Chapter 2
A Model-Free-Based Proportional Reduced-Order Observer Design for the Synchronization of Lorenz Systems

Abstract In this chapter, we deal with the synchronization problem of a Lorenz system using a proportional reduced-order observer design in the algebraic and differential settings. We prove the asymptotic stability of the resulting error system, and by means of algebraic manipulations, we obtain estimates of the current states (master system). In this chapter, the construction of a proportional reduced-order observer is the main ingredient in our approach. Finally, we present simulations to illustrate the effectiveness of the suggested approach.

2.1 Introduction

In recent years, synchronization of chaotic systems has received a great deal of attention among scientists in many fields [1–5]. As is well known, the study of the synchronization problem for nonlinear systems has been very important from the nonlinear sciences point of view, in particular the applications to biology, medicine, cryptography, secure data transmission, and so on. In general, synchronization research has been focused onto two areas. The first is related to the employment of state observers, whereby the main application lies in the synchronization of nonlinear oscillators. On the other hand, the use of control laws makes it possible to achieve synchronization between nonlinear oscillators with different structures and orders [6]. Of particular interest is the connection between the observers for nonlinear systems and synchronization, which is also known as a master–slave configuration. Thus, the chaos synchronization problem can be posed as a one-observer design in which the coupling signal is viewed as the output, and the slave system as the (reduced-order) observer. In other words, basically, the chaos synchronization problem can be formulated as follows. Given a chaotic system that is considered the master (or driving) system, and another identical system, given as the slave (or response) system, the aim is to force the response of the slave system to synchronize with the master system [1, 2]. The idea is to use the output of the master system to control the slave system so that the states of the slave system follow the states of the master system asymptotically.

In this procedure, the construction of a full-order observer is unnecessary; that is, we construct a reduced-order observer (so-called model-free based observer) using

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the algebraic observability condition (AOC) applied to the estimation problem. The methodology proposed consists in the following. Define first a function as an additional state of the original system; this function is given in terms of the states. The dynamics of this new state is not known. The original system is then converted to an extended system in which the dynamics of the additional state is not known and supposed to be bounded. The original problem is then an observation problem, where the aim is to observe this additional state of the system. Since the dynamics of this new state is not known, a reduced-order observer for the unknown part of the system is proposed. In particular for its simplicity, a model-free-based proportional reduced-order observer is presented. Finally, we illustrate the effectiveness of the suggested approach with a Lorenz system.

The main contribution in this chapter is to present a technique for the synchronization problem. We propose a proportional reduced-order observer structure to synchronize with a Lorenz system. As far as we know, this class of observers has not been used in the literature. We introduce the algebraic observability concept in the differential-algebraic setting to construct a reduced-order observer for the synchronization problem. The methodology proposed is very simple and flexible. In our procedure, the new proposed reduced-order observer employs neither the well-known Kalman filter nor the Luenberger observer. Both require a copy of the system and a proportional correction given by the measurement of the error (the difference between the actual observed signal and its estimate). This reduced-order proportional observer does not require an accurate model of the system, since its structure just contains a proportional correction of the measurement of the error and a so-called derivator of adjustable gain. It should be noted that the reduced-order observer proposed is an alternative technique to the "observers" structure given in the literature. In addition, some algebraic manipulations are required with this technique. We note that some authors propose estimators without observer's gain, but we think it more natural to define an observer with a gain that attenuates the observation error (the difference between the actual observed signal and its estimate), and this observer must be able to be tuned through the observer's gain.

Our intention in choosing as an example the Lorenz system is to clarify the proposed methodology. However, it is worth mentioning that this method is applicable not only to the Lorenz system. It can be applied to a class of systems that satisfy the state representation described in Sect. 2.2. Additionally, it is clear that this class of systems should satisfy Definition 2.1 and some hypotheses given in Sects. 2.2.1 and 2.2.2. Among this class of systems we can mention some examples such as the Duffing system, Chen's chaotic system, Chua's chaotic circuit, Rössler's chaotic system, and the Colpitts chaotic circuit.

The remainder of this chapter is organized as follows: In Sect. 2.2, we describe the Lorenz system and introduce a basic definition on algebraic observability in a differential-algebraic framework. A reduced-order observer structure to synchronize with a Lorenz system is given as well. Section 2.3 presents some numerical results applied to the Lorenz system. Finally, in Sect. 2.4, we will close the chapter with some concluding remarks.

2.2 Synchronization of Lorenz System

Consider the Lorenz chaotic system described by the following set of differential equations:

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1), \\ \dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 &= x_1 x_2 - \beta x_3, \\ y &= x_1. \end{aligned} \tag{2.1}$$

With the positive parameters $\sigma,\ \rho,\ \beta > 0$, the system (2.1) exhibits chaotic behavior. In the classical synchronization scheme, we assume that the system (2.1) runs at the transmitter end and the state x_1 (output system) is sent to the receiver via the communication channel as the synchronization signal [1]. Furthermore, it is assumed that the receiver has exact knowledge of the parameters $\sigma,\ \rho,\ \beta > 0$ (i.e., there is no parametric uncertainty). The receiver's task is to construct a dynamical system to estimate or reconstruct the unknown signals x_2 and x_3 using the available signal x_1 and the known parameters. The basic practical question whether it is possible to construct the signals x_2 and x_3. In what follows, we give an answer to this question by introducing a basic definition related to the construction of the states.

2.2.1 Algebraic Observability Condition

Before proposing the reduced-order observer, a definition concerning on AOC is given. Consider the nonlinear system described by the following dynamic equations:

$$\begin{aligned} \dot{x}(t) &= f(x), \\ y(t) &= h(x), \end{aligned} \tag{2.2}$$

where $f \in \mathbb{R}^n$ is continuously differentiable and satisfies $f(0) = 0$, $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ is a state vector, and $y \in \mathbb{R}$ is a smooth nonsingular output.

Definition 2.1 The system (2.2) is said to be algebraically observable if the valued vector

$$x(t) = \phi(y, \dot{y}, \ddot{y}, \ldots, y^{(\mu)})(t). \tag{2.3}$$

is defined on $\mathbb{R}^{\mu+1} \to \mathbb{R}^n$ for a positive integer μ (ϕ is a polynomial with unknown $y, \dot{y}, \ddot{y}, \ldots, y^{(\mu)}$). The above condition will be called the algebraic observability condition (AOC).

In subsequent chapters, we shall generalize this condition to cases such as fractional and Liouvillian systems. The unknown state of the system can be included in a new variable $\eta(x)$. Then the augmented system (nonlinear systems immersion [7, 8]) can be considered instead of the original one, because we want to work on the dynamics η, which is unknown. The immersion can be realized as follows:

$$\begin{aligned} \dot{x}(t) &= f(x, \eta(x)), \\ \dot{\eta}(x) &= \Delta(x), \\ y &= h(x), \end{aligned} \tag{2.4}$$

where $\Delta(x)$ is a function of the states. The problem now is to construct the variable $\eta(x)$ and, once it is known, determine the value of the desired state.

Remark 2.1 In practice, identification of the variable depends on the variable choice to be estimated.

We obviously need to impose certain conditions on $\eta(x)$ and $\Delta(x)$. We propose a procedure to solve the problem stated above. We will assume that the following hypotheses are satisfied:

H1: $\eta(x)$ satisfies AOC property.
H2: the auxiliary variable γ is a C^1 real-valued function.
H3: $\Delta(x)$ is bounded, i.e., $\exists\ M \in \mathbb{R}^+$ such that $\|\Delta(x)\| \leq M, \forall x \in \Omega \subset \mathbb{R}^n$.

The following equation represents the dynamics of the unknown state:

$$\dot{\eta}(x) = \Delta(x). \tag{2.5}$$

Remark 2.2 In our case, $\eta(x)$ can be chosen as x_2 or x_3.

The hypothesis H1 is satisfied, since the Lorenz system satisfies the AOC, that is,

$$\dot{x}_1 = \sigma(x_2 - x_1) \Rightarrow x_2 = \phi(y, \dot{y}) = (1/\sigma)(\sigma y + \dot{y}), \tag{2.6}$$

where $\mu = 1, \sigma > 0$, and $y = x_1$. In the same manner, for x_3, we have

$$x_3 = -(1/x_1)(x_2 + \dot{x}_2) + \rho. \tag{2.7}$$

It is worth noting that x_3 loses the algebraic observability property when $x_1 = 0$. In other words, we do not have the synchronization signal x_1. We suppose, therefore, that $x_1 \neq 0$. Then

$$x_3 = \phi(y, \dot{y}, \ddot{y}) = (-\ddot{y} - \dot{y}(\sigma + 1))/\sigma y + \rho - 1,\ \ \mu = 2, \sigma > 0, y = x_1. \tag{2.8}$$

Remark 2.3 From (2.6) and (2.7), it is clear that x_2 and x_3 satisfy the AOC, and thus x_2 and x_3 are algebraically observable. Since the time derivatives $\dot{y}$ and $\ddot{y}$ are not available, we will design, with the help of some auxiliary variables (γ satisfying H2), a reduced-order observer.

We propose a proportional reduced-order observer in order to estimate the variable $\eta(x)$ and determine the value of the desired state.

2.2.2 Observer Synthesis

We are now in a position to establish the following lemma, which describes the construction of a proportional reduced-order observer for the system (2.5), which is algebraically observable (see Definition 2.1).

Lemma 2.1 *The system*

$$\dot{\hat{\eta}} = k(\eta - \hat{\eta}) \tag{2.9}$$

is an asymptotic model-free-based proportional reduced-order observer for system (2.5), where $\hat{\eta}$ denotes the estimate of η, $k \in R^+$ determines the desired convergence rate of the observer if the following hypothesis is satisfied (in general, k can be chosen as a time-varying function, for instance in a Hardy field [9]):

H4: $\lim_{t\to t_0} e^{-\int kdt} = 0$ *with t_0 sufficiently large and* $\lim_{t\to t_0} \sup(M/|k|)) = 0$.

Proof Let us define the estimation error as follows:

$$e(t) = \eta(x) - \hat{\eta}(x). \tag{2.10}$$

This yields the nonlinear dynamics of the estimation error given by

$$\dot{e}(t) + ke(t) = \Delta(x). \tag{2.11}$$

Solving the above equation, we have

$$e(t) = e^{-\int kdt}\left[e_0 + \int_0^t e^{\int kdt}\Delta(s)ds\right], \tag{2.12}$$

where e_0 is an initial condition. Then with the assumptions H1–H4 and using the triangle and Cauchy–Schwarz inequalities from expression (2.12), we obtain

$$0 \le |e(t)| \le e^{-\int kdt}|e_0| + e^{-\int kdt}\int_0^t |e^{\int kdt}\Delta(s)|ds. \tag{2.13}$$

Thus, as $t \to t_0$ with t_0 sufficiently large,

$$0 \le \lim_{t\to t_0} \sup |e_0| \le |e_0| \lim_{t\to t_0} \sup [e^{-\int kdt}] + \lim_{t\to t_0} \sup \frac{[\int_0^t |e^{\int kdt}\Delta(s)|ds]}{|e^{\int kdt}|}. \tag{2.14}$$

From H1 and H4, we obtain

$$0 \leq \limsup_{t \to t_0} |e_0| \leq \limsup_{t \to t_0} \frac{[M \int_0^t |e^{\int kdt}| ds]}{|e^{\int kdt}|}. \tag{2.15}$$

This means that we can apply L'Hôpital's rule for the case $\dfrac{\infty}{\infty}$ as follows:

$$0 \leq \limsup_{t \to t_0} |e_0| \leq \limsup_{t \to t_0} \frac{[M \int_0^t |e^{\int kdt}| ds]}{|e^{\int kdt}|} = \limsup_{t \to t_0} \frac{M}{|k|}. \tag{2.16}$$

From H4, we obtain

$$0 \leq \limsup_{t \to t_0} |e(t)| \leq \limsup_{t \to t_0} \frac{M}{|k|} = 0. \tag{2.17}$$

Then

$$\lim_{t \to t_0} |e(t)| = 0. \tag{2.18}$$

Therefore, the reduced-order observer given by (2.9) exhibits asymptotic behavior.

□

Corollary 2.1 *The dynamical system (2.9) along with*

$$\dot{\gamma} = \zeta(\gamma, x), \gamma_0 = \gamma(0) \tag{2.19}$$

constitutes a proportional asymptotic reduced-order observer for the system (2.5), where γ is a change of variable that depends on the estimated and state variables.

Remark 2.4 In practice, since γ is a variable that depends on x and η, its complexity depends on the form selected for η.

Remark 2.5 It should be noted that an integral action can be added in the proportional asymptotic reduced-order observer to achieve robustness in the observation process.

Using Lemma 2.1, we have the following proportional reduced-order observer:

$$\dot{\hat{x}}_2 = k_{x_2}(x_2 - \hat{x}_2), k_{x_2} > 0, \tag{2.20}$$

where $\hat{x}_2$ denotes the estimate of x_2, and $k_{x_2} \in R^+$ determines the desired convergence rate of the observer.

Replacing (2.6) in (2.20) leads to

$$\dot{\hat{x}}_2 = k_{x_2}\left(\frac{\dot{y}}{\sigma} + y\right) - k_{x_2}\hat{x}_2, \sigma > 0. \tag{2.21}$$

Since the time derivative $\dot{y}$ is not available, observer (2.21) cannot be implemented. In order to overcome this problem, let us consider the following auxiliary variable γ_{x_2}:

$$\gamma_{x_2} = -\frac{k_{x_2}}{\sigma} y + \hat{x}_2, \sigma > 0. \tag{2.22}$$

Then

$$\hat{x}_2 = \gamma_{x_2} + \frac{k_{x_2}}{\sigma} y, \sigma > 0. \tag{2.23}$$

The time derivative of (2.23) is

$$\dot{\hat{x}}_2 = \dot{\gamma}_{x_2} + \frac{k_{x_2}}{\sigma} \dot{y}, \sigma > 0. \tag{2.24}$$

Then from (2.21), (2.23), and (2.24), it can be easily shown that the time derivative $\dot{\gamma}_{x_2}$ is given by

$$\dot{\gamma}_{x_2} = -k_{x_2}\gamma_{x_2} + \left(1 - \frac{k_{x_2}}{\sigma} k_{x_2}\right) y, \gamma_{x_2}(0) = \gamma_{x_{2_0}}, \sigma, k_{x_2} > 0. \tag{2.25}$$

Then the reduced-order observer is given by Eqs. (2.23) and (2.25).

Remark 2.6 It is clear that the solution of the dynamics of the auxiliary variable is exponentially stable.

On the other hand, using the above technique, we estimate the variable x_3. From (2.1) and (2.6), we obtain

$$\dot{x}_3 = y\left(\frac{\dot{y} + \sigma y}{\sigma}\right) - \beta x_3. \tag{2.26}$$

Or in other words,

$$x_3 = \frac{-1}{\beta}\left(\dot{x}_3 - y\left(\frac{\dot{y} + \sigma y}{\sigma}\right)\right). \tag{2.27}$$

Then the following observer is suggested:

$$\dot{\hat{x}}_3 = k_{x_3}(x_3 - \hat{x}_3), k_{x_3} > 0. \tag{2.28}$$

Replacing (2.27) in (2.28), we obtain

$$\dot{\hat{x}}_3 = k_{x_3}\left[\frac{-1}{\beta}\left(\dot{x}_3 - y\left(\frac{\dot{y} + \sigma y}{\sigma}\right)\right) - \hat{x}_3\right]. \tag{2.29}$$

It should be noted that $\dot{x}_3$ is unknown and can be approximated by its estimate $\dot{\hat{x}}_3$, which is valid only in a region where $\|\dot{x}_3 - \dot{\hat{x}}_3\| < \epsilon, \epsilon > 0$. This yields the following relationship:

$$\dot{\hat{x}}_3 = k_{x_3}\left[\frac{-1}{\beta}\left(\dot{\hat{x}}_3 - y\left(\frac{\dot{y}+\sigma y}{\sigma}\right)\right) - \hat{x}_3\right]. \tag{2.30}$$

Or in other words,

$$\dot{\hat{x}}_3\left(1+\frac{k_{x_3}}{\beta}\right) - \frac{k_{x_3}}{\beta\sigma} y\dot{y} = k_{x_3}\left[\frac{1}{\beta}y^2 - \hat{x}_3\right]. \tag{2.31}$$

By making some algebraic manipulations, we obtain

$$\begin{aligned}\dot{\hat{x}}_3 - \frac{k_{x_3}}{(\beta+k_{x_3})\sigma} y\dot{y} = \frac{k_{x_3}\beta}{\beta+k_{x_3}}&\left[\left(\frac{1}{\beta} - \frac{k_{x_3}}{2(\beta+k_{x_3})\sigma}\right) y^2\right.\\ &\left.-\left(\hat{x}_3 - \frac{k_{x_3}}{2(\beta+k_{x_3})\sigma} y^2\right)\right].\end{aligned} \tag{2.32}$$

Then from (2.32) and the following change of variable follows

$$\gamma_{x_3} = \hat{x}_3 - \frac{k_{x_3}}{2(\beta+k_{x_3})\sigma} y^2, \tag{2.33}$$

$$\dot{\gamma}_{x_3} = \frac{k_{x_3}\beta}{\beta+k_{x_3}}\left[\left(\frac{1}{\beta} - \frac{k_{x_3}}{2(\beta+k_{x_3})\sigma}\right) y^2 - \gamma_{x_3}\right]. \tag{2.34}$$

That is, the reduced-order observer to calculate $\hat{x}_3$ is given by the following relationship:

$$\hat{x}_3 = \gamma_{x_3} + \frac{k_{x_3}}{2(\beta+k_{x_3})\sigma} y^2; \beta, k_{x_3}, \sigma > 0, \tag{2.35}$$

$$\dot{\gamma}_{x_3} = \frac{k_{x_3}\beta}{\beta+k_{x_3}}\left[\left(\frac{1}{\beta} - \frac{k_{x_3}}{2(\beta+k_{x_3})\sigma}\right) y^2 - \gamma_{x_3}\right]; \gamma_{x_3}(0) = \gamma_{x_{3,0}}. \tag{2.36}$$

Remark 2.7 The proposed model-free-based proportional reduced-order observer employs neither the well-known Kalman filter nor the Luenberger observer. Both require a copy of the system and a proportional correction given by the measurement of the error (the difference between the actual observed signal and its estimate). This reduced-order proportional observer does not require an accurate model of the system, since its structure just contains a proportional correction of the measurement of the error and a so-called derivator of adjustable gain.

In what follows, we illustrate the performance of the reduced-order proportional observer by means of numerical simulations.

2.3 Numerical Results

In order to verify the effectiveness of the proposed methodology, we show the convergence of the estimates to the current signals. We have considered the initial conditions to the master system $x_1 = 1$, $x_2 = 0$, $x_3 = -5$, and the initial conditions to the slave system $\hat{x}_2 = -5$, $\hat{x}_3 = 8$. The parameters of the system are $\sigma = 10$, $\rho = 28$, $\beta = 8/3$; the initial conditions of the auxiliary functions of the reduced-order observer are $\gamma_{x_2} = -20$, $\gamma_{x_3} = 8$; and finally, the gain parameters in the proportional reduced-order observer are fixed as $k_{x_2} = 150$, $k_{x_3} = 250$. The convergence of the estimates to the true signals is shown in Fig. 2.1, and the error synchronization is shown in Fig. 2.2.

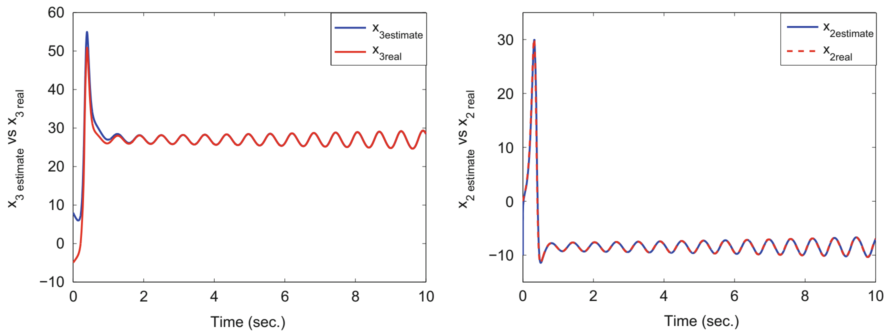

Fig. 2.1 Convergence of the estimates by means of a model-free-based proportional reduced-order observer

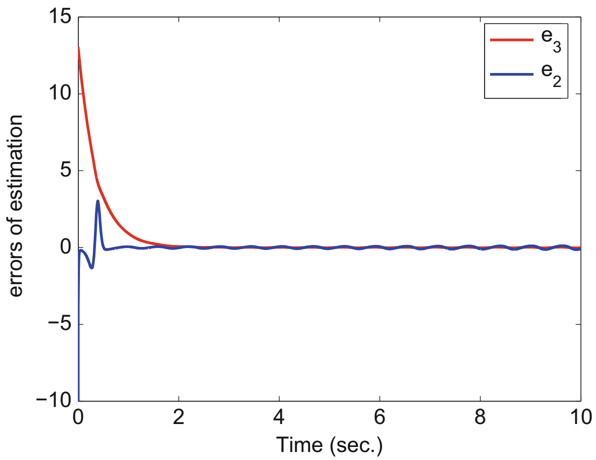

Fig. 2.2 Synchronization errors

2.4 Concluding Remarks

In this chapter, we have presented a model-free-based proportional reduced-order observer to address the synchronization problem of the Lorenz system. A model-free observer has the advantage of not requiring a copy of the system. In the next chapter, a model-free sliding-mode observer is developed. In order to achieve robustness against perturbations to the system and noisy measurements, we have proven the asymptotic stability of the resulting error system, and by means of simple algebraic manipulations, we construct the estimates of the slave system, which asymptotically converge to the current states of the master system. Finally, we have presented a simulation to illustrate the effectiveness of the suggested approach.

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