

Chapter 2

Basics of Reliability

2.1 Introduction

In today's competitive world reliability has become a major concern in almost all the industries. Reputation of a company is very closely associated with the reliability of its products. The more reliable a product is, the more likely the company is to have a favorable reputation. The reliability of a product also affects the customer satisfaction. An unreliable product will affect customer satisfaction severely. When it comes to the customer satisfaction quality of the product also matters. At this point one should understand the difference between quality and reliability. Quality of a product is the degree of conformance to applicable specifications and it is dependent on the manufacturing process. Whereas reliability is maintaining the quality of a product over some period of time.

Nowadays, it has become very common to mention the warranty periods for the products by the companies. This has got both advantage and disadvantage. For an example, a person goes to a shop for purchasing a mobile phone. Assume that there are three companies offering the same product for the same price but with different warranty periods. It is obvious that the person may choose the product for which warranty period is more. Some people may go by the company's reputation. Whichever the company has got more reputation people may choose those products irrespective of their cost and warranty periods. If the person is going by the warranty period, then whichever the company specifies more warranty period it will capture the market. This may be the advantage, on the contrary if the product quality is poorer and the product fails to perform its function within the warranty period, the replacement and repair costs will negatively affect company's profits. Hence, it is very much important to maintain the quality of a product for longer times.

In these discussions one should note that the time factor is mentioned only for certain duration. Because, no component in this world last for infinity time. Even the human beings have their time limitation on this earth. When the human beings have got the time limitation, the products produced by them will certainly have the time limitation. In certain cases it may be much longer but not infinity time. With this discussion one can define the reliability as “the probability that a component or unit performs its intended function for a given period of time under specified environmental conditions.” This definition is governed by four factors namely:

1. Probability
2. Intended function
3. Time duration
4. Environmental conditions.

One can always ask a question that why reliability is a probabilistic concept? Reliability is associated with probability because one cannot tell when a component is going to fail. Consider a case of centrifugal pump that is put into the operation, if the pump is of high quality and reliable it can function for longer period, even then one cannot judge when this component will cease from functioning. This can fail immediately after put into the operation or may fail within an hour or in a day or in a year, etc. Here the time to failure of a component is not certain, hence, reliability should be dealt with probability concepts. In the above definition the second term is given as intended function because every component is designed for performing certain task.

Coming to the time duration, as discussed earlier no component will last for infinity time or forever. Hence, there is always some mission time associated with each and every component. This mission time can vary from component to component and also from industry to industry. Consider the case of satellites and their launching vehicles. The mission time of the components used in the space launching vehicles will be until it keeps the satellites in their respective orbital. However, satellite mission time may vary from one launching to another depending on their purpose.

One more factor that is important is environmental conditions, because not all the components will perform under all the environmental conditions. Each and every component has got its own limitations. As an example some components will be designed for performing at very low temperatures and some components may be designed for higher temperatures. Consider a simple case of thermometer which is used for measuring the temperature of a human body. This can well measure the temperature of human body in its specified limits. If the same thermometer is used for measuring the temperature of hot water it will certainly fail because it is not designed for those temperatures, in certain cases that bulb of the thermometer will rupture. Here the question comes can it be possible to design a component working under all environmental conditions. It may be possible in certain cases but it will cost more from manufacturing side.

Finally, is there any component that exists in this world that can deliver its intended function under all environmental conditions for infinity time or forever?

The answer is 'No.' If at all any component exists that is manufactured and has not failed so far will certainly fail at some point of time in the future but that time may be in some cases very longer. The specific causes of failures of components and equipments in a system can be many. Some are known and others are unknown due to the complexity of the system and its environment. A few of them are listed below.

- (a) Poor design (component or system),
- (b) Wrong manufacturing techniques,
- (c) Lack of total knowledge and experience,
- (d) Complexity of equipment,
- (e) Poor maintenance policies, and
- (f) Human errors, etc.

As was discussed earlier the basis for reliability is probability theory, hence, it is very much important to learn the concepts of probability and is explained in the following sections.

2.2 Probability Theory

Probability theory is used where it is not possible to predict the exact outcome of any experiment. The following terms are generally used in the probability theory. These terms are just defined for the sake of completeness of the theory, but for detailed study the reader is requested to refer some statistics books [1–3].

2.2.1 *Random Experiment*

An experiment that can result in to different outcomes, even though it is repeated in the same manner every time, is called a **random experiment**.

2.2.2 *Sample Space*

The set of all possible outcomes of a random experiment is called the **sample space** of the experiment. The sample space is denoted as S .

- A sample space is **discrete** if it consists of a finite or countable infinite set of outcomes.
- A sample space is **continuous** if it contains an interval (either finite or infinite) of real numbers.

2.2.3 Event

An **event** is a subset of the sample space of a random experiment.

- The **union** of two events is the event that consists of all outcomes that are contained in either of the two events. Consider the two events as E_1 and E_2 then one can denote the union as $E_1 \cup E_2$.
- The **intersection** of two events is the event that consists of all outcomes that are contained in both of the two events. One can denote the intersection as $E_1 \cap E_2$.
- The **complement** of an event in a sample space is the set of outcomes in the sample space that are not in the event. The complement of the event E can be represented as E^c .
- Two events, denoted as E_1 and E_2 , such that $E_1 \cap E_2 = \phi$ are said to be **mutually exclusive**. These terms are illustrated in Fig. 2.1.

2.2.4 Probability

Probability is used to quantify the likelihood, or chance, that an outcome of a random experiment will occur.

- Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $1/N$.
- For a discrete sample space, the *probability of an event E* , denoted as $P(E)$, equals the sum of the probabilities of the outcomes in E .

Example 2.1 A random experiment can result in one of the outcomes $\{a, b, c, d\}$ with probabilities 0.1, 0.3, 0.5, and 0.1, respectively. Let A denote the event $\{a, b\}$, B the event $\{b, c, d\}$, and C the event $\{d\}$. Then find $P(A)$, $P(B)$, and $P(C)$.

Solution:

$$P(A) = 0.1 + 0.3 = 0.4$$

$$P(B) = 0.3 + 0.5 + 0.1 = 0.9$$

$$P(C) = 0.1$$

2.2.5 Axioms of Probability

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

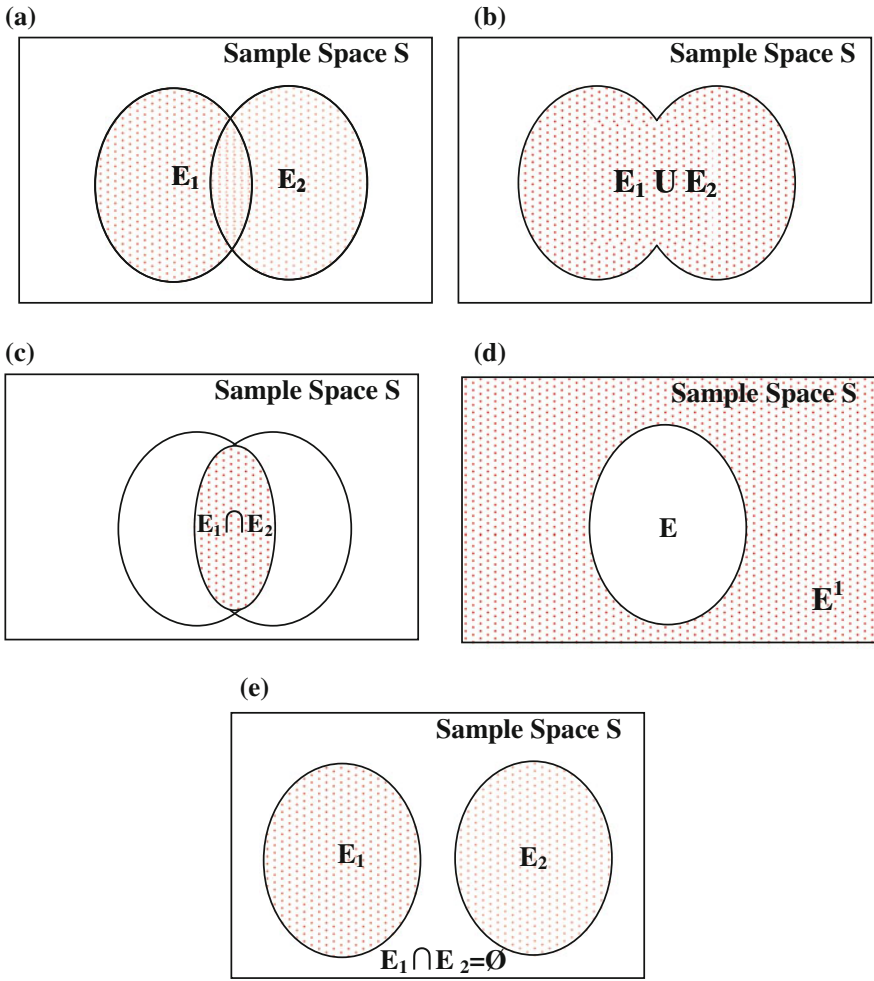


Fig. 2.1 Illustration of **a** sample space and events. **b** Union of two events. **c** Intersection of two events. **d** Complement of an event. **e** Mutually exclusive events

If S is the sample space and E is any event in a random experiment,

- (1) $P(S) = 1$
- (2) $0 \leq P(E) \leq 1$
- (3) For two events E_1 and E_2 with $E_1 \cap E_2 = \Phi$ [$P(\Phi) = 0$]

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) \tag{2.1}$$

Example 2.2 Consider Example 2.1. Find $P(S)$, $P(A \cap C)$, $P(A \cup C)$. Where S is the sample space.

Solution:

$$S = \{a, b, c, d\}$$

$$P(S) = P(a) + P(b) + P(c) + P(d)$$

$$P(S) = 0.1 + 0.3 + 0.5 + 0.1 = 1.0$$

$$A = \{a, b\} \Rightarrow P(A) = P(a) + P(b) = 0.1 + 0.3 = 0.4$$

$$C = \{d\} \Rightarrow P(C) = P(d) = 0.1$$

$$A \cap C = \Phi$$

$$P(A \cap C) = 0$$

$$P(A \cup C) = P(A) + P(C) - P(A \cap C) = P(A) + P(C)$$

$$P(A \cup C) = 0.4 + 0.1 = 0.5$$

2.3 Random Variable

A variable that associates a number with the outcome of a random experiment is referred to as a **random variable**. A random variable is a function that assigns a real number to each outcome in the sample space of a random experiment. A random variable is denoted by an uppercase letter such as X . After an experiment is conducted, the measured value of the random variable is denoted by a lowercase letter such as $x = 60$ MPa.

- A **discrete** random variable is a random variable with a finite (or countably infinite) range. Examples of **discrete** random variables: number of scratches on a surface, proportion of defective parts among 1000 tested. Consider a random variable N which represents number of cracks on a pipe. This can be treated as a discrete random variable whose sample space can take finite range such as 1, 2, 3, 4, etc. This is shown in Fig. 2.2.
- A **continuous** random variable is a random variable with an interval (either finite or infinite) of real numbers for its range. Examples of **continuous** random variables: Strength, length, pressure, temperature, time, voltage, weight. Consider a random variable S , which represents tensile strength of a material. This can be treated as a continuous random variable and can take any value on a real number range (e.g., 300–500 MPa). This is graphically represented as shown in Fig. 2.3.

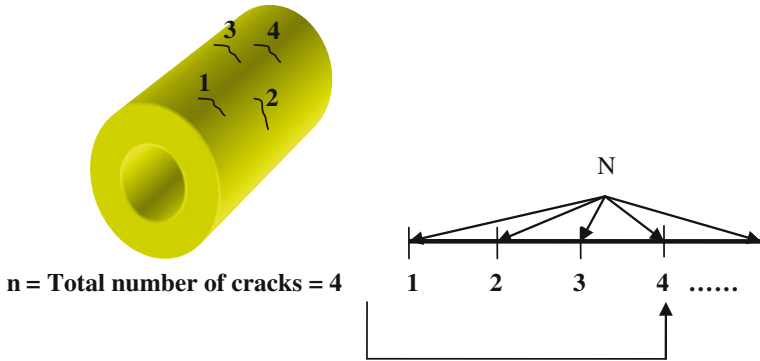


Fig. 2.2 Graphical representation of a discrete random variable

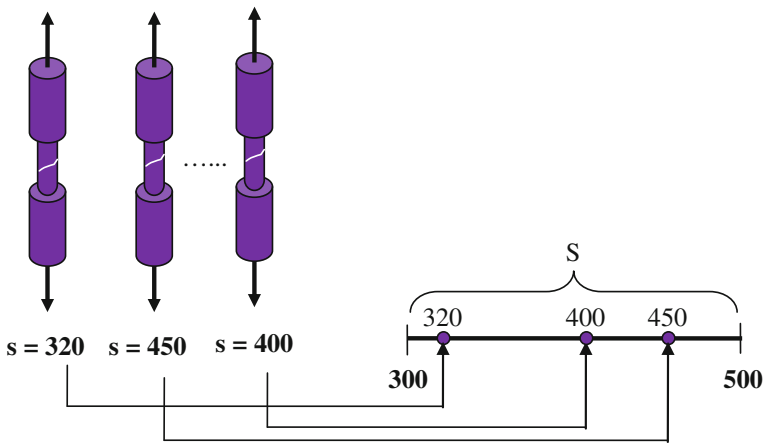


Fig. 2.3 Graphical representation of a continuous random variable

2.3.1 Discrete Random Variable

2.3.1.1 Probability Distribution

The **probability distribution** of a random variable X is a description of the probabilities associated with the possible values of X . For a discrete random variable, the distribution is often specified by just a list of the possible values along with the probability of each. In some cases, it is convenient to express the probability in terms of a formula.

2.3.1.2 Probability Mass Function

Consider a cantilever beam as shown in Fig. 2.4. The beam is subjected to the external loading in which the load is placed at discrete points. The load on the remaining points where load is not mentioned is zero (neglecting the self weight of the beam). Now the loading can be described by a function that specifies the mass at each of the discrete points. Similarly, for a discrete random variable X , its distribution can be described by a function that specifies the probability at each of the possible discrete values for X .

For a discrete random variable X with possible values x_1, x_2, \dots, x_n , a **probability mass function** is a function such that

$$(1) \quad f(x_i) \geq 0$$

$$(2) \quad \sum_{i=1}^n f(x_i) = 1 \tag{2.2}$$

$$(3) \quad f(x_i) = P(X = x_i)$$

Here $P(X = x)$ represents the probability that the discrete random variable X takes the value x . Probability mass function of a discrete random variable is shown in Fig. 2.5.

Example 2.3 X is a discrete random variable having the probabilities as given in Table 2.1 for different values of x , calculate (a) $P(X > -2)$ (b) $P(-1 \leq X \leq 1)$ (c) $P(X \leq -1 \text{ or } X = 2)$. Plot the probability mass function (Fig. 2.6).

Solution:

(a)	$P(X > -2)$	$= P(X = -1) + P(X = 0) + P(X = 1) + P(X = 2)$	7/8
(b)	$P(-1 \leq X \leq 1)$	$= P(X = -1) + P(X = 0) + P(X = 1)$	3/4
(c)	$P(X \leq -1 \text{ or } X = 2)$	$= P(X = -2) + P(X = -1) + P(X = 2)$	1/2

Fig. 2.4 Loadings at discrete points on a long thin beam

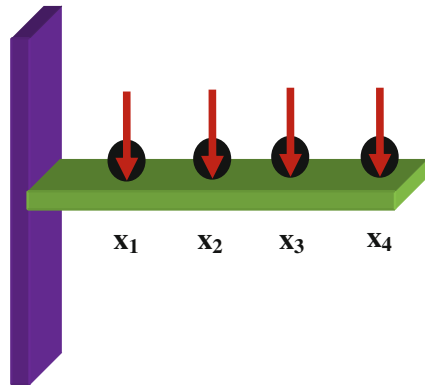


Fig. 2.5 Probability mass function of a discrete random variable

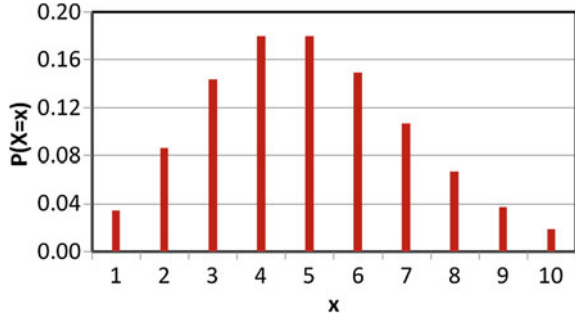
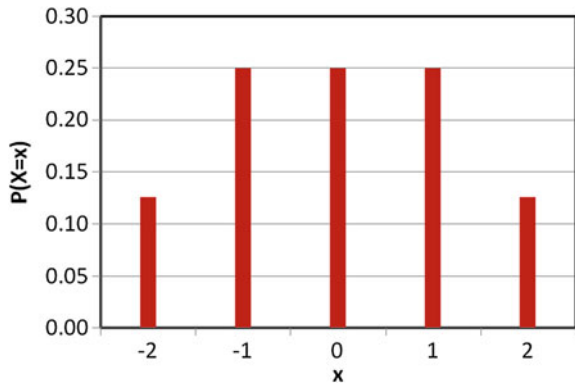


Table 2.1 Probability data of random variable X

x	-2	-1	0	1	2
$f(x)$	1/8	2/8	2/8	2/8	1/8

Fig. 2.6 Probability mass function of x



2.3.1.3 Cumulative Distribution Function

The cumulative distributive function (CDF) of variable X, denoted as $F(x)$, can be given as

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i) \tag{2.3}$$

For a discrete random variable X, $F(x)$ satisfies the following properties.

- (1) $F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$
- (2) $0 \leq F(x) \leq 1$
- (3) if $x \leq y$, then $F(x) \leq F(y)$ (2.4)

Fig. 2.7 Cumulative distribution function of a discrete random variable

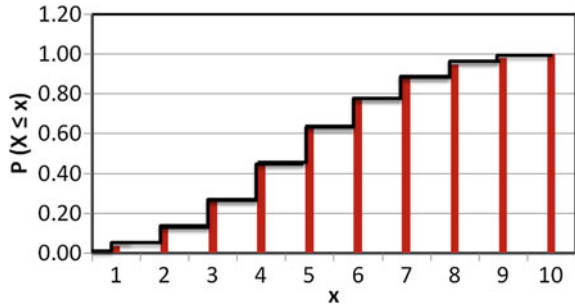
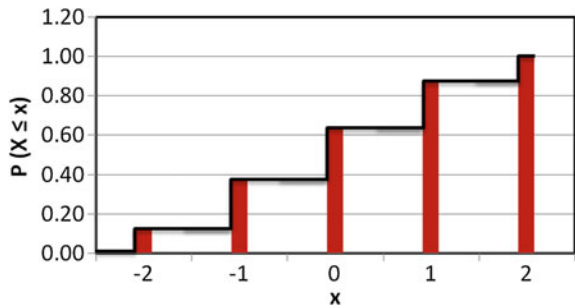


Fig. 2.8 Cumulative distribution function of x



The CDF represents the probability that the random variable assumes a value less than or equal to any specified value. Probability distribution function of a discrete random variable is shown in Fig. 2.7.

Example 2.4 Consider Example 2.3, plot cumulative distribution function (Fig. 2.8) and calculate

- (a) $P(X \leq -1)$ (b) $P(X \leq 0)$
- (c) $P(X \leq 1)$ (d) $P(X \leq 2)$

Solution:

(a)	$P(X \leq -1) = P(X = -2) + P(X = -1)$	3/8
(b)	$P(X \leq 0) = P(X = -2) + P(X = -1) + P(X = 0)$	5/8
(c)	$P(X \leq 1) = P(X = -2) + P(X = -1) + P(X = 0) + P(X = 1)$	7/8
(d)	$P(X \leq 2) = P(X = -2) + P(X = -1) + P(X = 0) + P(X = 1) + P(X = 2)$	1

2.3.1.4 Mean

The **mean** or **expected value** of the discrete random variable X , denoted as μ or $E(X)$, can be written as

$$\mu = E(X) = \sum_x xf(x) \tag{2.5}$$

The mean value generally represents the average of all the possible outcome of an experiment.

2.3.1.5 Variance

The variance of X , denoted as σ^2 or $V(X)$, is given as

$$\begin{aligned} \sigma^2 &= V(X) = E(X - \mu)^2 \\ \sigma^2 &= \sum_x (x - \mu)^2 f(x) = \sum_x x^2 f(x) - \mu^2 \end{aligned} \tag{2.6}$$

The standard deviation of X is represented as σ which is equal to the square root of the variance. The variance or standard deviation represents the spread in the data. If the standard deviation is zero means there is no spread in the data, that is irrespective of number of times one conducts the experiment all the time one observes a single outcome. Thus, in this case it is not necessary to apply probability theory, the outcome is deterministic in nature.

Example 2.5 Consider Example 2.3, calculate mean and variance for the given data as shown in Table 2.2

The mean and variance of the given data are obtained as 0 and 12/8, respectively.

Example 2.6 The probability mass function of a discrete random variable X is given as follows, find the following values.

Table 2.2 Calculation of mean and variance

x	$f(x)$	$xf(x)$	$(x - \mu)^2$	$(x - \mu)^2 f(x)$
-2	1/8	-2/8	4	4/8
-1	2/8	-2/8	1	2/8
0	2/8	0	0	0
1	2/8	2/8	1	2/8
2	1/8	2/8	4	4/8
$\mu = \sum xf(x) = 0$			$\sigma^2 = \sum (x - \mu)^2 f(x) = 12/8$	

$$f(x) = \frac{2x+1}{22} \quad x = 0, 1, 3, 5$$

(i) $P(X = 3)$	(ii) $P(X \leq 1)$
(iii) $P(1 \leq X < 4)$	(iv) $P(X > -5)$
(v) Mean(μ)	(vi) Variance(σ^2)

Solution:

$$P(X = 0) = \frac{2 \times 0 + 1}{22} = \frac{1}{22} \quad P(X = 1) = \frac{2 \times 1 + 1}{22} = \frac{3}{22}$$

$$P(X = 3) = \frac{2 \times 3 + 1}{22} = \frac{7}{22} \quad P(X = 5) = \frac{2 \times 5 + 1}{22} = \frac{11}{22}$$

(i) $P(X = 3) = \frac{7}{22}$

(ii) $P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{1}{22} + \frac{3}{22} = \frac{4}{22}$

(iii) $P(1 \leq X < 4) = P(X = 1) + P(X = 3) = \frac{3}{22} + \frac{7}{22} = \frac{10}{22}$

(iv) $P(X > -5) = P(X = 0) + P(X = 1) + P(X = 3) + P(X = 5)$
 $= \frac{1}{22} + \frac{3}{22} + \frac{7}{22} + \frac{11}{22} = \frac{22}{22} = 1$

(v) Mean(μ) $= \sum xf(x)$
 $= 0f(0) + 1f(1) + 3f(3) + 5f(5)$
 $= 0P(X = 0) + 1P(X = 1) + 3P(X = 3) + 5P(X = 5)$
 $\mu = 0 \times \frac{1}{22} + 1 \times \frac{3}{22} + 3 \times \frac{7}{22} + 5 \times \frac{11}{22} = \frac{79}{22} = 3.59$

(vi) Variance(σ^2) $= \sum (x - \mu)^2 f(x)$
 $\sigma^2 = (0 - 3.59)^2 f(0) + (1 - 3.59)^2 f(1) + (3 - 3.59)^2 f(3) + (5 - 3.59)^2 f(5)$
 $\sigma^2 = 12.89 \times \frac{1}{22} + 6.71 \times \frac{3}{22} + 0.35 \times \frac{7}{22} + 1.99 \times \frac{11}{22} = \frac{57.36}{22} = 2.61$

Some generally used discrete distributions and their corresponding parameters are briefly explained in the following subsections:

2.3.1.6 Uniform Distribution

- Each outcome of an experiment has equal probability of occurrence.
- Consider a discrete random variable X follows uniform distribution. Then each of the outcome in its range, say, x_1, x_2, \dots, x_n , has equal probability of occurrence. Then the probability mass function can be written as

$$f(x_i) = P(X = x_i) = \frac{1}{n} \quad (2.7)$$

- Let 1, 2, 3, 4, ..., 10 are the outcomes of a random experiment, then the probability of occurrence of each outcome will be

$$f(x_i) = P(X = x_i) = \frac{1}{10} = 0.1$$

- The probability mass function of random variable is shown in Fig. 2.9.

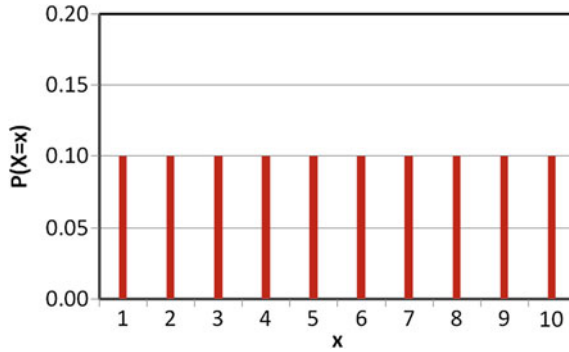


Fig. 2.9 Probability mass function of a discrete uniform random variable

- If X is a discrete uniform random variable on the consecutive integers $a, a + 1, a + 2, \dots, b$ for $a \leq b$. Then

$$\begin{aligned} \text{Mean} &= \frac{b + a}{2} \\ \text{Variance} &= \frac{(b - a + 1)^2 - 1}{12} \end{aligned} \tag{2.8}$$

2.3.1.7 Binomial Distribution

A random experiment consists of n Bernoulli trials such that

- The trials are considered to be independent
- There are only two outcomes of each trial either “success” or “failure”
- The probability of a success in each trial, denoted as p , is considered as constant
- If a random variable X follows binomial distribution then probability mass function of X can be given as

$$f(x) = \binom{n}{x} p^x (1 - p)^{n-x} \quad x = 0, 1, \dots, n \tag{2.9}$$

- where n is the total number of trials, x is the number of success, p is the probability of success, $(1 - p)$ is the probability of failure. For the case of $n = 10$, $p = 0.7$ the probability mass function is shown in Fig. 2.10.
- The mean and variance of binomial distribution can be given as follows:

$$\begin{aligned} \text{Mean} &= np \\ \text{Variance} &= np(1 - p) \end{aligned} \tag{2.10}$$

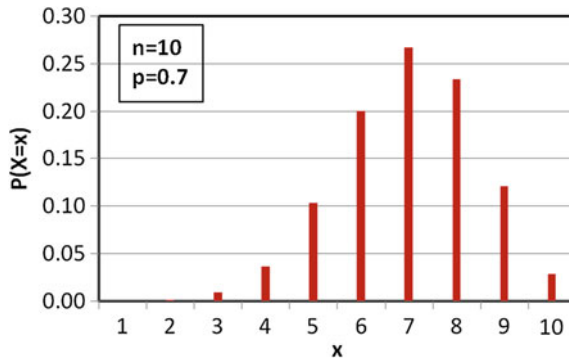


Fig. 2.10 Probability mass function of binomial random variable

2.3.1.8 Poisson Distribution

- Consider a random variable X follows Poisson distribution then probability mass function of X can be given as

$$f(x) = \frac{e^{-\lambda t} (\lambda t)^x}{x!} \quad x = 0, 1, 2, \dots \quad (2.11)$$

- where x is the total number of occurrences of an event in a given interval of time t . λ is the rate of occurrence of events.
- Following are the characteristics of Poisson distribution:
 - the probability of occurrence of more than one event in a small interval of time is zero,
 - The rate of occurrence of events is constant
 - The probability of occurrence of one event is independent of other event
- The mean and variance of Poisson distribution are same and are given as

$$\begin{aligned} \text{Mean} &= \lambda t \\ \text{Variance} &= \lambda t \end{aligned} \quad (2.12)$$

- The probability mass function is shown in Fig. 2.11.

2.3.2 Continuous Random Variable

2.3.2.1 Probability Density Function

Consider a cantilever beam as shown in Fig. 2.12. The beam is subjected to the external loading (Kg/m) in which the load is uniformly distributed along the length

Fig. 2.11 Probability mass function of Poisson distribution

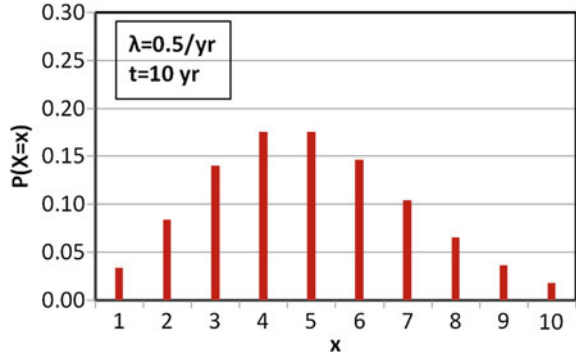
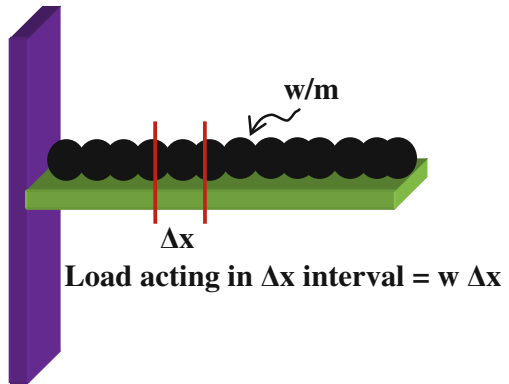


Fig. 2.12 Density function of loading on a long thin beam

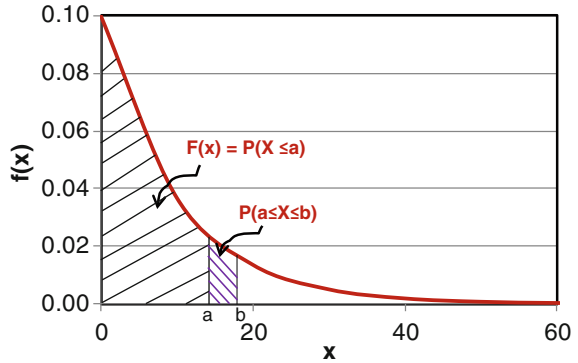


of the beam. From this it is not possible to find the amount of loading that is acting at a point. But if one considers the small section of the beam (of length Δx), then one can find easily the total loading acting in that section. It is simply the summation of all the loadings in that interval. Now the loading can be described by a function that specifies the density, i.e., load per unit length of the beam. Similarly, for a continuous random variable X , its distribution can be described by a function that specifies the probability per unit interval of x , i.e., probability density function. For a continuous random variable X , a probability density function (PDF) is a function such that

$$\begin{aligned}
 (1) \quad & f(x_i) \geq 0 \\
 (2) \quad & \int_{-\infty}^{\infty} f(x) dx = 1 \\
 (3) \quad & P(a \leq X \leq b) = \int_a^b f(x) dx = \text{area under } f(x) \text{ from } a \text{ to } b
 \end{aligned}
 \tag{2.13}$$

for any a and b

Fig. 2.13 Probability density function of a continuous random variable



A probability density function provides a simple description of the probabilities associated with a random variable. Figure 2.13 shows the probability density function of a continuous random variable.

2.3.2.2 Cumulative Distribution Function

The cumulative distributive function (CDF) of a continuous random variable X can be written as follows:

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(x)dx \quad \text{for } -\infty \leq x \leq \infty \quad (2.14)$$

As shown in Fig. 2.13 CDF is the area under the PDF curve in the interval between lower limit of the random variable to the specified x value. The probability density function of a continuous random variable can be determined from the cumulative distribution function by differentiation as follows

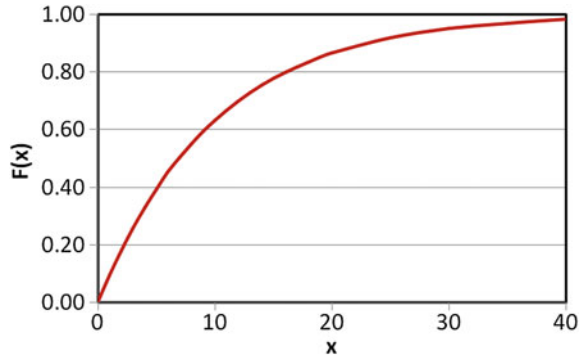
$$f(x) = \frac{dF(x)}{dx} \quad (2.15)$$

The CDF of a continuous random variable is shown in Fig. 2.14.

2.3.2.3 Mean

Suppose X is a continuous random variable with probability density function $f(x)$. The mean or expected value of X can be denoted as $E(X)$ and mathematically can be represented as

Fig. 2.14 Cumulative distribution function of a continuous random variable



$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx \tag{2.16}$$

2.3.2.4 Variance

The variance of X can be denoted as σ^2 or $V(X)$ and is given as follows:

$$\begin{aligned} \sigma^2 &= V(X) = E(X - \mu)^2 \\ &= \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \\ \sigma^2 &= \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2 \end{aligned} \tag{2.17}$$

The standard deviation of X can be denoted as σ and is equivalent to square root of variance.

Example 2.7 Consider X as a continuous random variable whose probability density function is given as $f(x) = 1/18 x^2$ for $-3 \leq X \leq 3$. Determine the following probabilities.

- (a) $P(X > 0)$
- (b) $P(X > 1)$
- (c) $P(-2 \leq X \leq 2)$
- (d) $P(X < -2)$
- (e) Determine x such that $P(X > x) = 0.05$
- (f) Mean
- (g) Variance

Solution:

$$\begin{aligned}
 f(x) &= \frac{1}{18}x^2 & -3 \leq X \leq 3 \\
 \text{(a) } P(X > 0) &= \int_0^3 f(x)dx = \int_0^3 \frac{1}{18}x^2 dx \\
 &= \left[\frac{1}{18} \frac{x^3}{3} \right]_0^3 = \frac{1}{2} \\
 \text{(b) } P(X > 1) &= \int_1^3 f(x)dx = \int_1^3 \frac{1}{18}x^2 dx \\
 &= \left[\frac{1}{18} \frac{x^3}{3} \right]_1^3 = \frac{26}{54} \\
 \text{(c) } P(-2 \leq X \leq 2) &= \int_{-2}^2 f(x)dx = \int_{-2}^2 \frac{1}{18}x^2 dx \\
 &= \left[\frac{1}{18} \frac{x^3}{3} \right]_{-2}^2 = \frac{16}{54} \\
 \text{(d) } P(X < -2) &= \int_{-3}^{-2} f(x)dx = \int_{-3}^{-2} \frac{1}{18}x^2 dx \\
 &= \left[\frac{1}{18} \frac{x^3}{3} \right]_{-3}^{-2} = \frac{19}{54} \\
 \text{(e) } P(X > x) &= 0.05 \\
 \int_x^3 f(x)dx &= 0.05 \\
 \int_x^3 \frac{1}{18}x^2 dx &= 0.05 \\
 \left[\frac{1}{18} \frac{x^3}{3} \right]_x^3 &= 0.05 \\
 \frac{27-x^3}{54} &= 0.05 \\
 x^3 &= 27 - 54 \times 0.05 = 24.3 \\
 x &= (24.3)^{1/3} = 2.8964 \\
 \text{(f) Mean} = \mu &= E(X) = \int_{-\infty}^{\infty} xf(x)dx \\
 \mu &= \int_{-3}^3 xf(x)dx = \int_{-3}^3 x \frac{1}{18}x^2 dx \\
 &= \int_{-3}^3 \frac{1}{18}x^3 dx \\
 \mu &= \left[\frac{1}{18} \frac{x^4}{4} \right]_{-3}^3 = \frac{9}{4} = 2.25 \\
 \text{(g) Variance} = \sigma^2 &= \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 \\
 \sigma^2 &= \int_{-3}^3 x^2 f(x)dx - \mu^2 \\
 &= \int_{-3}^3 x^2 \frac{1}{18}x^2 dx - \mu^2 = \int_{-3}^3 \frac{1}{18}x^4 dx - \mu^2 \\
 \sigma^2 &= \left[\frac{1}{18} \frac{x^5}{5} \right]_{-3}^3 - \left(\frac{9}{4} \right)^2 = \frac{586}{90} - \frac{81}{16} = 1.4486
 \end{aligned}$$

Important note: Here one should understand the difference between probability mass function (PMF) and PDF. In PMF one can calculate probability that the

random variable takes a particular value. Whereas, in case of PDF it is not possible to find the probability that the random variable takes a particular value. Since the continuous random variables can take any value in the real number range it is very difficult to find the probability at a particular value. But one can find the probability value for a given interval. That is one can find what is the probability that the random variable will take a value in between certain range or interval. For example in case of continuous random variable one can calculate $P(a \leq X \leq b)$, which is nothing but the area under the PDF curve in between this range. But it is not possible to calculate $P(X = a)$. In this case it will be simply **Zero**. These two can be mathematically represented as follows:

$$P(a \leq X \leq b) = \int_a^b f(x) dx \tag{2.18}$$

$$P(X = a) = 0$$

‘PDF’ simply represents probability per unit interval. The total area under any PDF curve will be always one. Similarly, the summation of all the probabilities in PMF will be one. Some generally used continuous distributions are briefly discussed in the following subsections.

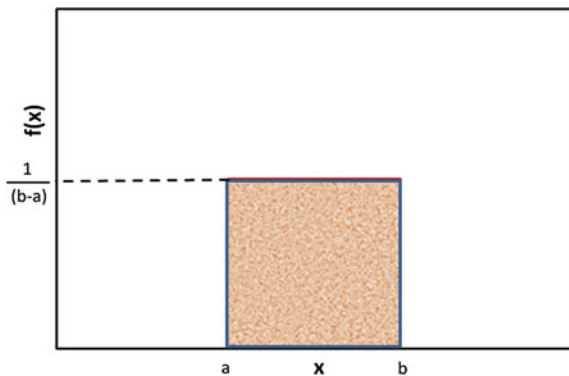
2.3.2.5 Uniform Distribution

- Consider a continuous random variable X follows uniform distribution. Then each of the outcome in its range, say, from a to b , will have equal probability density value. Then the probability density function can be written as

$$f(x_i) = \frac{1}{b - a} \quad a \leq X \leq b \tag{2.19}$$

- The probability density function of X is shown in Fig. 2.15.

Fig. 2.15 Probability density function of uniform distribution



- The cumulative distribution function of X can be obtained as follows

$$\begin{aligned}
 F_X(x) &= \int_{-\infty}^x f(x)dx \\
 &= \int_a^x f(x)dx = \int_a^x \frac{1}{b-a}dx \\
 &= \frac{1}{b-a} \int_a^x dx \\
 &= \frac{1}{b-a} [x]_a^x \\
 F_X(x) &= \frac{x-a}{b-a}
 \end{aligned} \tag{2.20}$$

- The mean and variance of X can be calculated as follows:

$$\begin{aligned}
 \text{Mean} = \mu = E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\
 \mu &= \int_a^b xf(x)dx = \int_a^b x \frac{1}{b-a}dx \\
 &= \frac{1}{b-a} \int_a^b xdx \\
 &= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b \\
 \mu &= \frac{b^2 - a^2}{2(b-a)} \\
 \mu &= \frac{(b+a)(b-a)}{2(b-a)} \\
 \mu &= \frac{b+a}{2}
 \end{aligned} \tag{2.21}$$

$$\begin{aligned}
 \text{Variance} = \sigma^2 &= \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2 \\
 \sigma^2 &= \int_a^b x^2 f(x) dx - \mu^2 \\
 &= \int_a^b x^2 \frac{1}{b-a} dx - \mu^2 = \frac{1}{b-a} \int_a^b x^2 dx - \mu^2 \\
 &= \frac{1}{b-a} \left[\frac{x^3}{3} \right]_a^b - \left(\frac{b+a}{2} \right)^2 = \frac{b^3 - a^3}{3(b-a)} - \left(\frac{b+a}{2} \right)^2 \\
 \sigma^2 &= \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} - \left(\frac{b+a}{2} \right)^2 \\
 \sigma^2 &= \frac{b^2 + ab + a^2}{3} - \frac{b^2 + 2ab + a^2}{4} \\
 &= \frac{b^2 - 2ab + a^2}{12} \\
 \sigma^2 &= \frac{(b-a)^2}{12} \tag{2.22}
 \end{aligned}$$

2.3.2.6 Exponential Distribution

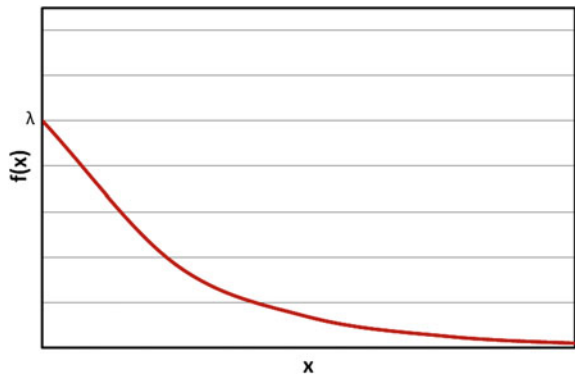
- Consider a continuous random variable X follows exponential distribution. Then the probability density function of X can be written as

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } 0 \leq x < \infty \tag{2.23}$$

where λ is parameter of the distribution

- The probability density function of X is shown in Fig. 2.16.

Fig. 2.16 Probability density function of exponential distribution



- The CDF of exponential distribution function can be calculated as

$$\begin{aligned}
 F_X(x) &= P(X \leq x) = \int_0^x f(x) dx \quad \text{for } 0 \leq X \leq \infty \\
 &= \int_0^x \lambda e^{-\lambda x} dx \\
 &= \lambda \int_0^x e^{-\lambda x} dx \\
 &= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^x \\
 &= -[e^{-\lambda x}]_0^x \\
 &= -[e^{-\lambda x} - 1] \\
 F_X(x) &= 1 - e^{-\lambda x} \tag{2.24}
 \end{aligned}$$

- The mean and variance of the exponential distribution are given as

$$\begin{aligned}
 \text{Mean} = \mu &= E(X) = \int_{-\infty}^{\infty} xf(x) dx \\
 &= \int_0^{\infty} xf(x) dx = \int_0^{\infty} x \lambda e^{-\lambda x} dx \\
 &= \lambda \int_0^{\infty} x e^{-\lambda x} dx \\
 &= \lambda \left[x \frac{e^{-\lambda x}}{-\lambda} \right]_0^{\infty} - \lambda \int_0^{\infty} \frac{e^{-\lambda x}}{-\lambda} dx \\
 &= \lambda [0] - \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^{\infty} \\
 \text{Mean} &= \frac{1}{\lambda} \tag{2.25}
 \end{aligned}$$

$$\begin{aligned}
 \text{Variance} = \sigma^2 &= \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2 \\
 &= \int_0^{\infty} x^2 f(x) dx - \mu^2 \\
 &= \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx - \mu^2 = \lambda \int_0^{\infty} x^2 e^{-\lambda x} dx - \mu^2 \\
 &= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 \\
 \text{Variance} = \sigma^2 &= \frac{1}{\lambda^2} \\
 \sigma &= \frac{1}{\lambda}
 \end{aligned} \tag{2.26}$$

2.3.2.7 Normal Distribution

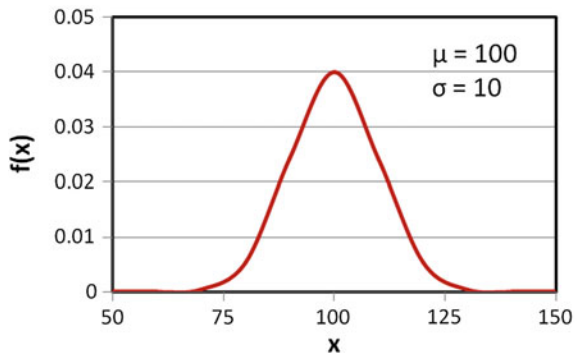
- Consider a continuous random variable X follows normal distribution. Then the probability density function of X can be written as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \leq X \leq \infty \tag{2.27}$$

where μ, σ are the parameters of the distribution

- The probability density function of X is shown in Fig. 2.17.

Fig. 2.17 Probability density function of normal distribution



- The mean and variance of the normal distribution are given as

$$\text{Mean} = \mu$$

$$\text{Variance} = \sigma^2$$

$$\text{Standard deviation} = \sigma$$

- Here mean (μ) represents the location parameter and standard deviation (σ) represents the scaling parameter. For a given μ and σ values the PDF of X is shown in Fig. 2.17. The distribution is symmetric about the mean value.
- As the mean value changes the PDF curve also moves either to the left or right as shown in Fig. 2.18. That means the location of the curve is shifting without changing the shape of the curve.
- As the standard deviation changes the scaling of the curve changes without changing the mean value. Here scaling means height and width of the curve changes.
- For the same value of mean, when the standard deviation increases the width of the PDF curve increases and height of the curve decreases which represents there is more spread in the data. Similarly, when the standard deviation decreases width of the curve reduces and height of the curve increases which represents there is less spread in the data. This can be seen in Fig. 2.19.

Fig. 2.18 Shifting of PDF curve with change in mean value

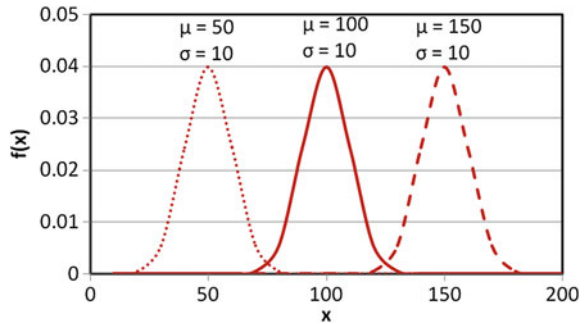
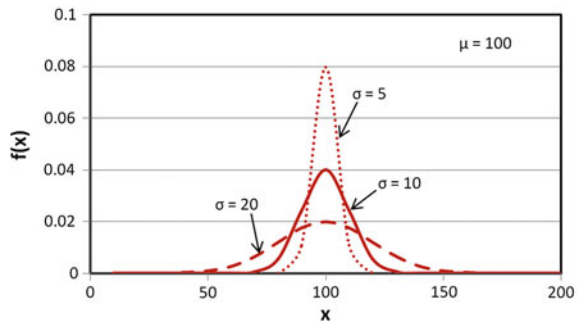


Fig. 2.19 Scaling of PDF curve with change in standard deviation value

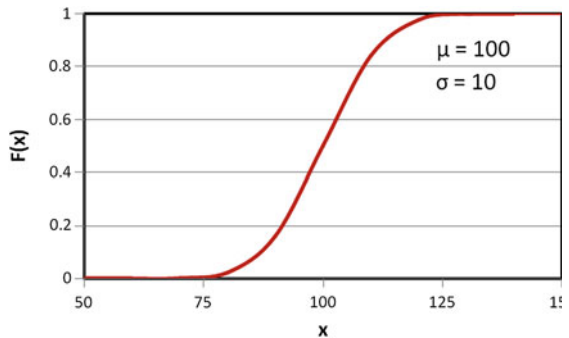


- As the standard deviation approaches zero, that is, there is no spread in the data, then the PDF curve looks like simply a straight line at the mean value. That means irrespective of the number of experiments the outcome is same, that is, its mean value.
- The cumulative distribution function of normal random variable can be derived as follows:

$$\begin{aligned}
 F_X(x) &= P(X \leq x) = \int_{-\infty}^x f(x)dx \quad \text{for } -\infty \leq X \leq \infty \\
 &= \int_{-\infty}^x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
 &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
 \text{Let } z &= \frac{x-\mu}{\sigma} \Rightarrow \frac{dz}{dx} = \frac{1}{\sigma} \\
 F_X(x) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{z^2}{2}} \sigma dz \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{z^2}{2}} dz \\
 F_X(x) &= \Phi(z) = \Phi\left(\frac{x-\mu}{\sigma}\right) = P(Z \leq z) = F_Z(z) \tag{2.28}
 \end{aligned}$$

- The cumulative distribution function of X is shown in Fig. 2.20.

Fig. 2.20 Cumulative distribution function of normal distribution



- Here z is called standard normal random variable whose mean and standard deviation can be calculated as

$$z = \frac{x - \mu}{\sigma}$$

$$\mu_z = E\left(\frac{x - \mu}{\sigma}\right)$$

$$= \frac{E(x) - E(\mu)}{\sigma}$$

$$= \frac{\mu - \mu}{\sigma}$$

$$\mu_z = 0$$

$$z = \frac{x - \mu}{\sigma}$$

$$V_z = V\left(\frac{x - \mu}{\sigma}\right)$$

$$= \frac{1}{\sigma^2} V(x - \mu)$$

$$= \frac{1}{\sigma^2} [V(x) - V(\mu)]$$

$$V_z = \frac{1}{\sigma^2} [\sigma^2 - 0]$$

$$\sigma_z^2 = 1 \Rightarrow \sigma_z = 1$$

- $\Phi(z)$ represents the cumulative distribution function value of z . For calculating CDF values there are standard normal tables existed from which one can directly get the CDF value for a given 'z' value.
- The total area under the standard normal density function also equals to one. Depending on the σ limits the area under the PDF curve will change and is tabulated in the following Table 2.3 and is also shown in Fig. 2.21.

Table 2.3 Area under PDF curve for different σ limits

σ limits	Range	Area (%)
1 σ	$(\mu - 1\sigma)$ to $(\mu + 1\sigma)$	67.73
2 σ	$(\mu - 2\sigma)$ to $(\mu + 2\sigma)$	95.00
3 σ	$(\mu - 3\sigma)$ to $(\mu + 3\sigma)$	99.73

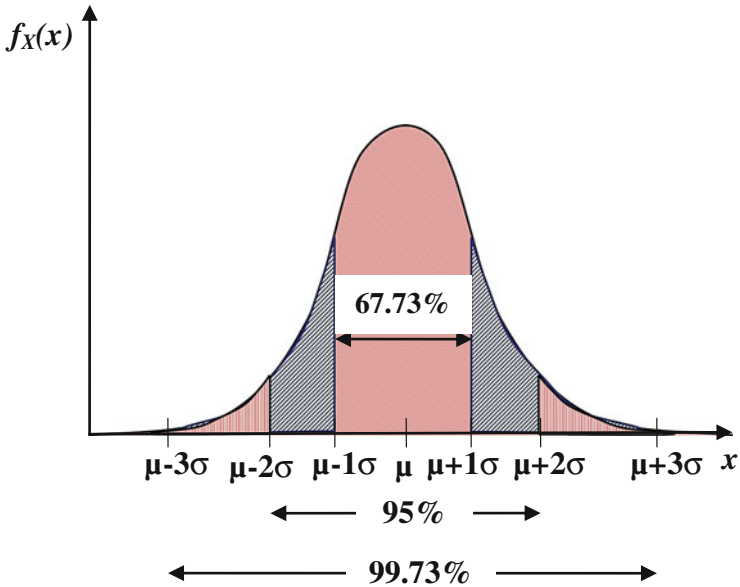


Fig. 2.21 Area under PDF curve for different σ limits

2.3.2.8 Lognormal Distribution

- Consider a continuous random variable Y follows normal distribution and the parameters of the distribution are μ and σ . If Y follows normal distribution then $\exp(Y)$ follows lognormal distribution. This is mathematically given as follows:

Y Normal
 $\exp(Y) = e^Y$ Lognormal
 Let $X = e^Y \Rightarrow Y = \ln(X)$
 Y Normal
 X Lognormal

- The probability density function of Y can be given as follows:

$$f(y) = \frac{1}{\sigma_y \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{y - \mu_y}{\sigma_y} \right)^2} \quad -\infty \leq Y \leq \infty \quad (2.29)$$

- Now the PDF of X can be obtained by applying transformation rule as follows:

$$\begin{aligned}
 f(x)dx &= f(y)dy \\
 f(x) &= f(y) \frac{dy}{dx} \\
 y &= \ln(x) \\
 \frac{dy}{dx} &= \frac{1}{x} \\
 f(x) &= \frac{1}{x} f(y) \\
 f(x) &= \frac{1}{\sigma_y \sqrt{2\pi} x} e^{-\frac{1}{2} \left(\frac{y-\mu_y}{\sigma_y} \right)^2} \tag{2.30}
 \end{aligned}$$

X limits

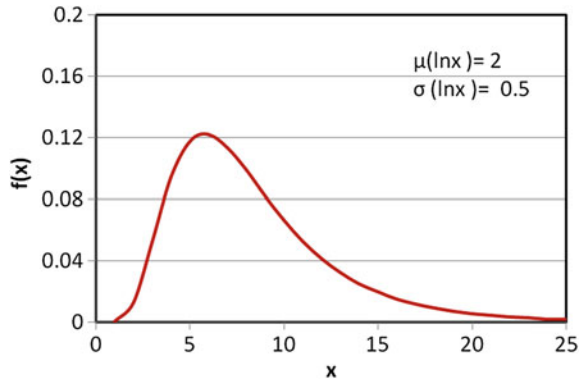
$$\begin{aligned}
 y = \ln(x) &\Rightarrow x = e^y \\
 y = -\infty &\Rightarrow x = e^{-\infty} = 0 \\
 y = \infty &\Rightarrow x = e^{\infty} = \infty
 \end{aligned}$$

By substituting $y = \ln(x)$

$$f(x) = \frac{1}{\sigma_y \sqrt{2\pi} x} e^{-\frac{1}{2} \left(\frac{\ln x - \mu_y}{\sigma_y} \right)^2} \quad 0 \leq X \leq \infty$$

- The PDF of lognormal distribution is shown in Fig. 2.22.

Fig. 2.22 Probability density function of lognormal distribution

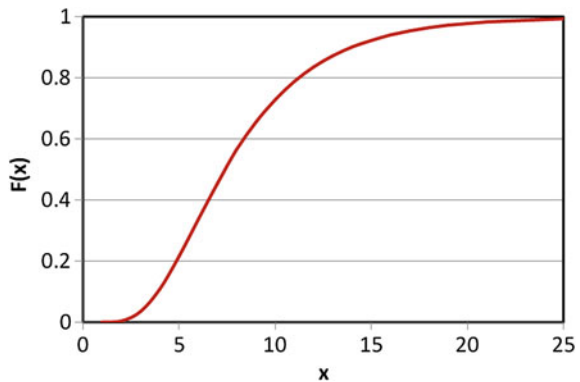


- The cumulative distribution of lognormal distribution can be obtained in the similar lines of normal distribution as follows:

$$\begin{aligned}
 F_X(x) &= P(X \leq x) = \int_0^x f(x) dx \quad \text{for } 0 \leq X \leq \infty \\
 &= \int_0^x \frac{1}{\sigma_y \sqrt{2\pi x}} e^{-\frac{1}{2} \left(\frac{\ln x - \mu_y}{\sigma_y} \right)^2} dx \\
 &= \frac{1}{\sigma_y \sqrt{2\pi}} \int_0^x \frac{1}{x} e^{-\frac{1}{2} \left(\frac{\ln x - \mu_y}{\sigma_y} \right)^2} dx \\
 \text{Let } z &= \frac{\ln x - \mu_y}{\sigma_y} \Rightarrow \frac{dz}{dx} = \frac{1}{\sigma_y x} \tag{2.31} \\
 F_X(x) &= \frac{1}{\sigma_y \sqrt{2\pi}} \int_{-\infty}^z \frac{1}{x} e^{-\frac{z^2}{2}} \sigma_y x dz \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{z^2}{2}} dz \\
 F_X(x) &= \Phi(z) = \Phi\left(\frac{\ln x - \mu_y}{\sigma_y}\right) = P(Z \leq z) = F_Z(z)
 \end{aligned}$$

- The CDF of lognormal distribution is shown in Fig. 2.23.
- As it is seen from the probability density function lognormal distribution is defined with the parameters of the normal distribution parameters, i.e., the parameters of the random variable Y (μ_Y, σ_Y).
- Now the question comes what are the parameters of lognormal distribution?

Fig. 2.23 Cumulative distribution function of lognormal distribution



- The lognormal distribution parameters are also similar to normal distribution parameters such as μ_X , σ_X . These parameters can be estimated as follows. We know that

$$X = e^Y$$

$$E(X) = E(e^Y)$$

From the definition of characteristic function

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

$$E(e^Y) = \int_0^{\infty} e^y f(y)dy$$

$$\text{But } f(y) = \frac{1}{\sigma_y \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{y-\mu_y}{\sigma_y} \right)^2}$$

$$E(e^Y) = \int_{-\infty}^{\infty} e^y \frac{1}{\sigma_y \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{y-\mu_y}{\sigma_y} \right)^2} dy$$

$$= \frac{1}{\sigma_y \sqrt{2\pi}} \int_{-\infty}^{\infty} e^y e^{-\frac{1}{2} \left(\frac{y-\mu_y}{\sigma_y} \right)^2} dy$$

$$\text{Let } z = \frac{y - \mu_y}{\sigma_y} \Rightarrow \frac{dz}{dy} = \frac{1}{\sigma_y}$$

$$y = z\sigma_y + \mu_y$$

$$E(e^Y) = \frac{1}{\sigma_y \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{z\sigma_y + \mu_y} e^{-\frac{z^2}{2}} \sigma_y dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mu_y} e^{z\sigma_y - \frac{z^2}{2}} dz$$

$$E(e^Y) = \frac{1}{\sqrt{2\pi}} e^{\mu_y} \int_{-\infty}^{\infty} e^{z\sigma_y - \frac{z^2}{2}} dz$$

$$= e^{\mu_y} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{z\sigma_y - \frac{z^2}{2}} dz$$

$$\begin{aligned}
 z\sigma_y - \frac{z^2}{2} &= \frac{2z\sigma - z^2}{2} \\
 &= -\frac{(z^2 - 2z\sigma)}{2} \\
 &= -\frac{1}{2}(z^2 - 2z\sigma + \sigma^2 - \sigma^2) \\
 z\sigma_y - \frac{z^2}{2} &= -\frac{1}{2}[(z - \sigma)^2 - \sigma^2]
 \end{aligned}$$

$$\begin{aligned}
 E(e^Y) &= \frac{1}{\sqrt{2\pi}} e^{\mu_y} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(z-\sigma)^2 - \sigma^2]} dz \\
 &= e^{\mu_y} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-\sigma)^2}{2}} e^{\frac{\sigma^2}{2}} dz \\
 &= e^{\mu + \frac{\sigma^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-\sigma)^2}{2}} dz
 \end{aligned}$$

Let $u = z - \sigma$

$$\frac{du}{dz} = 1$$

$$f(u)du = f(z)dz$$

$$f(u) = f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$\int_{-\infty}^{\infty} f(u)du = \int_{-\infty}^{\infty} f(z)dz = 1$$

$$E(e^Y) = e^{\mu + \frac{\sigma^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du$$

$$E(e^Y) = e^{\mu + \frac{\sigma^2}{2}} \tag{2.32}$$

- **Standard deviation (σ_X)**

we know that

$$\begin{aligned} V(X) &= E[(X - \mu_X)^2] \\ &= E(X^2) - \mu_X^2 \end{aligned}$$

But $X = e^Y$

$$\begin{aligned} V(X) &= E(e^{2Y}) - \mu_X^2 \\ \mu_X &= E(X) = E(e^Y) = e^{\mu + \frac{\sigma^2}{2}} \end{aligned}$$

$$\begin{aligned} V(X) &= E(e^{2Y}) - \left(e^{\mu + \frac{\sigma^2}{2}}\right)^2 \\ &= E(e^{2Y}) - e^{2\left(\mu + \frac{\sigma^2}{2}\right)} \\ &= E(e^{2Y}) - e^{2\mu + \sigma^2} \end{aligned}$$

- As we have $E(e^Y) = e^{\mu + \frac{\sigma^2}{2}}$
- Similarly $E(e^{tY}) = e^{\mu t + \frac{t^2 \sigma^2}{2}}$
- If $t = 2$ then $E(e^{2Y}) = e^{2\mu + \frac{4\sigma^2}{2}} = e^{2(\mu + \sigma^2)}$

$$\begin{aligned} V(e^Y) &= e^{2(\mu + \sigma^2)} - e^{2\mu + \sigma^2} \\ &= e^{2\mu + \sigma^2} \left[\frac{e^{2(\mu + \sigma^2)}}{e^{2\mu + \sigma^2}} - 1 \right] \\ &= e^{2\mu + \sigma^2} \left[e^{2(\mu + \sigma^2) - 2\mu - \sigma^2} - 1 \right] \\ V(e^Y) &= e^{2\mu + \sigma^2} \left[e^{\sigma^2} - 1 \right] \end{aligned}$$

$$\begin{aligned} E(e^Y) &= e^{\mu + \frac{\sigma^2}{2}} \\ \mu_X &= e^{\mu + \frac{\sigma^2}{2}} \end{aligned}$$

$$\begin{aligned} V(e^Y) &= e^{2\mu + \sigma^2} \left[e^{\sigma^2} - 1 \right] \\ \sigma_X^2 &= e^{2\mu + \sigma^2} \left[e^{\sigma^2} - 1 \right] \end{aligned}$$

$$\sigma_X = \sqrt{e^{2\left(\mu + \frac{\sigma^2}{2}\right)} \left[e^{\sigma^2} - 1 \right]}$$

$$\begin{aligned}
\sigma_X &= e^{\left(\mu + \frac{\sigma^2}{2}\right)} \sqrt{e^{\sigma^2} - 1} \\
\sigma_X &= \mu_X \sqrt{e^{\sigma^2} - 1} \\
\mu_X &= e^{\mu_Y + \frac{\sigma_Y^2}{2}} \\
\sigma_X &= \mu_X \sqrt{e^{\sigma_Y^2} - 1}
\end{aligned} \tag{2.33}$$

- Similarly, if we know μ_X and σ_X we can find out μ_Y and σ_Y as follows:

$$\begin{aligned}
\sigma_Y^2 &= \ln \left[\left(\frac{\sigma_X}{\mu_X} \right)^2 + 1 \right] \\
\mu_Y &= \ln \mu_X - \frac{1}{2} \sigma_Y^2 \quad \text{where } Y = \ln X
\end{aligned} \tag{2.34}$$

2.3.2.9 Weibull Distribution

- Consider the random variable X follows Weibull distribution [4]. Then the probability density function of X can be written as

$$f(x) = \frac{\beta}{\alpha} \left(\frac{x}{\alpha} \right)^{\beta-1} e \left[-\left(\frac{x}{\alpha} \right)^\beta \right], \quad \text{for } x > 0 \tag{2.35}$$

- where α is the scale parameter and β is the shape parameter.
- The PDF of Weibull distribution is shown in Fig. 2.24. For different values of β the shape of the PDF curve changes as shown in Fig. 2.24.
- When $\beta = 1$ Weibull distribution will be similar to exponential distribution as shown below

$$f(x) = \frac{\beta}{\alpha} \left(\frac{x}{\alpha} \right)^{\beta-1} e \left[-\left(\frac{x}{\alpha} \right)^\beta \right], \quad \text{for } x > 0$$

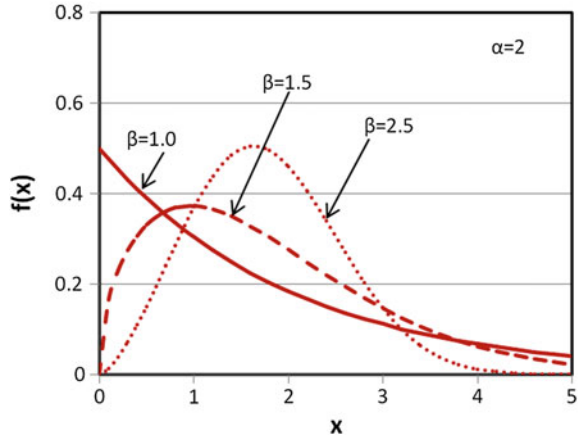
$$\text{when } \beta = 1$$

$$f(x) = \frac{1}{\alpha} \left(\frac{x}{\alpha} \right)^{1-1} e \left[-\left(\frac{x}{\alpha} \right)^1 \right] = \frac{1}{\alpha} e \left[-\left(\frac{x}{\alpha} \right) \right]$$

$$\text{let } \lambda = \frac{1}{\alpha}$$

$$f(x) = \lambda e^{-\lambda x}$$

Fig. 2.24 Probability density function of Weibull distribution



- The cumulative distribution can be obtained as

$$F_X(x) = P(X \leq x) = \int_0^x f(x) dx \quad \text{for } 0 \leq X \leq \infty$$

$$= \int_0^x \frac{\beta}{\alpha} \left(\frac{x}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^\beta} dx$$

$$= \frac{\beta}{\alpha} \frac{1}{\alpha^{\beta-1}} \int_0^x x^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^\beta} dx$$

Let $z = \left(\frac{x}{\alpha}\right)^\beta \Rightarrow \frac{dz}{dx} = \frac{1}{\alpha^\beta} \beta x^{\beta-1}$

$$dx = \frac{\alpha^\beta}{\beta x^{\beta-1}} dz$$

$$F_X(x) = \frac{\beta}{\alpha^\beta} \int_0^z x^{\beta-1} e^{-z} \frac{\alpha^\beta}{\beta x^{\beta-1}} dz$$

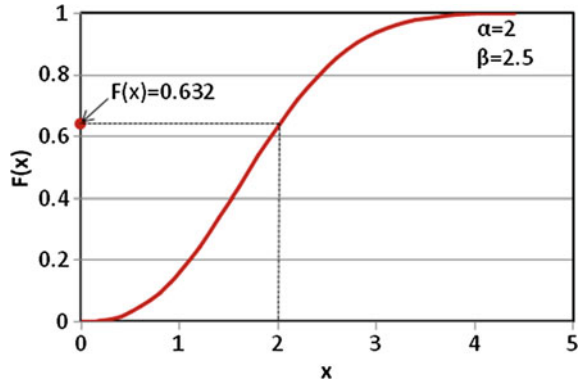
$$= \int_0^z e^{-z} dz$$

$$= \left[\frac{e^{-z}}{-1} \right]_0^z = 1 - e^{-z}$$

$$F_X(x) = 1 - e^{-\left(\frac{x}{\alpha}\right)^\beta} \tag{2.36}$$

- The CDF of Weibull distribution is shown in Fig. 2.25.

Fig. 2.25 Cumulative distribution function of Weibull distribution



- When $F(x) = 0.632$ then

$$F_X(x) = 1 - e^{-\left(\frac{x}{\alpha}\right)^\beta} = 0.632$$

$$e^{-\left(\frac{x}{\alpha}\right)^\beta} = 1 - 0.632 = 0.368$$

$$e^{\left(\frac{x}{\alpha}\right)^\beta} = \frac{1}{0.368} = 2.7174$$

$$\ln\left(e^{\left(\frac{x}{\alpha}\right)^\beta}\right) = \ln(2.7174) = 1$$

$$\left(\frac{x}{\alpha}\right)^\beta = 1$$

$$\frac{x}{\alpha} = 1^{\frac{1}{\beta}} = 1$$

$$x = \alpha$$

when CDF = $F(x) = 0.632$

$$\alpha = x$$

- This can be seen in the CDF curve as shown in Fig. 2.25.

2.4 The Reliability Function

In this section the mathematical representation of reliability as a function of time is derived. As was discussed earlier reliability is a probabilistic concept and is also associated with time. To derive the reliability function first one should know what type of random variable that will enter into the analysis. Since reliability is also associated with the functioning of component it is not possible to tell when exactly the component will cease to function or fails. Hence, the times to failure of a component is a random variable and this can take any value in the positive real

number range. So in the derivation of reliability function time to failure of a component is considered as a random variable and it is continuous in nature.

Consider the time to failure of a component is a continuous random variable and is represented as 'T.' The probability distribution can be well represented with its probability density function $f_T(t)$. Now the reliability function can be derived by using the definition of the CDF. From the definition of the CDF, the probability of a unit failing by time t is given by [5–7]:

$$F(t) = P(T \leq t) = \int_0^t f(t)dt \quad \text{for } 0 \leq t \leq \infty \quad (2.37)$$

Since this function defines the probability of failure by a certain time, one can consider this as the unreliability function or failure probability function.

This can be explained with a simple example. Consider a component (light bulb) which is kept under testing to study its performance. Assume total number of components put under testing is 100. All the components are identical but they fail independently with each other. That is failure of one component does not affect the failure of the other component. The test is continued until all the bulbs have failed and the time to failure of each bulb is collected. The data can be grouped into intervals. That is, 0–10, 10–20, 20–30 h. In this way from the data one can find out how many components have failed in each interval. Now if one wants to find the fraction of components that is failed in each interval, this can be easily find out by dividing the total number of components failed in that interval with the total number of components. If one wants to find number of components failed till a particular time, this is simply counting all the components failed until that time. Now the fraction of these failed components with the total number of components will give the cumulative distribution function value. That means CDF will tell how many number of components have failed till a particular time in terms of probability.

Now the question comes what is the reliability of the component for a particular time? This is nothing but total number of components that have survived till that time in terms of probability. In other words this will be total number of components that have failed after a particular time in probability scale. This will represent fraction of components that have failed after a particular point of time with the total number of components. In terms of PDF this can be represented as

$$R(t) = P(T > t) = \int_t^{\infty} f(t)dt \quad \text{for } 0 \leq t \leq \infty \quad (2.38)$$

The CDF and reliability functions can be graphically represented as shown in Fig. 2.26.

In Fig. 2.26 the area 'Left' to the particular time t represents fraction of total number of components that have failed till that time and is nothing but CDF or failure probability function. The area 'Right' to the time t represents the fraction of

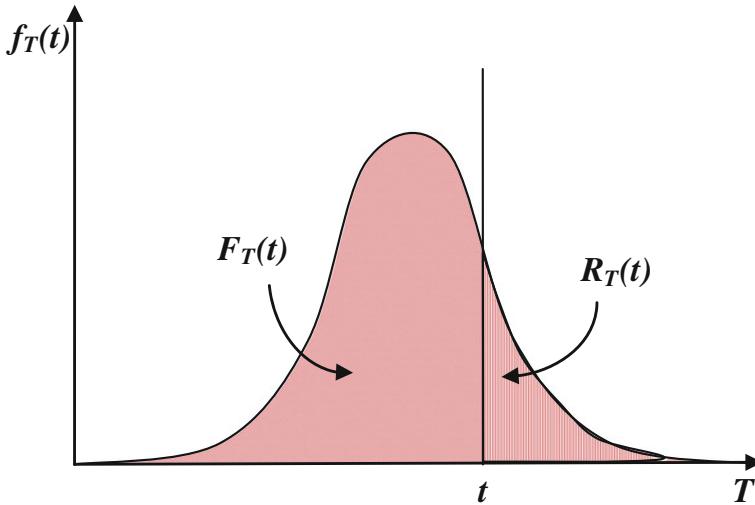


Fig. 2.26 Graphical representation of failure probability and reliability function

total number of components failed after that time and is the reliability function. As we know the total area under the PDF curve is one, then reliability function is complement to the failure probability function or CDF and also they are mutually exclusive events. This can be mathematically represented as

$$\begin{aligned}
 F(t) + R(t) &= 1 \\
 \int_0^t f(t)dt + \int_t^\infty f(t)dt &= 1 \quad \text{for } 0 \leq t \leq \infty \\
 F(t) &= 1 - R(t) \\
 R(t) &= 1 - F(t)
 \end{aligned}
 \tag{2.39}$$

From the above the PDF can be represented in terms of cumulative distribution function or reliability function as follows:

$$\begin{aligned}
 f(t) &= \frac{dF(t)}{dt} = -\frac{dR(t)}{dt} \\
 \int_0^\infty f(t)dt &= 1 \\
 0 &\leq F(t) \leq 1 \\
 0 &\leq R(t) \leq 1 \\
 F(0) &= 0 \quad \lim_{t \rightarrow \infty} F(t) = 1 \\
 R(0) &= 1 \quad \lim_{t \rightarrow \infty} R(t) = 0
 \end{aligned}
 \tag{2.40}$$

From the above expressions at $t = 0$ the failure probability is given as zero and reliability is given as 1. That means all the components are survived at $t = 0$ and

there are no failures less than this time. Similarly, as the time tends to infinity the failure probability approaches to 1 and reliability approaches to zero. That means no component will be survived or all the components will be failed by the infinity time. Here infinity means very long time.

Example 2.8 Consider the failure data of a component follows exponential distribution function whose PDF is given as follows:

$$f_T(t) = \lambda e^{-\lambda t} \quad \text{for } 0 \leq t \leq \infty$$

where λ is parameter of the distribution and is a constant which represents the failure rate of a component. Find the CDF and reliability function?

$$F(t) = P(T \leq t) = \int_0^t f(t) dt \quad \text{for } 0 \leq t \leq \infty$$

$$= \int_0^t \lambda e^{-\lambda t} dt = 1 - e^{-\lambda t}$$

$$R(t) = P(T > t) = \int_t^\infty f(t) dt \quad \text{for } 0 \leq t \leq \infty$$

$$= \int_t^\infty \lambda e^{-\lambda t} dt = e^{-\lambda t}$$

The PDF, CDF, and reliability functions of exponential distribution function are shown in Figs. 2.27, 2.28 and 2.29.

Fig. 2.27 PDF of exponential distribution

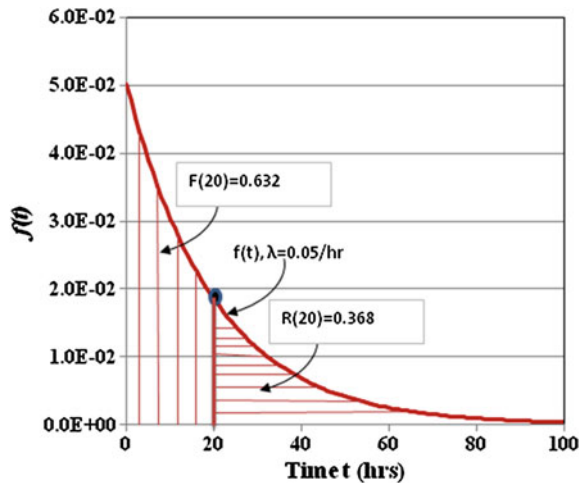


Fig. 2.28 CDF of exponential distribution

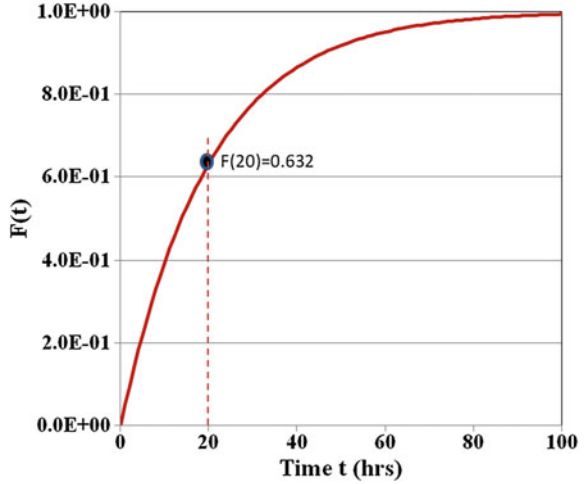
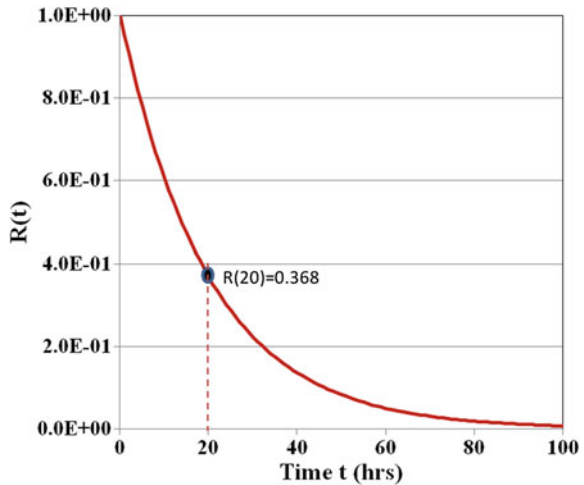


Fig. 2.29 Reliability function of exponential distribution



2.5 Measures of Reliability

Once the failure data of a component is available one can estimate the different parameters from the data. These parameters indirectly give the measure of reliability. These are namely

- Mean time to failure
- Median time to failure
- Mode
- Variance (standard deviation)

These measures are briefly explained below.

2.5.1 Mean Time to Failure

The mean value represents the average time to failure or expected time to failure of a component. This is same as the mean value of a continuous random variable as discussed earlier. This can be mathematically represented as follows:

$$\text{MTTF} = E(t) = \int_0^{\infty} t f(t) dt \quad \text{for } 0 \leq t \leq \infty \quad (2.41)$$

Mean Time to Failure (MTTF) can also be represented as a function of reliability by simplifying the Eq. 2.41 as follows:

$$\begin{aligned} \text{MTTF} &= \int_0^{\infty} t f(t) dt \quad \text{for } 0 \leq t \leq \infty \\ f(t) &= -\frac{dR(t)}{dt} \\ \text{MTTF} &= \int_0^{\infty} t \left[-\frac{dR(t)}{dt} \right] dt \\ &= -\int_0^{\infty} t dR(t) \\ \text{MTTF} &= -[tR(t)]_0^{\infty} + \int_0^{\infty} R(t) dt \quad (\text{integration by parts}) \\ \text{at } t=0 \quad R(0) &= 1 \\ t=\infty \quad R(\infty) &= 0 \\ \therefore \text{MTTF} &= \int_0^{\infty} R(t) dt \quad (2.42) \end{aligned}$$

The integration of above expression will be simpler compared to the previous expression which is a function of PDF.

2.5.2 Median Time to Failure

Median time to failure represents the median value of the failure data. In other words by this time 50 % of the components under operation would have failed or only 50 % of the component would survive. This can be derived from the

cumulative distribution function or reliability function. Since the 50 % of the components would have failed by this time, hence, probability of failure or reliability at median time will be 0.5. This can be represented as

$$\begin{aligned} F(t_{50}) &= P(T \leq t_{50}) = 0.5 \\ R(t_{50}) &= P(T > t_{50}) = 0.5 \end{aligned} \quad (2.43)$$

It divides the distribution into two halves, with 50 % of the failures occurring before the median time to failure and 50 % occurring after the median value. Median is preferred to the mean when the distribution is highly skewed.

2.5.3 Mode

The mode of the failure data represents the most likely observed failure time. This can be calculated from the PDF of the data. For the continuous distributions wherever the $f(t)$ attains maximum in the entire range that value will be the mode for that data. This also can be represented as follows:

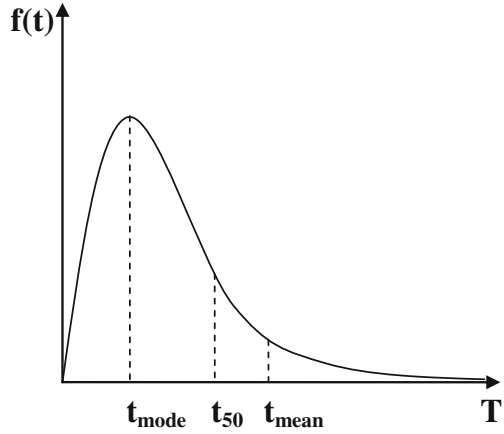
$$\begin{aligned} f(t_{mode}) &= \max_{0 \leq t \leq \infty} f(t) \\ \frac{df(t)}{dt} &= 0 \quad \text{at } t = t_{mode} \end{aligned} \quad (2.44)$$

This means at the peak value slope of the PDF curve is zero. It is applicable only if the PDF has both +ve and -ve slope. For a monotonically decreasing function t_{mode} will be lower limit of t . Similarly, for the continuously increasing function t_{mode} will be upper limit of t . In the case of normal distribution MTTF, median time to fail and mode all are same. Whereas in the case of lognormal distribution (distribution which is skewed to the right) the reliability measures will appear in the following order:

$$\begin{aligned} t_{mode} &< t_{median} < t_{mean} \\ t_{mode} &< t_{50} < \text{MTTF} \end{aligned}$$

The same is shown in Fig. 2.30.

Fig. 2.30 Reliability measures for lognormal distribution



Even if two reliability functions have the same mean, their reliabilities may be quite different for the same operating time.

2.5.4 Variance

It is a measure of the spread or dispersion of the failure times about the mean. An average squared distance, a failure time will be from the MTTF. Square root of the variance will give standard deviation.

$$\begin{aligned} \sigma^2 &= \int_0^{\infty} (t - \text{MTTF})^2 f(t) dt \\ &= \int_0^{\infty} t^2 f(t) dt - \text{MTTF}^2 \end{aligned} \tag{2.45}$$

Example 2.9 Consider the time to failure data of a component follows exponential distribution function whose PDF is given as follows:

$$f_T(t) = \lambda e^{-\lambda t} \quad \text{for } 0 \leq t \leq \infty$$

Find the MTTF, median, mode, and variance?

$$F(t) = P(T \leq t) = \int_0^t f(t) dt \quad \text{for } 0 \leq t \leq \infty$$

$$= \int_0^t \lambda e^{-\lambda t} dt = 1 - e^{-\lambda t}$$

$$R(t) = e^{-\lambda t}$$

$$\text{MTTF} = \mu$$

$$\mu = \int_0^{\infty} t f(t) dt = \int_0^{\infty} R(t) dt$$

$$\mu = \int_0^{\infty} e^{-\lambda t} dt = \frac{1}{\lambda}$$

at $t = t_{50}$

$$F(t_{50}) = 0.5 = 1 - e^{-\lambda t_{50}}$$

$$e^{-\lambda t_{50}} = 0.5$$

$$\ln(e^{-\lambda t_{50}}) = \ln(0.5)$$

$$-\lambda t_{50} = -0.693$$

$$t_{50} = \frac{0.693}{\lambda}$$

at $t = 0$

$$f(0) = \lambda = f_{\max}$$

$$\therefore t_{\text{mode}} = 0$$

$$\text{Variance} = \sigma^2$$

$$\sigma^2 = \int_0^{\infty} t^2 f(t) dt - \mu^2$$

$$= \int_0^{\infty} t^2 \lambda e^{-\lambda t} dt - \mu^2$$

$$\sigma^2 = \frac{1}{\lambda^2}$$

$$\sigma = \frac{1}{\lambda}$$

Example 2.10 The failure distribution of a component is defined by

$$f_T(t) = \frac{4t^3}{a} \quad \text{for } 0 \leq t \leq 100 \text{ h}$$

- (a) Find the constant 'a'
- (b) Compute the MTTF
- (c) Find the design life for a reliability of 0.99.
- (d) Find the mode for the given function.

Solution:

- (a) According to the axioms of probability the area under the PDF curve should be equal to '1' in its defined range.

$$f_T(t) = \frac{4t^3}{a} \quad \text{for } 0 \leq t \leq 100 \text{ h}$$

$$\int_0^{100} f_T(t) dt = 1$$

$$\int_0^{100} \frac{4t^3}{a} dt = 1$$

$$\left[\frac{4t^4}{4a} \right]_0^{100} = 1$$

$$\frac{10^8}{a} = 1$$

$$\therefore a = 10^8$$

$$f_T(t) = \frac{4t^3}{10^8} \quad \text{for } 0 \leq t \leq 100 \text{ h}$$

(b) MTTF can be calculated as follows:

$$\begin{aligned}
 \text{MTTF} = \mu &= \int_0^{\infty} tf(t)dt \\
 \mu &= \int_0^{100} tf(t)dt \\
 &= \int_0^{100} t \frac{4t^3}{10^8} dt \\
 \mu &= \frac{4}{10^8} \int_0^{100} t^4 dt \\
 \mu &= \frac{4}{10^8} \left[\frac{t^5}{5} \right]_0^{100} \\
 &= \frac{4}{10^8} \frac{10^{10}}{5} = \frac{400}{5} \\
 \mu &= 80 \text{ h}
 \end{aligned}$$

(c) Find the design life for a reliability of 0.99. We know that

$$\begin{aligned}
 R(t) &= \int_t^{\infty} f(t)dt \\
 &= \int_t^{100} \frac{4t^3}{10^8} dt \\
 &= \int_t^{100} \frac{4t^3}{10^8} dt \\
 &= \frac{4}{10^8} \int_t^{100} t^3 dt \\
 &= \frac{4}{10^8} \left[\frac{t^4}{4} \right]_t^{100} \\
 &= \frac{4}{10^8} \left[\frac{10^8 - t^4}{4} \right] \\
 R(t) &= \frac{1}{10^8} [10^8 - t^4]
 \end{aligned}$$

But $R(t) = 0.99$

$$\frac{1}{10^8} [10^8 - t^4] = 0.99$$

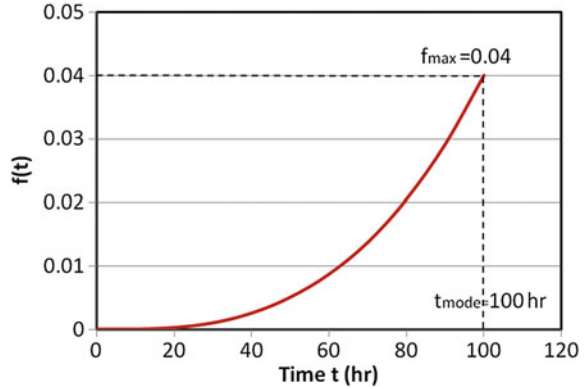
$$t^4 = 10^8 - 0.99 \times 10^8 = 0.01 \times 10^8 = 10^6$$

$$t = 10^{6/4} = 31.623 \text{ h}$$

Hence, the design life of the component for the reliability of 0.99 is 31.623 h.

(d) Find the mode for the given function

Fig. 2.31 Probability density function of T



The mode of the function will be time at which PDF will be maximum. From the density function it is clear that it is a continuously increasing function of time. Hence, the f_{\max} will occur at its upper limit, i.e., at $t = 100$ h. Hence mode for the present function is 100 h. This can be seen in the PDF curve as shown in Fig. 2.31.

2.6 Hazard Rate Function

One more important function that is used often in reliability is hazard rate function. This can be sometimes used as failure rate depending on the context of use. It gives the instantaneous failure rate of a component. In simple terms it is nothing but number of failures per unit time. The hazard rate function can be derived based on the conditional probability concepts. Assume that a component has survived up to time ' t .' Now what is the probability that the component will fail immediately after time t . This can be probabilistically represented as below. Consider the component will fail in a finite time interval Δt , then probability that the component will fail in the time interval t and $t + \Delta t$ can be given as

$$P[t \leq T \leq t + \Delta t] = F(t + \Delta t) - F(t) \quad (2.46)$$

But the component has survived till time t , this can be represented with conditional probability as follows:

$$\begin{aligned}
 P[t \leq T \leq t + \Delta t | T \geq t] &= \frac{P[t \leq T \leq t + \Delta t \cap T \geq t]}{P[T \geq t]} \\
 \therefore P(A|B) &= \frac{P(A \cap B)}{P(B)} \\
 P[t \leq T \leq t + \Delta t | T \geq t] &= \frac{P[t \leq T \leq t + \Delta t]}{P[T \geq t]} \\
 P[t \leq T \leq t + \Delta t | T \geq t] &= \frac{F(t + \Delta t) - F(t)}{R(t)} \tag{2.47}
 \end{aligned}$$

Now divide the above function with the interval size Δt and set the limits as $\Delta t \rightarrow 0$, then this can be written as

$$\begin{aligned}
 h(t) &= \lim_{\Delta t \rightarrow 0} \frac{F(t + \Delta t) - F(t)}{R(t)} \times \frac{1}{\Delta t} \\
 &= \lim_{\Delta t \rightarrow 0} \frac{F(t + \Delta t) - F(t)}{\Delta t} \times \frac{1}{R(t)} \\
 &= \lim_{\Delta t \rightarrow 0} \frac{\Delta F(t)}{\Delta t} \times \frac{1}{R(t)} = \frac{dF(t)}{dt} \times \frac{1}{R(t)} \\
 \text{but } f(t) &= \frac{dF(t)}{dt} \\
 \therefore h(t) &= \frac{f(t)}{R(t)} \tag{2.48}
 \end{aligned}$$

Further simplifying the above function

$$\begin{aligned}
 h(t) &= \frac{f(t)}{R(t)} \\
 \text{but } f(t) &= -\frac{dR(t)}{dt} \\
 \therefore h(t) &= -\frac{dR(t)}{dt} \times \frac{1}{R(t)} \\
 h(t) dt &= -\frac{dR(t)}{R(t)} \\
 \int_0^t h(t) dt &= \int_0^t -\frac{dR(t)}{R(t)} = -\int_0^t \frac{dR(t)}{R(t)} \\
 \int_0^t h(t) dt &= -[\ln(R(t))]_0^t \quad \text{at } t=0 \quad R(0) = 1
 \end{aligned}$$

$$\ln(R(t)) = - \int_0^t h(t) dt$$

$$\therefore R(t) = e^{-\int_0^t h(t) dt} \quad (2.49)$$

From the above reliability function is a function of hazard rate function. The hazard function itself is a function of time. The hazard rate function can be decreasing function of time or can be constant or it can be increasing function of time. If the hazard rate function is constant then the reliability function will take the following form:

$$h(t) = \lambda = \text{Constant}$$

$$R(t) = e^{-\int_0^t h(t) dt} = e^{-\int_0^t \lambda dt} = e^{-\lambda t} \quad (2.50)$$

This is same as the reliability function of an exponential distribution. This can be derived as follows:

$$h(t) = \frac{f(t)}{R(t)}$$

$$f(t) = \lambda e^{-\lambda t} \quad \text{and} \quad R(t) = e^{-\lambda t} \quad (2.51)$$

$$h(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

Hence, if the component failure data follows exponential distribution then its hazard rate is constant and is equivalent to its distribution parameter λ , i.e., failure rate. As discussed earlier hazard rate is a function of time and it can be decreasing or constant or increasing function of time. Based on this there are different types of hazard rate models existed. This has got beautiful application in life characteristic curve of components and is explained in the following section.

2.7 Life Characteristic Curve

Life characteristic curve (bathtub curve) is the graph between the hazard rate and the component lifetime. There are three stages of failures in the life of a product: early stage (infant mortality), operating stage, and wear out stage as shown in Fig. 2.32 which is called life characteristic curve or bathtub curve because of its shape.

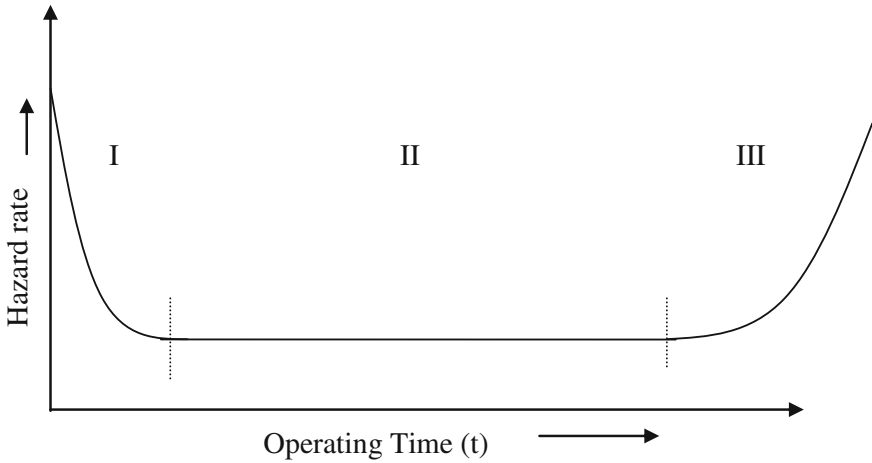


Fig. 2.32 Life characteristic curve

- (1) Early failure region (infant mortality, burn-in)
- (2) Useful life region (hazard rate constant)
- (3) Wear out region.

When the equipment is put into use for the first time any inherently weak parts normally fail soon. Thus early hazard rate is very high. But once the weak parts are replaced the hazard rate falls and fairly constant and finally hazard rate rises again as parts start to wear out.

The region (1) suggests that no item be used unless it has survived this period. Some of the reputed manufacturers sell only those components which have survived this period. The region (2) is useful life period where hazard rate is governed by chance failure and is fairly constant. The region (3) indicates that the component should be replaced or scrapped. Table 2.4 gives the differences between the three phases.

Example 2.11 Times to failure of a component are obtained as follows:

Time interval	No. of failures
0–20	95
20–40	50
40–60	23
60–80	12
80–100	6
100–120	4
>120	10

Table 2.4 Different phases of bathtub curve

	Phase I	Phase II	Phase III
Characterized by	Decreasing failure rates	Constant failure rate	Increasing failure rate
	Infant mortality, burn-in	Useful life	Wear out
Caused by	Manufacturing defects, welding flaws, cracks, defective parts, poor quality control, contamination, poor workmanship	Environment, random loads, human error, chance events	Fatigue, corrosion, aging, friction, cyclical loading
Reduced by	Burn-in testing, screening, quality control acceptance testing	Redundancy, excessive strength	Derating, preventive maintenance, parts replacement

- (a) Find the probability density function $f(t)$, cumulative distribution function $F(t)$, reliability function $R(t)$, and the hazard rate function $h(t)$?
 (b) In which phase of the bathtub curve the above data fits?

Solution:

To solve the problem we first list down all the relevant formulas

$$f(t) = \frac{dF(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{F(t + \Delta t) - F(t)}{\Delta t}$$

$$F(t) = \frac{N_f(t)}{N}$$

$$R(t) = \frac{N_s(t)}{N} = 1 - F(t)$$

$$h(t) = \frac{f(t)}{R(t)}$$

where

$N_f(t)$ is Total number of components failed till time 't'

$N_s(t)$ is Total number of components survived till time 't'

N is the Total number of components

- From the Table each time interval has lower and upper limit and are denoted as t_L and t_U . In the first interval $t_L = 0$ and $t_U = 20$ h.
- Interval size $\Delta t = t_U - t_L = 20$ h
- Total number of components (N) = 200
- For the first interval total number of components failed till $t = 0$ is 'Zero' and total number of components survived till $t = 0$ is 200.

Table 2.5 Calculation of CDF and reliability

t_L	$N_f(t_L)$	$N_s(t_L)$	$F(t_L) = \frac{N_f(t_L)}{N}$	$R(t_L) = \frac{N_s(t_L)}{N} = 1 - F(t_L)$
0	0	200	$0/200 = 0.000$	$200/200 = 1.000$
20	95	105	$95/200 = 0.475$	$105/200 = 0.525$
40	145	55	$145/200 = 0.725$	$55/200 = 0.275$
60	168	32	$168/200 = 0.840$	$32/200 = 0.160$
80	180	20	$180/200 = 0.900$	$20/200 = 0.100$
100	186	14	$186/200 = 0.930$	$14/200 = 0.070$
120	190	10	$190/200 = 0.950$	$10/200 = 0.050$
>120	200	0	$200/200 = 1.000$	$0/200 = 0.0000$

- For the second interval total number of components failed till $t = 20$ is 95 and total number of components survived till $t = 20$ is 105. This is shown in Table 2.5 below.
- $F(t)$ and $R(t)$ are calculated at each and every point of time and are also tabulated in Table 2.5 and are also shown in Figs. 2.33 and 2.34, respectively.
- $f(t)$ is calculated by using the following formula

$$f(t) = \frac{F(t + \Delta t) - F(t)}{\Delta t}$$

where t is the lower limit of the interval and Δt is the interval size.

Here $f(t)$ is calculated for the entire interval, i.e., average value for the interval. This is because we do not have the enough failure information at each and every point of the interval. Calculation is shown in Table 2.6. This can be seen in Fig. 2.35.

- Similarly $h(t)$ is also calculated (Table 2.7) for the entire interval, which will represent average value for the interval (Fig. 2.36).

Fig. 2.33 Cumulative distribution function

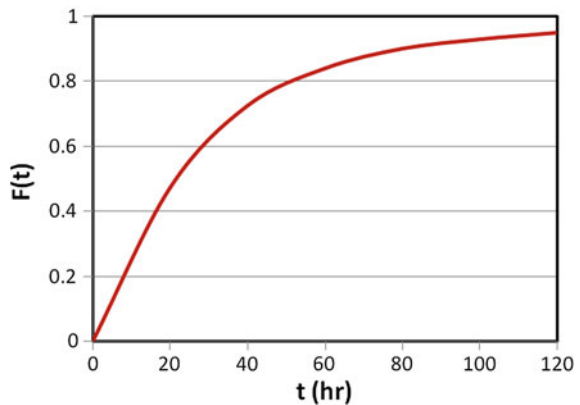


Fig. 2.34 Reliability function

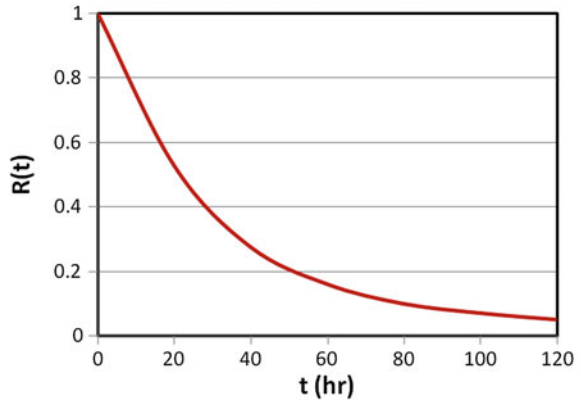
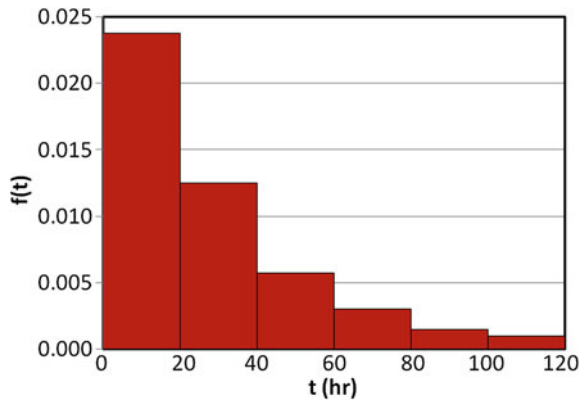


Table 2.6 Calculation of PDF

Time interval	t_L	t_U	$F(t_L)$	$F(t_L + \Delta t)$	$f(t) = \frac{F(t_L + \Delta t) - F(t_L)}{\Delta t}$
		$t_L + \Delta t$			
0–20	0	20	0.000	0.475	$(0.475 - 0)/20 = 0.02375$
20–40	20	40	0.475	0.725	$(0.725 - 0.475)/20 = 0.0125$
40–60	40	60	0.725	0.840	$(0.84 - 0.725)/20 = 0.00575$
60–80	60	80	0.840	0.900	$(0.9 - 0.84)/20 = 0.003$
80–100	80	100	0.900	0.930	$(0.93 - 0.9)/20 = 0.0015$
100–120	100	120	0.930	0.950	$(0.95 - 0.93)/20 = 0.001$

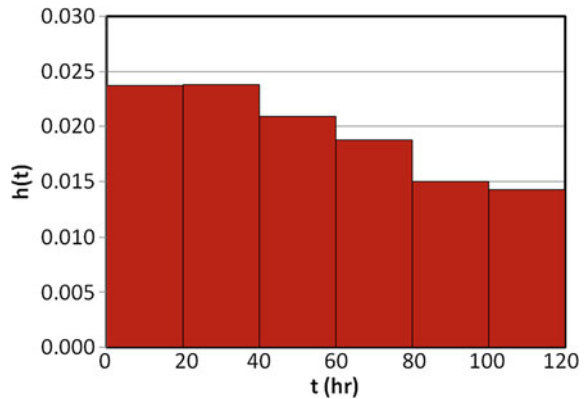
Fig. 2.35 Probability density function interval wise



- From Fig. 2.36 it is clear that the hazard rate is decreasing with respect to time. Hence, the failure data will represent the phase I in the bathtub curve, i.e., infant mortality region.

Table 2.7 Calculation of $h(t)$

Time interval	$f(t)$	$R(t)$	$h(t) = \frac{f(t)}{R(t)}$
0–20	0.02375	1.000	$0.02375/1.0 = 0.02375$
20–40	0.01250	0.525	$0.0125/0.525 = 0.02381$
40–60	0.00575	0.275	$0.00575/0.275 = 0.0209$
60–80	0.00300	0.160	$0.003/0.160 = 0.0188$
80–100	0.00150	0.100	$0.0015/0.1 = 0.0150$
100–120	0.00100	0.070	$0.001/0.07 = 0.0143$

Fig. 2.36 Hazard rate function intervalwise

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