

Chapter 2

Fractional Heat Conduction and Related Theories of Thermoelasticity

*Some play at chess, some at cards, some at the Stock Exchange.
I prefer to play Cause and Effect.*

Ralph Emerson

Abstract This chapter is devoted to time- and space-nonlocal generalizations of the standard Fourier law, the corresponding generalizations of the classical heat conduction equation and formulation of associated theories of fractional thermoelasticity. Different kinds of boundary conditions for the time-fractional heat conduction equation are analyzed including the Dirichlet, mathematical and physical Neumann and Robin conditions, the conditions of perfect thermal contact and the moving interface boundary conditions at the solid-liquid interface. Representations of stresses in terms of the displacement potential, biharmonic Galerkin vector and biharmonic Love function are discussed.

2.1 Time and Space Nonlocality

The local dependence of one physical quantity (an effect \mathbf{E}) at a point \mathbf{x} at time t ($\mathbf{E}(x, y, z, t)$) on a field of another physical quantity (a cause \mathbf{C}) at the same point \mathbf{x} and at the same time t ($\mathbf{C}(x, y, z, t)$) has the following form

$$\mathbf{E}(x, y, z, t) = F(\mathbf{C}(x, y, z, t)). \quad (2.1)$$

For materials with time nonlocality the effect \mathbf{E} at a point \mathbf{x} at time t depends on the histories of causes at a point \mathbf{x} at all past and present times. If $t = -\infty$ is chosen as a “starting point”, then

$$\mathbf{E}(x, y, z, t) = \int_{-\infty}^t K(t - \tau) F(\mathbf{C}(x, y, z, \tau)) d\tau, \quad (2.2)$$

where $K(t - \tau)$ is the corresponding weight function (the time-nonlocality kernel). If $t = 0$ is chosen as a “starting point”, then

$$\mathbf{E}(x, y, z, t) = \int_0^t K(t - \tau) F(\mathbf{C}(x, y, z, \tau)) d\tau. \quad (2.3)$$

Space nonlocality means that the effect \mathbf{E} at a point \mathbf{x} at time t depends on causes at all the points \mathbf{x}' at time t

$$\mathbf{E}(x, y, z, t) = \int_V \beta(|\mathbf{x} - \mathbf{x}'|) F(\mathbf{C}(x', y', z', t)) dx' dy' dz'. \quad (2.4)$$

Here $\beta(|\mathbf{x} - \mathbf{x}'|)$ is the corresponding weight function (the space-nonlocality kernel).

When time nonlocality is accompanied by space nonlocality, we have dependence of the effect \mathbf{E} at a point \mathbf{x} at time t on causes at all the points \mathbf{x}' and at all the times prior to and at time t

$$\mathbf{E}(x, y, z, t) = \int_0^t \int_V \gamma(|\mathbf{x} - \mathbf{x}'|, t - \tau) F(\mathbf{C}(x', y', z', \tau)) dx' dy' dz' d\tau, \quad (2.5)$$

where $\gamma(|\mathbf{x} - \mathbf{x}'|, t - \tau)$ is the weight function (the space-time-nonlocality kernel).

Time nonlocality describes memory (history) effects, space nonlocality deals with the long-range interaction, represents attempts to extend the continuum approach to smaller length scales and to link some aspects of lattice mechanics to continuum theories. General properties of nonlocality kernels are discussed, for example, in [13].

2.2 Nonlocal Generalizations of the Fourier Law

The conventional theory of heat conduction is based on the classical (local) Fourier law [14], which relates the heat flux vector \mathbf{q} to the temperature gradient

$$\mathbf{q} = -k \text{grad } T, \quad (2.6)$$

where k is the thermal conductivity of a solid. In combination with a law of conservation of energy

$$C \frac{\partial T}{\partial t} = -\text{div } \mathbf{q} \quad (2.7)$$

with C being the heat capacity, the Fourier law (2.6) leads to the parabolic heat conduction equation

$$\frac{\partial T}{\partial t} = a \Delta T, \quad (2.8)$$

where $a = k/C$ is the thermal diffusivity coefficient.

Nonclassical theories, in which the Fourier law and the standard heat conduction equations are replaced by more general equations, constantly attract the attention of the researchers.

The general time-nonlocal constitutive equations for the heat flux were considered in [9, 19, 33, 35, 38, 40]. In the theory of heat conduction proposed by Gurtin and Pipkin [19] the law of heat conduction is given by the general time-nonlocal dependence

$$\mathbf{q}(t) = -k \int_0^\infty K(\tau) \operatorname{grad} T(t - \tau) d\tau. \quad (2.9)$$

Substitution $\tau = t - u$ leads to the following equation

$$\mathbf{q}(t) = -k \int_{-\infty}^t K(t - \tau) \operatorname{grad} T(\tau) d\tau. \quad (2.10)$$

Choosing 0 instead of $-\infty$ as a “starting point”, we obtain

$$\mathbf{q}(t) = -k \int_0^t K(t - \tau) \operatorname{grad} T(\tau) d\tau \quad (2.11)$$

and the heat conduction equation with memory [35, 36]

$$\frac{\partial T}{\partial t} = a \int_0^t K(t - \tau) \Delta T(\tau) d\tau. \quad (2.12)$$

Now let us consider several particular cases of general time-nonlocal constitutive equation for the heat flux. The classical Fourier law (2.6) and the standard heat conduction equation (2.8) are obtained for “instantaneous memory” with the kernel being Dirac’s delta.

“Full sclerosis” corresponds to the choice of the kernel as the time derivative of Dirac’s delta or

$$\mathbf{q}(t) = -k \frac{\partial}{\partial t} \operatorname{grad} T(t), \quad (2.13)$$

thus leading to the Helmholtz equation for temperature

$$T = a \Delta T. \quad (2.14)$$

“Full memory” [18, 35] means that there is no fading of memory, the weight function is constant and

$$\mathbf{q}(t) = -k \int_0^t \operatorname{grad} T(\tau) d\tau. \quad (2.15)$$

As a results we have the wave equation for temperature

$$\frac{\partial^2 T}{\partial t^2} = a \Delta T. \quad (2.16)$$

The time-nonlocal dependence between the heat flux vector and the temperature gradient with the “long-tail” power kernel [46, 48, 50, 51, 65]

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\tau)^{\alpha-1} \text{grad } T(\tau) d\tau, \quad 0 < \alpha \leq 1, \quad (2.17)$$

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha-1)} \int_0^t (t-\tau)^{\alpha-2} \text{grad } T(\tau) d\tau, \quad 1 < \alpha \leq 2, \quad (2.18)$$

can be interpreted in terms of fractional integrals and derivatives

$$\begin{aligned} \mathbf{q}(t) &= -k D_{RL}^{1-\alpha} \text{grad } T(t), & 0 < \alpha \leq 1, \\ \mathbf{q}(t) &= -k I^{\alpha-1} \text{grad } T(t), & 1 < \alpha \leq 2. \end{aligned} \quad (2.19)$$

The constitutive equations (2.19) yield the time-fractional heat conduction equation with Caputo derivative:

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \Delta T, \quad 0 < \alpha \leq 2. \quad (2.20)$$

The details of obtaining Eq. (2.20) from the constitutive equations (2.19) can be found in [55].

Equation (2.20) describes the whole spectrum from localized heat conduction (the Helmholtz equation (2.14) when $\alpha \rightarrow 0$) through the standard heat conduction (2.8) ($\alpha = 1$) to the ballistic heat conduction (the wave equation (2.16) when $\alpha = 2$).

“Short-tail memory” with the exponential kernel

$$\mathbf{q}(t) = -\frac{k}{\zeta} \int_0^t \exp\left(-\frac{t-\tau}{\zeta}\right) \text{grad } T(\tau) d\tau, \quad (2.21)$$

where ζ is a nonnegative constant, or in the Cattaneo form [4, 5]

$$\mathbf{q} + \zeta \frac{\partial \mathbf{q}}{\partial t} = -k \text{grad } T, \quad (2.22)$$

leads to the telegraph equation for temperature

$$\frac{\partial T}{\partial t} + \zeta \frac{\partial^2 T}{\partial t^2} = a \Delta T. \quad (2.23)$$

Compte and Metzler [8] considered four possible generalizations of the Cattaneo telegraph equation (2.23), three of them supported by a different scheme: continuous time random walks, nonlocal transport theory, and delayed flux-force relation. The fourth equation was discarded from their discussion, as the authors of [8] believed that none of the other approaches leads to this generalization. These four possible generalizations are the following:

$$D_{RL}^\alpha T + \zeta D_{RL}^{2\alpha} T = a \frac{\partial^2 T}{\partial x^2}, \quad \text{GCE I} \quad (2.24)$$

$$D_{RL}^{2-\gamma} T + \zeta \frac{\partial^2 T}{\partial t^2} = a \frac{\partial^2 T}{\partial x^2}, \quad \text{GCE II} \quad (2.25)$$

$$D_{RL}^\alpha T + \zeta D_{RL}^{1+\alpha} T = a \frac{\partial^2 T}{\partial x^2}, \quad \text{GCE III} \quad (2.26)$$

$$\frac{\partial T}{\partial t} + \zeta D_{RL}^{1+\alpha} T = a \frac{\partial^2 T}{\partial x^2}. \quad (2.27)$$

It was shown in [52, 54, 58] that all four fractional generalizations of telegraph equation can be obtained from time-nonlocal generalizations of the Fourier law with different kernels being the functions of Mittag-Leffler type. We consider the Caputo fractional derivative, but if care is taken, the results obtained using the Caputo formulation can be recast to the Riemann-Liouville version.

Consider the following generalization of Eq. (2.22)

$$I^{1-\alpha} \mathbf{q} + \zeta \frac{\partial^{2\alpha-1} \mathbf{q}}{\partial t^{2\alpha-1}} = -k \text{grad } T, \quad 1/2 < \alpha \leq 1. \quad (2.28)$$

In combination with a law of conservation of energy (2.7) this equation leads to the generalized Cattaneo equation I

$$\frac{\partial^\alpha T}{\partial t^\alpha} + \zeta \frac{\partial^{2\alpha} T}{\partial t^{2\alpha}} = a \Delta T, \quad 1/2 < \alpha \leq 1. \quad (2.29)$$

It should be noted that derivation of equations of such a type needs care, as fractional derivatives, in general, do not satisfy neither the semigroup property nor the commutative property.

The Laplace transform of (2.28) with respect to time t (neglecting the initial value $\mathbf{q}(0^+)$) gives

$$\frac{1}{s^{1-\alpha}} \mathbf{q}^* + \zeta s^{2\alpha-1} \mathbf{q}^* = -k \text{grad } T^*(s), \quad 1/2 < \alpha \leq 1, \quad (2.30)$$

or

$$\mathbf{q}^* = -\frac{k}{\zeta} \frac{s^{1-\alpha}}{s^\alpha + 1/\zeta} \text{grad } T^*(s). \quad (2.31)$$

To invert the Laplace transform we use the following formula [15, 30, 42]:

$$\mathcal{L}^{-1} \left\{ \frac{s^{\alpha-\beta}}{s^\alpha + b} \right\} = t^{\beta-1} E_{\alpha,\beta}(-bt^\alpha), \quad (2.32)$$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0, \beta > 0, z \in \mathbb{C}. \quad (2.33)$$

Hence, the convolution law leads to the time-nonlocal dependence

$$\mathbf{q}(t) = -\frac{k}{\zeta} \int_0^t (t-\tau)^{2\alpha-2} E_{\alpha,2\alpha-1} \left[-\frac{(t-\tau)^\alpha}{\zeta} \right] \text{grad } T(\tau) d\tau, \quad 1/2 < \alpha \leq 1. \quad (2.34)$$

Similarly, the constitutive equation

$$\frac{\partial^{1-\gamma} \mathbf{q}}{\partial t^{1-\gamma}} + \zeta \frac{\partial \mathbf{q}}{\partial t} = -k \text{grad } T, \quad 0 \leq \gamma \leq 1, \quad (2.35)$$

results in the generalized Cattaneo equation II

$$\frac{\partial^{2-\gamma} T}{\partial t^{2-\gamma}} + \zeta \frac{\partial^2 T}{\partial t^2} = a \Delta T, \quad 0 \leq \gamma \leq 1. \quad (2.36)$$

The Laplace transform allows us to rewrite (2.35) as a time-nonlocal constitutive equation

$$\mathbf{q}(t) = -\frac{k}{\zeta} \int_0^t E_\gamma \left[-\frac{(t-\tau)^\gamma}{\zeta} \right] \text{grad } T(\tau) d\tau, \quad 0 \leq \gamma \leq 1, \quad (2.37)$$

where $E_\gamma(z) \equiv E_{\gamma,1}$ is the Mittag-Leffler function being the generalization of the exponential function. Equation (2.37) was obtained by Compte and Metzler [8].

Next, we consider the generalization of Eq. (2.22) in the form

$$I^{1-\alpha} \mathbf{q} + \zeta \frac{\partial^\alpha \mathbf{q}}{\partial t^\alpha} = -k \text{grad } T, \quad 0 < \alpha \leq 1, \quad (2.38)$$

giving rise to the generalized Cattaneo equation III

$$\frac{\partial^\alpha T}{\partial t^\alpha} + \zeta \frac{\partial^{1+\alpha} T}{\partial t^{1+\alpha}} = a \Delta T, \quad 0 < \alpha \leq 1. \quad (2.39)$$

In this instance the time-nonlocal constitutive equation reads

$$\mathbf{q}(t) = -\frac{k}{\zeta} \int_0^t (t-\tau)^{\alpha-1} E_{1,\alpha} \left(-\frac{t-\tau}{\zeta} \right) \text{grad } T(\tau) d\tau, \quad 0 < \alpha \leq 1. \quad (2.40)$$

In the similar scheme, from equation

$$\mathbf{q} + \zeta \frac{\partial^\alpha \mathbf{q}}{\partial t^\alpha} = -k \text{grad } T, \quad 0 < \alpha \leq 1, \quad (2.41)$$

we obtain the generalized Cattaneo equation

$$\frac{\partial T}{\partial t} + \zeta \frac{\partial^{1+\alpha} T}{\partial t^{1+\alpha}} = a \Delta T, \quad 0 < \alpha \leq 1, \quad (2.42)$$

whereas the fractional differential equation (2.41) has the solution

$$\mathbf{q}(t) = -\frac{k}{\zeta} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha} \left[-\frac{(t-\tau)^\alpha}{\zeta} \right] \text{grad } T(\tau) d\tau, \quad 0 < \alpha \leq 1. \quad (2.43)$$

Atanacković et al. [2] investigated a generalized telegraph equation with two Caputo fractional derivatives

$$\frac{\partial^\alpha T}{\partial t^\alpha} + \zeta \frac{\partial^\gamma T}{\partial t^\gamma} = a \frac{\partial^2 T}{\partial x^2}, \quad (2.44)$$

where $0 < \alpha < \gamma \leq 2$.

In this case, the fractional generalization of the Cattaneo equation (2.22) has the following form [58]:

$$I^{1-\alpha} \mathbf{q} + \zeta \frac{\partial^{\gamma-1} \mathbf{q}}{\partial t^{\gamma-1}} = -k \text{grad } T, \quad 0 < \alpha \leq 1, \quad 1 < \gamma \leq 2, \quad (2.45)$$

$$\mathbf{q}(t) = -\frac{k}{\zeta} \int_0^t (t-\tau)^{\gamma-2} E_{\gamma-\alpha,\gamma-1} \left[-\frac{(t-\tau)^{\gamma-\alpha}}{\zeta} \right] \text{grad } T(\tau) d\tau \quad (2.46)$$

and

$$\frac{\partial^\alpha T}{\partial t^\alpha} + \zeta \frac{\partial^\gamma T}{\partial t^\gamma} = a \Delta T, \quad 0 < \alpha \leq 1, \quad 1 < \gamma \leq 2. \quad (2.47)$$

Space-nonlocal constitutive equations for the heat flux,

$$\mathbf{q}(\mathbf{x}) = -k \int_V \beta(|\mathbf{x} - \mathbf{x}'|) \text{grad } T(x', y', z') d\mathbf{x}' dy' dz', \quad (2.48)$$

were also discussed. Demiray and Eringen [10] and Eringen [12] considered the exponential kernel $\beta(|\mathbf{x} - \mathbf{x}'|)$:

$$\beta(|\mathbf{x} - \mathbf{x}'|) = \delta(|\mathbf{x} - \mathbf{x}'|) + c_1 \exp(-c_2|\mathbf{x} - \mathbf{x}'|), \quad (2.49)$$

where c_1 and c_2 are nonnegative constants.

Another type of space-nonlocality is based on power kernels resulting in fractional differential operators in space coordinates. For example, the fractional operators described in Sect. 1.3 lead to the space-fractional heat conduction equation [16]:

$$\frac{\partial T}{\partial t} = a \frac{\partial^\beta T}{\partial |x|^\beta}, \quad 0 < \beta \leq 2, \quad (2.50)$$

or in the case of higher dimensions [20]

$$\frac{\partial T}{\partial t} = -a(-\Delta)^{\beta/2} T, \quad 0 < \beta \leq 2. \quad (2.51)$$

The general space-time-fractional heat conduction equation has the form [17, 32, 67]:

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \frac{\partial^\beta T}{\partial |x|^\beta}, \quad 0 < \alpha \leq 2, \quad 0 < \beta \leq 2 \quad (2.52)$$

and [21]

$$\frac{\partial^\alpha T}{\partial t^\alpha} = -a(-\Delta)^{\beta/2} T, \quad 0 < \alpha \leq 2, \quad 0 < \beta \leq 2. \quad (2.53)$$

Sometimes, based on additional physical arguments, the range $1 \leq \beta \leq 2$ is considered.

Similarly, the space-time-fractional telegraph equation in the case of one space variable is written as

$$\frac{\partial^\alpha T}{\partial t^\alpha} + \zeta \frac{\partial^\gamma T}{\partial t^\gamma} = a \frac{\partial^\beta T}{\partial |x|^\beta}, \quad 0 < \alpha \leq 1, \quad 1 < \gamma \leq 2, \quad (2.54)$$

and in the general case as

$$\frac{\partial^\alpha T}{\partial t^\alpha} + \zeta \frac{\partial^\gamma T}{\partial t^\gamma} = -a(-\Delta)^{\beta/2} T, \quad 0 < \alpha \leq 1, \quad 1 < \gamma \leq 2. \quad (2.55)$$

2.3 Theories of Fractional Thermoelasticity

In this section, we formulate equations of fractional thermoelasticity, i.e. thermoelasticity based on the heat conduction described by differential operators of fractional order. To compare the results presented in this book with results obtained in the framework of classical thermoelasticity or in its earlier generalizations, the interested reader is referred to the recent books by Hetnarski and Eslami [23], Ignaczak and Ostoja-Starzewski [26], Noda et al. [37] and the comprehensive Encyclopedia of Thermal Stresses [22], where historical notes and extensive literature on the subject can be found.

The thermoelastic state of a solid is governed by the equation of motion in terms of displacements

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \text{grad div } \mathbf{u} - \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \beta_T K_T \text{grad } T, \quad (2.56)$$

the stress-strain-temperature relation (the Duhamel–Neumann equation; see the pioneering works [11, 34])

$$\boldsymbol{\sigma} = 2\mu \mathbf{e} + (\lambda \text{tr } \mathbf{e} - \beta_T K_T \mathbf{I}), \quad (2.57)$$

the geometrical relations

$$\mathbf{e} = \frac{1}{2} (\nabla \mathbf{u} + \mathbf{u} \nabla), \quad (2.58)$$

where \mathbf{u} is the displacement vector, $\boldsymbol{\sigma}$ the stress tensor, \mathbf{e} the linear strain tensor, T the temperature, ρ the mass density, λ and μ are Lamé constants, $K_T = \lambda + 2\mu/3$, β_T is the thermal coefficient of volumetric expansion, \mathbf{I} denotes the unit tensor; in (2.58) ∇ stands for the gradient operator.

It should be noted that the balance equation (2.7) is obtained from the balance equation for entropy S

$$T_0 \frac{\partial S}{\partial t} = -\text{div } \mathbf{q} \quad (2.59)$$

and the constitutive equation

$$S = \frac{C}{T_0} T, \quad (2.60)$$

where T_0 is the reference temperature. If the effect of deformation on the thermal state of a solid is considered, then the constitutive equation for entropy becomes

$$S = \frac{C}{T_0} T + \beta_T K_T \text{tr } \mathbf{e}. \quad (2.61)$$

Taking into account the heat source Q in the balance equation (2.59) and substituting (2.61) into (2.59), we arrive at

$$C \frac{\partial T}{\partial t} + \beta_T K_T T_0 \frac{\partial \text{tr} \mathbf{e}}{\partial t} = -\text{div} \mathbf{q} + Q. \quad (2.62)$$

Assuming different constitutive equations for the heat flux \mathbf{q} , Eqs. (2.56)–(2.58) are supplemented by one of the following equations:

Classical thermoelasticity

$$\frac{\partial T}{\partial t} + \gamma_T \frac{\partial \text{tr} \mathbf{e}}{\partial t} = a \Delta T + W. \quad (2.63)$$

The coefficient $\gamma_T = \beta_T K_T T_0 / C$ describes the effect of deformation on the thermal state of a solid, $W = Q / C$.

Localized thermoelasticity

$$T + \gamma_T \text{tr} \mathbf{e} = a \Delta T + I^1 W. \quad (2.64)$$

Thermoelasticity without energy dissipation

$$\frac{\partial^2 T}{\partial t^2} + \gamma_T \frac{\partial^2 \text{tr} \mathbf{e}}{\partial t^2} = a \Delta T + \frac{\partial W}{\partial t}. \quad (2.65)$$

Theory of thermal stresses based on the wave equation for temperature was introduced by Green and Naghdi [18].

Thermoelasticity based on time-fractional heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} + \gamma_T \frac{\partial^\alpha \text{tr} \mathbf{e}}{\partial t^\alpha} = a \Delta T + \begin{cases} I^{1-\alpha} W, & 0 < \alpha \leq 1, \\ \frac{\partial^{\alpha-1} W}{\partial t^{\alpha-1}}, & 1 < \alpha \leq 2. \end{cases} \quad (2.66)$$

This theory (in the case $\gamma_T = 0$) was proposed by the author in [46].

Generalized thermoelasticity of Lord and Shulman

$$\frac{\partial T}{\partial t} + \gamma_T \frac{\partial \text{tr} \mathbf{e}}{\partial t} + \zeta \frac{\partial^2 T}{\partial t^2} + \zeta \gamma_T \frac{\partial^2 \text{tr} \mathbf{e}}{\partial t^2} = a \Delta T + W + \zeta \frac{\partial W}{\partial t}. \quad (2.67)$$

The first generalized theory of thermal stresses was put forward by Lord and Shulman [31] in 1967. The interested reader is also referred to reviews [6, 7, 24, 25, 29, 70] and books [26, 43].

Thermoelasticity based on time-fractional telegraph equation

$$\begin{aligned} \frac{\partial^\alpha T}{\partial t^\alpha} + \gamma_T \frac{\partial^\alpha \text{tr} \mathbf{e}}{\partial t^\alpha} + \zeta \frac{\partial^\gamma T}{\partial t^\gamma} + \zeta \gamma_T \frac{\partial^\gamma \text{tr} \mathbf{e}}{\partial t^\gamma} = a \Delta T + I^{1-\alpha} W \\ + \zeta \frac{\partial^{\gamma-1} W}{\partial t^{\gamma-1}}, \quad 0 < \alpha \leq 1, \quad 1 < \gamma \leq 2. \end{aligned} \quad (2.68)$$

Several particular cases of thermoelasticity based on the time-fractional telegraph equation for different values of α and γ were considered in [52, 54, 58, 68, 72].

Thermoelasticity based on space-time-fractional heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} + \gamma_T \frac{\partial^\alpha \text{tr} \mathbf{e}}{\partial t^\alpha} = -a (-\Delta T)^{\beta/2} + \begin{cases} I^{1-\alpha} W, & 0 < \alpha \leq 1, \\ \frac{\partial^{\alpha-1} W}{\partial t^{\alpha-1}}, & 1 < \alpha \leq 2. \end{cases} \quad (2.69)$$

This theory of thermal stresses was studied in [47–51].

Thermoelasticity based on space-time-fractional telegraph equation

$$\begin{aligned} \frac{\partial^\alpha T}{\partial t^\alpha} + \gamma_T \frac{\partial^\alpha \text{tr} \mathbf{e}}{\partial t^\alpha} + \zeta \frac{\partial^\gamma T}{\partial t^\gamma} + \zeta \gamma_T \frac{\partial^\gamma \text{tr} \mathbf{e}}{\partial t^\gamma} = -a (-\Delta T)^{\beta/2} + I^{1-\alpha} W \\ + \zeta \frac{\partial^{\gamma-1} W}{\partial t^{\gamma-1}}, \quad 0 < \alpha \leq 1, \quad 1 < \gamma \leq 2. \end{aligned} \quad (2.70)$$

This type of fractional thermoelasticity was investigated in [52, 58].

In the present book we will consider various theories of fractional thermoelasticity with the emphasis on thermoelasticity based on the time-fractional heat conduction equation (2.66). In what follows we will restrict ourselves to the quasi-static uncoupled theory neglecting the inertia term in Eq. (2.56) and the coupling term in Eq. (2.66). From physical point of view neglecting the inertia term in Eq. (2.56) means that no account has been taken of mechanical oscillations. The quasi-static statement of the thermoelasticity problem is possible if the relaxation time of mechanical oscillations is considerable less than relaxation time of heat conduction process. Neglecting in the heat conduction equations the term which describes coupling of thermal and mechanical effects ($\gamma_T = 0$) is likely for all problems except when thermoelastic energy dissipation is of special interest.

In the quasi-static case, applying the operator div to the both sides of Eq. (2.56), we get

$$\Delta \text{div } \mathbf{u} = \Delta \text{tr } \mathbf{e} = \frac{\beta_T K_T}{\lambda + 2\mu} \Delta T. \quad (2.71)$$

Integrating (2.71) and setting the integration constants equal to zero (which is easy to do in the case of unbounded body), we obtain the dependence

$$\text{tr } \mathbf{e} = m T, \quad (2.72)$$

where

$$m = \frac{1 + \nu}{1 - \nu} \frac{\beta_T}{3}, \quad (2.73)$$

and ν is the Poisson ratio.

Hence, for infinite medium taking into account coupling of thermal and mechanical effects only changes the thermal diffusivity coefficient a in the heat conduction equation:

$$a' = \frac{a}{1 + m\gamma_T}. \quad (2.74)$$

Additional discussion concerning the validity of quasi-static uncoupled statement of thermoelasticity problems can be found in [3, 28, 45], among others.

2.4 Initial and Boundary Conditions

For dynamic problems with taking into account the inertia term in (2.56), initial values of displacement and velocity are imposed:

$$t = 0 : \mathbf{u} = \mathbf{u}_i(\mathbf{x}), \quad (2.75)$$

$$t = 0 : \frac{\partial \mathbf{u}}{\partial t} = \mathbf{v}_i(\mathbf{x}). \quad (2.76)$$

In a quasi-static statement of the thermoelasticity problem, neglecting the inertia term in (2.56), initial values of mechanical quantities are not considered.

For bounded domains, the boundary conditions should be imposed. The first fundamental boundary problem in terms of stresses is characterized by the conditions of traction at the boundary Σ

$$\mathbf{n} \cdot \boldsymbol{\sigma} \big|_{\Sigma} = \mathbf{S}_0(\mathbf{x}_{\Sigma}, t), \quad (2.77)$$

where \mathbf{n} is the outer unit normal to the boundary and \mathbf{x}_{Σ} is a point at the surface Σ .

The second fundamental boundary problem incorporates the condition for displacements

$$\mathbf{u}|_{\Sigma} = \mathbf{u}_0(\mathbf{x}_{\Sigma}, t). \quad (2.78)$$

In the case of classical thermoelasticity, the Cauchy problem for the heat conduction equation implies that the initial value of the temperature is given:

$$t = 0 : \quad T = f(\mathbf{x}). \quad (2.79)$$

For more general equations (2.66) and (2.69) for $1 < \alpha \leq 2$, as well as for (2.65), (2.67), (2.68), and (2.70), the initial value of the first time derivative of temperature should also be imposed

$$t = 0 : \quad \frac{\partial T}{\partial t} = F(\mathbf{x}). \quad (2.80)$$

If Eq. (2.66) is considered in a bounded domain, the corresponding boundary conditions should be imposed (see [53, 56, 57, 59–65]).

The Dirichlet boundary condition (the boundary condition of the first kind) specifies the temperature over the surface of the body

$$T|_{\Sigma} = g(\mathbf{x}_{\Sigma}, t). \quad (2.81)$$

For fractional heat conduction equation, two types of Neumann boundary condition (the boundary condition of the second kind) can be considered: the mathematical condition with the prescribed boundary value of the normal derivative

$$\left. \frac{\partial T}{\partial n} \right|_{\Sigma} = g(\mathbf{x}_{\Sigma}, t) \quad (2.82)$$

and the physical condition with the prescribed boundary value of the heat flux

$$D_{RL}^{1-\alpha} \left. \frac{\partial T}{\partial n} \right|_{\Sigma} = g(\mathbf{x}_{\Sigma}, t), \quad 0 < \alpha \leq 2, \quad (2.83)$$

where $\partial/\partial n$ denotes differentiation along the outward-drawn normal at the boundary surface Σ .

Similarly, the mathematical Robin boundary condition (the boundary condition of the third kind) is a specification of a linear combination of the values of temperature and the values of its normal derivative at the boundary of the domain

$$\left(c_1 T + c_2 \frac{\partial T}{\partial n} \right) \Big|_{\Sigma} = g(\mathbf{x}_{\Sigma}, t) \quad (2.84)$$

with some nonzero constants c_1 and c_2 , while the physical Robin boundary condition specifies a linear combination of the values of temperature and the values of the heat flux at the boundary. For example, the Newton condition of convective heat exchange between a body and the environment with the temperature T_e

$$\mathbf{q} \cdot \mathbf{n} \Big|_{\Sigma} = H \left(T \Big|_{\Sigma} - T_e \right), \quad (2.85)$$

where H is the convective heat transfer coefficient, leads to

$$\left(HT + k D_{RL}^{1-\alpha} \frac{\partial T}{\partial n} \right) \Big|_{\Sigma} = g(\mathbf{x}_{\Sigma}, t), \quad 0 < \alpha \leq 2, \quad (2.86)$$

where $g(\mathbf{x}_{\Sigma}, t) = HT_e(\mathbf{x}_{\Sigma}, t)$ and $D_{RL}^{-\alpha} f(t)$ for $\alpha > 0$ is understood as the Riemann-Liouville fractional integral $I^{\alpha} f(t)$ (see (1.22)).

In the case of the classical heat conduction equation ($\alpha = 1$) the mathematical and physical Neumann boundary conditions coincide as well as the mathematical and physical Robin boundary conditions, but for the fractional heat conduction equation ($\alpha \neq 1$) they are essentially different.

If the surfaces of two solids are in perfect thermal contact, the temperatures on the contact surface and the heat fluxes through the contact surface are the same for both solids, and we obtain the boundary conditions of the fourth kind [61, 62]:

$$T_1 \Big|_{\Sigma} = T_2 \Big|_{\Sigma}, \quad (2.87)$$

$$k_1 D_{RL}^{1-\alpha} \frac{\partial T_1}{\partial n} \Big|_{\Sigma} = k_2 D_{RL}^{1-\beta} \frac{\partial T_2}{\partial n} \Big|_{\Sigma}, \quad 0 < \alpha \leq 2, \quad 0 < \beta \leq 2, \quad (2.88)$$

where subscripts 1 and 2 refer to solids 1 and 2, respectively, and n is the common normal at the contact surface.

It should be emphasized that for other generalized heat conduction equations presented above the proper boundary conditions should be formulated in terms of the corresponding heat flux (not in terms of the normal derivative of temperature alone).

Next, we investigate moving interface boundary conditions (in the case of classical heat conduction such conditions were considered, for example, in the books by Arpaci [1] and Jiji [27]). Consider a two-dimensional surface Σ moving in a three-dimensional space. The position vector \mathbf{r} of such a surface can be written as

$$\mathbf{r} = \mathbf{r}(\xi, \eta, t), \quad (2.89)$$

where ξ and η are the curvilinear coordinates on a surface.

For a geometrical point “fixed” at a surface, the normal component of the velocity vector is calculated as [44]

$$v_{\Sigma}^{(n)} = \frac{\partial \mathbf{r}(\xi, \eta, t)}{\partial t} \cdot \mathbf{n}. \quad (2.90)$$

In the case of change of phase (for example, the solidification of a liquid), two boundary conditions are prescribed at the solid-liquid interface. Continuity of temperature at the interface requires that

$$T_1 \Big|_{\Sigma} = T_2 \Big|_{\Sigma}. \quad (2.91)$$

In what follows the thermal properties of the liquid and solid are marked by the subscripts 1 and 2, respectively.

The energy balance at the moving interface allows us to calculate the velocity $v_{\Sigma}^{(n)}$. The general expression of energy balance is written in terms of the heat fluxes and does not depend on the considered constitutive equation for the heat flux. For classical heat conduction, the energy balance equation can be found, for example, in [1, 27] and reads

$$-\rho_2 h_{12} v_{\Sigma}^{(n)} = \mathbf{q}_2 \cdot \mathbf{n} - \mathbf{q}_1 \cdot \mathbf{n}, \quad (2.92)$$

where ρ_2 is the density of a solid, h_{12} is the latent heat of fusion. Taking into account the constitutive equations for the heat fluxes, we get

$$-\rho_2 h_{12} v_{\Sigma}^{(n)} = -k_2 D_{RL}^{1-\alpha} \left(\frac{\partial T_2}{\partial n} \right)_{\Sigma} + k_1 D_{RL}^{1-\beta} \left(\frac{\partial T_1}{\partial n} \right)_{\Sigma}, \quad (2.93)$$

assuming that in the general case heat conduction in the solid and liquid is governed by the time-fractional heat conduction equations with the Caputo derivatives of order α and β , respectively.

Another type of boundary condition is obtained when the heat flux \mathbf{q}_1 is described by the Newton condition of convective heat exchange (2.85):

$$-\rho_2 h_{12} v_{\Sigma}^{(n)} = -k_2 D_{RL}^{1-\alpha} \left(\frac{\partial T_2}{\partial n} \right)_{\Sigma} \pm H (T_{\Sigma} - T_{\infty}), \quad (2.94)$$

where T_{Σ} is the temperature of solidification, T_{∞} is the temperature of the liquid far from the moving boundary. The plus sign in (2.94) corresponds to melting, whereas the minus sign is appropriate to solidification.

The Stefan boundary condition is obtained when the liquid is at the solidification temperature:

$$\rho_2 h_{12} v_{\Sigma}^{(n)} = k_2 D_{RL}^{1-\alpha} \left(\frac{\partial T_2}{\partial n} \right)_{\Sigma}. \quad (2.95)$$

The foregoing boundary conditions simplify in the case of one space variable:

$$\rho_2 h_{12} \frac{ds}{dt} = k_2 D_{RL}^{1-\alpha} \left(\frac{\partial T_2}{\partial x} \right)_{x=s(t)} - k_1 D_{RL}^{1-\beta} \left(\frac{\partial T_1}{\partial x} \right)_{x=s(t)}, \quad (2.96)$$

$$\rho_2 h_{12} \frac{ds}{dt} = k_2 D_{RL}^{1-\alpha} \left(\frac{\partial T_2}{\partial x} \right)_{x=s(t)} \mp H(T_\Sigma - T_\infty), \quad (2.97)$$

$$\rho_2 h_{12} \frac{ds}{dt} = k_2 D_{RL}^{1-\alpha} \left(\frac{\partial T_2}{\partial x} \right)_{x=s(t)}, \quad (2.98)$$

where $x = s(t)$ is the solidification front.

Usually, zero initial conditions for $s(t)$ and the heat flux are imposed. In this case the difference between the Riemann-Liouville and Caputo derivatives disappears and (2.98) can be written as [66, 71]

$$\rho_2 h_{12} \frac{d^\alpha s}{dt^\alpha} = k_2 \left(\frac{\partial T_2}{\partial x} \right)_{x=s(t)}. \quad (2.99)$$

The sign in the right-hand side of (2.99) depends on whether solidification or melting is considered.

2.5 Representation of Thermal Stresses

Just as in the classical theory of thermal stresses [39, 41], we can introduce the displacement potential Φ

$$\mathbf{u} = \text{grad } \Phi. \quad (2.100)$$

In the quasi-static case, from the equilibrium equation (2.56) with $\rho = 0$ we get

$$\Delta \Phi = mT, \quad (2.101)$$

where m is defined by (2.73).

The part of stresses due to the displacement potential Φ describes the influence of the temperature field and is given as

$$\sigma^{(1)} = 2\mu (\nabla \nabla \Phi - \mathbf{I} \Delta \Phi). \quad (2.102)$$

The part of the stress field expressed in terms of the biharmonic Galerkin vector \mathbf{G} [39]

$$\sigma^{(2)} = 2\mu \left[(\nu \mathbf{I} \Delta - \nabla \nabla) \nabla \cdot \mathbf{G} + (1 + \nu) \Delta (\nabla \mathbf{G} + \mathbf{G} \nabla) \right], \quad (2.103)$$

where

$$\Delta \Delta \mathbf{G} = 0, \quad (2.104)$$

allows us to satisfy the prescribed boundary conditions for the components of the total stress tensor $\sigma = \sigma^{(1)} + \sigma^{(2)}$.

In cylindrical coordinates in the case of axial symmetry

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z^2} = mT \quad (2.105)$$

and

$$\sigma_{rr}^{(1)} = 2\mu \left[\frac{\partial^2 \Phi}{\partial r^2} - \Delta \Phi \right], \quad (2.106)$$

$$\sigma_{\theta\theta}^{(1)} = 2\mu \left[\frac{1}{r} \frac{\partial \Phi}{\partial r} - \Delta \Phi \right], \quad (2.107)$$

$$\sigma_{zz}^{(1)} = 2\mu \left[\frac{\partial^2 \Phi}{\partial z^2} - \Delta \Phi \right], \quad (2.108)$$

$$\sigma_{rz}^{(1)} = 2\mu \frac{\partial^2 \Phi}{\partial r \partial z}. \quad (2.109)$$

If the displacement potential $\Phi(r)$ depends only on the radial coordinate r , then

$$\sigma_{zz}^{(1)} = \sigma_{rr}^{(1)} + \sigma_{\theta\theta}^{(1)} = -2\mu \Delta \Phi = -2\mu mT, \quad (2.110)$$

$$\sigma_{rr}^{(1)} - \sigma_{\theta\theta}^{(1)} = 2\mu \left(\frac{\partial^2 \Phi}{\partial r^2} - \frac{1}{r} \frac{\partial \Phi}{\partial r} \right). \quad (2.111)$$

The Hankel transform of order n with respect to the radial coordinate

$$\mathcal{H}_{(n)} \{f(r)\} = \int_0^\infty f(r) J_n(r\xi) r \, dr \quad (2.112)$$

with the inverse

$$f(r) = \int_0^\infty \mathcal{H}_{(n)} \{f(r)\} J_n(r\xi) \xi \, d\xi, \quad (2.113)$$

where ξ is the transform variable, is often used for solving problems in cylindrical coordinates.

The following formulae [69] are helpful in applications

$$\mathcal{H}_{(n)} \left\{ \frac{d^2 f(r)}{dr^2} + \frac{1}{r} \frac{df(r)}{dr} - \frac{n^2}{r^2} f(r) \right\} = -\xi^2 \mathcal{H}_{(n)} \{f(r)\}, \quad (2.114)$$

$$\mathcal{H}_{(1)} \left\{ \frac{df(r)}{dr} \right\} = -\xi \mathcal{H}_{(0)} \{f(r)\}, \quad (2.115)$$

$$\mathcal{H}_{(2)} \left\{ \frac{d^2 f(r)}{dr^2} - \frac{1}{r} \frac{df(r)}{dr} \right\} = \xi^2 \mathcal{H}_{(0)} \{f(r)\}. \quad (2.116)$$

In the case $n = 0$, simultaneously with the notation $\mathcal{H}_{(0)} \{f(r)\}$, we will use the notation (see (1.44) and (1.45))

$$\mathcal{H}_{(0)} \{f(r)\} = \widehat{f}(\xi) = \int_0^\infty f(r) J_0(r\xi) r dr, \quad (2.117)$$

$$\mathcal{H}_{(0)}^{(-1)} \{\widehat{f}(\xi)\} = f(r) = \int_0^\infty \widehat{f}(\xi) J_0(r\xi) \xi d\xi. \quad (2.118)$$

It follows from (2.110) and (2.111) that

$$\mathcal{H}_{(0)} \left\{ \sigma_{rr}^{(1)} + \sigma_{\theta\theta}^{(1)} \right\} = 2\mu \xi^2 \widehat{\Phi}, \quad (2.119)$$

$$\mathcal{H}_{(2)} \left\{ \sigma_{rr}^{(1)} - \sigma_{\theta\theta}^{(1)} \right\} = 2\mu \xi^2 \widehat{\Phi}. \quad (2.120)$$

The biharmonic Love function

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right)^2 L = 0 \quad (2.121)$$

can be obtained as a particular case of the Galerkin vector having only the z -component $\mathbf{G} = (0, 0, L)$ and results in the following representation [39]:

$$\sigma_{rr}^{(2)} = 2\mu \frac{\partial}{\partial z} \left[\nu \Delta L - \frac{\partial^2 L}{\partial r^2} \right], \quad (2.122)$$

$$\sigma_{\theta\theta}^{(2)} = 2\mu \frac{\partial}{\partial z} \left[\nu \Delta L - \frac{1}{r} \frac{\partial L}{\partial r} \right], \quad (2.123)$$

$$\sigma_{zz}^{(2)} = 2\mu \frac{\partial}{\partial z} \left[(2 - \nu) \Delta L - \frac{\partial^2 L}{\partial z^2} \right], \quad (2.124)$$

$$\sigma_{rz}^{(2)} = 2\mu \frac{\partial}{\partial r} \left[(1 - \nu) \Delta L - \frac{\partial^2 L}{\partial z^2} \right]. \quad (2.125)$$

In spherical coordinates in the case of central symmetry we have

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi}{\partial r} = mT \quad (2.126)$$

and

$$\sigma_{rr}^{(1)} = -\frac{4\mu}{r} \frac{\partial \Phi}{\partial r}, \quad (2.127)$$

$$\sigma_{\theta\theta}^{(1)} = \sigma_{\phi\phi}^{(1)} = 2\mu \left(\frac{1}{r} \frac{\partial \Phi}{\partial r} - \Delta \Phi \right). \quad (2.128)$$

References

1. Arpaci, V.S.: Conduction Heat Transfer. Addison-Wesley, Reading (1966)
2. Atanacković, T.M., Pilipović, S., Zorica, D.: Diffusion wave equation with two fractional derivatives of different order. *J. Phys. A: Math. Theor.* **40**, 5319–5333 (2007)
3. Boley, B.A., Weiner, J.H.: Theory of Thermal Stresses. Wiley, New York (1960)
4. Cattaneo, C.: Sulla conduzione del calore. *Atti Sem. Mat. Fis. Univ. Modena* **3**, 83–101 (1948)
5. Cattaneo, C.: Sur une forme de l'équation de la chaleur éliminant le paradoxe d'une propagation instantanée. *C. R. Acad. Sci.* **247**, 431–433 (1958)
6. Chandrasekharaiah, D.S.: Thermoelasticity with second sound: a review. *Appl. Mech. Rev.* **39**, 355–376 (1986)
7. Chandrasekharaiah, D.S.: Hyperbolic thermoelasticity: a review of recent literature. *Appl. Mech. Rev.* **51**, 705–729 (1998)
8. Comte, A., Metzler, R.: The generalized Cattaneo equation for the description of anomalous transport processes. *J. Phys. A: Math. Gen.* **30**, 7277–7289 (1997)
9. Day, W.: The Thermodynamics of Simple Materials with Fading Memory. Springer, Berlin (1972)
10. Demiray, H., Eringen, A.C.: On nonlocal diffusion of gases. *Arch. Mech.* **30**, 65–77 (1978)
11. Duhamel, J.-M.-C.: Second mémoire sur les phénomènes thermo-mécanique. *J. Ecole Polytech.* **15**, 1–57 (1837)
12. Eringen, A.C.: Theory of nonlocal thermoelasticity. *Int. J. Eng. Sci.* **12**, 1063–1077 (1974)
13. Eringen, A.C.: Vistas of nonlocal continuum physics. *Int. J. Eng. Sci.* **30**, 1551–1565 (1992)
14. Fourier, J.B.J.: Théorie analytique de la chaleur. Firmin Didot, Paris (1822)
15. Gorenflo, R., Mainardi, F.: Fractional calculus: integral and differential equations of fractional order. In: Carpinetti, A., Mainardi, F. (eds.) *Fractals and Fractional Calculus in Continuum Mechanics*, pp. 223–276. Springer, Wien (1997)
16. Gorenflo, R., Mainardi, F.: Random walk models for space-fractional diffusion processes. *Fract. Calc. Appl. Anal.* **1**, 167–191 (1998)
17. Gorenflo, R., Iskenderov, A., Luchko, Yu.: Mapping between solutions of fractional diffusion-wave equations. *Fract. Calc. Appl. Anal.* **3**, 75–86 (2000)
18. Green, A.E., Naghdi, P.M.: Thermoelasticity without energy dissipation. *J. Elast.* **31**, 189–208 (1993)
19. Gurtin, M.E., Pipkin, A.C.: A general theory of heat conduction with finite wave speeds. *Arch. Ration. Mech. Anal.* **31**, 113–126 (1968)
20. Hanyga, A.: Multidimensional solutions of space-fractional diffusion equations. *Proc. R. Soc. Lond. A* **457**, 2993–3005 (2001)
21. Hanyga, A.: Multidimensional solutions of space-time-fractional diffusion equations. *Proc. R. Soc. Lond. A* **458**, 429–450 (2002)
22. Hetnarski, R.B. (ed.): *Encyclopedia of Thermal Stresses*, in 11 vols. Springer, New York (2014)
23. Hetnarski, R.B., Eslami, M.R.: *Thermal Stresses—Advanced Theory and Applications*. Springer, New York (2009)
24. Hetnarski, R.B., Ignaczak, J.: Generalized thermoelasticity. *J. Therm. Stress.* **22**, 451–476 (1999)

25. Hetnarski, R.B., Ignaczak, J.: Nonclassical dynamical thermoelasticity. *Int. J. Solids Struct.* **37**, 215–224 (2000)
26. Ignaczak, J., Ostoja-Starzewski, M.: *Thermoelasticity with Finite Wave Speeds*. Oxford University Press, Oxford (2009)
27. Jiji, L.M.: *Heat Conduction*, 3rd edn. Springer, Berlin (2009)
28. Jordan, P.M., Puri, P.: Thermal stresses in a spherical shell under three thermoelastic models. *J. Therm. Stress.* **24**, 47–70 (2001)
29. Joseph, D.D., Preziosi, L.: Heat waves. *Rev. Mod. Phys.* **61**, 41–73 (1989)
30. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam (2006)
31. Lord, H.W., Shulman, Y.: A generalized dynamical theory of thermoelasticity. *J. Mech. Phys. Solids* **15**, 299–309 (1967)
32. Mainardi, F., Luchko, Yu., Pagnini, G.: The fundamental solution of the space-time fractional diffusion equation. *Fract. Calc. Appl. Anal.* **4**, 153–192 (2001)
33. Moodi, T.B., Tait, R.J.: On thermal transients with finite wave speeds. *Acta Mech.* **50**, 97–104 (1983)
34. Neumann, F.: *Vorlesung über die Theorie des Elasticität des festen Körper und des Lichtäthers*. Teubner, Leipzig (1885)
35. Nigmatullin, R.R.: To the theoretical explanation of the “universal response”. *Phys. Status Solidi (B)* **123**, 739–745 (1984)
36. Nigmatullin, R.R.: On the theory of relaxation for systems with “remnant” memory. *Phys. Status Solidi (B)* **124**, 389–393 (1984)
37. Noda, N., Hetnarski, R.B., Tanigawa, Y.: *Thermal Stresses*, 2nd edn. Taylor and Francis, New York (2003)
38. Norwood, F.R.: Transient thermal waves in the general theory of heat conduction with finite wave speeds. *J. Appl. Mech.* **39**, 673–676 (1972)
39. Nowacki, W.: *Thermoelasticity*, 2nd edn. PWN-Polish Scientific Publishers, Warsaw and Pergamon Press, Oxford (1986)
40. Nunziato, J.W.: On heat conduction in materials with memory. *Q. Appl. Math.* **29**, 187–204 (1971)
41. Parkus, H.: *Instationäre Wärmespannungen*. Springer, Wien (1959)
42. Podlubny, I.: *Fractional Differential Equations*. Academic Press, San Diego (1999)
43. Podstrigach, Ya.S., Kolyano, Yu.M.: *Generalized Thermomechanics*. Naukova Dumka, Kiev (1976) (in Russian)
44. Podstrigach, Ya.S., Povstenko, Y.Z.: *Introduction to Mechanics of Surface Phenomena in Deformable Solids*. Naukova Dumka, Kiev (1985) (in Russian)
45. Podstrigach, Ya.S., Shvets, R.N.: The quasi-static problem in coupled thermoelasticity. *Int. Appl. Mech.* **5**, 33–39 (1969)
46. Povstenko, Y.: Fractional heat conduction equation and associated thermal stresses. *J. Therm. Stress.* **28**, 83–102 (2005)
47. Povstenko, Y.: Thermoelasticity based on fractional heat conduction equation. In: Ziegler, F., Heuer, R., Adam, C. (eds.) *Proceedings of the 6th International Congress on Thermal Stresses*, Vienna, Austria, 26–29 May 2005, vol. 2, pp. 501–504. Vienna University of Technology, Vienna (2005)
48. Povstenko, Y.: Thermoelasticity which uses fractional heat conduction equation. *Math. Methods Phys.-Mech. Fields* **51**(2), 239–246 (2008)
49. Povstenko, Y.: Space-time-fractional heat conduction equation and the theory of thermoelasticity. In: *3rd IFAC Workshop on Fractional Differentiation and its Applications*, Ankara, Turkey, 5–7 Nov 2008
50. Povstenko, Y.: Thermoelasticity which uses fractional heat conduction equation. *J. Math. Sci.* **162**, 296–305 (2009)
51. Povstenko, Y.: Theory of thermoelasticity based on the space-time-fractional heat conduction equation. *Phys. Scr. T* **136**, 014017-1-6 (2009)

52. Povstenko, Y.: Theories of thermoelasticity based on space-time-fractional Cattaneo-type equation. In: Podlubny, I., Vinagre Jara, M.B., Chen, Y.Q., Felin Batlle, V., Tejado Balsera, I. (eds.) *Proceedings of the 4th IFAC Workshop on Fractional Differentiation and Its Applications*, Badajoz, Spain, 18–20 Oct 2010, Article No. FDA10-014
53. Povstenko, Y.: Different formulations of Neumann boundary-value problems for time-fractional diffusion-wave equation in a half-plane. In: Podlubny, I., Vinagre Jara, M.B., Chen, Y.Q., Felin Batlle, V., Tejado Balsera, I. (eds.) *Proceedings of the 4th IFAC Workshop on Fractional Differentiation and Its Applications*, Badajoz, Spain, 18–20 Oct 2010, Article No. FDA10-015
54. Povstenko, Y.: Fractional Cattaneo-type equations and generalized thermoelasticity. *J. Therm. Stress.* **34**, 97–114 (2011)
55. Povstenko, Y.: Non-axisymmetric solutions to time-fractional diffusion-wave equation in an infinite cylinder. *Fract. Calc. Appl. Anal.* **14**, 418–435 (2011)
56. Povstenko, Y.: Different formulations of Neumann boundary-value problem for time-fractional heat conduction equation in a half-space. In: *Proceedings of the 9th International Congress on Thermal Stresses*, Budapest, Hungary, 5–9 June 2011
57. Povstenko, Y.: Neumann boundary-value problems for a time-fractional diffusion-wave equation in a half-plane. *Comput. Math. Appl.* **64**, 3183–3192 (2012)
58. Povstenko, Y.: Theories of thermal stresses based on space-time-fractional telegraph equations. *Comput. Math. Appl.* **64**, 3321–3328 (2012)
59. Povstenko, Y.: Axisymmetric solutions to time fractional heat conduction equation in a half-space under Robin boundary conditions. *Int. J. Differ. Equ.* **2012**, 154085-1-13 (2012)
60. Povstenko, Y.: Different kinds of boundary problems for fractional heat conduction equation. In: Petráš, I., Podlubny, I., Kostúr, K., Kačur, J., Mojžišová, A. (eds.) *Proceedings of the 13th International Carpathian Control Conference*, Podbanské, Hight Tatras, Slovak Republic, 28–31 May 2012, pp. 588–591. Institute of Electrical and Electronics Engineers, Košice (2012)
61. Povstenko, Y.: Fractional heat conduction in infinite one-dimensional composite medium. *J. Therm. Stress.* **36**, 351–363 (2013)
62. Povstenko, Y.: Fractional heat conduction in an infinite medium with a spherical inclusion. *Entropy* **15**, 4122–4133 (2013)
63. Povstenko, Y.: Axisymmetric solutions to fractional diffusion-wave equation in a cylinder under Robin boundary condition. *Eur. Phys. J. Spec. Top.* **222**, 1767–1777 (2013)
64. Povstenko, Y.: Fundamental solutions to time-fractional heat conduction equations in two joint half-lines. *Cent. Eur. J. Phys.* **11**, 1284–1294 (2013)
65. Povstenko, Y.: Fractional thermoelasticity. In: Hetnarski, R.B. (ed.) *Encyclopedia of Thermal Stresses*, vol. 4, pp. 1778–1787. Springer, New York (2014)
66. Roscani, S., Marcus, E.S.: Two equivalent Stefan's problems for the time-fractional diffusion equation. *Fract. Calc. Appl. Anal.* **16**, 802–815 (2013)
67. Saichev, A.I., Zaslavsky, G.M.: Fractional kinetic equations: solutions and applications. *Chaos* **7**, 753–764 (1997)
68. Sherief, H.H., El-Sayed, A.M.A., Abd El-Latief, A.M.: Fractional order theory of thermoelasticity. *Int. J. Solids Struct.* **47**, 269–275 (2010)
69. Sneddon, I.N.: *The Use of Integral Transforms*. McGraw-Hill, New York (1972)
70. Tamma, K.K., Zhou, X.: Macroscale and microscale thermal transport and thermo-mechanical interactions: some noteworthy perspectives. *J. Therm. Stress.* **21**, 405–449 (1998)
71. Voller, V.R., Falcini, F., Garra, R.: Fractional Stefan problem exhibiting lumped and distributed latent-heat memory effects. *Phys. Rev. E* **87**, 042401-1-6 (2013)
72. Youssef, H.M.: Theory of fractional order generalized thermoelasticity. *J. Heat Transf.* **132**, 061301-1-7 (2010)

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