_0 = \tau_0, \tau_1, \dots: \tau_0 < \tau_1 < \tau_2 < \dots\}$ is a given unbounded and discrete set with no finite accumulation points. We assume that $g_k(0) = 0$ and there exists a constant $d_k > 0$ such that

$$\|g_k(\varphi)\|_{H^1} \leq d_k \|\varphi\|_{H^1} \quad (2.64)$$

for all $\varphi \in X$, $k \in \mathbb{N}$.

Associated with (2.63) we have a family of *initial and boundary value problems* determined by

$$\begin{cases} \frac{\partial u}{\partial t} = a_k \Delta u, & (t, x) \in [\tau_k, \infty) \times \Omega \\ u(\tau_k, x) = \varphi_k(x), & x \in \Omega \\ u(t, x) = 0, & (t, x) \in [\tau_k, \infty) \times \partial\Omega, \end{cases} \quad (2.65)$$

$k \in \mathbb{N}$. It has been shown (e.g., [39]) that for each $(\tau_k, \varphi_k) \in \mathbb{R}^+ \times X$, the initial and boundary value problem (2.65) has a unique solution $u_k = u_k(t, x)$, $t \geq \tau_k, x \in \Omega$, such that $u_k(t, \cdot) \in X$ for each fixed $t \geq \tau_k$ and $u_k(t, \cdot) \in X$ is a continuously differentiable function from $[\tau_k, \infty)$ to X with respect to the H^1 -norm given in (2.62).

It now follows that for every $\varphi_0 \in X$, (2.63) possesses a unique solution $u(t, \cdot)$ that exists for all $t \geq \tau_0 \geq 0$, given by

$$u(t, \cdot) = \begin{cases} u_k(t, \cdot), & \tau_k \leq t < \tau_{k+1} \\ g_k(u_k(t^-, \cdot)) \triangleq \varphi_{k+1}(\cdot), & t = \tau_{k+1}, k \in \mathbb{N} \end{cases} \quad (2.66)$$

with $u(t_0, x) = \varphi_0(x)$. Notice that $u(t, \cdot)$ is continuous with respect to t on the set $[t_0, \infty) - \{\tau_1, \tau_2, \dots\}$, and that at $t = \tau_k, k = 1, 2, \dots$, $u(t, \cdot)$ may be discontinuous (depending on the properties of $g_k(\cdot)$).

For each $k \in \mathbb{N}$, (2.65) can be cast as an *initial value problem* in the space X with respect to the H^1 -norm, letting $u_k(t, \cdot) = U_k(t)$,

$$\begin{cases} \dot{U}_k(t) = A_k U_k(t), & t \geq \tau_k \\ U_k(\tau_k) = \varphi_k \in X \end{cases} \quad (2.67)$$

where $A_k = a_k \sum_{i=1}^n \partial^2 / \partial x_i^2$ and $U_k(t, \varphi_k), t \geq \tau_k$, denotes the solution of (2.67) with $U(\tau_k, \varphi_k) = \varphi_k$. It has been shown (see, e.g., [39]) that (2.67) determines a C_0 -semigroup $T_k(t - \tau_k): X \rightarrow X$, where for any $\varphi_k \in X$, $U_k(t, \varphi_k) = T(t - \tau_k)\varphi_k$.

Letting $u_k(t, \cdot) = T_k(t - \tau_k)u_k(\tau_k)$ in (2.66), system (2.63) can now be characterized as

$$\begin{cases} u(t, \cdot) = T_k(t - \tau_k)u_k(\tau_k, \cdot), & \tau \leq t < \tau_{k+1} \\ u(t, \cdot) = g_k(u_k(t^-, \cdot)), & t = \tau_{k+1}, k \in \mathbb{N}. \end{cases} \quad (2.68)$$

Finally, it is clear that (2.63) (resp., (2.68)) determines a DDS which is a special case of the DDS (SH) . \square

2.13 Notes and References

Depending on the applications, different variants of dynamical systems have been employed (e.g., Hahn [18], Willems [44], and Zubov [48]). Our concept of dynamical system (Definition 2.2.3) was first used in [35], [36] and extensively

further refined in [34] in the study of the role of stability-preserving mappings in stability analysis of dynamical systems. In the special case when X is a normed linear space and each motion $p(t, a, t_0)$ is assumed to be continuous with respect to a , t , and t_0 , the definition of a dynamical system given in Definition 2.2.3 reduces to the definition of a dynamical system used in Hahn [18, pp. 166–167] (called a *family of motions* in [18]). When the motions satisfy additional requirements that we do not enumerate, Definition 2.2.3 reduces to the definition of a dynamical system, defined on metric space, used by Zubov [48, p. 199] (called a *general system* in [48]). The notion of a dynamical system employed in [44] is defined on normed linear space and involves variations to Definition 2.2.3 which we do not specify here.

In the problem section we provide hints on how to prove the results given in Section 2.3. For the complete proofs of these results (except Theorem 2.3.3) and for additional material on ordinary differential equations, refer to Miller and Michel [37]. Our treatment of the continuation of solutions (Theorem 2.3.3) is not conventional, but very efficient, inasmuch as it involves Lyapunov results developed in subsequent chapters.

Ordinary differential inequalities (and ordinary difference inequalities) play an important role in the qualitative analysis of dynamical systems (see, e.g., [26]) and are employed throughout this book.

Good sources on ordinary difference equations with applications to control systems and signal processing include Franklin and Powell [13] and Oppenheim and Schaffer [38], respectively.

For the complete proofs of Theorems 2.7.1–2.7.3, and additional material on functional differential equations, refer to Hale [19]. Hale is perhaps the first to treat Volterra integrodifferential equations as functional differential equations with infinite delay [20]. For a proof of Theorem 2.8.1, refer to Barbu and Grossman [3].

For the proofs of Theorems 2.9.1–2.9.4 and for additional material concerning C_0 -semigroups, refer to Hille and Phillips [21], Krein [23] (Chapter 1), and Pazy [39]. For the proof of Theorem 2.9.5, refer to Slemrod [42]. For the proofs of Theorems 2.9.6 and 2.9.7 and for additional material concerning nonlinear semigroups and differential inclusions defined on Banach spaces, refer to Crandall [8], Crandall and Liggett [9], Brezis [5], Kurtz [25], Godunov [15], Lasota and Yorke [27], and Aubin and Cellina [1]. Our presentation in Section 2.9 on semigroups and differential inclusions defined on Banach spaces (see also Section 2.6) is in the spirit of the presentation given in Michel and Miller [31] (Chapter 5), and Michel *et al.* [34].

For the proofs of Theorems 2.10.1–2.10.3, and additional material concerning partial differential equations, refer to Krein [23], Friedman [14], and Pazy [39]. Additional sources on partial differential equations include Hörmander [22] and Krylov [24]. Our presentation on partial differential equations in Section 2.10 is in the spirit of Michel and Miller [31, Chapter 5] and Michel *et al.* [34, Chapter 2].

Our presentation on composite dynamical systems in Section 2.11 is primarily based on material from Michel and Miller [31], Michel *et al.* [34, Chapter 6], and Rasmussen and Michel [40], and Section 2.12 on discontinuous dynamical systems relies primarily on material from Michel [29], Michel and Hu [30], Michel and

Sun [32], Michel *et al.* [33], Sun *et al.* [43], and Ye *et al.* [46]. Finally, for a general formulation of a hybrid dynamical system defined on a metric space (involving a notion of generalized time), refer to Ye *et al.* [45] with subsequent developments given in Ye *et al.* [46], Sun *et al.* [43], Michel *et al.* [33], Michel and Sun [32], Michel and Hu [30], and Michel [29].

2.14 Problems

Problem 2.14.1. Consider a class of scalar n th-order ordinary differential equations given by

$$y^{(n)} = g(t, \dot{y}, \dots, y^{(n-1)}) \quad (E_n)$$

where $t \in J \subset \mathbb{R}$, J is a finite or an infinite interval, $y \in \mathbb{R}$, $\dot{y} = y^{(1)} = dy/dt, \dots$, $y^{(n)} = d^n y/dt^n$, and $g \in C[J \times \mathbb{R}^n, \mathbb{R}]$. Initial value problems associated with (E_n) are given by

$$\begin{cases} y^{(n)} = g(t, \dot{y}, \dots, y^{(n-1)}) \\ y(t_0) = y_0, \quad \dot{y}(t_0) = y_1, \quad \dots, \quad y^{(n-1)}(t_0) = y_{n-1} \end{cases} \quad (I_{E_n})$$

where $t_0 \in J$ and $y_0, y_1, \dots, y_{n-1} \in \mathbb{R}$.

Show that (E_n) determines a dynamical system (in the sense of Definition 2.2.3) that we denote by S_{E_n} .

Hint: Show that (E_n) (and (I_{E_n})) can equivalently be represented by a system of n first-order ordinary differential equations. \square

Problem 2.14.2. Consider a class of n th-order ordinary scalar difference equations given by

$$y(k) = g(k, y(k-1), \dots, y(k-n)) \quad (D_n)$$

where $k \in \mathbb{N}_n \triangleq [n, \infty) \cap \mathbb{N}$, $n \in \mathbb{N}$, $y: \mathbb{N} \rightarrow \mathbb{R}$, and $g: \mathbb{N}_n \times \mathbb{R}^n \rightarrow \mathbb{R}$. Associated with (D_n) , consider *initial value problems* given by

$$\begin{cases} y(k) = g(k, y(k-1), \dots, y(k-n)) \\ y(0) = y_0, \quad y(1) = y_1, \quad \dots, \quad y(n-1) = y_{n-1} \end{cases} \quad (I_{D_n})$$

where $y_0, y_1, \dots, y_{n-1} \in \mathbb{R}$.

Show that (D_n) determines a dynamical system (in the sense of Definition 2.2.3) which we denote by S_{D_n} .

Hint: Show that (D_n) (and (I_{D_n})) can equivalently be represented by a system of n first-order ordinary difference equations. \square

Problem 2.14.3. Let D denote a fixed Dini derivative and let $g \in C[J \times (\mathbb{R}^+)^n, \mathbb{R}^n]$ where $g(t, 0) \geq 0$ for all $t \in J$. Consider differential inequalities given by

$$Dx \geq g(t, x) \quad (2.69)$$

and define a *solution* of (2.69) as a function $\varphi \in C[[t_0, t_1), (\mathbb{R}^+)^n]$ that satisfies $(D\varphi)(t) \geq g(t, \varphi(t))$ for all $t \in [t_0, t_1) \subset J$. Associated with (2.69), we consider *initial value problems* given by

$$Dx \geq g(t, x), \quad x(t_0) = x_0 \quad (2.70)$$

where $t_0 \in J$ and $x_0 \in (0, \infty)^n \cup \{0\}$. We say that $\varphi \in C[[t_0, t_1), (\mathbb{R}^+)^n]$ is a solution of (2.70) if $\varphi(t_0) = x_0$.

Show that (2.69) determines a dynamical system that we denote by $S_{(2.69)}$. \square

Problem 2.14.4. Consider *ordinary difference inequalities* given by

$$x(k+1) \geq g(k, x(k)) \quad (2.71)$$

where $k \in \mathbb{N}$ and $g: \mathbb{N} \times (\mathbb{R}^+)^n \rightarrow (\mathbb{R}^+)^n$ with $g(k, 0) \geq 0$ for all $k \in \mathbb{N}$. A function $\varphi: \mathbb{N}_{k_0} \rightarrow (\mathbb{R}^+)^n$ is a *solution* of (2.71) if

$$\varphi(k+1) \geq g(k, \varphi(k))$$

for all $k \in \mathbb{N}_{k_0}$. In this case $\varphi(k_0)$ is an *initial value*.

Show that (2.71) determines a dynamical system that we denote by $S_{(2.71)}$. \square

Problem 2.14.5. (a) In Figure 2.3, M_1 and M_2 denote point masses, K_1, K_2, K denote spring constants, and x_1, x_2 denote displacements of the masses M_1 and M_2 , respectively. Use the Hamiltonian formulation of dynamical systems described in Example 2.3.7 to derive a system of first-order ordinary differential equations that characterize this system. Verify your answer by using Newton's second law of motion to derive the same system of equations. By specifying $x_1(t_0), \dot{x}_1(t_0), x_2(t_0)$, and $\dot{x}_2(t_0)$, the above yields an initial value problem.

(b) Show that the above mechanical system determines a dynamical system in the sense of Definition 2.2.3. \square

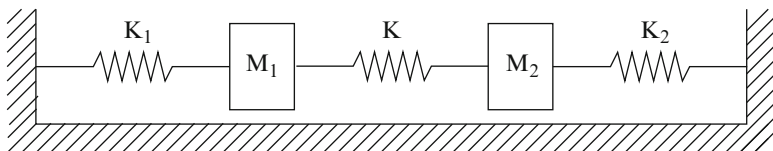


Fig. 2.3 Example of a conservative dynamical system.

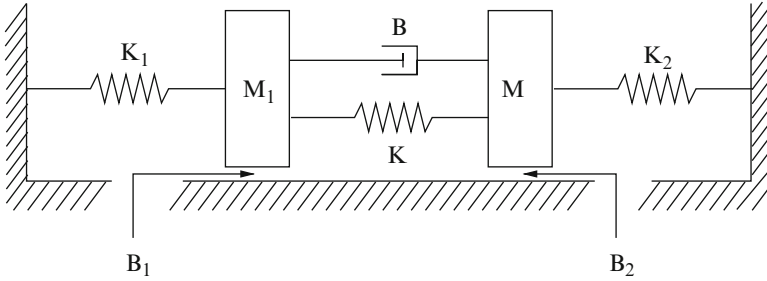


Fig. 2.4 Example of a mechanical system with energy dissipation.

- Problem 2.14.6.** (a) In Figure 2.4, K_1 , K_2 , K , M_1 , and M_2 are the same as in Figure 2.3 and B_1 , B_2 , and B denote viscous damping coefficients. Use the Lagrange formulation of dynamical systems described in Example 2.3.8 to derive two second-order ordinary differential equations that characterize this system. Transform these equations into a system of first-order ordinary differential equations. Verify your answer by using Newton's second law of motion to derive the same system of equations. By specifying $x_1(t_0)$, $\dot{x}_1(t_0)$, $x_2(t_0)$, and $\dot{x}_2(t_0)$, the above yields an initial value problem.
- (b) Show that the above mechanical system determines a dynamical system in the sense of Definition 2.2.3. \square

Problem 2.14.7. The following result, called the *Ascoli–Arzela Lemma*, is required in the proof of Problem 2.14.8 given below.

Let D be a closed and bounded subset of \mathbb{R}^n and let $\{f_m\}$ be a sequence of functions in $C[D, \mathbb{R}^n]$. If $\{f_m\}$ is *equicontinuous* and *uniformly bounded* on D , then there is a subsequence $\{f_{mk}\}$ and a function $f \in C[D, \mathbb{R}^n]$ such that $\{f_{mk}\}$ converges to f uniformly on D . Recall that $\{f_m\}$ is equicontinuous on D if for every $\varepsilon > 0$ there is a $\delta > 0$ (independent of x , y , and m) such that

$$|f_m(x) - f_m(y)| < \varepsilon \text{ whenever } |x - y| < \delta$$

for all $x, y \in D$ and for all m . Recall also that $\{f_m\}$ is uniformly bounded if there is a constant $M > 0$ such that $|f_m(x)| \leq M$ for all $x \in D$ and for all m .

Hint: To prove the Ascoli–Arzela Lemma, let $\{r_k\}$, $k \in \mathbb{N}$, be a dense subset of D . Determine a subsequence $\{f_{km}\}$ and a function f defined on $\{r_k\}$ such that $f_{km}(r_k) \rightarrow f(r_k)$ as $m \rightarrow \infty$ for all $k \in \mathbb{N}$. Next, prove that the subsequence $\{f_{mm}\}$ converges to $\{f\}$ on $\{r_k\}$, $k \in \mathbb{N}$, uniformly as $m \rightarrow \infty$. Conclude, by extending the domain of f from $\{r_k\}$ to D .

For a complete statement of the proof outlined above, refer to Miller and Michel [37]. \square

Problem 2.14.8. Prove Theorem 2.3.1.

Hint: First, show that for every $\varepsilon > 0$ there exists a piecewise linear function $\varphi_\varepsilon: J \rightarrow \mathbb{R}^n$ such that $\varphi_\varepsilon(t_0) = x_0$, $(t_0, x_0) \in D$, and $|\dot{\varphi}_\varepsilon(t) - f(t, \varphi_\varepsilon(t))| < \varepsilon$ for all $t \in [t_0, t_0 + c]$ (where $\dot{\varphi}_\varepsilon$ is defined) for some $c > 0$ and $(t, \varphi_\varepsilon(t)) \in D$ for all $t \in [t_0, t_0 + c]$. (φ_ε is called an ε -approximate solution of (I_E) .)

Next, let φ_m be an ε -approximate solution of (I_E) with $\varepsilon_m = 1/m$. Show that the sequence $\{\varphi_m\}$ is uniformly bounded and equicontinuous.

Finally, apply the Ascoli–Arzela Lemma to show that there is a subsequence $\{\varphi_{m_k}\}$ of $\{\varphi_m\}$ given above and a $\varphi \in C[[t_0, t_0 + c], \mathbb{R}^n]$ such that $\{\varphi_{m_k}\}$ converges to φ uniformly on $[t_0, t_0 + c]$, and such that φ satisfies

$$\varphi(t) = x_0 + \int_{t_0}^t f(s, \varphi(s)) ds$$

for $t \in [t_0, t_0 + c]$. Therefore, φ is a solution of (I_E) .

For a complete statement of the proof outlined above, refer to Miller and Michel [37]. \square

Problem 2.14.9. The following result, called the *Gronwall Inequality*, is required in the proof of Problem 2.14.10 given below.

Let $r, k \in C[[a, b], \mathbb{R}^+]$ and let $\delta \geq 0$ such that

$$r(t) \leq \delta + \int_a^t k(s)r(s)ds, \quad a \leq t \leq b. \quad (2.72)$$

Then

$$r(t) \leq \delta \exp\left[\int_a^t k(s)ds\right], \quad a \leq t \leq b. \quad (2.73)$$

Hint: For $\delta > 0$, integrate both sides of

$$\frac{k(s)r(s)}{\delta + \int_a^s k(\eta)r(\eta)d\eta} \leq k(s)$$

from a to t . Use inequality (2.72) to conclude the result when $\delta \neq 0$. When $\delta = 0$, consider a positive sequence $\{\delta_n\}$ such that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$ and apply it to (2.73).

For a complete statement of the proof outlined above, refer to Miller and Michel [37]. \square

Problem 2.14.10. Prove Theorem 2.3.2

Hint: Apply the Gronwall inequality given above in Problem 2.14.9.

For a complete statement of the proof, refer to Miller and Michel [37]. \square

Problem 2.14.11. The following result is required in the proof of Problem 2.14.12 given below.

Let $D \subset \mathbb{R} \times \mathbb{R}^n$ be a domain. Let $f \in C[D, \mathbb{R}^n]$ with f bounded on D and let φ be a solution of (E) on the interval (a, b) . Show that

- (a) The two limits $\lim_{t \rightarrow a^+} \varphi(t) = \varphi(a^+)$ and $\lim_{t \rightarrow b^-} \varphi(t) = \varphi(b^-)$ exist.
- (b) If $(a, \varphi(a^+)) \in D$ (resp., $(b, \varphi(b^-)) \in D$), then the solution φ can be continued to the left past the point $t = a$ (resp., to the right past the point $t = b$).

(A complete statement of the proof of the above result can be found in Miller and Michel [37].) □

Problem 2.14.12. Prove Theorem 2.3.3.

Hint: Use the result given in Problem 2.14.11. □

Problem 2.14.13. Prove Theorem 2.7.1.

Hint: To prove this result, use *Schauder's Fixed Point Theorem*: A continuous mapping of a compact convex set in a Banach space X into itself has at least one fixed point. Let T be the operator defined by (2.19). Find a compact convex set $X \subset C[[t_0 - r, t_0 + c], \mathbb{R}^n]$ for some $c > 0$ such that $T(X) \subset X$. Now apply Schauder's Fixed Point Theorem. A possible choice of X is given by

$$X = \{x \in C[[-r + t_0, t_0 + c], \mathbb{R}^n] : x_{t_0} = \psi, \|x_t - \psi\| \leq d \text{ for all } t \in [t_0, t_0 + c]\},$$

where $0 < c \leq d/M$, $d > 0$ sufficiently small, with $M \geq |f(t, \varphi)|$ for all (t, φ) in a fixed neighborhood of (t_0, ψ) in Ω .

For the complete proof of Theorem 2.7.1 outlined above, refer to Hale [19]. □

Problem 2.14.14. Prove Theorem 2.7.2.

Hint: Let $x(t)$ and $y(t)$ be two solutions of (I_F) . Then

$$x(t) - y(t) = \int_{t_0}^t [f(s, x_s) - f(s, y_s)] ds, \quad t \geq t_0, \quad x_{t_0} - y_{t_0} = 0.$$

Using the above, show that there exists a $c_0 > 0$ such that $x(t) = y(t)$ for all $t \in [t_0 - r, t_0 + c_0]$. To complete the proof, repeat the above for successive intervals of length c_0 .

For the complete proof of Theorem 2.7.2 outlined above, refer to Hale [19]. □

Problem 2.14.15. The following result is required in the proof of Problem 2.14.16 given below.

Let Ω be an open set in $\mathbb{R} \times C_r$ and let $F: \Omega \rightarrow \mathbb{R}^n$ be completely continuous. Assume that $p \in C[[t_0 - r, b), \mathbb{R}^n]$ is a noncontinuable solution of (F) . Show that for any bounded closed set U in $\mathbb{R} \times C_r$, $U \subset \Omega$, there exists a $t_U \in (t_0, b)$ such that $(t, p_t) \notin U$ for every $t \in [t_U, b]$.

Hint: The case $b = \infty$ is clear. Suppose that b is finite. The case $r = 0$ reduces to an ordinary differential equation. So assume that $r > 0$. Now prove the assertion by contradiction, assuming that $b < \infty$ and $r > 0$. □

Problem 2.14.16. Prove Theorem 2.7.3.

Hint: Apply the result given in Problem 2.14.15. For the complete proof, refer to Hale [19]. \square

Problem 2.14.17. Prove Theorem 2.8.1.

Hint: Using the theory of C_0 -semigroups, refer to Example 2.9.3 for a choice of the infinitesimal generator for the C_0 -semigroup (refer to [3]). \square

Problem 2.14.18. Consider the initial value problem

$$\begin{cases} \dot{x} = A(t)x \\ x(t_0) = x_0 \end{cases} \quad (LH)$$

where $A \in C[\mathbb{R}^+, \mathbb{R}^{n \times n}]$.

- (a) Show that the set of solutions obtained for (LH) by varying (t_0, x_0) over $(\mathbb{R}^+, \mathbb{R}^n)$ determines a dynamical system in the sense of Definition 2.2.3.
- (b) Show that in general, (LH) does not determine a C_0 -semigroup.
- (c) Show that when $A(t) \equiv A$, (LH) determines a C_0 -semigroup. \square

Problem 2.14.19. Prove the assertion made in Example 2.9.1 that the initial value problem (2.32) determines a quasi-contractive semigroup. \square

Problem 2.14.20. Consider the initial value problem for the *heat equation*

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \Delta u, x \in \mathbb{R}^n, & t \in \mathbb{R}^+ \\ u(0, x) = \varphi(x), & x \in \mathbb{R}^n \end{cases} \quad (2.74)$$

where $a > 0$, $\Delta = \sum_{i=1}^n \partial^2 / \partial x_i^2$, and $\varphi \in C[\mathbb{R}^n, \mathbb{R}]$ is bounded.

- (a) Verify that the unique solutions of (2.74) are given by *Poisson's formula*,

$$u(t, x) = \frac{1}{(2a\sqrt{\pi t})^n} \int_{\mathbb{R}^n} e^{-|x-y|^2/(4a^2 t)} \varphi(y) dy.$$

- (b) Show that the operators $T(t), t \in \mathbb{R}^+$, determined by $u(t, \cdot) = T(t)\varphi$, determine a C_0 -semigroup. \square

Problem 2.14.21. Consider the initial value problem for the *one-dimensional wave equation*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R}, t \in \mathbb{R}^+ \\ u(0, x) = \varphi(x), \frac{\partial u}{\partial t}(0, x) = \psi(x), & x \in \mathbb{R} \end{cases} \quad (2.75)$$

where $c > 0$, $\varphi \in C^2[\mathbb{R}, \mathbb{R}]$, and $\psi \in C^1[\mathbb{R}, \mathbb{R}]$.

(a) Verify that the unique solution of (2.75) is given by *d'Alembert's formula*

$$u(t, x) = \frac{1}{2} [\varphi(x - ct) + \varphi(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\eta) d\eta.$$

- (b) Let $\psi \equiv 0$. For $\varphi \in C^2[\mathbb{R}, \mathbb{R}]$, define the operators $T(t)$, $t \in \mathbb{R}^+$, by $T(t)\varphi = u(t, \cdot)$. Show that $T(t)$, $t \in \mathbb{R}^+$, do not satisfy the semigroup property (specifically, they do not satisfy the property $T(t)T(s) = T(t+s)$, $t, s \in \mathbb{R}^+$).
- (c) Now let $u(t, \varphi, t_0)$ denote the solutions of

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R}, t \geq t_0 \\ u(t_0, x) = \varphi(x), \quad \frac{\partial u}{\partial t}(t_0, x) = 0, & x \in \mathbb{R} \end{cases} \quad (2.76)$$

where $t_0 \in \mathbb{R}^+$ and $\varphi \in C^2[\mathbb{R}, \mathbb{R}]$. Show that for all $\varphi \in C^2[\mathbb{R}, \mathbb{R}]$, the resulting solutions $u(t, \varphi, t_0)$ form a dynamical system in the sense of Definition 2.2.3 with $T = \mathbb{R}^+$, $X = A = C^2[\mathbb{R}, \mathbb{R}]$ where we assume that X is equipped with some norm (e.g., $\|\varphi\| = \max_{x \in \mathbb{R}} |\varphi(x)|$). \square

Problem 2.14.22. We now consider a specific class of *multirate digital feedback control systems*. The *plant* is described by

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 u_{1c}(t) + B_2 u_{2c}(t) \\ y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} D_1 x(t) \\ D_2 x(t) \end{bmatrix} \end{cases} \quad (2.77)$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $B_1 \in \mathbb{R}^{n \times n_1}$, $B_2 \in \mathbb{R}^{n \times n_2}$, $D_1 \in \mathbb{R}^{m_1 \times n}$, $D_2 \in \mathbb{R}^{m_2 \times n}$, $u_{1c} \in \mathbb{R}^{n_1}$, $u_{2c} \in \mathbb{R}^{n_2}$, and

$$\begin{cases} u_{1c}(t) = u_1(k), & kT_b \leq t < (k+1)T_b, & k \in \mathbb{N}, \\ u_{2c}(t) = u_2(2k), & 2kT_b \leq t < 2(k+1)T_b, & k \in \mathbb{N}. \end{cases} \quad (2.78)$$

In (2.78), $T_b > 0$ is the *basic sampling period* whereas $u_1(k)$ and $u_2(2k)$ are specified by output feedback equations of the form

$$\begin{aligned} u_1(k+1) &= F_1 u_1(k) + K_1 y_1(kT_b) \\ &= F_1 u_1(k) + K_1 D_1 x(kT_b), & k \in \mathbb{N} \\ u_2(2(k+1)) &= F_2 u_2(2k) + K_2 y_2(2kT_b) \\ &= F_2 u_2(2k) + K_2 D_2 x(2kT_b), & k \in \mathbb{N} \end{aligned} \quad (2.79)$$

where K_1 , K_2 , F_1 , and F_2 are matrices of appropriate dimensions. The system inputs $u_{1c}(t)$ and $u_{2c}(t)$ are realized by *multirate zero-order hold elements*.

Similarly as in Example 2.12.1, show that the above hybrid system can equivalently be represented by a system of discontinuous ordinary differential equations that generate a discontinuous dynamical system. \square

Bibliography

1. J.P. Aubin, A. Cellina, *Differential Inclusions* (Springer, New York, 1984)
2. D.D. Bainov, P.S. Simeonov, *Systems with Impulse Effects: Stability Theory and Applications* (Halsted Press, New York, 1989)
3. V. Barbu, S.I. Grossman, Asymptotic behavior of linear integrodifferential systems. *Trans. Am. Math. Soc.* **171**, 277–288 (1972)
4. R. Bellman, K.L. Cooke, *Differential-Difference Equations* (Academic, New York, 1963)
5. H. Brezis, *Opérateurs Maximaux Monotones* (North-Holland, Amsterdam, 1973)
6. R.W. Brockett, Hybrid models for motion control systems, in *Essays on Control Perspectives in the Theory and Its Applications*, ed. by H.L. Trentelman, J.C. Willems (Birkhäuser, Boston, 1993), pp. 29–53
7. B. Brogliato, S.I. Niculescu, P. Orhant, On the control of finite-dimensional systems with unilateral constraints. *IEEE Trans. Autom. Control* **42**, 200–215 (1997)
8. M.G. Crandall, Semigroups of nonlinear transformations on general Banach spaces, in *Contributions to Nonlinear Functional Analysis*, ed. by E.H. Zarantonello (Academic, New York, 1971)
9. M.G. Crandall, T.M. Liggett, Generation of semigroups of nonlinear transformations on general Banach spaces. *Am. J. Math.* **93**, 265–298 (1971)
10. R. DeCarlo, S. Branicky, S. Pettersson, B. Lennartson, Perspectives and results on the stability and stabilizability of hybrid systems. *Proc. IEEE* **88**, 1069–1082 (2000)
11. J. Dieudonné, *Foundations of Modern Analysis* (Academic, New York, 1960)
12. N. Dunford, J.T. Schwarz, *Linear Operators*, Part I (Wiley, New York, 1958)
13. G.F. Franklin, J.D. Powell, *Digital Control of Dynamical Systems* (Addison-Wesley, Reading, 1980)
14. A. Friedman, *Partial Differential Equations of the Parabolic Type* (Prentice Hall, Englewood Cliffs, 1964)
15. A.N. Godunov, Peano's theorem in Banach spaces. *Funct. Anal. Appl.* **9**, 53–56 (1975)
16. A. Gollu, P.P. Varaiya, Hybrid dynamical systems, in *Proceedings 28th IEEE Conference on Decision and Control*, Tampa, December 1989, pp. 2708–2712
17. R. Grossman, A. Nerode, A. Ravn, H. Rischel (eds.), *Hybrid Systems* (Springer, New York, 1993)
18. W. Hahn, *Stability of Motion* (Springer, Berlin, 1967)
19. J.K. Hale, *Functional Differential Equations* (Springer, Berlin, 1971)
20. J.K. Hale, Functional differential equations with infinite delays. *J. Math. Anal. Appl.* **48**, 276–283 (1974)
21. E. Hille, R.S. Phillips, *Functional Analysis and Semigroups*. American Mathematical Society Colloquium Publication, vol. 33 (American Mathematical Society, Providence, 1957)
22. L. Hörmander, *Linear Partial Differential Equations* (Springer, Berlin, 1963)
23. S.G. Krein, *Linear Differential Equations in Banach Spaces*. Translation of Mathematical Monographs, vol. 29 (American Mathematical Society, Providence, 1970)
24. N.V. Krylov, *Nonlinear Elliptic and Parabolic Equations of the Second Order* (D. Reidel, Boston, 1987)
25. T. Kurtz, Convergence of sequences of semigroups of nonlinear equations with applications to gas kinetics. *Trans. Am. Math. Soc.* **186**, 259–272 (1973)

26. V. Lakshmikantham, S. Leela, *Differential and Integral Inequalities*, vols. 1 and 2 (Academic, New York, 1969)
27. A. Lasota, J.A. Yorke, The generic property of existence of solutions of differential equations in Banach space. *J. Diff. Equ.* **13**, 1–12 (1973)
28. D. Liberzon, A.S. Morse, Basic problems in stability and design of switched systems. *IEEE Control Syst. Mag.* **19**, 59–70 (1999)
29. A.N. Michel, Recent trends in the stability analysis of hybrid dynamical systems. *IEEE Trans. Circuits Syst.* **46**, 120–134 (1999)
30. A.N. Michel, B. Hu, Towards a stability theory of general hybrid dynamical systems. *Automatica* **35**, 371–384 (1999)
31. A.N. Michel, R.K. Miller, *Qualitative Analysis of Large Scale Dynamical Systems* (Academic, New York, 1977)
32. A.N. Michel, Y. Sun, Stability of discontinuous Cauchy problems in Banach space. *Nonlinear Anal.* **65** 1805–1832 (2006)
33. A.N. Michel, Y. Sun, A.P. Molchanov, Stability analysis of discontinuous dynamical systems determined by semigroups. *IEEE Trans. Autom. Control* **50**, 1277–1290 (2005)
34. A.N. Michel, K. Wang, B. Hu, *Qualitative Theory of Dynamical Systems – The Role of Stability Preserving Mappings*, 2nd edn. (Marcel Dekker, New York, 2001)
35. A.N. Michel, K. Wang, K.M. Passino, Qualitative equivalence of dynamical systems with applications to discrete event systems, in *Proceedings of the 31st IEEE Conference on Decision and Control*, Tucson, December 1992, pp. 731–736
36. A.N. Michel, K. Wang, K.M. Passino, Stability preserving mappings and qualitative equivalence of dynamical systems, Part I. *Avtomatika i Telemekhanika* **10**, 3–12 (1994)
37. R.K. Miller, A.N. Michel, *Ordinary Differential Equations* (Dover, Mineola, 2007)
38. A.V. Oppenheim, R.W. Schaffer, *Digital Signal Processing* (Prentice Hall, Englewood Cliffs, 1975)
39. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations* (Springer, New York, 1963)
40. R.D. Rasmussen, A.N. Michel, Stability of interconnected dynamical systems described on Banach spaces. *IEEE Trans. Autom. Control* **21**, 464–471 (1976)
41. C. Rui, I. Kolmanovsky, N.H. McClamroch, Hybrid control for stabilization of a class of cascade nonlinear systems, in *Proceedings of American Control Conference*, pp. 2800–2804, Albuquerque, NM, June 1997
42. M. Slemrod, Asymptotic behavior of C_0 -semigroups as determined by the spectrum of the generator. *Indiana Univ. Math. J.* **25** 783–791 (1976)
43. Y. Sun, A.N. Michel, G. Zhai, Stability of discontinuous retarded functional differential equations with applications. *IEEE Trans. Autom. Control*, **50**, 1090–1105 (2005)
44. J.C. Willems, Paradigms and puzzles in the theory of dynamical systems. *IEEE Trans. Autom. Control* **36**, 259–294 (1991)
45. H. Ye, A.N. Michel, P.J. Antsaklis, A general model for the qualitative analysis of hybrid dynamical systems, in *Proceedings of the 34th IEEE Conference on Decision and Control*, New Orleans, December 1995, pp. 1473–1477
46. H. Ye, A.N. Michel, L. Hou, Stability theory for hybrid dynamical systems. *IEEE Trans. Autom. Control* **43**, 461–474 (1998)
47. T. Yoshizawa, *Stability Theory by Lyapunov's Second Method* (Mathematical Society of Japan, Tokyo, 1966)
48. V.I. Zubov, *Methods of A.M. Lyapunov and Their Applications* (P. Noordhoff, Groningen, 1964)

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