1 Introduction

To determine specific contents of area and volume as well as integrals is a very old theme in mathematics. Unsurpassed are the achievements of Archimedes, in particular his computation of the volume of the unit ball as $4\pi/3$ and of the area of the unit sphere as 4π. Starting from Euler, problems like determining the value of $\int_0^\infty \frac{\sin x}{x}\,dx$ (which is $\pi/2$) have kept the analysts busy.

To the end of the nineteenth century, this subject became less and less important, as there was not much left to be discovered. At that moment, measure and integration theory entered the stage. It, too, deals with contents or (as we will call it in the following) *measures* of sets, as well as with integrals of functions, but the question has changed. It no longer reads "what is the measure of this or that set?" but rather "which sets can be measured, which functions can be integrated?". To which sets one thus can assign a measure, to which functions an integral? Their specific value becomes secondary, general rules of integration come to the fore. The relation to differential calculus, which for a long period since Newton and Leibniz was in the foreground, loses its dominant role.

Such a change of perspective is not uncommon in mathematics. In our case, it arose in the context that one no longer considered integrals on their own, but rather needed them as tools in other mathematical investigations. Historically one should mention in particular the Fourier analysis of functions, the decomposition of real-valued functions into sinusoidal oscillations. Their coefficients (amplitudes) can be expressed by certain integrals—soon, one realized that for this purpose one needed properties of integration which could not be provided by the notions of integrals being available at that time.

Measure and integration theory according to Lebesgue arose by and large between the years 1900 and 1915, based on essential preliminary work of Borel[1]

[1]Émile Borel, 1871–1956, born in Saint-Affrique, active in Paris at the École Normale Supérieure and the Sorbonne. He significantly contributed not only to the foundations of measure theory, but also to complex analysis, set theory, probability theory, and to applications of

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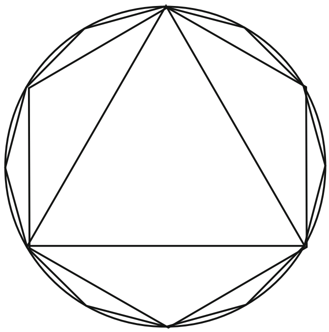
from 1894. Right from the start, the pioneers during that time directed their attention towards the fundamental properties of measure and integral. Borel was the first to demand that measures should be not only additive, but also σ-additive. This means that not only for *finitely many* disjoint measurable sets $B_1, B_2, \ldots \subset \mathbb{R}^d$ with measures $\lambda(B_1), \lambda(B_2), \ldots$ the union $B = B_1 \cup B_2 \cup \cdots$ is measurable and has measure

$$\lambda(B) = \lambda(B_1) + \lambda(B_2) + \cdots ,$$

but that moreover this property holds for every *infinite* sequence $B_1, B_2, \ldots$ of disjoint measurable sets. Borel realized that only under this assumption a fertile mathematical theory arises. Particular cases like the circle in the figure



of course do not yield anything new. Lebesgue,[2] the founder of modern integration theory, in his fundamental treatise on integration from the year 1901 started from six properties that integrals should reasonably satisfy.

Measure and integration theory is based on set theory and cannot dispense with its ways of reasoning. Only with the aid of set theory a path was found leading to the full system of measurable subsets of $\mathbb{R}^d$ and of other spaces. Yet this approach is comparatively abstract and indirect. To realize that it is justified, for a start it is perhaps appropriate to take a look at other more descriptive approaches, even though they finally were not conclusive.

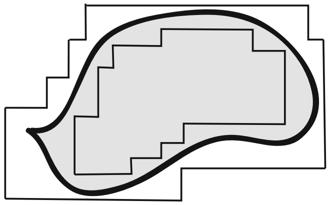
mathematics. He combined this work with a political career as member of the parliament, minister of the navy, and finally member of the Résistance.

[2] HENRI LEBESGUE, 1875–1941, born in Beauvais, active in Paris at the Sorbonne and the Collège de France. His foundation of integration theory is a landmark in mathematics, he could resort to preliminary work of Borel and Baire. With his methods he then obtained results on Fourier series.

Let us look at the approach due to Jordan.[3] His idea is intuitively appealing: Let $V = \bigcup_{j=1}^{k} I_j$ be a union of finitely many disjoint d-dimensional intervals $I_j \subset \mathbb{R}^d$, thus $I_j = [a_{j1}, b_{j1}) \times \cdots \times [a_{jd}, b_{jd})$ (it turns out to be useful, though not strictly necessary, to work with semi-open intervals). One obtains its measure $\lambda(V)$ by adding the products of the edge lengths of the individual intervals:

$$\lambda(V) := \sum_{j=1}^{k} (b_{j1} - a_{j1}) \cdots (b_{jd} - a_{jd}) .$$

Following Jordan, the exterior and the interior measure of a subset $B \subset \mathbb{R}^d$ result from covering resp. exhausting B by a union of intervals:



Expressed in formulas,

$$\lambda^*(B) := \inf\{\lambda(V) : V \supset B\} , \quad \lambda_*(B) := \sup\{\lambda(V) : V \subset B\} .$$

If both expressions have the same value, then the set B is called a Jordan set, and $\lambda(B) := \lambda^*(B) = \lambda_*(B)$ is called the Jordan measure of B. This definition is analogous to that of the Riemann integral of a function.

Without a doubt, this approach assigns to a Jordan set its "correct" measure. The deficiency of this approach lies elsewhere, on the structural level. Indeed, finite unions, finite intersections, and complements of Jordan sets are again Jordan sets. But it turns out that, in general, a countable union of Jordan sets is not necessarily a Jordan set. One easily sees, for example, that every set which consists of just a single element is a Jordan set of measure 0, while the set of rational numbers in [0, 1] is not a Jordan set (its inner and outer measures are 0 resp. 1). The σ-additivity is lacking.

This deficiency is fatal. All attempts to modify Jordan's definition in order to remove this deficiency have failed.

But perhaps it is not really necessary to define measurability of sets through an explicit construction. Is it maybe possible to assign a measure to *each* subset of

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[3] Camille Jordan, 1838–1922, born in Lyon, active in Paris at the École Polytechnique and the Collège de France. Better known than his contributions to measure theory is his work on group theory. The Jordan normal form of matrices as well as Jordan curves demonstrate his wide mathematical interests.

$\mathbb{R}^d$ in a reasonable manner, no matter whether in a direct or an indirect fashion? Already Lebesgue posed that question. The answer is negative, as was discovered by Vitali[4] and Hausdorff.[5] Later, Hausdorff's result was extended by Banach[6] and Tarski.[7] It is somewhat perplexing and thus nowadays known as the *Banach-Tarski paradox*. These two mathematicians proved in 1924: Any two bounded subsets B and B' of $\mathbb{R}^d$, $d \geq 3$, with nonempty interior, for example two balls of different radii, can be decomposed into an equal number of disjoint subsets $B = C_1 \cup \cdots \cup C_k$ and $B' = C'_1 \cup \cdots \cup C'_k$ such that all the parts $C_1, \ldots, C_k, C'_1, \ldots, C'_k$ are pairwise congruent, that is, they can be transformed into each other by translations, rotations and reflexions. One then is inclined to conclude that all parts have the same measure due to congruency, and therefore B and B' have the same measure by virtue of additivity. This would be paradoxical. How can one realise such decompositions? Intuitively this is inconceivable.

The answer is the following: The theorem of Banach-Tarski is a result of set theory, and set theory (in particular, when the axiom of choice is employed) admits the formation of rather exotic subsets of $\mathbb{R}^d$ which are no longer accessible through imagination. This is the meaning of the theorem: The system of all subsets of $\mathbb{R}^d$ is so extensive that it is impossible to assign measures to every subset such that they are invariant under congruency as well as additive. Therefore, the conclusion mentioned above cannot be drawn. Thus the paradox dissolves. These results due to Vitali, Hausdorff, Banach and Tarski are significant in the history of measure theory; nowadays they rather are a special theme.

Let us record: Attempting to view measurable subsets as single items does not lead to a sound mathematical theory. Therefore, we no longer look at individual subsets, but focus instead on systems $\mathcal{B}$ of measurable subsets. Their properties are simple. Following Borel, two properties are indispensable:

$$B \in \mathcal{B} \Rightarrow B^c \in \mathcal{B} \quad \text{and} \quad B_1, B_2, \ldots \in \mathcal{B} \Rightarrow \bigcup_{n\geq 1} B_n \in \mathcal{B}$$

[4]Giuseppe Vitali, 1875–1932, born in Ravenna, active in Modena, Padova and Bologna. He provided distinguished contributions to measure theory, but also to complex analysis.

[5]Felix Hausdorff, 1868–1942, born in Breslau, active in Leipzig, Greifswald, and Bonn. Hausdorff made fundamental contributions to set theory, topology, and measure theory. His monograph on set theory had enormous influence. Under the alias *Paul Mongré* he published essayistic and literary works. Due to his Jewish origin, Hausdorff was forced to retire in 1935. To escape deportation he took his own life in 1942.

[6]Stefan Banach, 1892–1945, born in Krakow, active in Lemberg. He established modern functional analysis. The Lemberg school of mathematicians formed around him and Hugo Steinhaus.

[7]Alfred Tarski, 1902–1983, born in Warsaw, active in Warsaw and Berkeley. He is regarded as one of the most famous logicians due to, for instance, his papers on model theory. He also contributed to set theory, measure theory, algebra, and topology. Because of his Jewish origin, after the German invasion of Poland he remained in the United States.

must hold for the complement B^c of B and for finite as well as infinite sequences $B_1, B_2, \ldots$ Such systems of sets are of fundamental importance in measure theory; following Hausdorff, they are called σ-algebras. Now the task arises to exhibit a σ-algebra which is large enough and is such that σ-additivity holds when assigning a measure to its elements.

This task can be tackled in different ways. One possibility is to start from a system $\mathcal{E}$ of sets to which a measure can be assigned in an obvious manner. For this, the system of all (semi-open) intervals of $\mathbb{R}^d$ qualifies. One then enlarges $\mathcal{E}$ to the system $\mathcal{E}'$ of all countable unions of sets from $\mathcal{E}$ together with the complements of those unions. Using σ-additivity, a measure can be assigned to all elements of $\mathcal{E}'$. If $\mathcal{E}'$ is not yet a σ-algebra, one repeats this step until a σ-algebra $\mathcal{B}^d$ has emerged. This path can be (and initially has been) entered, however it turns out that uncountably many steps are required to attain the goal. This not only stresses our intuition, but moreover one has to utilize advanced methods of set theory, namely, the theory of well-ordered sets and transfinite induction. No view emerges of how a typical measurable set looks like.

Fortunately, an elementary and much simpler approach was found soon: one directly focuses on $\mathcal{B}^d$ by characterizing it as the *smallest* σ-algebra which contains $\mathcal{E}$. It is called the *Borel σ-algebra*, and its elements $B \subset \mathbb{R}^d$ are called *Borel sets*. We will see how one assigns a measure to every Borel set so that σ-additivity holds, and how an integration theory is established whose rules are transparent and easy to apply.

One has to pay a price: in order to smoothly manipulate measurable sets and integrable functions one also has to deal with sets and functions, which in no way conform to classical perceptions. Back then, leading mathematicians faced this development in a reserved or even hostile manner, Hermite,[8] for example, spoke about the “deplorable plague” of functions not possessing derivatives. Nevertheless, the ideas of Borel and Lebesgue prevailed. Their theory is one of the most important accomplishments of set theory.

As individual elements, measurable sets can hardly be controlled, one gets hold of them only through their affiliation to systems of sets. This also means that nobody can say how a “typical” Borel set looks like. In contrast, one may imagine of a typical Jordan set as the above figure suggests. Nevertheless, in the following we will no longer bring up Jordan sets, while Borel sets will remain in the focus of our considerations. In measure and integration theory one has to get used to operate with systems of sets and of functions, not with individual sets and functions.

Since its emergence, during the age of Newton and Leibniz, the integral has evolved into a fundamental tool to be employed in many areas within and outside of mathematics. Among them are the description of processes taking place in the continuum—e.g. the space-time continuum—in the corresponding areas of

[8] CHARLES HERMITE, 1822–1901, born in Dieuze, active in Paris at the École Polytechnique and at the Sorbonne. He significantly contributed to algebra and number theory, orthogonal polynomials, and elliptic functions.

(mathematical) analysis, the description of random phenomena in probability theory, as well as the description of algorithms for computer approximation and simulation of such processes in numerical mathematics and scientific computing.

In all those contexts the Lebesgue integral has turned out to be the most adequate notion of an integral. Concerning analysis and numerical mathematics, the main reason is that the functions whose p-th power possesses a Lebesgue integral form a complete space (that is, every Cauchy sequence converges) with respect to the integral norm. In the case $p = 2$ the integral moreover yields a scalar product, and we obtain a Hilbert space. These spaces, called L_p spaces, and their descendants—for example, the Sobolev spaces—provide the predominant mathematical framework for problems in the continuum.

While Lebesgue integration theory does not concern itself with the computation of specific integrals, some of its results nevertheless assist this purpose. The results pertaining to the interchange of integrals and limits (on monotone and dominated convergence) have manifold applications, for example they clarify under which circumstances derivatives and integrals can be interchanged. Analogously, this is true for the theorems of Fubini[9] and Tonelli[10] concerning interchanging the order of integration for multiple integrals. Some specific important integrals will be dealt with in the text.

[9]GUIDO FUBINI, 1879–1943, born in Venice, active in Catania, Turin, and Princeton. He worked on real analysis, differential geometry, and complex analysis. 1939 he emigrated to the USA after he had lost his chair in Turin in the course of the antisemitic politics under Mussolini.

[10]LEONIDA TONELLI, 1885–1946, born in Gallipoli near Lecce, active in Cagliari, Parma, Bologna, and Pisa. He worked in many areas of analysis and is known mainly for his contributions to the calculus of variations.

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