Chapter 2
An Overview of Drillstring Models

Modeling of drillstring dynamics constitutes the basis for system analysis and control of harmful vibrations. Over the last half-century, extensive research effort has been conducted to mathematically describe the physical phenomena occurring in real wells.

Existing drilling models can be classified into the following general categories:

- **Lumped parameter models**. The drillstring is regarded as a mass-spring-damper system which can be described by an ordinary differential equation. This finite-dimensional system representation provides a rough description of the dynamics taking place at different levels of the string; it can be of one to several degrees of freedom.
- **Distributed parameter models**. The drillstring is considered as a beam subject to axial and/or torsional efforts. A system of partial differential equations provides a characterization of the drilling variables in an infinite-dimensional setting. The price paid for the model accuracy is the complexity involved in its analysis and simulations.
- **Neutral-type time-delay models**. These models, which are directly derived from the distributed parameter ones (when damping is considered negligible), provide an input–output system description. The involved time delays (which are dependent on the string length) are related to the speed of the oscillatory waves traveling throughout the rod. This type of model provides a good trade-off between system representation accuracy and complexity of the description. Furthermore, it offers some good matching with data transmission models that use delays in their mathematical representation.

This chapter presents a brief compilation of the most popular modeling strategies allowing the analysis and control of a vertical oilwell drilling system.

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2.1 Lumped Parameter Models

2.1.1 Torsional Dynamics

The use of reduced models for vibration analysis is motivated by the need to define a simple description of drilling dynamics. Roughly speaking, the continuous system, consisting of drillpipes and of the bottom hole assembly, is regarded as a torsional pendulum described by a lumped parameter model with one or multiple degrees-of-freedom (DOF).

Figure 2.1 shows the simplified two-degree-of-freedom torsional model of a conventional vertical drillstring proposed in [209]. The inertial masses I_p and I_b, locally damped by d_p and d_b, are connected one to each other by a linear spring with torsional stiffness k and torsional damping c. The equations of motion can be represented as follows:

$$\begin{cases} I_p\ddot{\Phi}_p + c(\dot{\Phi}_p - \dot{\Phi}_b) + k(\Phi_p - \Phi_b) + d_p\dot{\Phi}_p = u_T, \\ I_b\ddot{\Phi}_b - c(\dot{\Phi}_p - \dot{\Phi}_b) - k(\Phi_p - \Phi_b) + d_b\dot{\Phi}_b \;\; = -T(\dot{\Phi}_b), \end{cases} \tag{2.1}$$

where Φ_p and Φ_b are the angular displacements of the rotary table and of the BHA, respectively. The control signal u_T is the drive torque coming from the rotary table transmission box used to regulate the rotary angular velocity $\dot{\Phi}_b$. The frictional torque T represents the torque on bit and the nonlinear frictional forces along the drill collars.

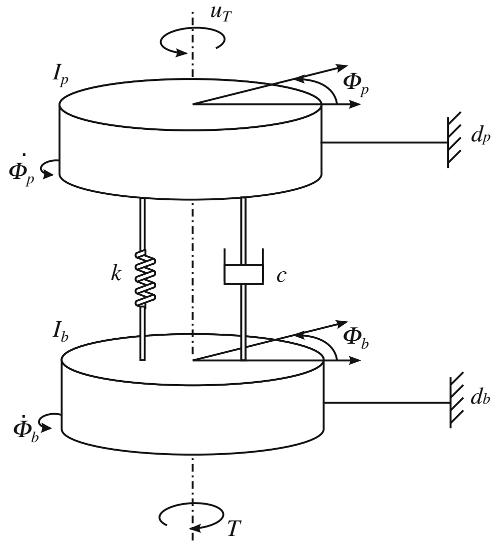

Fig. 2.1 Lumped parameter model of a rotary drilling rig

2.1.2 Axial Dynamics

A simplified model described by an ordinary differential equation is presented in [53]; the modeling strategy is inspired by the fact that any mass subject to a force in stable equilibrium acts as a harmonic oscillator for small vibrations. More precisely, the damped harmonic oscillator model describing the longitudinal drillstring motion is:

$$m_0 \ddot{\nu} + c_0 \left(\dot{\nu} + \rho(t)\right) + k_0 \nu = -\mu_1 T(\dot{\Phi}_b), \tag{2.2}$$

where the variable ν is defined as: $\nu = U_b - \rho_0 t$.

Notations m_0, c_0, and k_0 represent the mass, damping, and spring constant; U_b, $\dot{U}_b$ and $\ddot{U}_b$ denote the bottom hole axial variables: position, velocity, and acceleration, respectively. The system is controlled through the rate of penetration $\rho(t)$, which is an axial speed imposed at the surface, ρ_0 is a constant nominal value. The torque on bit function T depending on the bit angular velocity couples the axial model with the torsional one.

The coefficient μ_1 is given by the following relation [72]:

$$\mu_1 = 2 \left(R_b \mu_{bit} c_{bit}\right)^{-1},$$

where R_b is the bit radius, μ_{bit} is the friction coefficient at the bit-rock contact, and c_{bit} is the so-called bit coefficient. For a bladed bit, c_{bit} is equal to the dimensionless length of the cutting edge (and independent of the number of blades). For a flat bit, c_{bit} is computed as follows:

$$c_{bit} = \frac{6 + 4\rho_{bit}}{6 + 3\rho_{bit}}, \tag{2.3}$$

where ρ_{bit} is the radial rate of increase of cutter density. Notice that in (2.3), c_{bit} varies between 1 and 4/3 and may be considered as a constant.

2.1.3 Torsional-Axial Dynamics

A discrete model that takes into consideration the axial and torsional vibration modes of a rotary drilling system with drag bit[1] is presented in [243]. Figure 2.2 shows the physical model as a two degrees-of-freedom system (one-axial and one-torsional) to describe the axial and torsional vibrations of the drillstring. The torsional part of

[1] A drag bit is a specific type of cutting device consisting of n identical blades symmetrically distributed around the axis of revolution.

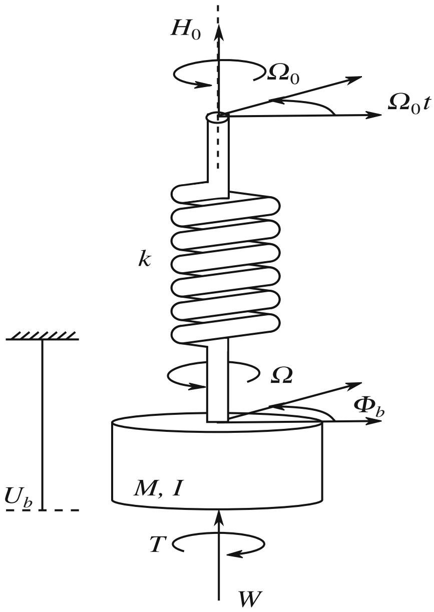

Fig. 2.2 Simplified model of the drilling system proposed in [243]

the model idealizes the drillstring as a torsional pendulum, and the axial part of the model idealizes the BHA and the drillstring as a lumped mass M.

The governing equations of motion are given by:

$$\begin{cases} I\frac{d^2\Phi_b}{dt^2} + k(\Phi_b - \Phi_{b0}) = T_0 - T \\ M\frac{d^2U_b}{dt^2} = W_0 - W, \end{cases} \tag{2.4}$$

where the variables U_b and Φ_b stand for the vertical and angular positions of the bit, respectively. The mechanical elements representing the BHA are: M, the point mass; I, the moment of inertia; and k, the spring stiffness representing the torsional rigidity of the drill pipe. The torque and the weight on bit are denoted by T and W, respectively. The stationary variables Φ_{b0}, T_0 and W_0, associated with the trivial solution of (2.4), satisfy:

$$\Phi_{b0} = \Omega_0 t - \frac{T_0}{k},$$

$$W_0 = W_s - H_0$$

where Ω_0 is a prescribed angular velocity, H_0 is a constant upward tension and W_s is the submerged weight of the drillstring.

A shortcoming of model (2.4) is that the dissipation in axial and torsional components is omitted, which may result unrealistic. Furthermore, the axial compliance of the drillpipes is not considered.

Some of these deficiencies were accounted for in the model proposed in [31], given by:

$$\begin{cases} I\frac{d^2\Phi_b}{dt^2} + k_t(\Phi_b - \Omega_0 t) = T_0 - T \\ M\frac{d^2U_b}{dt^2} + d\frac{dU_b}{dt} + k_a\,(U_b - \rho_0 t) = W_0 - W, \end{cases} \tag{2.5}$$

where k_t, k_a denote the torsional and axial spring stiffness and the friction parameter d characterizes viscous friction along the BHA.

An improved version of the lumped parameter model describing the coupled torsional-axial drilling dynamics is proposed in [207]. The suggested governing equations are given by:

$$\begin{cases} I\frac{d^2\Phi_b}{dt^2} + d_t\frac{d\Phi_b}{dt} + k_t(\Phi_b - \Omega_0 t) = \quad T_0 - T \\ M\frac{d^2U_b}{dt^2} + d_a\frac{dU_b}{dt} + k_a\,(U_b - \rho_0 t) = W_0 - W, \end{cases} \tag{2.6}$$

where d_t and d_a represent the viscous damping coefficients accounting for dissipation in the torsional and axial directions, respectively.

Model (2.6) is used for developing a local stability analysis to determine the stable and unstable operating regimes in the WoB-rotary speed parameter plane yielding some operating guidelines for practical purposes [207].

2.1.4 Torsional-Axial-Lateral Dynamics

A discrete system model for analytical and numerical studies of the coupled axial-torsional-lateral motions of the drillstring is developed in [181]. The discretized model of the drilling system considers two disks connected together over one constant length span, each disk with different translational inertia and rotary inertia properties. The inertia properties of the drill pipes are lumped into the first disk, while the inertial properties for the BHA are lumped into the second one. The dynamics of each disk is described collectively by a total of four DOF, with one DOF for axial motions, one DOF for torsional motions, and two DOF for lateral motions. The discretized system is driven by a motor with a constant angular speed Ω_0 and the two discrete inertial elements are connected through two identical massless, elastic elements which have associated axial, torsional, and bending potential energy contributions that reflect the elastic properties of the drill pipes.

The positions of each disk are described by the lateral, longitudinal, and torsional coordinates $(x_1, y_1, z_1, \varphi_1)$ and $(x_2, y_2, z_2, \varphi_2)$.

The governing equations of motion for the eight degree-of-freedom system are given by:

$$\begin{cases} m_1\ddot{x}_1 + c_l\dot{x}_1 + 2k_l\left(x_1 - \frac{x_2}{2}\right) = m_1 e\dot{\varphi}_1^2\cos\varphi_1 + F_{1x} \\ m_1\ddot{y}_1 + c_l\dot{y}_1 + 2k_l\left(y_1 - \frac{y_2}{2}\right) = m_1 e\dot{\varphi}_1^2\sin\varphi_1 + F_{1y} \\ m_1\ddot{z}_1 + c_a\dot{z}_1 + k_a\left(z_1 - z_2\right) \;= m_1 g - H_0 \\ J_1\ddot{\varphi}_1 + c_t\dot{\varphi}_1 + k_t\left(2\varphi_1 - \varphi_2\right) = -T_1 + k_t\Omega_0 t \\ m_2\ddot{x}_2 + c_l\dot{x}_2 + k_l\left(\frac{x_2}{2} - x_1\right) \;= m_2 e\dot{\varphi}_2^2\cos\varphi_2 + F_{2x} \\ m_2\ddot{y}_2 + c_l\dot{y}_2 + k_l\left(\frac{y_2}{2} - y_1\right) \;= m_2 e\dot{\varphi}_2^2\sin\varphi_1 + F_{2y} \\ m_2\ddot{z}_2 + c_a\dot{z}_2 + k_a\left(z_2 - z_1\right) \;= m_2 g - F_{2z} \\ J_2\ddot{\varphi}_2 + c_t\dot{\varphi}_2 + k_t\left(\varphi_2 - \varphi_1\right) \;= -T_2 \end{cases}$$

where m and J are the mass and the rotary inertia of each disk, e is the eccentricity which is assumed to be the same, and the influence of torsional acceleration on the lateral centrifugal force has been neglected. The stiffness and damping are denoted by k and c, respectively. The subscripts a, t, and l correspond to the direction in which these properties have influence: axial, torsional, and lateral. The quantities F_{1x}, F_{1y}, and T_1 are the respective force components and the torque that arises due to interactions between the drill pipe and wellbore. Similarly, the quantities F_{2x}, F_{2y}, F_{2z}, and T_2 are the respective force components and torque that arise due to interactions between the drill bit and rock. The axial, torsional, and lateral vibrations of the discrete system are coupled through these interaction forces and torques, see [181].

2.2 Distributed Parameter Models

To the best of the author's knowledge, the very first model of drilling vibrations was developed in the 1960s at the Shell Development Company, by Bailey [20] and Finnie [89]. They proposed a wave equation model to analytically treat the longitudinal and torsional drilling vibrations. Due to the accuracy of this modeling approach in reproducing the drilling behavior, regarding its distributed nature, it constitutes the basis of several recent contributions on drilling analysis and control.

In order to have a better understanding of the system' dynamics, the following section explains the physical aspects of a flexible bar giving rise to the wave equation model.

2.2.1 Model Derivation

Let us analyze the axial dynamics of a flexible metal bar of length L and of cross-section σ_0. Let $q(x,t)$ be the displacement of a point x of the bar with respect to

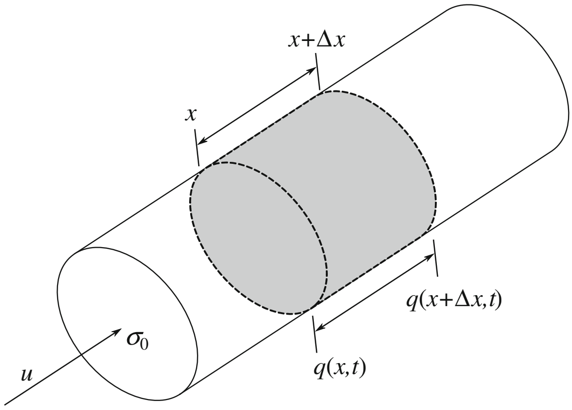

Fig. 2.3 Flexible bar

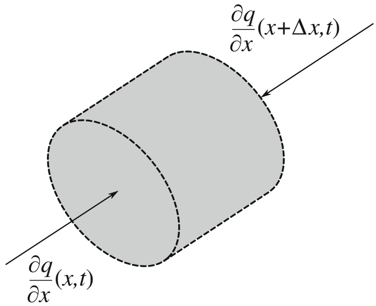

Fig. 2.4 Tension applied to a short segment of the bar

its equilibrium position, and $T(x,t)$, the tension applied at the point x at time "t" (Figs. 2.3 and 2.4).

Consider an element of length l_0 under the mean tension T_0.

The *fundamental elasticity law* establishes the following relation between the elongation $dl := l - l_0$ and the infinitesimal tension $dT := T - T_0$:

$$\frac{dT}{\sigma_0} = E_0 \frac{dl}{l_0}, \tag{2.7}$$

where E_0 is the Young modulus, or elasticity factor under the tension T_0. This law only applies for a sufficiently small relative elongation dl/l_0. At time "t", the segment $(x, x+\Delta x)$ has a static length of l_0 and takes the position $(x+q(x,t),\ x+\Delta x+q(x+\Delta x,t))$. Under a tension force, the segment length increases from $l_0 = \Delta x$ to $l = l_0+dl = \Delta x+(\partial q/\partial x)\Delta x$, we then have $dl/l_0 = \partial q/\partial x$. The elasticity law implies:

$$T - T_0 = E_0\sigma_0 \frac{\partial q}{\partial x}. \tag{2.8}$$

Let ρ_0 be the linear density (mass per unit length) of the bar.

The fundamental principle of dynamics states:

$$\rho_0 \frac{\partial^2 q}{\partial t^2} \Delta x = \frac{\partial T}{\partial x} dx, \tag{2.9}$$

by introducing (2.8), we obtain:

$$\rho_0 \frac{\partial^2 q}{\partial t^2} = E_0 \sigma_0 \frac{\partial^2 q}{\partial x^2}, \tag{2.10}$$

which constitutes the wave equation with propagation speed: $c = \sqrt{E_0 \sigma_0 / \rho_0}$.

The normalized model can be written as:

$$\frac{\partial^2 q}{\partial t^2}(x, t) = c^2 \frac{\partial^2 q}{\partial x^2} \tag{2.11a}$$

$$\frac{\partial q}{\partial x}(0, t) = -u(t) \qquad \frac{\partial q}{\partial x}(1, t) = 0 \tag{2.11b}$$

$$q(x, 0) = q_0(x) \qquad \frac{\partial q}{\partial t}(x, 0) = q_{t_0}(x) \tag{2.11c}$$

where $x \in [0, 1]$. Expression (2.11a) is known as the wave equation with boundary conditions (2.11b) and initial conditions (2.11c).

2.2.2 Drillstring Oscillatory Behavior

The propagation of torsional waves along a drillstring of length L can be modeled by the following hyperbolic partial differential equation [53]:

$$GJ \frac{\partial^2 \Phi}{\partial s^2}(s, t) - I \frac{\partial^2 \Phi}{\partial t^2}(s, t) - \gamma \frac{\partial \Phi}{\partial t}(s, t) = 0, \qquad s \in (0, L), \tag{2.12}$$

where the twist angle Φ depends on the length coordinate "s" and time "t." The shear modulus and the second moment of area (also known as geometric moment of inertia) are denoted by G and J, respectively. The inertia I is such that $I = \rho_a J$, where ρ_a is the area density. A viscous damping $\gamma \geqslant 0$ is assumed along the structure.

Since most of the energy dissipation in drilling systems is taking place at the bit-rock interface, we may consider that the damping γ is null.

Thus, the distributed parameter model (2.12) reduces to the one-dimensional wave equation:

$$\frac{\partial^2 \Phi}{\partial s^2}(s, t) = \tilde{c}^2 \frac{\partial^2 \Phi}{\partial t^2}(s, t), \quad s \in (0, L), \tag{2.13}$$

where $\tilde{c} = \sqrt{I/GJ} = \sqrt{\rho_a / G}$.

An appropriate choice of boundary conditions allows characterizing the propagating torsional waves along the drillstring. Several equations have been proposed to describe the dynamics taking place at the upper and lower ends of the drillstring; the main ones are presented in the sequel.

2.2.2.1 Kinematic Boundary Conditions

In [53], the following boundary conditions are considered:

$$\Phi(0,t) = \Omega t, \tag{2.14a}$$

$$GJ\frac{\partial \Phi}{\partial s}(L,t) + I_B\frac{\partial^2 \Phi}{\partial t^2}(L,t) = -T\left(\frac{\partial \Phi}{\partial t}(L,t)\right). \tag{2.14b}$$

A lumped inertia I_B is chosen to represent the assembly at the bottom hole. It is assumed that the speed at the surface ($s = 0$) is restricted to a constant value Ω, the other extremity ($s = L$), is subject to a torque T, which is a function of the bit speed. The model of T includes a linear term $c_b\partial\Phi/\partial t(L,t)$ representing the viscous damping torque which approximates the influence of the mud drilling and a nonlinear term $F\,(\partial\Phi/\partial t(L,t))$ representing the dry friction torque which models the bit-rock contact, i.e.,

$$T\left(\frac{\partial \Phi}{\partial t}(L,t)\right) = c_b\frac{\partial \Phi}{\partial t}(L,t) + F\left(\frac{\partial \Phi}{\partial t}(L,t)\right). \tag{2.15}$$

2.2.2.2 Boundary Conditions Considering a Speed Difference at the Top Extremity

The boundary condition (2.14b) satisfactorily reproduces the behavior at the ground level; however, (2.14a) is only a pure kinematic description of the drillstring upper part. The angular velocity coming from the rotor Ω does not match the rotational speed of the load $\partial\Phi/\partial t(0,t)$, this sliding speed results in the local torsion of the drillstring. In order to take into account this phenomenon, the following boundary conditions are introduced in [253]:

$$GJ\frac{\partial \Phi}{\partial s}(0,t) = \beta\left(\frac{\partial \Phi}{\partial t}(0,t) - \Omega(t)\right) = \beta\frac{\partial \Phi}{\partial t}(0,t) - u_T(t) \tag{2.16a}$$

$$GJ\frac{\partial \Phi}{\partial s}(L,t) = -I_B\frac{\partial^2 \Phi}{\partial t^2}(L,t) - T\left(\frac{\partial \Phi}{\partial t}(L,t)\right), \tag{2.16b}$$

where β denotes the angular momentum at the top extremity.

2.2.2.3 Newtonian Boundary Conditions

The following equation constitutes an alternative top boundary condition inspired by the Newton's second law of motion, see for instance [64]:

$$I_T \frac{\partial^2 \Phi}{\partial t^2}(0,t) = -T_T(t) + u_T(t).$$

The effective moment of inertia of the top-drive is denoted by I_T, u_T corresponds to the external torque delivered by the rotary table taken as a control input, and the function $T_T(t)$, describing the transmitted torque and damping due to viscous effects, is given by:

$$T_T(t) = -GJ\frac{\partial \Phi}{\partial s}(0,t) + \beta \frac{\partial \Phi}{\partial t}(0,t).$$

The boundary conditions are then written as:

$$GJ\frac{\partial \Phi}{\partial s}(0,t) = I_T \frac{\partial^2 \Phi}{\partial t^2}(0,t) + \beta \frac{\partial \Phi}{\partial t}(0,t) - u_T(t) \tag{2.17a}$$

$$GJ\frac{\partial \Phi}{\partial s}(L,t) = -I_B \frac{\partial^2 \Phi}{\partial t^2}(L,t) - T\left(\frac{\partial \Phi}{\partial t}(L,t)\right). \tag{2.17b}$$

2.2.3 Coupled Axial-Torsional Vibrations

It is well known that torsional vibrations contribute to the excitation of axial oscillations, in fact, these self-excited oscillations are intimately coupled together and may occur simultaneously.

Coupled axial-torsional excitations of a drillstring of length L, described by the rotary angle $\Phi(s,t)$ and the longitudinal position $U(s,t)$ can be modeled by the wave equations [39]:

$$\frac{\partial^2 \Phi}{\partial s^2}(s,t) = \tilde{c}^2 \frac{\partial^2 \Phi}{\partial t^2}(s,t), \tag{2.18a}$$

$$\frac{\partial^2 U}{\partial s^2}(s,t) = c^2 \frac{\partial^2 U}{\partial t^2}(s,t), \tag{2.18b}$$

with boundary conditions

$$GJ\frac{\partial \Phi}{\partial s}(0,t) = \beta \frac{\partial \Phi}{\partial t}(0,t) - u_T(t) \tag{2.19a}$$

$$GJ\frac{\partial \Phi}{\partial s}(L,t) = -I_B \frac{\partial^2 \Phi}{\partial t^2}(L,t) - T\left(\frac{\partial \Phi}{\partial t}(L,t)\right), \tag{2.19b}$$

and

$$E\Gamma \frac{\partial U}{\partial s}(0,t) = \alpha \frac{\partial U}{\partial t}(0,t) - u_H(t) \tag{2.20a}$$

$$E\Gamma \frac{\partial U}{\partial s}(L,t) = -M_B \frac{\partial^2 U}{\partial t^2}(L,t) - T\left(\frac{\partial \Phi}{\partial t}(L,t)\right). \tag{2.20b}$$

The spatial variable "s" is chosen such that $s = 0$ denotes the top of the drillstring and $s = L$ its bottom. The propagation speeds of the axial and torsional waves v_U, v_Φ, defined as: $v_U = c^{-1}$ and $v_\Phi = \tilde{c}^{-1}$ can be computed from material parameters, namely Young modulus E, the shear modulus G and the density ρ_a, by means of

$$c = \sqrt{\frac{\rho_a}{E}} \quad \text{and} \quad \tilde{c} = \sqrt{\frac{\rho_a}{G}}. \tag{2.21}$$

In the boundary condition (2.20a), u_H is the brake motor control (upward hook force) and $\alpha \partial U/\partial t(0,t)$ represents a friction force of viscous type (where α is the viscous friction coefficient). In (2.19a), u_T represents the torque produced by the rotary table motor and $\beta \partial \Phi/\partial t(0,t) - u_T(t)$ designates the difference between the motor speed and rotational speed of the first pipe. It is assumed that the drilling system can be controlled by the boundary force u_H and the boundary torque u_T.

The model contains geometrical parameters of the drill string, that are assumed to be spatially and timely constant. These comprise the drillstring's cross-section Γ and its second moment of area J, as well as the mass M_B and the inertia moment[2] of the drill bit I_B. The function T considered in the bottom boundary conditions accounts for the frictional torque resulting from the interaction between the drill bit and the rock.

Notice that the boundary conditions (2.14b), (2.19b) and (2.20b), corresponding to the bottom of the rod, involve a frictional torque arising from the bit-rock interaction. The modeling of the torque on the bit constitutes a crucial aspect of the system description since it allows to reproduce the vibrational phenomena; this subject will be discussed in Chap. 3.

2.3 Neutral-Type Time-Delay Models

The wave equation model provides a realistic description of the distributed system variables; however, under certain circumstances, it is convenient to deal with a relatively simpler model involving just the primary interest variables. This section presents a direct procedure to derive, from the wave equations, equivalent input-output

[2] The inertia moment is such that $I_B = M_B r^2$, where r is taken as the averaged radius of drillpipe.

models described by neutral-type time-delay equations relating the variables at both ends of the drilling rod.

Integration along characteristics of the hyperbolic PDE allows the association of a certain system of functional differential equations to the mixed problem, more precisely, a one-to-one correspondence may be established and proved between the solutions of the mixed problem for hyperbolic PDE and the initial value problem for the associated system of functional equations [238].

By reducing a boundary value problem to a neutral-type time-delay equation we are able to exploit techniques from delay systems theory to gain insight into the complexity involved in the analysis and simulation of PDE models.

The *method of d'Alembert* provides a solution to the one-dimensional wave equation. Introducing the variables $\gamma = t + \tilde{c}s$ and $\eta = t - \tilde{c}s$, the general solution of the undamped wave equation (2.13), describing the torsional drilling behavior, is given by:

$$\Phi(s,t) = \varphi(\gamma) + \psi(\eta), \tag{2.22}$$

where φ and ψ are arbitrary continuously differentiable real-valued functions, with φ representing an arbitrary up-traveling wave and ψ an arbitrary down-traveling wave.

The boundary conditions (2.19a, 2.19b) can be rewritten as:

$$\tilde{c}\frac{\partial \varphi}{\partial \gamma}(t) - \tilde{c}\frac{\partial \psi}{\partial \eta}(t) = \frac{\beta}{GJ}\left(\frac{\partial \varphi}{\partial \gamma}(t) + \frac{\partial \psi}{\partial \eta}(t)\right) - \frac{1}{GJ}u_T(t), \tag{2.23}$$

$$\begin{aligned}\tilde{c}\frac{\partial \varphi}{\partial \gamma}(t+\tau) - \tilde{c}\frac{\partial \psi}{\partial \eta}(t-\tau) = &-\frac{I_B}{GJ}\left(\frac{\partial^2 \varphi}{\partial \gamma^2}(t+\tau) + \frac{\partial^2 \psi}{\partial \eta^2}(t-\tau)\right) \\ &-\frac{1}{GJ}T\left(\frac{\partial \varphi}{\partial \gamma}(t+\tau) + \frac{\partial \psi}{\partial \eta}(t-\tau)\right),\end{aligned} \tag{2.24}$$

where $\tau = \tilde{c}L$.

We define $\dot{\Phi}_b(t)$ as the angular velocity at the bottom extremity of the rod:

$$\dot{\Phi}_b(t) = \frac{\partial \Phi}{\partial t}(L,t) = \dot{\varphi}(t+\tau) + \dot{\psi}(t-\tau). \tag{2.25}$$

Equations (2.23) and (2.24) can be rewritten as:

$$\tilde{c}\dot{\varphi}(t) - \tilde{c}\dot{\psi}(t) = \frac{\beta}{GJ}\left(\dot{\varphi}(t) + \dot{\psi}(t)\right) - \frac{1}{GJ}u_T(t), \tag{2.26}$$

$$\tilde{c}\dot{\varphi}(t+\tau) - \tilde{c}\dot{\psi}(t-\tau) = -\frac{I_B}{GJ}\ddot{\Phi}_b(t) - \frac{1}{GJ}T\left(\dot{\Phi}_b(t)\right). \tag{2.27}$$

Equation (2.25) gives

$$\dot{\varphi}(t) = -\dot{\psi}(t-2\tau) + \dot{\Phi}_b(t-\tau), \tag{2.28}$$

substituting (2.28) into (2.27) yields

$$\dot{\psi}(t-\tau) = \frac{1}{2}\dot{\Phi}_b(t) + \frac{I_B}{2\tilde{c}GJ}\ddot{\Phi}_b(t) + \frac{1}{2\tilde{c}GJ}T\left(\dot{\Phi}_b(t)\right). \tag{2.29}$$

Substituting (2.29) into (2.28) gives

$$\dot{\varphi}(t) = \frac{1}{2}\dot{\Phi}_b(t-\tau) - \frac{I_B}{2\tilde{c}GJ}\ddot{\Phi}_b(t-\tau) - \frac{1}{2\tilde{c}GJ}T\left(\dot{\Phi}_b(t-\tau)\right). \tag{2.30}$$

We can write (2.26) as

$$\left(\tilde{c} - \frac{\beta}{GJ}\right)\dot{\varphi}(t) - \left(\tilde{c} + \frac{\beta}{GJ}\right)\dot{\psi}(t) = -\frac{1}{GJ}u_T(t). \tag{2.31}$$

Substituting the expressions for $\dot{\psi}$ and $\dot{\varphi}$ given in (2.29) and (2.30) into (2.31) yields:

$$\left(\tilde{c} - \frac{\beta}{GJ}\right)\left(\frac{1}{2}\dot{\Phi}_b(t-2\tau) - \frac{I_B}{2\tilde{c}GJ}\ddot{\Phi}_b(t-2\tau) - \frac{1}{2\tilde{c}GJ}T\left(\dot{\Phi}_b(t-2\tau)\right)\right)$$
$$-\left(\tilde{c} + \frac{\beta}{GJ}\right)\left(\frac{1}{2}\dot{\Phi}_b(t) + \frac{I_B}{2\tilde{c}GJ}\ddot{\Phi}_b(t) + \frac{1}{2\tilde{c}GJ}T\left(\dot{\Phi}_b(t)\right)\right) = -\frac{1}{GJ}u_T(t-\tau).$$

Simplifying, we get a torsional drilling model described by the neutral-type time-delay equation:

$$\begin{aligned}\ddot{\Phi}_b(t) - \Upsilon\ddot{\Phi}_b(t-2\tau) = &-\Psi\dot{\Phi}_b(t) - \Upsilon\Psi\dot{\Phi}_b(t-2\tau) - \frac{1}{I_B}T\left(\dot{\Phi}_b(t)\right)\\ &+\frac{1}{I_B}\Upsilon T\left(\dot{\Phi}_b(t-2\tau)\right) + \Pi_\Omega u_T(t-\tau),\end{aligned} \tag{2.32}$$

where $\dot{\Phi}_b(t)$ is the angular velocity at the bottom extremity, and

$$\Pi_\Omega = \frac{2\Psi}{\beta + \tilde{c}GJ}, \quad \Upsilon = \frac{\beta - \tilde{c}GJ}{\beta + \tilde{c}GJ}, \quad \Psi = \frac{\tilde{c}GJ}{I_B}, \quad \tau = \tilde{c}L. \tag{2.33}$$

Similarly, the wave equation (2.18b) with boundary conditions (2.20a, 2.20b), corresponding to the axial drilling dynamics is transformed into the following neutral-type equation:

$$\begin{aligned}\ddot{U}_b(t) - \tilde{\Upsilon}\ddot{U}_b(t-2\tilde{\tau}) = &-\tilde{\Psi}\dot{U}_b(t) - \tilde{\Upsilon}\tilde{\Psi}\dot{U}_b(t-2\tilde{\tau}) - \frac{1}{M_B}T\left(\dot{\Phi}_b(t)\right)\\ &+\frac{1}{M_B}\tilde{\Upsilon}T\left(\dot{\Phi}_b(t-2\tilde{\tau})\right) + \tilde{\Pi}_\rho u_H(t-\tilde{\tau}),\end{aligned} \tag{2.34}$$

where $\dot{U}_b(t)$ is the axial velocity at the bottom extremity, and

$$\tilde{\Pi}_\rho = \frac{2\tilde{\Psi}}{\alpha + cE\Gamma}, \quad \tilde{\Upsilon} = \frac{\alpha - cE\Gamma}{\alpha + cE\Gamma}, \quad \tilde{\Psi} = \frac{cE\Gamma}{M_B}, \quad \tilde{\tau} = cL. \tag{2.35}$$

It is important to point out that the delays τ and $\tilde{\tau}$ represent the time that the torsional and the axial waves take to travel from one to the other extremity of the drillstring. They are computed as follows: $\tau = L/v_\Phi$ and $\tilde{\tau} = L/v_U$, where L is the length of the drilling rod and v_Φ, v_U are the speeds of propagation of the rotational and longitudinal waves.

2.4 Notes and References

The most commonly used models for describing the drillstring behavior are the lumped parameter ones. The model accuracy is related to the involved DOF number. In [178], a one DOF model is used to investigate the effects of viscous damping, rotary speed, and natural frequency on the stick-slip phenomenon. A similar model is considered in [128] to design a control strategy allowing a smooth bit rotation to reduce axial and lateral drilling vibrations. A two DOF lumped parameter model is used in [43] to analyze torsional vibrations resulting from the perforation characteristics when drilling with a polycrystalline diamond compact (PDC) bit. This model is also used in [138, 266] to design control strategies (active damping and $\mathscr{H}_\infty$ control, respectively) to tackle torsional drilling oscillations. The empirical methods studied in [209] and the D-OSKILL control technique proposed in [51] are tested through a similar two DOF model. The sliding-mode control proposed in [210] is based on a discontinuous lumped parameter torsional model of four DOF. In [181], a discrete model of a drillstring system of eight DOF is proposed and used to study their nonlinear motions. Coupled axial-torsional dynamics of a drill string are studied in [182] through a discrete 32-segment model with 128 states that considers nonlinearities such as dry friction, loss of contact, and state-dependent time delays; the normal strain contours of the spatial-temporal system demonstrate the existence of strain wave propagation along the drill string. Different reduced-order models to describe the drilling behavior can be found in [177].

It is well known that lumped parameter models constitute a simplified representation of the physical mechanism neglecting the distributed nature of the system. The wave equation model provides higher accuracy in reproducing the rod oscillatory behavior. The very first analytic studies on the subject were carried out using the classical wave equation to describe the torsional behavior of drillstring assemblies [20]. Several recent contributions have adopted this modeling approach. In [53], a distributed parameter model subject to mixed boundary conditions is used to investigate the drilling system stability. A quite similar mathematical description is considered in [284] to design drilling vibration controllers. It is worth mentioning that the wave equation model plays a key role in the analysis and control of torsional drilling oscillations developed in the forthcoming chapters.

As explained in the previous paragraphs, in our opinion, the complexity involved in the analysis and simulations of a nonlinear partial differential equation can be avoided by transforming it into a delay system of neutral type. To the best of the author's knowledge, the transformation process, developed via the d'Alembert method, was presented for the first time in [3] and investigated in [62]. Furthermore, it was employed in [91] to study the controllability and motion planning of a flexible rod, and in [19] to analyze the drilling oscillatory dynamics.

A similar transformation of PDE boundary value problems to time-delay equations plays a significant role in a wide range of other biological, physical, and engineering problems. These include cardiovascular system dynamics [136], homeostatic mechanisms for maintaining blood pressure (the system model is presented in [223] and a bifurcation analysis in [265]), laser optical fibers, power transmission line networks [42], sonar/radar ranging technologies [25], and many other applications [134].

A different modeling method for describing structural waves propagating in a drill system is proposed in [129]. The proposed strategy is based on a vibration transfer matrix approach in which a drill pipe section is modeled through an analytical vibration transfer matrix between two sets of structural wave variables at the two ends of the pipe section. This model requires significantly less computational resources compared to the conventional Finite Element Method (FEM) [34], which requires a large number of meshes, complicating the numerical analysis of the system.

There are some other modeling techniques to describe drilling vibrations. For instance, in [114], the drillstring vibrations are investigated using both a coupled nonlinear elastodynamic mathematical model and a dynamic FEM model. In [85], a mechanics model has been developed to analyze axial and torsional vibrations in the frequency domain and provide vibration indices indicative of dysfunction in these modes. The model makes use of transfer matrices solve harmonic perturbations around a baseline solution obtained from a torque-and-drag type analysis.

A drilling system model is not complete if the frictional interface between the bit and the rock formation is not considered. In order to reproduce the nonlinearities giving rise to the drillstring vibration phenomena, an appropriate torque on bit model must be built. The following chapter addresses this important subject. It provides first the basic concepts of tribology that allows describing the force resisting the motion between two surfaces in contact; then, a compilation of the most popular modeling techniques which approximate the frictional force between the cutting device and the drilling surface is presented.

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