Varying the Money Supply of Commercial Banks

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Abstract We consider the problem of financing two productive sectors in an economy through bank loans, when the sectors may experience independent demands for money but when it is desirable for each to maintain an independently determined sequence of prices. An idealized central bank is compared with a collection of commercial banks that generate profits from interest rate spreads and flow those through to a collection of consumer/owners who are also one group of borrowers and lenders in the private economy. We model the private economy as one in which both production functions and consumption preferences for the two goods are independent, and in which one production process experiences a shock in the demand for money arising from an opportunity for risky innovation of its production function. An idealized, profitless central bank can decouple the sectors, but for-profit commercial banks inherently propagate shocks in money demand in one sector into price shocks with a tail of distorted prices in the other sector. The connection of profits with efficiency-reducing propagation of shocks is mechanical in character, in that it does not depend on the particular way profits are used strategically within the banking system. In application, the tension between profits and reserve requirements is essential to enabling but also controlling the distributed perception and evaluation services provided by commercial banks. We regard the inefficiency inherent in the profit system as a source of costs that are paid for distributed perception and control in economies.

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1 Preamble

1.1 The Problems of Decoupling Scale and Structure in Plumbing

Consider a problem faced by designers of plumbing for hotels. Trunk lines supply hot and cold water to many taps in many guest rooms. Sinks, showers, and toilets draw water from the trunks in uncoordinated and unpredictable ways. The water flow demanded from a trunk is a variable that aggregates across users who tap the trunk, the scale of which is subjected to ongoing shocks in the course of normal usage. Water also has pressure, and the relative pressure in hot and cold lines allows the guest taking a shower to set the desired temperature by adjusting two valves. Pressure might be called a "structural" feature of the plumbing system. In good circumstances—which even in crude plumbing systems may be approached under conditions of constant demand—the pressure across the trunk is stable over time and may even be constant across taps.[1] Stability both in space and in time are essential to the system's providing its key services.

However, under shocks to the scale of flow, in a plumbing system without well-designed reservoirs of pressure, scale shocks create pressure shocks. Showering in hotels a generation or two ago offered a well-known adventure: a guest somewhere would flush a toilet, and the patron in the shower would briefly scald and then freeze. This sequence might repeat dozens of times in the course of a single shower in a large hotel on a busy morning. Plumbing designed with inadequate technology fails to buffer scale shocks in demand from propagating into the structure variables of the water flow: the time-dependent sequence of pressure values at all the valves.

1.2 Scale and Structure Problems in Money Supply and Prices

A mathematically analogous problem arises in economies. Multiple sectors demand funds for ongoing business and to cope with cycles, take risks, and respond to their unknown outcomes and other surprises. Many of these are uncoordinated and unpredictable demand shocks to the money supply, which is a scale variable of the economy. The price system in an economy is a structural property dual to its money supply [3]. Prices across sectors, or across time within a sector, need not be constant, but in a well-functioning economy they should reflect a consistent response to agents' marginal utilities or other relevant measures of valuation. In particular, it may be a design objective for a banking system that prices in a sector not be subject to ongoing ripples or other disturbances that arise purely through financial frictions,

[1]This is true for taps at the same elevation; we leave aside corrections for gravity which are not central to the point of this illustration.

due to demand shocks for money in other sectors. Governments and bankers face problems of system design of a purely mechanistic nature—meaning that they apply in a wide range of strategic contexts—analogous to the problems faced by plumbing engineers who deliver a different quantity (water) also subject to inertia and friction and by its nature not compressible.

1.2.1 Innovation as a Source of Shocks that Require Economics Beyond General Equilibrium

A pair of articles by Shubik and Sudderth [6, 7] considers innovation as a process that creates demand shocks through the problem recognized by Schumpeter [4], of "breaking the circular flow of funds". The financial design problems that arise from contexts with innovation are inherently dynamical. They offer perhaps the most direct widespread class of economic phenomena that require a robust theory falling essentially outside the General Equilibrium paradigm.

In this paper we use the *cost innovation* model of Shubik and Sudderth as a testbed to study the banker's problem of decoupling sectors in an economy. Under idealized theoretical conditions, a model banking system can both function as a strategic dummy and also decouple production and consumption sectors if they are not otherwise coupled through substitution effects. However, actual banking systems do not operate under idealized theoretical conditions. They face uncertainty throughout the economy, and they require distributed and scalable services of perception and evaluation. The limited monolithic structure of a central bank that is easily modeled in theory, is replaced in operation with an ecology of one or more central banks and a collection of competitive commercial banks operating in the private sector and responsive to its fluctuating demands for service and its geographic and demographic distribution [1].

The policy tools used to grant commercial banks the independence they require to fulfill their functions, while still controlling the quality of risk in their portfolios, include reserve requirements (set reserve ratios and possibly also minimum reserve quantities) and profits which guide the banks' strategic actions and provide a layer of abstraction between the commercial banks operational decisions and their owners' preferences. Profits create an incentive to make bank money available, while reserve requirements control its scarcity. The design problem is to balance the forces of incentive and constraint to achieve policy objectives for the banking system as a whole.

Shubik and Sudderth consider the general problem of control in strategic reserve banking. Here we do not address that higher-level problem, but instead consider the pre-strategic (more purely mechanical) question of whether the existence of profits creates inherent limits in the extent to which banking systems can decouple demands for money from propagation of price shocks. We consider an explicitly time-dependent economy with many periods of production and consumption, in which stability of the price system within a sector is essential to planning an optimal program of output and distribution. We consider only 100 % reserve banking, so that profits of commercial banks arise only through the opening of interest rate spreads between the rates on loans and on deposits.

The paper compares the interface that a simple central bank could present to an economy if there were no need for perception and control, with the interface that a comparable for-profit commercial bank presents. In cases where the central bank can decouple sectors, we find that the introduction of interest rate spreads, which are needed to create a profit motive, inherently cross-couples sector prices.[2] In the comparison, all interest rates are treated as parameters (rather than strategic variables), so this is a purely mechanistic effect, holding independent of the strategic context to which profits might be put in more elaborate models that seek to capture larger-scale regulatory dynamics.

1.2.2 Stock and Flow Distinctions as a Further Measure of Cross-Sector Propagation of Disturbances

We use a continuous-time analogue of the discrete period innovation model of Shubik and Sudderth [6, 7] because scaling analysis on the approach to this limit makes precise the distinction between stocks and flows. Stocks include outstanding (revolving) loan levels and inputs to production, while flows include streams of interest payment, velocity of money, and consumption rates of goods. We show below that in an idealized economy where money demands and prices are buffered between systems, the inter-sector loan levels within the economy, and the overall money supply and its exchange with the banks, are also distinguished in their scaling behavior. Inter-sector loans scale as stocks that remain finite in the continuous-time limit, while total money supply and net private-sector credits or debts on which interest payments change the money supply, go to zero in the continuous-time limit as the velocity of money (a flow variable) becomes the regular property of that limit.

The introduction of profits that couples cross-sector production decisions also couples inter-sector and aggregate debt levels, breaking their independent scaling behavior in the idealized efficient economy. Thus profits that couple scale shocks to structure shocks do so in several dimensions.

2 Innovation, Chance, Growth, Cycle and the Money Supply

2.1 Efficiency, Arbitrage and Equilibrium

The no arbitrage and the efficiency conditions do not coincide with incomplete markets, but the property of no arbitrage can still be defined and reflects the

[2] The coupling is in linear proportion to the spread at sufficiently small spreads.

individualistic behavior property of the noncooperative equilibrium. Once we give up the comforting fiction of complete markets we still have the definition of Pareto Optimality as an ideal and a clean picture of efficiency; but we have no individualistic solution that guarantees its attainment. A welter of theoretical problems appear in the construction of indices to measure efficiency with incomplete markets. It is well known that one can construct comparative measures between two mechanisms and possibly decide that one is more efficient than the other over a given parametric range. There is also the important empirical problem of trying to measure just how inefficient is a market structure with incomplete markets when compared with the same structure with complete markets.

If we accept the position that any market mechanism requires resources to operate it, then even Pareto optimality is challenged.

2.2 No Arbitrage and Varying the Money Supply

If prices in a monetary economy are to be consistent with competitive markets[3] there are several scenarios that call for the variation of the money supply. They are exogenous uncertainty, strategic uncertainty, the presence of growth and cycles in the economy. All call for a flexible money supply if cash flow constraints are to be avoided. Possibly the most interesting scenario involves innovation where the financing of the risk involved in innovation calls for a flexible money supply. We use this as the context for much of the investigation below. We note that the ability to vary the money supply confers considerable economic power on the agent able to do so.

We address specifically cost innovation and the breaking of the circular flow of funds.

2.3 A Closed Economy with Producers, Consumers, Commercial and Investment Banks and a Central Bank

We preface our mathematical analysis with a verbal discussion of both the modeling problems, simplifications and basic questions.

The minimal number of agent types we need to illustrate a mechanism that varies the money supply is three. They are an aggregate set of consumers; producers and a central bank.

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[3] An added condition is that prices are stationary when the real goods distribution is stationary. This raises further complications involving incentives and information conditions in an economy where all laws are not indexed against inflation or deflation. This problem is not considered further here.

The consumer/stockholder/passive saver is the one set of natural legal persons required. The others are corporate legal persons all owned by the "natural persons". They are the firms, and possibly a collection of commercial banks.
The central bank differs from the other legal persons as being part of government. We first describe the central bank.

2.3.1 The Central Bank

The central bank's powers may be modeled in many ways. The simplest is as a strategic dummy endowed with the ability to accept deposits or to make loans with unlimited issue of the only legal money in the system. A formal game can be defined if either the central bank sets interest rates at which it will lend or pay on deposits, or it sets a limit on the amount of money it offers in net supply.

- In virtually all of the existing national monetary systems, not only do central banks exist, but so do commercial banks. This raises the question: Why do commercial banks exist, if the central bank can vary the money supply by itself? As Bagehot noted [1] the commercial banks (and bill jobbers) perform as perceptors and evaluators of the state of business and the need for credit over the whole space of a nation. The Soviet Union did not utilize a commercial banking system internally. It utilized bureaucratically run branches. We do not consider their perception functions here; but observe that we may formulate the construction of a four agent model where there are consumers, producers, the central bank and commercial banks where the central bank has delegated much of the variation of the money supply to the commercial, for-profit banks. With this structure several questions must be answered:
- Can the commercial banking system be competitive?
- If so, in what dimensions do they compete?
- Can they be designed to transmit fully the policy of the central bank?
- Do reserve requirements play a role?
- What are the permitted strategies of the commercial banks?
- How are the banks' profits defined?

In our belief in the virtue of separating out problems we limit our analysis here to the influence of the commercial banks on shock transmission.

3 The Flexibility of Commercial Banks

In an enterprise economy the central bank may delegate the detailed adjustments of the money supply to a commercial banking system. The problems of economic coordination need to be resolved. The particular instruments and rules of this delegation, require a set of minimal models to demonstrate this systematically. For simplicity, to begin with, we consider the commercial banking system as a strategic dummy designed to provide a flexible money supply for an economy with variable monetary needs.

4 Preliminaries

> "The doorkeeper laughs and says: 'If you are so drawn to it, just try to go in despite my veto. But take note: I am powerful. And I am only the least of the door-keepers. From hall to hall there is one doorkeeper after another, each more powerful than the last. The third doorkeeper is already so terrible that even I cannot bear to look at him.' "
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> – Franz Kafka, *Before the Law*

Above we have presented a brief verbal sketch of why one may need a flexible money supply. We now provide a formal model to achieve this goal.

What might appear to be relatively simple mechanisms require computation or simulation of specific examples in order to illustrate even behaviorally simplistic economics.

We offer a quote from Kafka that we deem apposite in dealing with economic models where the equations of motion can be tightly defined over the whole state space.

The task of abstracting the reason why a variable money supply is needed, and the construction of the minimal institutions that fill that need, is to acknowledge the diversity of instantiations both have taken historically. The rise of fractional reserve banking in London in the last half of the ninetieth century, and the real bills doctrine in a range of conceptions from Jean-Baptiste Say to Adam Smith, were formulations of parts of this problem. Contemporary discussions of the feasibility (and consequences) of control of the money supply through interest rates versus open-market operations, and of desirable reserve levels for banks, are different mechanistically but should be understood as addressing the same fundamental questions in an age where money and credit diversity are much larger than they were in the age of Smith.

5 Varying the Money Supply with Credit

5.1 Sources of Need for a Flexible Money Supply

The different needs for a flexible money supply can be captured in formal models in a variety of ways. Often one-period models suffice to illustrate limitations in the quantity or distribution of money. In these cases, the difference between efficient and inefficient function of the financial system may be defined in terms of the alternative between interior and boundary solutions.

5.2 Separating Scale from Structure

We abstract the need for a variable money supply as a need to *separate scale from structure* in production and exchange economies. All societies undergo variations in the desired volumes of trade. These may be cyclical as in harvest seasonalities, episodic driven by good or bad harvests, immigration and emigration, innovation, etc., or progressive driven by growth or decline of population or productivity. All these variations in the capacity for production and consumption, which drive variations in the desirable volume of trade, we regard as *scale* fluctuations. A well-functioning economy also must converge on a range of price systems, both inter-sectorial and inter-temporal, including interest rates for money loans. These we regard as properties of the *structure* characterizing equilibria or near-equilibria. The stability of these price systems and the extent to which they can approximate reservation prices of agents determine the efficiency of the economy in extracting surplus, and are essential to any program of rational planning.

In its most basic abstraction, the goal of monetary policy is to accommodate the needs for scale fluctuations in an economy, without causing scale shocks to propagate to cause disruptions of structure *where such propagation can be avoided by monetary design or regulation*. Obviously, many scale shocks inherently result in shocks to prices, production, or consumption, as when innovations in substitute goods change which consumption bundles are preferred. We regard as "avoidable" propagations those that result entirely from limits on the volume and distribution of money, across sectors in which production technologies or consumption preferences are not inherently coupled.[4] Informally, a money supply that is too "rigid" or "incompressible", such as a fixed stock of gold in circulation, will generically propagate shocks in the production or consumption volume in any sector into ripples of price change across all sectors and through time, until money can be redistributed to approximate a new equilibrium for the circular flow. Alleviating this rigidity is a goal of varying the money supply that can be recognized in a variety of monetary mechanisms across societies and in different eras.

An important and general hazard and technical challenge for institutions that provide a variable money supply is ensuring consistency in the quality of credit and the pricing of risk. These are essential to the stability particularly of intertemporal price systems. The problem of credit risk evaluation is not easily centralized, and is a primary driver to grant the status of legal tender to privately created bank credit. The Real Bills Doctrine of Adam Smith may be understood as an early mechanism to permit open-ended variability in bank credit while providing criteria for credit quality that could be used by banks evaluating a range of distinct contracts. Reserve levels in modern central banking and commercial banking systems are another mechanism that attempts to regulate credit quality implicitly through lending prices and leverage.

[4]This abstraction is easy to define in models. Validating the abstraction for actual economies may be more or less difficult depending on the sectors considered.

5.3 A Class of Minimal Models

As in previous work comparing the functionality of alternative money systems [8, 9], we construct a single underlying model of production, consumption, and trade, which creates a template for a family of strategic market games (differing in their financial system models) for which explicit non-cooperative equilibria can be computed. A formal specification of the models is given below; here we give a brief summary in order to explain the main purpose of the construction. Two kinds of storable goods define two production sectors. Production in each sector occurs by a simple input/output function, which converts an initial stock of the good into more of the same good at a rate that depends on the size of the working stock.[5] Working stocks are ideally storable, though they can be wasted (so that the constraint on the quantity of goods available is an inequality rather than an equality). Each good is also consumable. In solutions without waste, goods persist from the time they are produced until the time they are consumed.

Production within each sector is performed by competing firms which are jointly owned by individuals who are also consumers of the produced goods. Production, trade, and consumption all occur in a sequence of many simply-structured, equivalent periods, and the establishment of a circular flow is a feature of time-stationary non-cooperative equilibria that balance output rates by the firms against marginal utilities of consumption by the consumer/owners.

Innovation is modeled as the possibility for one group of firms to attempt to change the production function in a single (particular) period, at the cost of one-time consumption of a fraction of their working stock. The attempted change succeeds with a probability $\xi < 1$. Although the cost of production is reduced and the limiting output rate is raised for firms that successfully innovate, the initially-reduced working stock cuts their output rates until that stock can be rebuilt from the output, which may require many periods. Firms that attempt to innovate and fail suffer the stock reduction but retain the pre-innovation production function. The problem of whether innovation is desirable can be posed in either of the two goods-sectors independently,[6] and the general structure of solutions for the depletion and

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[5]Our models resemble the von Neumann growth model, restricted to a single good. However, in our production function the rate of output is a non-linear rather than a linear function of the input stock.

[6]We do not digress to derive the solution for Robinson Crusoe here, because its important features are subsumed in the solutions we demonstrate. A more systematic introduction to this class of models, including a separate solution for Robinson Crusoe as a reference, will be given elsewhere.

There are essentially three levels of models that require consideration for a complete exposition of basic distinctions. They are

- Crusoe without money,
- the price-taking individual firm with money,
- the oligopolistic firm without money.

subsequent restoration of productive stocks serves as a reference for these sectors in a monetary economy.

The specific feature of this real-goods economy that allows us to measure efficiency of money and banking systems is that *only one good* undergoes the opportunity for innovation. The other good has time-stationary production and consumption parameters, which we choose to be separable. Therefore it has no intrinsic reason to be influenced by innovation in other sectors. We demonstrate, however, that buffering the two sectors in the economy becomes difficult if models are not permitted an unrealistic degree of fine-tuning, and this is a basis of the need for substructure within the banking sector.

5.3.1 Many-Period Models, and the Passage to Continuous-Time Limits

The Bellman equations for many-period strategic market game models, in which the non-cooperative equilibria are non-stationary, are generally difficult to solve if the periods cover non-infinitesimal quantities of goods produced, traded, and consumed (that is, if they correspond to non-infinitesimal intervals of real time).[7] Some of these difficulties diminish if we take model periods to correspond to infinitesimal time periods, and scale production, trade, and consumption to be infinitesimal accordingly. We will refer to this scaling limit as the *continuous time* (or "continuum") limit for a many-period strategic market game.

Singular events, such as the choice to innovate a firm's production function and the required consumption of stocks, remain events that occur within a single period, so in the continuum limit they become singular, but this creates no difficulties as long as the continuum is defined as a limit of discrete-period models.

5.3.2 Continuous Time Defined Through Equivalence Classes

Formally, we treat economic processes that occur in continuous time as processes that may be modeled with any of a sequence of discrete-time models, with time intervals Δt that go to zero along the sequence. One performs calculations in discrete time so that definitions of moves in the game are unambiguous, but then requires that all economically relevant structure in the solution does not depend on Δt. More formally: the continuous-time limit is defined if there is a *scaling* of the other quantities in the model with Δt for which observables evaluated at two different times t_1 and t_2, which are held fixed as Δt is varied, converge on finite

The first two should produce the same physical allocations but differ in the presence or absence of money.

[7]The source of the simplification is that difference equations and discrete series reduce to differential equations and integrals, though the structure and meaning of the Bellman equations remains unchanged.

limiting values as $\Delta t \to 0$. Therefore a continuous-time limit is associated with an *equivalence class* of discrete-time models.

The formalization of continuous time in terms of scaling and limits provides a systematic way to partition stocks from flows. Within any single discrete-time model, all quantities may be represented as stocks within periods or changes of stocks between periods. When an equivalence relation over Δt is introduced, those changes in stocks that are to be interpreted as flows are required to vanish in linear proportion to Δt. The constant of proportionality in this scaling relation– the *rate* of the flow—is held fixed and is one of the parameters that defines the equivalence class. By such scaling relations, in continuous-time models, stocks, flows, and shocks are distinguished *mathematically* as well as descriptively.

5.3.3 The Economic Meaning of Continuous-Time Limits: How Many Timescales Represent Economically Significant Commitments of a Model?

It is possible to take a more conceptual view of continuous-time limits than merely technical tricks that simplify Bellman equations. In conventional discrete-period models, the period length is a dynamically important time interval in the model. It interacts with other model features such as non-linear production functions or utilities, and this interaction is one source of complexity in Bellman equations. In a continuous-time limit, since stocks and flow converge on regular limits as $\Delta t \to 0$, the period length ceases to be a model property that influences economic dynamics. For problems such as shock and recovery in production, consumption, and the circular flow of funds, the natural timescales of economic dynamics are determined by production functions, utilities, and interest rates, and *only* by these model properties.

5.3.4 The Use of Continuum Limits to Separate Dimensions of Economic Dynamics

It is not necessary to use continuous-time limits only *at* the limit point $\Delta t \to 0$. The existence of a well-defined and regular limit ensures that solutions to discrete-period models at small but nonzero Δt also exist and that they are approximated (to various orders in Δt) by properties of the limiting solution. For many applications it is useful to approximate these short-period solutions in terms of the structural parameters at the limit point.

The most important pair of economic quantities in short-period models are the money supply and the money velocity. In the continuous-time limit, with production and consumption per period scaled in linear proportion to Δt, solutions with stable prices also have regular continuum limits for the velocity of money, and solutions for the money supply that scale as Δt times this velocity (by definition of the velocity of money).

When banks are introduced that can both inject or extract money in circulation, and also mediate loans between agents, the two quantities will generally scale differently. Changes in the money supply, in efficient or nearly-efficient solutions, scale as $\mathcal{O}(\Delta t)$, like the original money supply. Interest streams between agents in steady circular flows are rates, which thus approach regular limits as $\Delta t \to 0$. Hence any *inter-agent* balances at the bank likewise scale as $\mathcal{O}(1)$; that is: the debts accumulated between agents at the bank can become arbitrarily larger than the money in circulation, in efficient solutions.

A second way in which a money system can be inefficient is that it can couple inter-agent lending to changes in the whole-economy money supply. If such a coupling arises, it creates a severe instability. A drain of $\mathcal{O}(1)$ can deplete the money in circulation in a time of $\mathcal{O}(\Delta t)$. Conversely, an addition of money at $\mathcal{O}(1)$ can lead to prices that grow to $\mathcal{O}(1/\Delta t)$. The continuum limit therefore offers ways to test the monetary system's capacity to buffer quantities with different natural dependence on turnover time, as well as different sectors.

5.3.5 Relation of Consumer/Owners to Firms and Banks

The models provide a minimum level of distinction sufficient to define economic sectors for goods production, and centralized versus distributed banking activities. In order to make all strategic actors price-takers, each type is modeled on a continuum. In order to minimize strategic complexity in the relation of ownership to control, with respect to the risk of failed innovation, we distribute ownership through uniformly-distributed shares of firms or banks. We do, however, retain a distinction between owners of firms of the two types, so that the economy creates income consequences from innovation, which bear on the role the banks play.

The specific structure of firms, consumer/owners, and banks is:

Firms: The economy has two goods, and we index production or consumption associated with these with subscripts $i \in \{1, 2\}$. For each of the two goods, a continuum of firms exist, which we index with a coordinate in the continuous interval [0, 1]. (We will not denote this index explicitly to reduce notational clutter; any production function, consumption utility, working stock, etc., with subscript $i \in \{1, 2\}$ implicitly refers to a particular firm or individual.)

Consumer/owners: Each type of good is also associated with a group of consumer/owners, also indexed with a coordinate in the continuous interval [0, 1]. All owners of a given type own equal shares of all firms of that type, and no shares of firms of the other type. Share ownership determines how firms deliver profits to owners, and in the case of firms that can engage in risky innovation, uniform share distribution leads to the same decision (to innovate or not to innovate) for all firms of the same type, and distributes the profit risk over all owners of that type.

The central bank: In economies with a central bank, the central bank is an atomic actor and a strategic dummy. It is not owned by any agents in the economy, and does not define or deliver profits. Its function is both to define the rules of monetary function, and to control the injection or extraction of money (either directly, or through commercial banks).

Commercial banks: In economies with a commercial banking sector, a single kind of commercial bank exists. Commercial banks are in some cases modeled as strategic dummies acting according to fixed rules, but the purpose for which they are introduced ultimately requires that they be strategic profit-maximizers, so that the profit incentive in a context of regulatory constraint guides their function within the economy. Therefore from the start we introduce commercial banks (in the cases where they occur) as a continuum of competitive corporations, again indexed with a coordinate in the continuous interval [0, 1]. All consumer/owners (so, the owners of both types of firms) jointly own the commercial banks. Again each owner owns a uniform distribution of shares of all banks, so that each bank's profits are distributed uniformly among all owners.

5.3.6 The Different Models to be Considered

1. **Fixed money supply:** The minimal solution in the absence of banking assumes a fixed supply of money in circulation, without reserves. The money could be gold or government fiat. Its fixed supply causes the shock from innovation in one good to strongly impact prices and output levels in the other good. The non-cooperative equilibrium in this game is an inefficient outcome corresponding to the pre-institutional (with respect to banking) economy.
2. **An idealized central bank:** If the economy does not require distributed commercial banking, a benevolent central bank can vary the money supply and mediate borrowing and lending among agents internal to the economy without interest rate spreads or leveraging. We show that this solution, with finely tuned parameters, can perfectly decouple the two goods sectors, so that the shock from innovation in one sector does not affect output in the other. This outcome defines the efficient function of the monetary system, and shows that it is achievable in a constructive solution. The buffering of the two production sectors is possible despite the fact that agents of different types experience relative wealth variations, so their consumption of goods is altered by innovation.
3. **Commercial banks with interest rate spreads and 100 % reserves:** In a first step toward defining a profit-seeking commercial banking sector, we introduce commercial banks that borrow "fiat money" from the central bank, and are permitted to issue bank credit to consumers in 1:1 ratio[8] to their holdings of fiat. The commercial banks can still enable steady-state production outcomes with many of the scaling properties of the efficient solution, if interest rates are finely tuned. However, they necessarily transmit shocks from the innovated to the non-innovated good, in proportion to the size of the interest rate spread.

[8]We could introduce a k:1 gearing ratio here with a little extra work, but our illustration does not need it.

We formalize this model as though the central bank has set the spread parametrically thereby reducing the commercial banks to strategic dummies. We suggest that it is the unmodeled evaluation function followed by the decision to lend or not to lend where important competition enters into banking.

5.4 Formal Definition: The Production and Consumption Problem

This section defines the scaling relations for production and for utility of consumption, in which *rates* of production and consumption give the invariant functional relations.

All discrete-period games, from which the continuum limit is defined, consist of a long sequence of periods with a time index t. The index is incremented by Δt, and the maximal value taken by t is some number T, which is the last period of the game. (We will return below to the way this period is selected, in order to address problems of robustness and interpretation of terminal conditions.) The period in which innovation occurs is indexed $t = 0$. Dynamically equivalent games can be defined either by initiating the sequence of periods at some time $t = t_{\text{init}} \ll 0$, and allowing the economy to converge to a steady state by $t = 0$ (because the dynamics to be defined below does produce such convergence, as we will demonstrate), or we could take the starting period as $t = 0$. For simplicity we will use $t = 0$ as the initial period, and as initial conditions we will provide firms with working stocks of goods, and agents with quantities of money in hand, which equal the fixed-point values with pre-innovation production functions.

5.4.1 Production Functions in Continuous Time, and a Sell-Surplus Market for Goods

In order to associate a quantity of goods production with intervals of real time $[t_1, t_2]$, across a class of models which may have variable period length Δt, it is necessary to separate well-defined stock variables from well-defined flow variables. For any firm, we use a variable s_t (with further indices as needed to specify the firm's type, introduced in the next sub-section) to denote the firm's working stock at the beginning of the period indexed t. The firm's output is characterized fundamentally by a *rate of production*, which depends on the working stock, which we denote by $f(s_t)$ (with other indices as required to distinguish types). In discrete-period models with period length Δt, the *amount* of the good produced within a single period is therefore $f(s_t)\, \Delta t$. We take the incremental increase of goods through production to happen at the beginning of the period, following which firms may sell some of the goods, to be purchased and consumed by consumers within the same period.

Each firm chooses a quantity $q_t \Delta t$ of goods to offer at a buy-sell trading post [2], in which consumers bid money to purchase the good. This quantity is a strategic variable, and can be varied over the range $q_t \Delta t \in [0, s_t + f(s_t)\, \Delta t]$. However, we denote it with the factor Δt made explicit, because in non-cooperative equilibria, the quantity q_t will have a regular limit as an *offer rate* as $\Delta t \to 0$.

5.4.2 Two Production Sectors; One Can Innovate

Goods of types 1 and 2 are produced by firms having production functions denoted respectively f_1 and f_2. If we denote by $s_{i,t}$ the stock of a firm producing good i in period t, the forms we will assume for the production rates are[9]:

$$f_i(s_{i,t}) \equiv f_{i,\infty} - \rho_\pi e^{-2s_{i,t}}. \tag{1}$$

f_i has dimensions of a rate, so both $f_{i,\infty}$ and ρ_π are rates. We will choose ρ_π to equal the discount rate from the definition of firms' discounted profits (introduce below, after the market clearing rule has been defined), to simplify the forms of solutions in worked examples. Nothing apart from simplifying presentation depends on this choice. Well-defined models require that we choose $f_{i,\infty} \geqslant \rho_\pi$ so that production is non-negative for all $s_{i,t} \geqslant 0$. In order to use certain small-deviation approximations in examples below, we will set $f_{i,\infty}/\rho_\pi \gtrsim 1$, but nothing in the model depends on finely tuning the values of these parameters.

The production rate f_2 is assumed to be a fixed function at all periods in all models. The production function f_1 is eligible to change, in period $t = 0$, into a new production function

$$\tilde{f}_1(s_{1,t}) \equiv (1 + \theta) f_1(s_{1,t}), \tag{2}$$

for all periods $t \geqslant 0$, with $\theta > 0$ a fixed parameter. This change of form is the game's representation of successful innovation.[10]

If firms of type 1 try to innovate in period $t = 0$, they must consume a quantity $s^{(\text{cost})}$ from their stocks $s_{1,t}$ at $t = 0$. Innovation succeeds with probability $\xi < 1$.

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[9]These forms are smoothed versions of a linear production function with a limiting output and corner solutions, developed by Shubik and Sudderth [6, 7]. Corner solutions provided a convenient way to truncate discrete-period models to a single period, but in the continuous-time setting, the smoothed production rate produces a simple decomposition of solutions.

[10]The form (2) is the smoothed counterpart to a combination of "cost innovation" and "capacity innovation" in the terminology introduced by Shubik and Sudderth [6, 7]. The rate of production for $s_{1,t} \lesssim 1/2$ is larger by the factor $(1 + \theta)$, generating the same output at less input cost. The saturation level $f_{1,\infty}$ likewise increases by the factor $(1 + \theta)$, so that maximum output capacity likewise increases. This combination is simpler, for the smoothed production function, than either cost innovation or capacity innovation alone.

Firms that attempt to innovate and fail still consume the stock $s^{(\mathrm{cost})}$, but are left with the previous production function f_1.

5.4.3 The Carry-Forward of Goods by Firms

The carry-forward equation for the working stock $s_{i,t}$ held by any firm of type i, at all values of i and t aside from the innovation event by firms of type-1, is

$$s_{i,t+\Delta t} \leqslant s_{i,t} + f_i(s_{i,t})\,\Delta t - q_{i,t}\Delta t. \tag{3}$$

The inequality indicates that the working stock could be wasted but cannot increase except by means of production.

The continuous-time limit is obtained from Eq. (3) by dividing by Δt, and replacing the difference $(s_{i,t+\Delta t} - s_{i,t})\,/\Delta t$ by the derivative ds_i/dt, to obtain a differential equation relating stocks to flows:

$$\frac{ds_i}{dt} \to f_i(s_i) - q_i. \tag{4}$$

We return to the definition of firms' profits after defining the consumption and trade problem for consumers.

5.4.4 Consumption Utilities in Continuous Time

We first introduce the functional form of utility. As a dummy index, let c_1 (without further subscripts) be the rate of consumption of good-1 by any consumer in any particular time period, and let c_2 be the rate of consumption of good-2, by that consumer.[11] Utility for the period's consumption must likewise be defined in terms of a rate in order to permit a well-defined continuous-time limit. The utility rate is a function of the two consumption rates. In this cascade of models we take the separable form

$$u(c_1, c_2) = -\rho\left(e^{-c_1/\gamma_1} + e^{-c_2/\gamma_2}\right). \tag{5}$$

ρ is a constant related to the natural rate of discount, which we define below, needed to provide the correct dimensions for u,[12] and γ_1 and γ_2 are two scale factors that determine the relative price elasticities of the two goods. Note that since c_1 and c_2

[11] Thus, in the discrete-period model, the *amounts* consumed in one period are $c_1\Delta t$ and $c_2\Delta t$.

[12] The absolute magnitude of this constant does not matter for the definition of $u(c_1, c_2)$; only the dimension of a rate is required. We use the rate ρ in the discount factor as this avoids introducing a further arbitrary parameter.

are rates, γ_1 and γ_2 must likewise have dimensions of rates, since the input to the exponential function must be a pure number.

From this base form, which is the same for all consumers, we can introduce an indexed notation for utilities of each of the two types of consumers, in terms of the goods produced by the firms they own and the goods produced by the firms they do not own.

For a consumer of type i, we denote by $c_{i,t}$ the rate of consumption of the good that his own firms produce (now indexing the good *relative to* the consumer's type), and $\tilde{c}_{i,t}$ the rate of consumption of the good produced by firms of the other type.[13] To define a notation that will allow us to refer to agents of either type, denote by u_i the utility rate for a consumer of type i. In terms of Eq. (5), $u_{1,2}$ are given by

$$\begin{aligned} u_1(c_1, \tilde{c}_1) &\equiv u(c_1, \tilde{c}_1) \\ u_2(c_2, \tilde{c}_2) &\equiv u(\tilde{c}_2, c_2)\,. \end{aligned} \tag{6}$$

The variables that define any consumer's state at the beginning of each period are a supply of money-in-hand $m_{i,t}$, and in cases where consumers may make deposits or take out loans with either a central bank or a commercial bank, a balance $a_{i,t}$ at the bank. The account balance $a_{i,t}$ may be of either sign as long as the conditions on money and credit permit.

The consumer's strategic variables within any period are quantities $b_{i,t}\Delta t$ of money to bid on goods made by the firms of his own type, and $\tilde{b}_{i,t}\Delta t$ to bid on goods of the other type, along with deposits $d_{i,t}\Delta t$ to make to the bank. (We refer to them as "deposits" to define the sign convention for the transfer of money between the consumer and the bank; if some $d_{i,t}$ is negative it is a withdrawal.) Therefore, like consumption levels, $b_{i,t}$, $\tilde{b}_{i,t}$, and $d_{i,t}$ are denominated as *rates*.

5.4.5 Market Clearing

The rate at which total bids are made on good i in the buy-sell trading post in any period t is related to the rates of bidding by the two agent types as

$$B_{i,t} = b_{i,t} + \tilde{b}_{\tilde{\imath},t}. \tag{7}$$

The price of good i in period t is denoted $p_{i,t}$. From the clearing rule for the Dubey-Shubik buy/sell model [2], it is given by

$$p_{i,t} = \frac{B_{i,t}\Delta t}{q_{i,t}\Delta t} = \frac{B_{i,t}}{q_{i,t}} = \frac{b_{i,t} + \tilde{b}_{\tilde{\imath},t}}{q_{i,t}}. \tag{8}$$

[13] To express this more didactically, ˜ is used to indicate exclusion, or opposition in binary sets: $\tilde{\imath}$ means whichever value in $\{1, 2\}$ that is not the value taken by index i. $\tilde{c}_i$ indicates the consumption rate of the good that is not the consumption rate c_i.

The price is defined either as a ratio of per-period bid and offer quantities, or as a ratio of their corresponding rates, since factors of Δt cancel in the ratio. Thus price level can converge to a regular continuous-time limit if the bid and offer rates do so.

The rates at which goods are delivered to consumers from trading posts are their consumption rates, which evaluate in the buy/sell game to

$$\begin{aligned} c_{i,t} &= \frac{b_{i,t}}{p_{i,t}} \\ \tilde{c}_{i,t} &= \frac{\tilde{b}_{i,t}}{p_{\tilde{\imath},t}}. \end{aligned} \tag{9}$$

5.4.6 Profit Rates for Firms and (When Applicable) Commercial Banks

Firms are defined in these games to carry forward goods between periods to use as working stocks, and thus they have no money expenses.[14] Their profits equal their proceeds from sale. The amount of profit made by a firm of type i in period t is denoted

$$\pi_{i,t}\Delta t = p_{i,t}q_{i,t}\Delta t = \left(b_{i,t} + \tilde{b}_{\tilde{\imath},t}\right)\Delta t, \tag{10}$$

in which $\pi_{i,t}$ is the corresponding profit rate.

Each firm of type i distributes its profits uniformly among consumer/owners of type i as a source of income for those owners. Since both firms and owners are indexed on the same continuous interval $[0, 1]$, the rate $\pi_{i,t}$ at which profit is delivered by a firm of type i is the same as the rate of income to the consumer of type i.

The firm's total discounted profit, which it seeks to maximize, is the sum

$$\Pi_i = \sum_{t=0}^{T} \beta_\pi^{t/\Delta t} \left[\pi_{i,t}\Delta t - \eta_{i,t}\left(s_{i,t+\Delta t} - s_{i,t} - \Delta t f_i(s_{i,t}) + q_{i,t}\Delta t\right)\right]. \tag{11}$$

The Lagrange multipliers $\eta_{i,t}$ enforce the inequality (3), and the profit discount factor β_π is given in terms of the profit rate of discount ρ_π by

$$\beta_\pi \equiv \frac{1}{1 + \rho_\pi \Delta t}. \tag{12}$$

This is the same ρ_π used to set a scale in the production rate functions (1), for reasons explained where these were introduced.

[14]This construction avoids most of the concerns with corporate financing.

Bank profits, when they are defined, will be particular to models, so at present we simply introduce a notation $\pi_{i,t}^{(B)}$ for the *rate of income* delivered from bank profits to owners of type i. In models without banking or without bank profits, this term is zero. (Recall that commercial banks, when introduced, will be indexed on a continuous interval [0, 1], but they will distribute profits to *two* types of consumers, each type also indexed on an interval [0, 1]. Therefore we will need to be careful with factors of 2 in relating banks' income to profits delivered to owners.)

From the foregoing definitions and the clearing rules (8), (9), the update equation for a consumer of type i's money-in-hand between the beginnings of two successive rounds is

$$m_{i,t+\Delta t} = m_{i,t} - d_{i,t}\Delta t - \left(b_{i,t} + \tilde{b}_{i,t}\right)\Delta t + \left(\pi_{i,t} + \pi_{i,t}^{(B)}\right)\Delta t. \tag{13}$$

5.4.7 The Consumer's Utility Maximization Problem

Trading posts and banks both transact in explicitly represented money (whether gold, fiat, or bank notes). Therefore bids on consumables, and bank deposits, are limited by a budget constraint, which takes the form for a consumer of type i in period t

$$d_{i,t}\Delta t + \left(b_{i,t} + \tilde{b}_{i,t}\right)\Delta t \leqslant m_{i,t}. \tag{14}$$

The consumer maximizes a discounted utility across all periods' consumption against the sequence of constraints (14) at each period t.

To define terminal conditions for the multi-period game, and to produce a salvage value for money, we introduce a "day of reckoning" at period $t = T + \Delta t$, in which any negative bank balance is penalized with a linear deduction $\Pi \min\left(a_{i,T+\Delta t}, 0\right)$ from the total utility. The linear default penalty is enforced by means of a Kuhn-Tucker multiplier on a finite interval $\Lambda_i \in [0, \Pi]$, as in [9]. We return in Sect. 5.4.9 to discuss information conditions, including when agents know the value of T.

The Lagrangian for the optimization problem of a consumer of type i contains a discounted sum of utilities from the rates defined in Eq. (6), constraint terms for the budget constraints, and constraint terms for final conditions. An appropriate form to produce a regular continuous-time limit is given by

$$\begin{aligned} U_i \equiv \sum_{t=0}^{T} \beta^{t/\Delta t} \left\{u_i(c_{i,t}, \tilde{c}_{1,t})\,\Delta t + \lambda_{i,t}\left[m_{i,t} - d_{i,t}\Delta t - \left(b_{i,t} + \tilde{b}_{i,t}\right)\Delta t\right]\right\} \\ + \beta^{(T+\Delta t)/\Delta t}\Lambda_i a_{i,T+\Delta t}. \end{aligned} \tag{15}$$

In models where banking does not exist, the terms $d_{i,t}$ and $a_{T+\Delta t}$ are omitted. Note that the factor $\Lambda_i a_{i,T+\Delta t}$ is discounted by $\beta^{(T+\Delta t)/\Delta t}$.

We also have not incorporated any terms constraining $a_{i,t}$ at intermediate times, such as might arise from limits on reserve requirements.

The per-period discount factor β in the utility function (15) is related to the period length Δt and the natural rate of discount ρ by

$$\beta \equiv \frac{1}{1 + \rho \Delta t}. \tag{16}$$

This convention leads to regular limits for the utility in continuous time. The increment Δt becomes a measure dt, and the sum over index t becomes an integral $\int dt$. The integrand will be a function only of rate-valued quantities, which in the continuous-time limit take piecewise-smooth trajectories. The ratio $m_{i,t}/\Delta t$ must likewise scale as a rate-valued quantity, which has the interpretation of the contribution from an agent of type i to the velocity of money, as the money supply scales linearly toward zero with Δt.

5.4.8 The Consumer's Bank-Balance Dynamics

The first two banking models demonstrated here permit unlimited revolving loans. Technically this means two things. The first is that the bank keeps an account, the balance of which is updated at a pre-specified interest rate within each period. The second is that the amount consumers deposit or withdraw is an unconstrained variable, apart from the penalty on unrepaid bank debts in the terminal conditions.

The bank's carry-forward equation for accounts it thus

$$a_{i,t+\Delta t} = \left(a_{i,t} + d_{i,t} \Delta t\right) \left(1 + \rho_{B,it} \Delta t\right). \tag{17}$$

Deposits or withdrawals are made at the beginning of the period, and interest accrues at a rate $\rho_{B,it}$.[15]

The bank may lend or accept deposits at different rates, in which case the interest rate for either type i is a function of time, evaluated to equal a lending or borrowing rate according to the rule

$$\rho_{B,it} = \begin{cases} \rho_{B,L} & \text{if } a_{i,t} < 0 \\ \rho_{B,D} & \text{if } a_{i,t} > 0. \end{cases} \tag{18}$$

To regularize the discontinuity at $a_t = 0$, we may adopt some convention such as $\rho_{B,t} = \left(\rho_{B,L} + \rho_{B,D}\right)/2$.[16] For a central bank acting as a public service, there is no

[15]Many alternative rules are well-defined: interest on deposits could accrue one period later than interest charged on loans, etc. Nothing depends on the intra-temporal order of interest charges and payments, in the continuous-time limit.

[16]Under conditions when the bank is actively used, $a_t = 0$ occurs only on time intervals of measure zero, so the results are not sensitive to the way the interest rate is regularized. Because, in this model, we assume initial conditions prior to the accumulation of bank balances, it is convenient to choose a regularization condition that will be consistent with the other simplifying assumptions made in the model.

need for interest rate spreads, but for a commercial bank a spread $\rho_{B,L} - \rho_{B,G} > 0$ will generally be required.

Using the money carry-forward relation (13) to express $d_{i,t}$ in terms of the bids, profits, and changes in agents' money-holdings, the account-balance carry-forward relation (17) may be written

$$\frac{a_{i,t+\Delta t}}{(1+\rho_{B,it}\Delta t)} = a_{i,t} - (m_{i,t+\Delta t} - m_{i,t}) - \left(b_{i,t} + \tilde{b}_{i,t}\right)\Delta t + \left(\pi_{i,t} + \pi_{i,t}^{(B)}\right)\Delta t. \quad (19)$$

Using Eq. (10) for firms' profits, dividing Eq. (19) by Δt, and then taking $\Delta t \to 0$ produces the continuous-time expression for the bank balance in relation to the money-in-hand m_i of[17]

$$\left(\frac{d}{dt} - \rho_{B,i}\right) a_i \to \left(\tilde{b}_{\tilde{i}} - \tilde{b}_i\right) - \frac{dm_i}{dt} + \pi_i^{(B)} + \mathcal{O}(\Delta t)\,. \quad (20)$$

5.4.9 Terminal Conditions

The handling of terminal conditions in a class of extended-time games of this form, with lending at interest, a small number of events that can occur, and no stochasticity, is generally a somewhat artificial exercise as a model of decision making in real economies. On one hand, the attempt by consumers and firms to converge to a steady state that permits long-term regular behavior, and the degree to which monetary flexibility permits or impedes that attempt, is the aspect of decision making that the model probably captures robustly. On the other hand, the specification of terminal conditions is a requirement from the standpoint of experimental gaming, and this generally rules out a steady state. The artificial feature of a model that requires cancellation of all debts at a finite horizon, in an economy that has structurally changed in the interim in such a way that revolving debt permits it to accommodate the change, is that exponential growth of account balances can lead to sensitive and arbitrary coupling of terminal conditions to otherwise-negligible differences in interior solutions. A continuum of solutions to the first-order conditions exist with utilities and profits that differ by exponentially small factors in $\rho_\pi T$, but which involve very different response of the production decisions at the terminal conditions.

We resolve these ambiguities by making use of the following observation to single out the class of non-cooperative equilibria that robustly separate the responses to initial and terminal conditions in a non-arbitrary manner. These games possess non-cooperative equilibria that could be called "turnpike solutions". Consumers and

[17] The residual terms at $\mathcal{O}(\Delta t)$, which we denote explicitly despite the fact that they approach zero as $\Delta t \to 0$, come from time lags between the making of bids and the delivery of profits. As long as the rates are continuous (differentiable at order one) functions, these effects contribute terms $\sim (db_i/dt)\,\Delta t$ in Eq. (20).

firms, after a transient that occupies an interval in $\rho_\pi t$ much smaller than 1, can converge exponentially (in $\rho_\pi t$) toward stable production, trade, and consumption values that can be preserved indefinitely. In general, these solutions require non-zero bank balances, as some agents lend to others, with interest flows supporting asymmetries in their consumption that reflect the real structural changes in the production sector. These steady-state values are the turnpike values. The games also possess a class of unstable solutions, in which firms exponentially diverge from the turnpike values, depleting or hoarding stocks in response to exponentially diverging price levels created by consumer bidding, as consumers re-direct their money to return their bank balances to zero. The diverging solutions cannot be extended indefinitely because they become singular, so they never occur at intermediate times. They can be chosen, however, to accommodate a terminal condition that eliminates all debts to the banks. The turnpike solutions require a specific coordinated price-setting behavior by the two types of consumers, and of production decisions by the two types of firms (all of which can be computed non-cooperatively by each group of agents), in order that neither aggregate nor internal debt exist at the terminal time.

In addition to the turnpike solutions, a continuum of other solutions exist, in which very small uncanceled aggregate debts can grow exponentially, and require different behavior by the two types of consumers and the two types of firms, relative to turnpike solution, to cancel aggregate as well as internal debt. The final behavior of the agents in these solutions is sensitive to uncanceled aggregate debts that may be of order $e^{-\rho_\pi T}$ at the end of transient response to the initial conditions, and which constitute arbitrarily small deviations from the pure turnpike solution.

To isolate the turnpike solutions, the agents are not told the time T of the terminal round at the beginning of the game. Instead, they are told that, in each period Δt, a binary variable will be sampled. The first time t at which the variable equals 1, the terminal round will be announced to occur at a specified later time, such as $T = t + 5/\rho_\pi$ (so five times the discount horizon, out from the present). The probability to draw value 1 is made sufficiently small that the values of T will be Poisson distributed with a mean much longer than the discount horizon $1/\rho_\pi$. This look-ahead declaration provides sufficient time to implement the terminal behaviors starting from time t, with utility consequences of differing from the turnpike solution that are bounded above by $\mathcal{O}\left(e^{-(T-t)\rho_\pi}\right) \approx \mathcal{O}\left(e^{-5}\right)$. (We choose the look-ahead horizon $t + 5/\rho_\pi$ for convenience in examples below; this number may be chosen as large as desired to decouple the initial terminal intervals to any desired degree.) As long as the error $e^{-(T-t)\rho_\pi}$ is made $\ll \Delta t \rho_\pi$, it is a smaller correction than finite-period discretization effects that we are ignoring. Players who solve the initial transient to converge to the turnpike produce a solution that is within $\mathcal{O}\left(e^{-(T-t)\rho_\pi}\right)$ of any non-cooperative equilibrium solution for any large T. Any non-cooperative equilibrium not converging to the turnpike could be one of a range of exact solutions for a particular T, but which solution this would be would depend on aggregate debt levels of $\mathcal{O}\left(e^{-(T-t)\rho_\pi}\right)$, and the initial part of this trajectory would differ from any non-cooperative equilibrium, for any terminal time different from T by more than $\mathcal{O}(1/\rho_\pi)$, at more than $\mathcal{O}\left(e^{(T-t)\rho_\pi}\right)$.

We will not develop the full machinery of expected-utility maximization in this note, but will simply compute properties of the turnpike equilibria, with the understanding that all deviations from these by more than $\mathcal{O}\left(-e^{(T-t)\rho_\pi}\right)$ are incompatible with existence of any non-cooperative equilibrium over ranges of T where the terminal-condition sampling has large probability, and so will be ruled out by any generic expected-utility maximization.

We believe that this minimal use of a stochastic variable yields the kinds of solutions that would arise in an actual economy where money and banking are available to facilitate regular events of structural change in the production sector, and in which agents carry persistent debt and respond to new events of innovation by changing their debt structure as these arise, in ongoing sequences. The addition of *specific* finite-horizon debt could, of course, be introduced as a qualitative modification to these games, but it should then be justified by other criteria (lenders' limitations, etc.) besides the question whether a flexible money supply can alleviate constraints on the circular flow of funds, which is the topic addressed by the current class of games.

5.4.10 The Leading Contributions in ($\rho_B \Delta t$) to the Time-Course of Monetized Private Credit and the Net Account Balance of Agents at the Bank

Now for the first time we may use small but nonzero Δt to distinguish the behavior of two components of the credit supply. One component comes from lending effectively by one type of consumers to the other, mediated by the bank. Promises to pay by consumers (enforced by the default penalty at the day of reckoning) are privately issued credit. Banks' promises to pay (whatever interest plus principle accrues) are met with bank credit. The part of loans and deposits that cancel among the consumers are effectively private credit from one group to another, monetized by the bank when it accepts private promises to pay and issues bankers' promises to pay. The part of loans or deposits that does not cancel when consumers are aggregated is the net injection or extraction of money in circulation. Injected money is also in the form of bank credit, while extraction may be whatever form of money was given to the consumers in the initial conditions. The two coordinates we use to represent intra-economy lending, and aggregate-economy lending, are respectively $(a_{1,t} - a_{2,t})$ and $(a_{1,t} + a_{2,t})$.

In a continuous-time model with regular prices, the supply of money in circulation scales as $\mathcal{O}(\Delta t)$. If banking is to leave prices regular, the change in money supply, driven by the sum of balances $(a_{1,t} + a_{2,t})$, must also scale linearly in $\mathcal{O}(\Delta t)$. In contrast, as we show now, the monetized private credit will normally scale as $\mathcal{O}(1)$ in economies operating at or near monetary efficiency. Thus some agents have outstanding, at any time, debts that are larger by $\mathcal{O}(1/\Delta t)$ than all money in circulation.

Personal Credit Monetized by Bank Accounting

From Eq. (20), the equation for $(a_{1,t} - a_{2,t})$ is

$$\left[\frac{d}{dt} - \frac{(\rho_{B,1} + \rho_{B,2})}{2}\right](a_1 - a_2) \to 2\left(\tilde{b}_2 - \tilde{b}_1\right) + \frac{(\rho_{B,1} - \rho_{B,2})}{2}(a_1 + a_2) - \frac{d(m_1 - m_2)}{dt} + \mathcal{O}(\Delta t)\,. \tag{21}$$

As long as both $(a_1 + a_2)$ and $d(m_1 - m_2)/dt$ are $\mathcal{O}(\Delta t)$ like the terms that have been dropped—a requirement if prices are not to diverge in the continuous-time limit—any $\mathcal{O}(1)$ contribution to $(a_1 - a_2)$ can only come from the term in $\left(\tilde{b}_{\bar{\imath}} - \tilde{b}_i\right)$.

Reducing Eq. (21) to quadrature gives the expression for the credit monetized by the banks within the economy,

$$\frac{a_{1,t} - a_{2,t}}{2} = \int_0^t dt'\, e^{\int_{t'}^{t} dt''\left(\rho_{B,1t''} + \rho_{B,2t''}\right)/2}\left(\tilde{b}_{2,t'} - \tilde{b}_{1,t'}\right) + \mathcal{O}(\Delta t)\,. \tag{22}$$

To determine the conditions under which these bank balances can approach a steady state turnpike solution that can extend indefinitely, we integrate Eq. (22) by parts to obtain the equivalent expression

$$\frac{a_{1,t} - a_{2,t}}{2} = e^{\int_0^t dt'\left(\rho_{B,1t'} + \rho_{B,2t'}\right)/2}\left\{\frac{\left(\tilde{b}_{2,0} - \tilde{b}_{1,0}\right)}{\left(\rho_{B,10} + \rho_{B,20}\right)/2} + \int_0^t dt'\, e^{-\int_0^{t'} dt''\left(\rho_{B,1t''} + \rho_{B,2t''}\right)/2}\frac{d}{dt'}\frac{\left(\tilde{b}_{2,t'} - \tilde{b}_{1,t'}\right)}{\left(\rho_{B,1t'} + \rho_{B,2t'}\right)/2}\right\} - \frac{\left(\tilde{b}_{2,t} - \tilde{b}_{1,t}\right)}{\left(\rho_{B,1t} + \rho_{B,2t}\right)/2}. \tag{23}$$

Section "Steady Post-Innovation Output and Stable Money Supply Lead to Stable Bid Levels" in the Appendix shows that the intermediate-time bids $\left(\tilde{b}_{2,t'} - \tilde{b}_{1,t'}\right)$ converge on steady values as long as the money supply is asymptotically constant, which is the condition for a non-inflationary solution.[18] Hence the time derivative in the integral in Eq. (23) approaches zero for t' sufficiently large. We return in

[18]Without uncertainty it calls for the rate ρ defining the utilitarian rate of discount in Eq. (16) to equal the average of the two interest rates faced by the agents, as shown in Eq. (42) below. (In the worked example of the following sections, this will be the average of the borrowing and the lending rates.) With uncertainty there is a delicate correction depending on the variance.

Sect. 6.2.3 to the way this solution connects to a terminal transient that returns both of $(a_{1,t} \pm a_{2,t})$ to zero as $t \to T$.

The relation (23), which at large t is exponentially well-approximated by the vanishing of the term in curly braces, determines $\hat{\epsilon}$ from Eq. (31) and Eq. (30).[19] Because this equation is homogeneous of order one in the numéraire, it is not necessary to know the overall magnitude of the money supply to determine $\hat{\epsilon}$.

Aggregate Debt and Change in the Money Supply

The mechanism by which banks may change the money in circulation is lending to or accepting deposits from consumers at interest. For example, consumers may borrow an initial stock of money following the event in which innovation occurs, and over the course of restoring the principle to zero so that the money-in-circulation converges to a steady value, they pay some quantity of aggregate interest.

Summing Eq. (20) over both agent types gives the equation for $(a_{1,t} + a_{2,t})$:

$$\left[\frac{d}{dt} - \frac{(\rho_{B,1} + \rho_{B,2})}{2}\right](a_1 + a_2) \to -\frac{d}{dt}(m_1 + m_2) + \left(\pi_1^{(B)} + \pi_2^{(B)}\right) + \frac{(\rho_{B,1} - \rho_{B,2})}{2}(a_1 - a_2). \tag{24}$$

In models with interest rate spreads, we face the possibility that the term in $(\rho_{B,1} - \rho_{B,2})$ in the second line of Eq. (24) could destroy the stability of prices by coupling the quantity $(a_1 - a_2)$ which is $\mathcal{O}(1)$ to the change in the money supply which must scale as $\mathcal{O}(\Delta t)$ for prices to be stable. In appropriately defined models this potential instability will be avoided, because the total profits from commercial banks $\left(\pi_1^{(B)} + \pi_2^{(B)}\right)$ will be a revenue $-(\rho_{B,1}a_1 + \rho_{B,2}a_2)$, minus a stream paid to the central bank. As long as the stream to the central bank remains at $\mathcal{O}(\Delta t)$, the remaining revenue stream recirculates, canceling the term $(\rho_{B,1} - \rho_{B,2})(a_1 - a_2)/2$ to within $\mathcal{O}(\Delta t)$. Any component of $-(\rho_{B,1}a_1 + \rho_{B,2}a_2)$ that is $\mathcal{O}(1)$ is also assured to be positive, because it can only come from a difference $(a_1 - a_2)$ that is $\mathcal{O}(1)$,

[19] When the term in curly braces is exactly zero, the late-time steady-state relation becomes

$$\frac{(\rho_{B,1t} + \rho_{B,2t})}{2}\frac{(a_{1,t} - a_{2,t})}{2} = \left(\tilde{b}_{2,t} - \tilde{b}_{1,t}\right).$$

This expression is simply the interest paid to agents of type-1, plus their share of bank profits when profits are defined, which balances the deficit in the profits of type-1 firms relative to the bids made by type-1 agents (who will consume more). Thus a consistent circular flow is restored in the asymptotic steady state, in a context of asymmetric production, profits, depositing/borrowing, and consumption.

and the lending rate (on the negative account balance) will be higher than the rate on deposits (the positive balance).

The simplest case will be 100 % reserve banking, in which any aggregate loans a commercial bank makes to consumers cannot exceed supplies of "heavy money" the commercial bank borrows from the central bank and holds as reserves. In that case the total profit stream of the commercial bank takes the form

$$\pi^{(B)}_{1,t} + \pi^{(B)}_{2,t} = \rho_{C,t}\left(a_{1,t} + a_{2,t}\right) - \left(\rho_{B,1t}a_{1,t} + \rho_{B,2t}a_{2,t}\right), \tag{25}$$

where ρ_C is the interest rate charged by the central bank.

Substituting this into Eq. (24) gives

$$\left(\frac{d}{dt} - \rho_C\right)\left(a_1 + a_2\right) \to -\frac{d}{dt}\left(m_1 + m_2\right). \tag{26}$$

Note that, if there is no commercial bank, and the consumers borrow from or deposit into the central bank directly, Eq. (26) results directly from Eq. (24).

In the simplifying case where ρ_C is constant, Eq. (26) is integrated to give the result

$$\begin{aligned}\left(a_{1,t} + a_{2,t}\right) = e^{\rho_C t}\left\{\left(a_{1,0} + a_{2,0} + m_{1,0} + m_{2,0}\right) - \int_0^t dt' \, \rho_C e^{-\rho_C t'}\left(m_{1,t'} + m_{2,t'}\right)\right\} \\ - \left(m_{1,t} + m_{2,t}\right).\end{aligned} \tag{27}$$

Both in the model with only a central bank, and in the model with a commercial bank using 100 % reserves, we will set $a_{i,0} = 0$ as initial condition, and $\left(m_{1,0} + m_{2,0}\right) \equiv 2m_0$ to define the initial money supply. Agents may borrow an amount of money that scales as $\sim m_0$ from the bank in the period $t = 0$ when the innovation event occurs, changing both the initial money supply and the initial debt abruptly. Under any such borrowing, however, $\left(a_{1,t} + a_{2,t} + m_{1,t} + m_{2,t}\right)_{t \to 0^+} = \left(a_{1,0} + a_{2,0} + m_{1,0} + m_{2,0}\right)$. Therefore both the initial value and the integral in Eq. (27) involve no singular terms even in the continuous-time limit.

The vanishing of the steady-state principle $-\left(a_{1,t} + a_{2,t}\right)$ owed by the agents to the banks in Eq. (27) determines the initial borrowed amounts $\left(m_{1,t} + m_{2,t}\right)_{t \to 0^+} - 2m_0 = -\left(d_{1,0} + d_{2,0}\right)\Delta t$, because these set the scale for the quantity $\left(m_{1,t'} + m_{2,t'}\right)$ in the integral and the final term $\left(m_{1,t} + m_{2,t}\right)$ relative to the initial term $\left(a_{1,0} + a_{2,0} + m_{1,0} + m_{2,0}\right) = 2m_0$, which is fixed. The vanishing of the term in curly braces in Eq. (27), taken as $t \to \infty$, given a value of $\hat{\epsilon}$ fixed by vanishing of the similar term in curly braces in Eq. (23), defines the turnpike response to the initial shock created by the innovation opportunity and the need to borrow.

5.5 First-Order Conditions

5.5.1 The Consumer's Goods-Consumption Problem

The first-order condition for consumption results from variation of $b_{i,t}$ and $\tilde{b}_{i,t}$ in Eq. (15), and takes the form

$$\frac{1}{p_{i,t}}\frac{\partial u_i}{\partial c_{i,t}} = \frac{1}{p_{\tilde{\imath},t}}\frac{\partial u_i}{\partial \tilde{c}_{i,t}} = \sum_{t'=t}^{T} \beta^{(t'-t)/\Delta t}\lambda_{i,t'}. \tag{28}$$

Irrespective of how the Kuhn-Tucker multipliers for these constraints are set,[20] the ratios of first-order conditions (28) imply relations of relative consumption between the two types of agents, who purchase against a shared price system. In the remainder of this sub-section, we suppress the explicit time index, because the relations hold period-by-period at each t.

The two ratios of marginal utilities of consumption are both given in terms of prices by

$$\frac{\partial u_1/\partial c_1}{\partial u_1/\partial \tilde{c}_1} = \frac{p_1}{p_2} = \frac{\partial u_2/\partial \tilde{c}_2}{\partial u_2/\partial c_2}. \tag{29}$$

To solve for the consequences of this relation, we introduce a pair of coordinates to relate the consumption of the two types of agents to the offer levels $q_{i,t}$. Define

$$\begin{aligned} c_1 &\equiv \frac{q_1}{2} + \epsilon_1 & \tilde{c}_2 &\equiv \frac{q_1}{2} - \epsilon_1 \\ \tilde{c}_1 &\equiv \frac{q_2}{2} + \epsilon_2 & c_2 &\equiv \frac{q_2}{2} - \epsilon_2. \end{aligned} \tag{30}$$

The model choice of a separable exponential utility (5) leads to the result that the offsets $\epsilon_{1,2}$ from even division for the two goods are in a fixed proportion determined by the relative elasticities,

$$\frac{\epsilon_1}{\gamma_1} = \frac{\epsilon_2}{\gamma_2} \equiv \hat{\epsilon}. \tag{31}$$

The output rate q_i will always appear scaled by the factor γ_i in the utility, so we introduce a shorthand

$$\hat{q}_i \equiv \frac{q_i}{\gamma_i}. \tag{32}$$

[20] These multipliers are always nonzero, as the budget constraint is always tight.

Because prices are ratios of total bids to total outputs, Eq. (29) together with the condition (31) implies that

$$\frac{B_1}{B_2} = \frac{\hat{q}_1 e^{-\hat{q}_1/2}}{\hat{q}_2 e^{-\hat{q}_2/2}}. \tag{33}$$

The relation of the bid level for either good to the total money supply is then

$$\frac{B_i}{B_1 + B_2} = \frac{\hat{q}_i e^{-\hat{q}_i/2}}{\hat{q}_1 e^{-\hat{q}_1/2} + \hat{q}_2 e^{-\hat{q}_2/2}}. \tag{34}$$

Therefore prices are given in relation to the total money rate of circulation B_1+B_2 by

$$p_i = \frac{1}{\gamma_i} \frac{e^{-\hat{q}_1/2}}{\hat{q}_1 e^{-\hat{q}_1/2} + \hat{q}_2 e^{-\hat{q}_2/2}} \left(B_1 + B_2\right). \tag{35}$$

5.5.2 Consumer's Banking Problem (When Applicable)

If the economy is one in which borrowing and lending are possible, a second condition for deposits or withdrawals results from variation of $d_{i,t}$. It there is no limit on consumers' account balances, the only two classes of Kuhn-Tucker multipliers come from the per-period budget constraint ($\lambda_{i,t}$) and the terminal conditions (Λ_i).[21] The first-order condition for deposits is then

$$\sum_{t'=t}^{T} \beta^{(t'-t)/\Delta t} \lambda_{i,t'} = \Lambda_i \prod_{t'=t}^{T} \left[\beta \left(1 + \rho_{B,it'} \Delta t\right)\right]. \tag{36}$$

Combining Eq. (28) with Eq. (36), and taking $\Delta t \rightarrow 0$, we arrive at the continuous-time relation among prices, output, interest rates, and a single Kuhn-Tucker multiplier for the terminal constraint:

$$\frac{1}{p_{i,t}} \frac{\partial u_i}{\partial c_{i,t}} = \frac{1}{p_{\tilde{\imath},t}} \frac{\partial u_i}{\partial \tilde{c}_{i,t}} \rightarrow e^{\int_t^T dt' \left(\rho_{B,it'} - \rho\right)} \Lambda_i. \tag{37}$$

Using the relations (29), (30), which hold at each time, we can evaluate the consumption first-order conditions (37) for the two types explicitly, to give

$$\frac{1}{p_{1,t}\gamma_1} e^{-\hat{q}_{1,t}/2} = \frac{1}{p_{2,t}\gamma_2} e^{-\hat{q}_{2,t}/2} \rightarrow e^{\hat{\epsilon}_t + \int_t^T dt' \left(\rho_{B,1t'} - \rho\right)} \frac{\Lambda_1}{\rho},$$

[21] If bounds were placed on the account balances, additional multipliers could arise within each period as shadow prices associated with these constraints.

$$\frac{1}{p_{1,t}\gamma_1}e^{-\hat{q}_{1,t}/2} = \frac{1}{p_{2,t}\gamma_2}e^{-\hat{q}_{2,t}/2} \to e^{-\hat{\epsilon}_t + \int_t^T dt' \left(\rho_{B,2t'} - \rho\right)} \frac{\Lambda_2}{\rho}. \tag{38}$$

The consumption asymmetry $\hat{\epsilon}$ must then satisfy

$$\hat{\epsilon}_t = \hat{\epsilon}_T - \frac{1}{2}\int_t^T dt' \left(\rho_{B,1t'} - \rho_{B,2t'}\right). \tag{39}$$

The Kuhn-Tucker multipliers for the two types of agents are related to the final-time value $\hat{\epsilon}_T$ as

$$e^{\hat{\epsilon}_T}\Lambda_1 = e^{-\hat{\epsilon}_T}\Lambda_2 \equiv \Lambda. \tag{40}$$

A Note on the Setting of the Default Penalty

We will show that, in general, $\hat{\epsilon}_T$ cannot equal zero, because consumers of different types have different incomes and consume at different levels. Therefore the shadow prices Λ_1 and Λ_2 cannot both be equal; hence, even in a game with artificially fine-tuned parameters, they could not both be set equal to the limiting value Π of the default penalty. Interior solutions can therefore only be obtained when at least one of $\Lambda_1 < \Pi$ or $\Lambda_2 < \Pi$ holds, and when both $a_{1,T} = 0$ and $a_{2,T} = 0$. This permits us to set Π "sufficiently large" that both $\Lambda_1 < \Pi$ and $\Lambda_2 < \Pi$, and to consider interior solutions without default and also with no savings at the day of reckoning. These two requirements define the terminal conditions for interior solutions with banking. We will illustrate their consequences for prices and production in Sect. 6.2.3.

The pair of first-order conditions (38) evaluate to a relation between the two prices and output levels to a single multiplier Λ (jointly determined by the agents' non-cooperative equilibria) and the (possibly time-dependent) interest rates of the two types:

$$\frac{1}{p_{1,t}\gamma_1}e^{-\hat{q}_{1,t}/2} = \frac{1}{p_{2,t}\gamma_2}e^{-\hat{q}_{2,t}/2} \to e^{\int_t^T dt' \left[\frac{1}{2}\left(\rho_{B,1t'} + \rho_{B,2t'}\right) - \rho\right]} \frac{\Lambda}{\rho}. \tag{41}$$

It was necessary that the relation between the output level of either good and its price in Eq. (41) be the same for the two goods, because by Eq. (35) either of these equals a relation between both output levels and the total money supply. Combining the two equations gives

$$\frac{\hat{q}_1 e^{-\hat{q}_1/2} + \hat{q}_2 e^{-\hat{q}_2/2}}{B_1 + B_2} \to e^{\int_t^T dt' \left[\frac{1}{2}\left(\rho_{B,1t'} + \rho_{B,2t'}\right) - \rho\right]} \frac{\Lambda}{\rho}. \tag{42}$$

Taking logarithms, and then differentiating with respect to t, then gives the relation between outputs, money supply, and interest rates

$$\frac{d}{dt}\log\left(\frac{\hat{q}_1 e^{-\hat{q}_1/2}+\hat{q}_2 e^{-\hat{q}_2/2}}{B_1+B_2}\right)=\left(\rho-\frac{\rho_{B,1}+\rho_{B,2}}{2}\right). \tag{43}$$

5.5.3 The Firms' Output Levels in Response to Prices

Firms attempt to maximize profits (11) in which the price sequences $p_{i,t}$ appear as parameters from Eq. (10).

Firms may respond to prices in either of two ways. Either

$$p_{i,t} \leqslant \eta_{i,t}, \tag{44}$$

and they set $q_{i,t} \to 0$, or else $q_{i,t} > 0$, and

$$\eta_{i,t-\Delta t} = \eta_{i,t}\beta_\pi\left(1+f_i'(s_{i,t})\right) \tag{45}$$

The former case can be realized by successfully-innovating firms in the early periods following innovation, in which they are better off to sit out of markets and rebuild their working stocks, while the type-1 firms that attempted to innovate and failed, provide the total supply in markets. Firms that failed in innovating can maintain market prices lower than the reservation prices of the successful firms, because their steady-state allocations at late times are not as high (we demonstrate this below), so that using their entire output to rebuild stocks is not as valuable to them as it is to the successful firms.

In the latter case, faced by all firms at sufficiently late times, by the type-1 firms that try and fail to innovate, and by all type-2 firms all the time, these firms optimize their output against the particular sequence of prices.

The recursive relation (45) among K-T multipliers becomes, in the continuous-time limit,

$$\eta_{i,t} = \eta_{i,T}e^{\int_t^T dt'\left[f_i'(s_{i,t'})-\rho_\pi\right]}. \tag{46}$$

When $p_{i,t} = \eta_{i,t}$, Eq. (46) dictates an intertemporal relation between prices and output which is the consequence of the profit-maximization criterion.

Setting $p_{i,t} = \eta_{i,t}$ in Eq. (46), combining this with the consumers' price/output/interest relations (41), taking logarithms, and differentiating with respect to t, produces a three-way relation among output levels, the stocks of all firms that are active offering in markets, the interest rates faced by consumers of both types, and the total money supply, in the form

$$\frac{d}{dt}\log\left(\frac{\hat{q}_1 e^{-\hat{q}_1/2}+\hat{q}_2 e^{-\hat{q}_2/2}}{B_1+B_2}\right) = -\frac{1}{2}\frac{d}{dt}\hat{q}_i + \left[f_i'(s_i) - \rho_\pi\right]$$
$$= \left(\rho - \frac{\rho_{B,1}+\rho_{B,2}}{2}\right). \tag{47}$$

The equality in the second line applies only in the case that consumers set prices by varying the money in circulation through borrowing and lending.

5.5.4 Equation (47) Is the Main Relation

Equation (47) is the main relation that links output decisions by firms to the dynamics of the money supply. The right-hand side of the first equality is a second-order differential response function of the working stocks and aggregate output of firms of a given type, to a source term (the left-hand side of the first equality) which involves the output levels of both goods, and the total money supply $(B_{1,t} + B_{2,t})$. The $\hat{q}_i$ on the right-hand side represents a total output variable,[22] and originates in the separable exponential utility of consumption (5). In this respect the separability between the left and right-hand sides in an exact relation depends on the specific assumption of exponential utility, which we introduced in order to make the production/consumption model a sharp test case for monetary efficiency. If the second equality in Eq. (47) applies, it determines both the dynamics of the total money supply, and the source term for output decisions, in terms of the interest rates faced by the two kinds of consumers in relation to the natural rate of discount.

Working stock and output decisions for both firms are coupled to the same source term which is an aggregate property of the whole economy. Moreover, the production decisions in the two sectors are independent of one another *except* for this shared source term, and except for any initial and terminal conditions created, respectively, by the innovation-induced shock to the supplies of working stocks, and the requirement to nullify bank debts on the day of reckoning. The form imposed on these equations by a particular monetary system therefore determines whether that system can insulate production decisions in the two sectors from one another, and if it cannot, the manner and strength with which they are coupled.

[22]This term must be corrected with a measure term to relate it to individual firms' output levels if not all firms are active in markets, as we show below.

5.5.5 The Criterion of Monetary Efficiency

We may thus sharply define the criterion for efficiency of the monetary system. **If the banking system makes the supply of money in circulation sufficiently flexible** that the money supply $(B_{1,t} + B_{2,t})$ can exactly track the numerator term $\left(\hat{q}_1 e^{-\hat{q}_1/2} + \hat{q}_2 e^{-\hat{q}_2/2}\right)$, then the production decisions in the sectors for good-1 and good-2 are completely decoupled. Supply shocks in one sector do not affect production in the other. **Scale in the overall economy has been separated from the structure of production and consumption**, with the result that intertemporal coordination of production may be optimized for each good through a price system, delivering the same production profiles as if the two goods occupied two separate economies. Note that, in economies with banking, this is possible only if $\rho - (\rho_{B,1t} + \rho_{B,2t})/2$ is constant.

Although production decisions are decoupled in an efficient economy, the relative consumption levels of both types of agents, for both goods, may become responsive to the innovation shock because their relative incomes differ due to the dependence of profit rates $\pi_{i,t}$ on supply rates $q_{i,t}$, even in cases where the two price systems are decoupled.

The feature that production rates are coupled only through the total money supply and not through its instantaneous distribution depends on the exponential utility (5), through the cancellation (31) of $(\epsilon_1/\gamma_1 - \epsilon_2/\gamma_2)$. This kind of modeling choice is similar in spirit to the choice of strictly symmetric production technologies in the one-period models of [8]. It is a *minimal* form that permits the many functions of the price system as a separating hyperplane to be performed independently. The overall production sector is separated from the dynamics of consumption due to wealth effects by one variable (the total money supply), whether or not the production decisions by firms of different types are also separated from each other.

5.5.6 Expansions in Small Deviations About the Fixed-Point Production Rate

In order to produce simple approximate demonstrations of the behavior of models in this class, we consider innovation shocks that are small compared with background stock and production levels, and evaluate responses to leading order in small perturbations.

The steady-state condition for production stocks, with production function f_i and embedded in an economy with steady prices, is given by Eq. (46) as

$$f_i'(\bar{s}_i) \equiv \rho_\pi . \tag{48}$$

Whenever all firms of type i are offering in markets, the offer rate equals the total output rate for good i, so we can abuse notation and use $q_{i,t}$ for both quantities. If only a measure $(1-\xi)$ of firms are offering in markets, then the output level per firm equals $(1-\xi)$ times the offer rate of the active firms.

The *offer rate* is approximated at leading linear order in small departures $s_i - \bar{s}_i$ by

$$q_i \approx f_i(\bar{s}_i) + \left(f_i'(\bar{s}_i) - \frac{d}{dt} \right) (s_i - \bar{s}_i) = \bar{f}_i + \left(\rho_\pi - \frac{d}{dt} \right) (s_i - \bar{s}_i) \, . \tag{49}$$

The first-order expansion for the marginal productivity appearing in Eq. (46) is

$$f_i'(s_i) - \rho_\pi \approx f_i''(\bar{s}_i) \, (s_i - \bar{s}_i) \equiv \bar{f}_i'' \, (s_i - \bar{s}_i) \, . \tag{50}$$

Using Eq. (49) to approximate q_i, and Eq. (50) to approximate the marginal productivity, in the first line of Eq. (47) gives

$$\frac{d}{dt} \log \left(\frac{\hat{q}_1 e^{-\hat{q}_1/2} + \hat{q}_2 e^{-\hat{q}_2/2}}{B_1 + B_2} \right) \approx \frac{\mu_i}{2\gamma_i} \frac{d}{dt} \left(\frac{d}{dt} - \rho_\pi \right) (s_i - \bar{s}_i) + \bar{f}_i'' \, (s_i - \bar{s}_i) \tag{51}$$

Here we have introduced a measure term μ_i, which equals unity when all firms of type i are active in markets, and equals $(1-\xi)$ in the case when a measure ξ of type-1 firms that have successfully innovated are sitting out of markets.

The right-hand side of Eq. (51) is a *linear* second-order differential response function, which means that the responses to different source terms or within different time intervals can be constructed independently and added to produce the full solution for $(s_i - \bar{s}_i)$. Complex matching conditions only arise at points where the solution for the non-cooperative equilibrium changes structure in some way, as when a subset of firms first enters markets, or when a type of consumers switch from being borrowers to being lenders. These are economically meaningful changes that only occur at a few points in a continuous time interval corresponding formally to an infinite number of periods (each of infinitesimal duration), in contrast with period boundaries in discrete-period models, which create complex matching conditions in every period. This feature explains our statement that the continuous-time limit may be seen as one in which the model period length does not reflect an economically significant timescale, and therefore should not affect the structure of solutions.

6 Example Solutions

6.1 Exchange with Gold Money Only

In a gold economy without banking or any other reserve supply of gold, the circulation rate $(B_{1,t} + B_{2,t})$ is constant in every period. Consumers spend all money in their possession. Therefore the time derivative on the left-hand side of Eq. (47) cannot be zero if $q_{1,t}$ experiences the shock of the investment in innovation. The production decisions of the two goods that can appear on the right-hand side of Eq. (47) must be coupled. The failure of decoupling—which defines our criterion of optimal monetary performance—is the main result which shows that gold or any money with fixed supply provides poor support to an exchange economy in which the production functions and consumption utilities could otherwise be optimized separately. It is another realization of Schumpeter's general observation about difficulties in breaking the circular flow of funds.

6.2 Innovation and Recovery in Utopia

Having established in Sect. 6.1 that a fixed money supply couples the shock in good-1 to production decisions in good-2, we now consider the opposite case of banking that creates any required level of bank money and monetized private credit, to show that such a system can realize the ideal efficiency of decoupling the two production sectors by making the left-hand side of Eq. (51) equal zero. We call this economy "Utopia" because the constraint-functions of money and default penalties serve to coordinate the efficient allocation of goods, but money has no other explicit utility. Banking is likewise a public service, with the policy objective of maximizing monetary efficiency, no requirement for strategic action, and thus no need to produce profits.

6.2.1 Consumer Lending and Borrowing with a Central Bank that is a Strategic Dummy

A minimal bank for the Utopia model is an atomic central bank, which is a strategic dummy. It produces any desired quantity of central-bank credit (or effectively distributes government fiat), which is accepted in trading posts on par with gold, and it provides accounting services for both its own debt and private debt without cost. Its behavior is defined by two parameters, the central-bank interest rate ρ_C and the default penalty Π, which we take to be sufficiently severe to support whatever shadow price on consumption is required for solutions without strategic default.

The following two subsections show numerical solutions for 1) the initial transient that converges to the turnpike steady state with fixed bank balances and part of the circular flow conducted through interest payments, and 2) the terminal divergence from the turnpike that cancels bank balances.

The Utopia solution fully decouples the two production sectors only during the initial transient from the innovation shock to the long-term turnpike solution. The terminal transient, combined with a requirement (forced by our probabilistic announcement of the terminal time T) for agents to converge to the turnpike, breaks the decoupling of the two sectors. In order to cancel the intra-economy debts by one type of consumers to the others, without incurring a net debt of the consumers to the central bank, both types of consumers must bid in a way that induces both types of firms to alter their output levels, rather than just type-1 firms that experienced the innovation opportunity.[23] This is an economically appropriate solution property: the conditions of production are permanently changed in this economy; if the terminal conditions require the termination of bank loans under such changed production conditions, they cannot avoid distorting production because they create a condition of inflexible money supply and distribution. However, this distortion is limited to a finite horizon before the day of reckoning, and decouples from the main solution property of buffering the circular flow of funds.

6.2.2 Initial Transient: From the Innovation Shock to the Turnpike

The first-order conditions in Utopia begin with the general solutions derived in Sect. 5.5.

In order to permit a non-inflationary/non-deflationary price system, the central bank interest rate must be tuned relative to the utilitarian discount factor $\beta = 1/\left(1 + \rho\Delta t\right)$ so that

$$\beta\left(1 + \rho_C \Delta t\right) \to 1, \tag{52}$$

or $\rho_C = \rho$. Since the central bank is a public good, and the rate of discount is known, this is consistent with other assumptions of fine-tuning that define Utopia.

The parameter $\hat{\epsilon}$ determining the asymmetry in consumption by Eq. (30), (31) is constant in this model, and determined by the turnpike condition, which is vanishing of the two terms in curly braces in Eq. (23) and (27) for $t \to \infty$.

The shadow price on money from consumer purchases of goods is determined from Eq. (42), in the case where borrowing and lending rates are equal and both equal $\rho_C = \rho$. Since $(B_{1,t} + B_{2,t}) = (m_{1,t} + m_{2,t})$ in steady-state where outputs take their stationary production values (set by $f_i'(s_i) = \rho_\pi$), the shadow price is then given by

$$\frac{\hat{q}_1 e^{-\hat{q}_1/2} + \hat{q}_2 e^{-\hat{q}_2/2}}{B_1 + B_2} = \frac{\Lambda}{\rho}. \tag{53}$$

[23] A continuum of solutions to the first-order conditions exists, in which the type-1 and type-2 firms deplete or hoard stocks in differing degrees so as to cancel the intra-economy debt $\left(a_{1,T} - a_{2,T}\right)$. This continuum includes a solution in which the type-2 firms continue to produce at the pre-innovation level, so they are buffered at all times. That solution, however, does not lead to a net *aggregate* balance $\left(a_{1,T} + a_{2,T}\right) = 0$, if $\left(a_{1,t} + a_{2,t}\right)$ starts from a zero aggregate balance at $t \ll T$. Therefore the solution with $s_{2,t} = \bar{s}_2, \ \forall t$ can only be reached by leaving a finely tuned non-zero aggregate balance $\left(a_{1,t} + a_{2,t}\right)$ of $\mathcal{O}\left(e^{-(T-t)\rho_\pi}\right)$ at early times t following the transient. Such an initial condition would lead to a different terminal solution than $(s_{2,t} = \bar{s}_2, \ \forall t)$ at any slightly different value for T, and would be incompatible with any non-cooperative equilibrium solution at a value of T differing by more than $\mathcal{O}(1/\rho_\pi)$ from the value for T which $\left(a_{1,t} + a_{2,t}\right)$ was tuned.

The constancy of this ratio (equal to a constant shadow price), in Eq. (47), together with the initial condition $s_{2,t=0} = \bar{s}_2$ then gives $s_{2,t} = \bar{s}_2, \ \forall t$ as the unique turnpike solution, completing the proof that banking in Utopia buffers production of good-2 from the innovation shock in good-1.

A similar evaluation, starting from Eq. (41), produces the relation between prices and output levels in Utopia of

$$\frac{1}{p_{1,t}\gamma_1} e^{-\hat{q}_{1,t}/2} = \frac{1}{p_{2,t}\gamma_2} e^{-\hat{q}_{2,t}/2} \to \frac{\Lambda}{\rho}. \tag{54}$$

A Numerical Example

The following example evaluates the integrals (22), (27) and non-cooperative equilibrium conditions from in the preceding sections, to show how the characteristic recovery structure following innovation is realized and determines the monetary properties of the economy.

Input parameters are: asymptotic production rate $f_{1,\infty}/\rho_\pi = 2$; $\gamma_1/\rho_\pi = 1/2$; the probability of success for firms that try to innovate is $\xi = 1/5$; the innovation cost $j = 0.1$, and the innovation output multiplier $\theta = 1/5$. For convenience, to avoid introducing new parameters, we set $f_{2,\infty} = f_{1,\infty}$ and $\gamma_2 = \gamma_1$. The pre-innovation steady-state of supply is therefore $\hat{q}_1 = \hat{q}_2 = (f_{1,\infty} - \rho_\pi/2)/\gamma_1 = 3$. Section "Solutions for the Utopia Economy" in the Appendix computes details of the time constants and structure of the recovery trajectories.

The natural timescale in the model is set by the profit rate ρ_π, which determines the dynamics of production stocks and output levels by Eq. (47). Since revolving loans are permitted in any amount that agents demand, the bank interest rate ρ_C does not determine a dynamical timescale, though it does affect the quantities of borrowed money. For simplicity in the numerical example we also set $\rho_C = \rho_\pi$.

The asymmetry of consumption (30), (31) generated by the non-cooperative equilibria of this game as a consequence of innovation evaluates numerically to $\hat{\epsilon} \approx 0.0075751$. Relative to the similarly scaled pre-innovation rates of production $\hat{q}_1 = \hat{q}_2 = 3$, $\hat{\epsilon}$ provides a measure of the utilitarian asymmetry introduced by innovation in one good.

Properties of the solution are shown in the following series of figures.

The Two-Stage Recovery Involving Stocks of Successful and Failed Innovators

Figures 1 and 2 show that the type-1 firms undergo a two-stage recovery following the innovation event. Before period $t = 0$, all type-1 firms are equivalent, so when the average outcome of innovation leads to higher output, all firms attempt to innovate. The fraction $(1 - \xi)$ that fail continue to offer goods at market in all periods $t > 0$, and in an initial interval their offer rates exceed their production rates, so they deplete their working stocks $s_1^{(-)}$. The profit incentive for this strategy comes

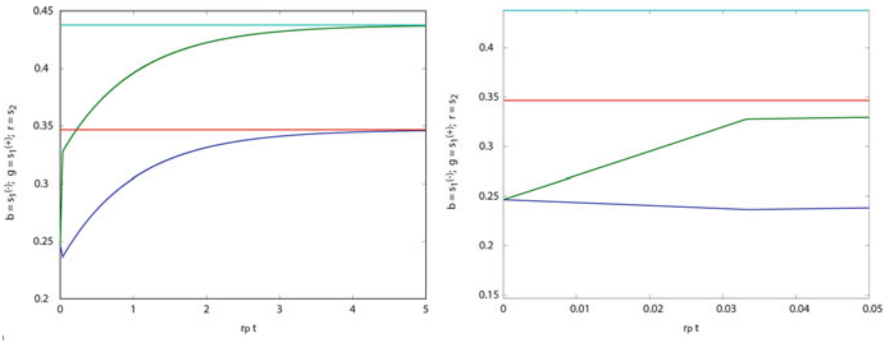


Fig. 1 The two-stage recovery associated with the cost and risk of failure in innovation. Firms of type-1 that try to innovate and fail follow recovery trajectories $s_1^{(-)}$ (*blue*) that initially deplete stocks while offering at an unsustainable rate in order to capture market share, by keeping prices below a level at which successful firms are willing to enter. When the stocks $s_1^{(+)}$ of successful firms (*green*) have grown and their shadow prices have decreased to equal market prices, both firms switch to offering at sustainable rates and converge with a fixed offset toward their late-time steady states (respectively *red and cyan*). Because of the choice (1) of functional form for f_i, the *red curve* is also the stock level $\bar{s}_2$, which is unaffected by innovation. *Left-hand panel* shows recovery over a long interval; *right-hand panel* gives a close-up of the interval following the innovation event

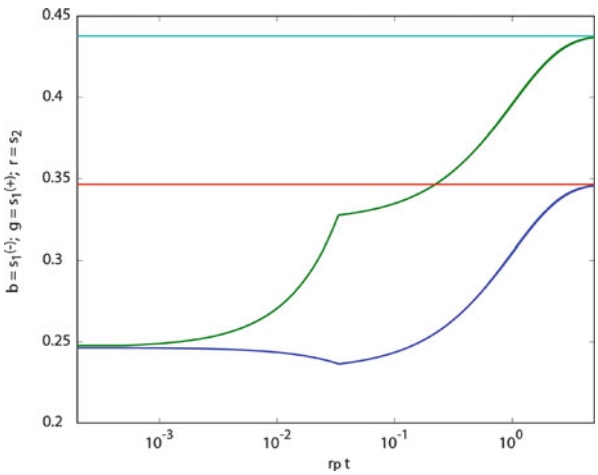


Fig. 2 Same timeseries as Fig. 1 with time $\rho_\pi t$ shown on log scale to make the initial phase more visible and to compress the subsequent recovery phase

from maintaining a price below the shadow price of the successfully-innovating firms, which will ultimately converge to a higher output level. The successful firms sit outside markets and accumulate stocks $s_1^{(+)}$, until their shadow prices fall to intersect the (rising) market prices maintained by the failed-innovation firms. After the two prices intersect, all firms offer in the markets, and the successful and failed type-1 firms both restore stocks to their (respective) steady-state production levels.

In the production functions (1), the steady-state stock is the same for both type-1 and type-2 firms, so the asymptotic level $\bar{s}_1$ to which failed-innovation type-1 firms recover is also the stock $\bar{s}_2$ maintained by type-2 firms throughout.

Rates at which Goods Are Delivered to Market for Consumption

Figure 3 shows the offer rates of the two groups of firms. Type-2 offer rates are constant. Type-1 offer rates are aggregated from the successful and failed-innovation firms. In the early interval, only a measure $(1 - \xi)$ of firms offer in markets, whereas in the later interval all firms offer. The discontinuous derivative in the stock $s_1^{(-)}$ visible in Fig. 1 exactly compensates for this jump in measure so that all of $s_1^{(-)}$, $s_1^{(+)}$, and q_1 are continuous through the transition.

Bid Levels and Money Supply in the Post-Innovation Interval

Figure 4 shows the bid levels on both types of goods by both groups of consumer/owners following the innovation shock. Total money supply in circulation $(B_{1,t} + B_{2,t})\,\Delta t$ is also shown (black curve) in the right-hand panel of the figure.

The amount of money in circulation per period is initially greater than $2m_0$ because agents of both types take out loans from the central bank. They borrow the maximum that they will be able to repay under the non-cooperative equilibrium trajectory. The money in circulation crosses (downward) through the pre-equilibrium value of $2m_0$ at $\rho_B t \approx 0.70050$ and continues to descend, as agents gradually pay down the principle.

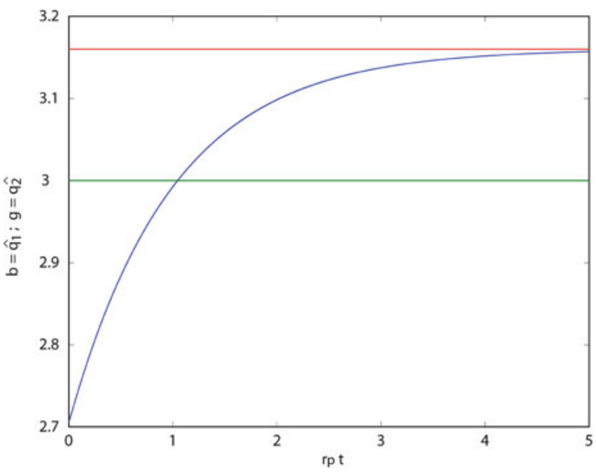


Fig. 3 Timeseries of the total rates $q_{1,t}/\gamma_1$ and $q_{2,t}/\gamma_2$ delivered to markets for consumption. $q_{2,t}$ (*green*) is constant at the pre-innovation solution over all time. $q_{1,t}$ (*blue*) begins in deficit relative to the pre-innovation solution, and ends in surplus relative to that solution

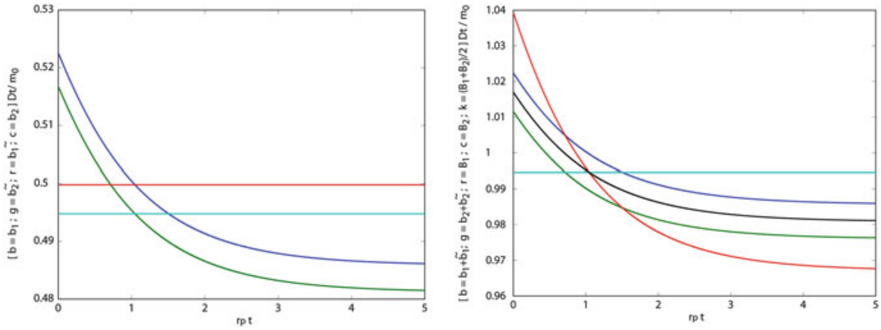


Fig. 4 Bid levels normalized by the pre-innovation money supply, $b_i \Delta t/m_0$, $\tilde{b}_i \Delta t/m_0$, aggregated in several ways. *Left panel*: by agents. *Blue* and *green* are bids on good-1 by consumers of types 1 and 2, respectively. *Red* and *cyan* are bids on good-2 by consumers of types 1 and 2 respectively. Type-1 consumers consume more of both goods, but in each period they pay out more than they receive in profits, a deficit that must be compensated by interest on bank savings. *Right panel*: by consumer-type or goods-type. Here *blue* and *green* are total expenditures by type-1 and type-2 consumers, respectively. *Red* and *cyan* are total bids offered on type-1 and type-2 goods, respectively. The *black curve* is $(B_1 + B_2)\,\Delta t/2m_0$, which is the total money in circulation normalized by the pre-innovation value

Monetized Credit From a Persistent Internal Loan

Figure 5 shows the solution to Eq. (21) for $(a_{1,t} - a_{2,t})/2$ in relation to the excess of payment rates made by type-1 agents over payment rates by type-2 agents $\left(\tilde{b}_{1,t} - \tilde{b}_{2,t}\right)$. The scale for the numéraire in this model is set by m_0, a quantity that scales $\sim \Delta t$, whereas the bid rates and inter-agent bank interest payment rates are regular quantities in the continuous-time limit. In order to normalize them to the numéraire, we compare the interest payments-per-period, which are $\rho_B\,(a_{1,t} - a_{2,t})\,\Delta t/2$, to m_0, and we likewise compare the excess bid amounts-per-period by type-1 over type-2 agents, which are $\left(\tilde{b}_{1,t} - \tilde{b}_{2,t}\right)\Delta t$, to m_0. These normalized curves are independent of Δt as $\Delta t \to 0$. The convergence of the two curves in Fig. 5 at late time verifies that the consumers converge to steady account balances at which interest payments via the bank provide part of the circular flow allowing type-1 agents to purchase and consume both goods at a constant excess rate ϵ over the rate of consumption by type-2 agents.

Aggregate Loan and Change in the Money Supply

Figure 6 shows the economy's aggregate balance with the banks $(a_1 + a_2)$ relative to the initial money supply of either agent type m_0. It also shows the excess money-in-circulation over the amount possible with the initial money supply, $(B_1 + B_2)\,\Delta t - 2m_0$, which is made possible by aggregate loans. Initially the two values are equal, but as the economy pays off the borrowed principle and also loses net money-in-

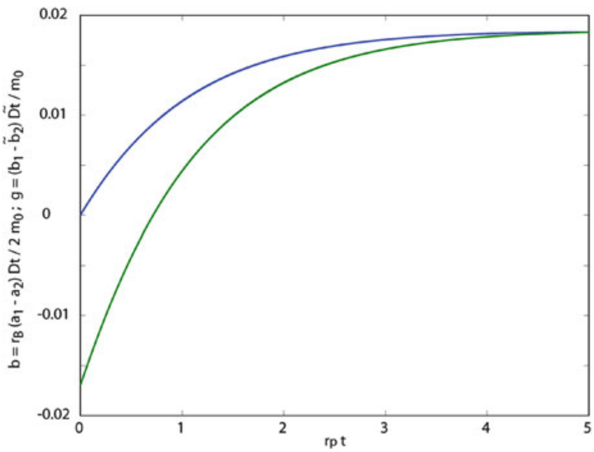


Fig. 5 Bank balance $\left(a_{1,t} - a_{2,t}\right)/2$ reflecting monetized private credit, normalized as $\rho_B \left(a_{1,t} - a_{2,t}\right) \Delta t / 2m_0$ (*blue*), which is the interest paid from type-2 agents to type-1 agents per period relative to the pre-innovation cash-per-agent, and excess bids per period of type-1 agents over type-2 agents similarly normalized, $\left(\tilde{b}_{1,t} - \tilde{b}_{2,t}\right) \Delta t / m_0$ (*green*), following the innovation event. The two converge to the same non-zero late-time steady state, as payment flows in the markets compensate interest flows through the bank. Note that the level of loans scales as $\left(a_{1,t} - a_{2,t}\right)/2 \sim \left(\tilde{b}_{1,t} - \tilde{b}_{2,t}\right)/\rho_B$, a quantity independent of Δt, which may be made arbitrarily larger than the money-in-circulation on the approach to the continuous-time limit

circulation to the payment of compounded interest, the money in circulation drops below $2m_0$. At late times, the principle is exactly repaid, and a new circular flow is established with asymptotically steady money supply $\left(B_{1,t} + B_{2,t}\right) \Delta t$ for $\rho_\pi t \gg 1$.

6.2.3 Terminal Conditions: Exiting the Turnpike to Cancel Bank Balances

A corresponding set of solutions for a terminal transient, which begins in the turnpike solutions for stocks, output, and prices, and terminates at a time T with zero bank balances, is shown in the next four figures. The overall behavior of the terminal transient is that type-1 consumers deplete their savings by increasing bids on goods, while type-2 consumers reduce their bids on goods to repay their outstanding account balances. These bids continue to respect all the non-cooperative equilibrium conditions, though now on an unstable diverging trajectory. In response to these changes in bidding behavior, the two types of firms either deplete or accumulate working stocks, altering their outputs to continue to maximize profits.

Working Stocks of the Firms

Figure 7 for the terminal transient may be compared with Fig. 1 for the behavior of stocks from the initial transient. Type-2 firms and failed-innovation type-1 firms

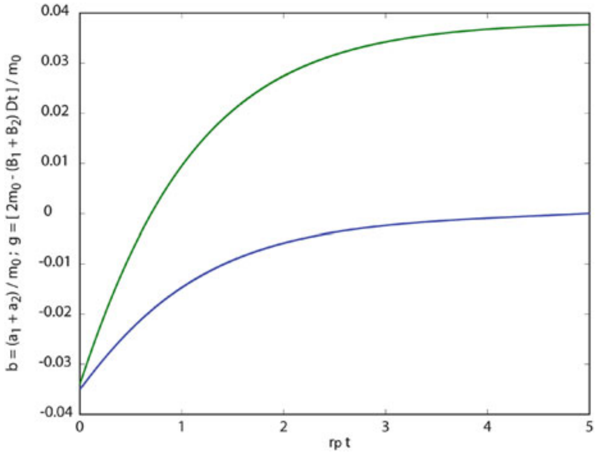


Fig. 6 Aggregate account balance the consumers hold at the bank, normalized as $(a_1 + a_2) / m_0$ (*blue*), and compared to total rate of money circulation in markets in excess of the circulation that would be possible with the initial gold-in-hand $2m_0$. The excess is normalized and plotted as $[2m_0 - (B_1 + B_2) \Delta t] / m_0$ (*green*). The amount borrowed in the period $t = 0$ when the innovation-cost is paid equals the excess of bids on goods over $2m_0$ (*green and blue curves* are equal at $t \to 0$ up to numerical imprecision). This loan amount is set using Eq. (27) so that the agents pay the principle to zero by time T. Note that $\left(B_{1,T} + B_{2,T}\right) \Delta t < 2m_0$, so gold has left the private economy and is being held by the bank. Note also that the net loan $(a_1 + a_2)$ is a few percent of $m_0 \sim \Delta t$, whereas the difference of balances $(a_1 - a_2)$ in Fig. 5, which is credit from one agent type to the other monetized by the bank, is several percent of $m_0 / (\rho_B \Delta t)$

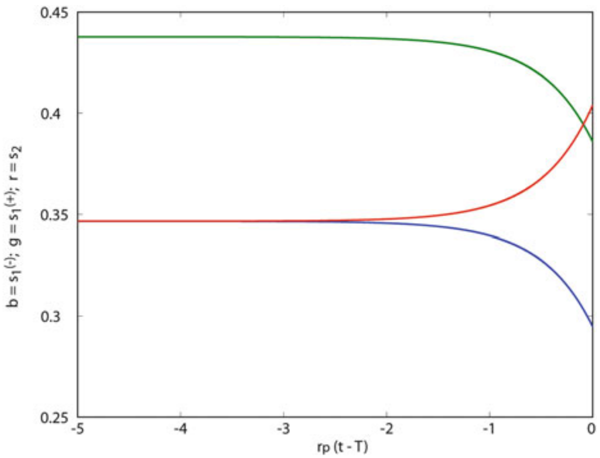


Fig. 7 Working stocks of the three types of firms in the terminal transient. Time is plotted as $\rho_\pi (t - T)$, which terminates at value 0. Trajectory $s_1^{(-)}$ (*blue*) and s_2 (*red*) begin at the same values but diverge in opposite directions. Trajectory $s_1^{(+)}$ of successful type-1 firms (*green*) moves in parallel to $s_1^{(-)}$ for unsuccessful type-1 firms. The initial working stocks of the terminal transient are the turnpike values to which the solutions in Fig. 1 converge at late times

both start with the same stocks $\bar{s}_2 = \bar{s}_1$, while successful innovation firms begin with stocks $\tilde{s}_1$. Because all type-1 firms optimize output against the same price system, both successful- and failed-innovation firms deplete stocks by the same amount, increasing output levels and lowering prices. Type-2 firms do the opposite, accumulating stocks and reducing outputs, and boosting prices.

Output Rates

Figure 8 shows the output rates produced by the stock trajectories from Fig. 7. Type-1 firms increase output rates, while type-2 firms reduce them. Recall that prices are given by Eq. (54).

Elimination of Intra-Economy Lending

Figure 9 shows the intra-economy debt, due to type-2 consumer borrowing from the central bank and type-1 consumer lending to the bank. The quantity $(a_{1,t} - a_{2,t})$ is plotted. Recall that this quantity is $\mathcal{O}(1)$ and thus generally much larger than the money in circulation. Hence, within $\mathcal{O}(\Delta t)$, $(a_{1,t} - a_{2,t})/2 \approx a_{1,t} \approx -a_{2,t}$.

In the initial steady state, the interest stream to/from the bank, $\rho_B (a_{1,t} - a_{2,t}) \Delta t/2$, equals the excess bids by type-1 agents per period relative to bids from type-2 agents, $\left(\tilde{b}_{1,t} - \tilde{b}_{2,t}\right) \Delta t$. The interest stream from bank accounts exactly provides the excess bids by type-1 agents to support their higher consumption levels. As the terminal transient develops, the bid excess by type-1 agents increases to deplete the

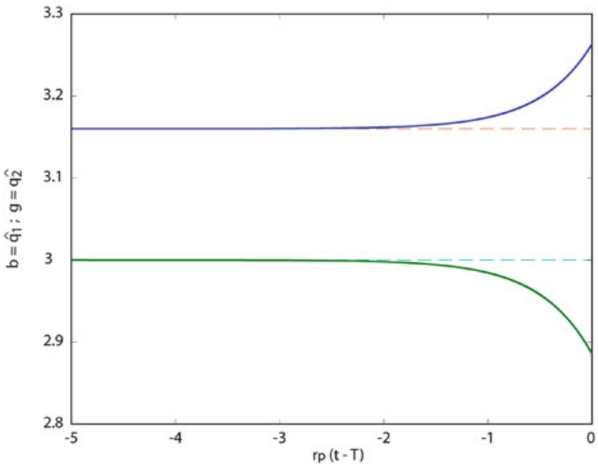
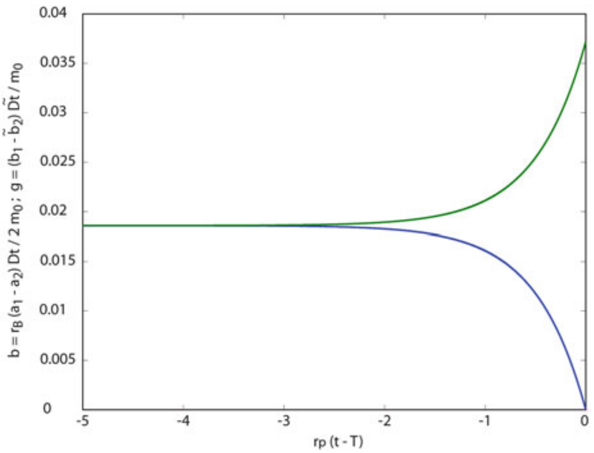


Fig. 8 Output rates $q_{1,t}/\gamma_1$ (*blue*) and $q_{2,t}/\gamma_2$ (*green*) delivered to markets for consumption in the terminal transient. The initial output levels for the terminal transient are the turnpike values to which the solutions in Fig. 3 converge, shown as *dashed lines*

Fig. 9 Difference of bank balances scaled as $\rho_B\left(a_{1,t}-a_{2,t}\right)\Delta t/2m_0$ (*blue*), and excess bids per period of type-1 agents over type-2 agents similarly normalized, $\left(\tilde{b}_{1,t}-\tilde{b}_{2,t}\right)\Delta t/m_0$ (*green*). The turnpike value of steady intra-economy loans $\left(a_{1,t}-a_{2,t}\right)$ to which the solutions in Fig. 5 converge is returned to zero in the terminal transient



principle in their account, at the same time as type-2 agents repay principle. These differences continue to respect the consumption relations (30) at fixed $\hat{\epsilon}$, because changes in the output levels by firms have adjusted the price levels consistently with the changes in bids.

Non-Accrual of Aggregate Debt by Either the Economy or the Bank

Finally, Fig. 10 shows the aggregate account balance $(a_{1,t}+a_{2,t})$ through the terminal transient. Because innovation has made the collection of type-1 firms distinct from the type-2 firms, it is not possible for them to maintain an exactly fixed money supply through the entire terminal transient. Therefore, the bids by the two types of agents, and the output levels by the two types of firms, must be adjusted so that any non-zero aggregate balance acquired early in the transient is repaid by time T, leading in general to a change in the money-in-circulation from the turnpike value. Because the innovation shock we have assumed in this example is small, the two types of firms remain broadly similar. In order for their net contribution to debt to cancel, their output levels must be roughly mirror images, and this is the reason for the opposite behavior of the stock transients in Fig. 7 and the output transients in Fig. 8. The changes in money supply throughout the transient therefore remain small relative to money-in-circulation.

Further properties of the economy in the terminal transient can be computed, along the same lines as those presented for the initial transient.

6.2.4 Summary of Banking in Utopia

The preceding model has used a context in which a rigid money supply leads to a failure of output efficiency, to illustrate how a simple banking scheme can

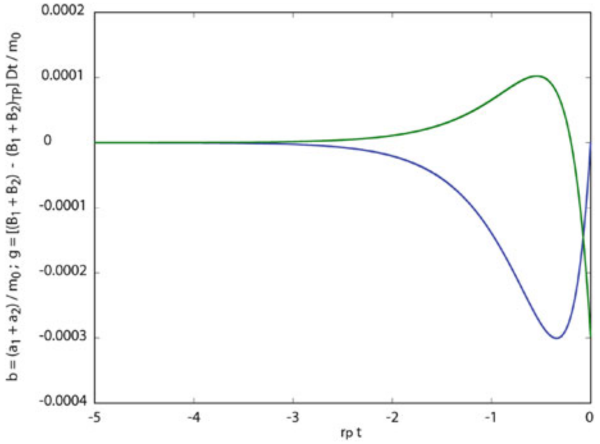


Fig. 10 Aggregate account balance the consumers hold at the bank, normalized as $(a_1 + a_2)/m_0$ (*blue*), and compared to total rate of money circulation in markets, now in excess of the turnpike money supply from the late-time asymptote in Fig. 6, which we denote $(B_1 + B_2)_{\mathrm{TP}} \Delta t$. The money supply is plotted as $[(B_1 + B_2) - (B_1 + B_2)_{\mathrm{TP}}]\,\Delta t/m_0$ (*green*). The total balance $\left(a_{1,T} + a_{2,T}\right) = 0$ as a property of the non-cooperative equilibrium solution

restore this efficiency. The main features of the Utopia model are that a single bank can change the money-in-circulation both transiently and persistently when this is required to stabilize the price system against which producers optimize, and can also monetize personal credit within the society to support emergent differences in purchasing power. The outcome of a many-period game is economically realistic: the owners of a technology that undergoes an innovative improvement in output capacity can become net holders of the debt of other members of the society, and the interest on this debt can support an indefinite increase in their relative purchasing power. It is an important feature of the banking model that members of the society can arrive at non-cooperative equilibria in which new steady states of money supply and the circular flow of funds are established, in which the bank withdraws from participation in the economy except as a keeper of its internal accounts.

6.3 Commercial Banking, Profit, and the Consequences of Interest Rate Spreads

In economies with distributed banking sectors, a criterion governing strategic action by the banks is profit maximization. Profits may come either from interest rate spreads or from permitting the banks to issue credit that receives the protection of law but is backed by only a fraction of its value in reserves of some form of "heavy money", which could be gold, government fiat, or central-bank credit. We consider first the introduction of interest rate spreads as a sole modification to the Utopia model, the resulting problems in the definition of profits, and the consequences of spreads for efficiency, which may be expressed in terms of the spread values independently of how (or whether) they are used strategically by the banks.

A non-zero spread exists whenever $\rho_{B,L} > \rho_{B,D}$ in Eq. (18), for the banks that serve consumers. In this section we consider the spread a fixed parameter and do not yet consider strategic action by banks.

The main features of (both transient and persistent) change in the money supply and monetization of private credit can be retained by profitable banks (if they are owned by the consumers and distribute their profits to consumers), but the efficiency of Utopia is lost in proportion to sizes of the spreads. We will show that the introduction of interest rate spreads inherently couples the innovation shock in good-1 to production and output decisions in good-2, with a strength proportional to the spread. Profit increases with increasing spreads, but so does cross-coupling among sectors and the consequent inefficiency. Therefore any regulatory system that requires spreads as a control mechanism carries an inherent efficiency cost.

6.3.1 The Continued Need for a Central Bank Even at 100 % Reserves

Even in the absence of fractional-reserve lending, purely mechanistic problems of defining profits from interest, using bank credit to vary the money supply, while also permitting asymptotically steady states in the absence of innovation, will require that we regard the banks with which consumers interact directly as *commercial banks*, and that we retain a *central bank* as a distinct entity.

In Utopia, it was important that the central bank *not* be merely a publicly-owned pass-through entity. One of its main functions was the injection or withdrawal of net quantities of money from the supply-in-circulation. This was achieved by accumulation of interest payments on (fully-repaid) net initial deposits or loans by the consumers. A feature of the Utopia model that makes a non-pass-through bank into a problem, however, when interest rate spreads are introduced, is that consumers in a two-good economy asymptotically make revolving loans from one type to the other, mediated by the bank, as shown in Fig. 5. If the bank collects a steady stream of payments proportional to $(\rho_{B,L} - \rho_{B,D})$ from these loans, and those are not passed back into the economy, no steady-state money supply and price system are possible. Yet the game must not pass all interest payments back to consumers, or else the banks lose the capability to vary the money supply.

To preserve both essential functions of the Utopian central bank when interest rate spreads are introduced, we must define one component of the net interest stream paid by consumers to the commercial bank as profit which is returned to the consumers, and a remainder that is not profit (because it is passed through to the central bank), with this remainder used to vary the money supply in circulation. The net interest paid by consumers to commercial banks will be $-(\rho_{B,1t}a_{1,t} + \rho_{B,2t}a_{2,t})$. The net interest paid by the commercial banks to the central bank, on money it must borrow to change the total money-in-circulation, is $-\rho_{C,t}(a_{1,t} + a_{2,t})$. The profit rate, which is a sum of two equal streams $\pi_{1,t}^{(B)} + \pi_{2,t}^{(B)}$ paid to the two types of agents, is then given in Eq. (25). As long as $\rho_{B,L} \geqslant \rho_C \geqslant \rho_{B,D}$, profits are never negative. In this section, we take the central bank rate ρ_C to be constant, as in Utopia.

For convenience of exposition here, since $\rho_{B,L}$ and $\rho_{B,D}$ are parameters, we take their average to equal the central bank rate, $(\rho_{B,L} + \rho_{B,D})/2 = \rho_C$. The central bank continues to be a public service, so we will set $\rho_C = \rho$ (the utilitarian rate of discount) to enable non-inflationary/non-deflationary turnpike solutions. More general solutions with steady-state production rates, but inflating or deflating prices, are also well-defined through Eq. (47), but are more complicated. The single new parameter for the commercial banks is then $(\rho_{B,L} - \rho_{B,D})/2$.

6.3.2 Interest Rate Spreads and Efficiency

For the single, simple event of innovation used in this class of games, the interest rates make a single transition at a time we may denote t_{split}. In terms of this transition time, instead of setting the left-hand side of Eq. (51) equal to zero as it is in Utopia, the equation satisfied by $s_{2,t}$ becomes

$$\left[\frac{d}{dt}\left(\frac{d}{dt} - \rho_\pi\right) + 2\gamma_2 \bar{f}_2''\right](s_2 - \bar{s}_2) = \pm\Theta\left(t_{\text{split}} - t\right)\gamma_2\left(\rho_{B,L} - \rho_{B,D}\right). \tag{55}$$

(Θ denotes the Heaviside function, which takes value one for $t < t_{\text{split}}$ and zero otherwise.) The boundary conditions for this second-order equation are that $s_2 = \bar{s}_2$ at $t = 0$ and again at $t \to \infty$. For the production function f_2 from Eq. (1), $\bar{f}_2'' = -2\rho_\pi$.[24] The value of t_{split} must be determined self-consistently with the signs of the bank balances in the solution that it yields. For small spreads, it is well approximated from the Utopia solutions shown in Fig. 5 and Fig. 6. We return to the determination of t_{split} in Sect. 6.3.3.

The solution to Eq. (55) is a sum of growing and decaying exponentials on the interval $0 \leqslant t \leqslant t_{\text{split}}$, and a decaying exponential for $t > t_{\text{split}}$. The magnitude of

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[24] Firms of type-1, in the period when both are offering in the markets, have equations identical in form to Eq. (55), for the *deviations* of their stocks from the Utopia solutions. For the firms that attempt to innovate and fail, we denote these deviations $\delta\left(s_1^{(-)} - \bar{s}_1\right)$, and for the firms that attempt to innovate and succeed, the corresponding quantity is $\delta\left(s_1^{(+)} - \tilde{s}_1\right)$. In the initial period, when firms that successfully innovated are sitting outside the markets, their inventory growth is governed only by internal production and they do not optimize against prices. The type-1 firms that failed to innovate satisfy a slightly modified equation given by

$$\left[(1-\xi)\frac{d}{dt}\left(\frac{d}{dt} - \rho_\pi\right) + 2\gamma_1 \bar{f}_1''\right](s_1 - \bar{s}_1) = \pm\Theta\left(t_{\text{split}} - t\right)\gamma_1\left(\rho_{B,L} - \rho_{B,D}\right),$$

because their measure is $(1-\xi)$ and the level of output than can contribute scales by the same factor.

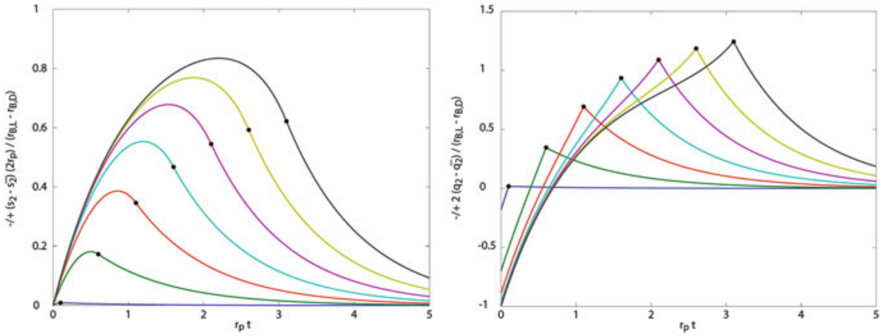

Fig. 11 Response of stocks and offers of good-2 to a discontinuity in interest rate by an amount $\pm\Theta\left(t_{\text{split}} - t\right)\left(\rho_{B,L} - \rho_{B,D}\right)/2$. *Left panel*: the response descaled by the strength of the spread, given by $\mp\left(s_2 - \bar{s}_2\right) \times \left(2\rho_\pi\right) / \left(\rho_{B,L} - \rho_{B,D}\right)$, with a range of times $\rho_\pi t_{\text{split}}$ (markers) from 0.1 to 3.1 in increments of 0.5. *Right panel*: the response of offers descaled by the spread, given by $\mp\left(q_2 - \bar{q}_2\right) \times 2/\left(\rho_{B,L} - \rho_{B,D}\right)$, for the same cases

the excursion can be determined by the condition that both the stocks and offer level be continuous through the transition. The matching conditions can always be met because the growing solution has a shorter time constant than the decaying solution.

Figure 11 shows the excursion in stock levels and offer rates by the type-2 firms in response to such a shock, at a sequence of increasing values of t_{split}. The quantities plotted in the figure are $\mp\left(s_2 - \bar{s}_2\right) \times \left(2\rho_\pi\right) / \left(\rho_{B,L} - \rho_{B,D}\right)$, and $\mp\left(q_2 - \bar{q}_2\right) \times 2/\left(\rho_{B,L} - \rho_{B,D}\right)$. The $\mp$ sign corresponds to the $\pm$ sign in Eq. (55), and thus determines the direction of the excursion in stocks and offers.

The Sign of the Excursion

If, in the immediate aftermath of the innovation, both types borrow from the bank, then $\rho_{B,1} = \rho_{B,2} = \rho_{B,L}$, and the sign in Eq. (55) is negative. The effect is that good-2 firms try to optimize production against a larger discount rate than ρ_π, which means increasing the target s_2. This is done transiently by reducing offers and accumulating. Later, when the bank rates split, and one group lends while the other borrows, the target stock level returns to $\bar{s}_2$, and offer rates are increased to return toward it.

6.3.3 Approximating the Effect on Output Using a Small-Parameter Expansion

We will not pursue a full self-consistent solution to the production/trade model with interest rate spreads. The major qualitative features that result from the introduction of spreads may be illustrated with an approximate solution. The approximation is valid if the spread $(\rho_{B,L} - \rho_{B,R})/\rho \ll 1$. In this limit, output, prices, and consumption allocation are dominated by the properties of the Utopia solution. If we choose a small but nonzero period length $\rho_\pi \Delta t \ll 1$, then the money in circulation $(B_{1,t} + B_{2,t})\,\Delta t$ (along with all changes in that money supply) scale as $\sim \rho_\pi \Delta t$ relative to the long-term indebtedness within the economy $(a_{1,t} - a_{2,t})$, for $\rho_\pi t \gg 1$.

Solution Part I: Relating Money Supply to Outstanding Private Debt and Determining t_{split}

In this solution, $\rho_\pi \Delta t$ is used to relate the quantity $\rho_\pi \Delta t\,(a_{1,t} - a_{2,t})/2m_0$ from Fig. 5, to $(a_{1,t} + a_{2,t})/m_0$ from Fig. 6. From these two, values a_1/m_0 and a_2/m_0 are obtained. The leading order approximation for the crossing time t_{split} is then its value in the Utopia solution. This approximation is then used in Eq. (55) and its counterparts for successful and failed innovating firms of type-1, to obtain the linear-order corrections to the production stocks and output rates. In an iterative solution, these profiles could then be fed back into equations for prices and allocations to update t_{split}, and the process could be repeated, but for this example we will stop with the leading-order approximation.

A Numerical Example

To provide a numerical example, we take a very coarse discretization $\rho_\pi \Delta t = 0.1$ to scale the money supply relative to the acquired internal debt. This number is of course much too large to be well-approximated with the continuous-time recovery trajectory in the Utopia example, and we use it only to produce effects in the plots that are large enough to see. The same methods we illustrate here continue to apply as $\rho_\pi \Delta t$ is made arbitrarily smaller, and the response sizes scale in proportion.

With this large value of $\rho_\pi \Delta t$, the crossing time when a_1 passes through zero (consumers of type-1 change from being net borrowers to net lenders) is given by $\rho_\pi t_{\text{split}} \approx 0.099$. The corresponding values of a_1/m_0 and a_2/m_0, and the perturbations in the goods-stocks, are shown in Fig. 12.

The solution combines three distinct output programs. Type-2 firms and type-1 firms that try to innovate and fail follow nearly the same trajectories of accumulation of goods $(s_2 - \bar{s}_2)$ and $\delta\left(s_1^{(-)} - \bar{s}_1\right)$ (see Footnote 24). Type-1 firms that successfully innovate do not optimize against the price system initially, so their accumulation of stocks is unaffected. After they enter markets, they follow a similar but less-extensive period of accumulation for $\delta\left(s_1^{(+)} - \tilde{s}_1\right)$, shown in the figure in cyan.

Correcting Stocks and Outputs, and Checking for Consistency

The modified stock trajectories for the three types of firms are shown in Fig. 13. For simplicity we take $f_{2,\infty} = f_{1,\infty}$ in Eq. (1), so that the two goods are completely

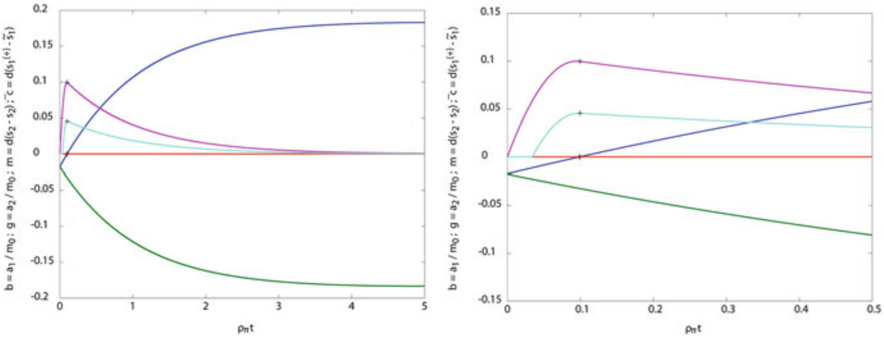

Fig. 12 Bank balances a_1/m_0 (*blue*) and a_2/m_0 (*green*) from Figs. 5 and 6 taking $\rho \Delta t = 0.1$. *Magenta curve* shows the change in response $\delta\,(s - \bar{s})$ due to the interest rate discontinuity, which applies both to s_2 and to $s_1^{(-)}$, since both types of firms optimize output against the price system at all times. *Cyan curve* shows the response $\delta\left(s_1^{(+)} - \tilde{s}\right)$, which is zero in the interval when the successfully innovating firms are not optimizing their output against the price system, and nonzero when these firms enter the market. The two curves are shown to scale, and normalized so that the maximum of $\delta\,(s_2 - \bar{s})$ is set to 0.1 for viewing purposes. The time when a_1 crosses through zero, and the interest rate $\rho_{B,1}$ changes from $\rho_{B,L}$ to $\rho_{B,D}$ is the time used for the matching conditions of both stock s_i and output q_i, marked with a *black cross*. *Left panel* is an extended recovery interval; *right panel* is a close-up of the initial interval following the innovation shock, during which balances are accumulated

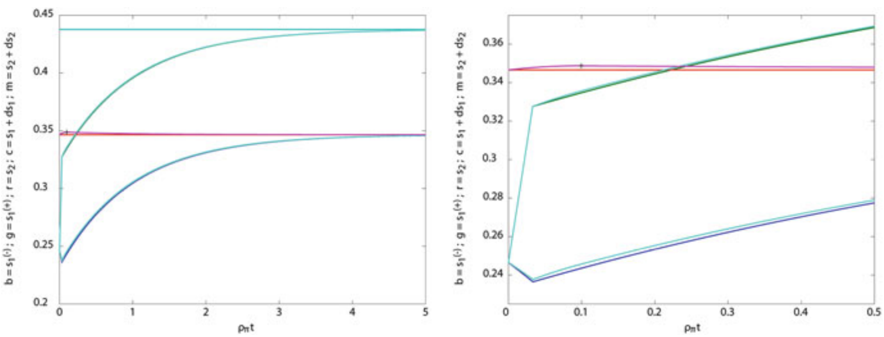

Fig. 13 Time-course of goods-stocks for the three kinds of firms. The Utopia solution of Fig. 1 is the leading order approximation for $s_1^{(-)}$ (*blue*), $s_1^{(+)}$ (*green*), and (in the simple case where $f_{1,\infty} = f_{2,\infty}$) s_2 (*red*). The perturbed stock levels taking $\left(\rho_{B,L} - \rho_{B,D}\right)/2\rho = 0.25$ are shown in *cyan* for both of $s_1^{(\pm)}$, and in *magenta* for s_2. Profile $s_1^{(+)}$ shows no change while successfully-innovating firms sit out of the market, and then undergoes a smooth deviation in output between the time it enters and the time the interest rates shift to their asymptotic late-time values. (Although the interest rate spread is set very large in order to produce a visible effect on output, the corrections to s_2 remain small, justifying the small-parameter approximations used)

equivalent in their production characteristics before the innovation event. The production-stock trajectories are obtained by adding the corrections from Eq. (55) (and its counterparts for type-1 firms) to the Utopia solution.

In the figure, we have again chosen a very coarse perturbation, $(\rho_{B,L} - \rho_{B,D}) / 2\rho = 0.25$, so the interest rate spread is fully *one half* of the average rate charged by the central bank. Again we do this to obtain results that are large enough to see easily in plots; for more realistic spreads the corrections scale proportionally. Even so, the figure shows that the perturbations to the histories of maintained stocks are small. A plot of the same recovery solutions with time on a logarithmic scale is shown in Fig. 14.

The offer levels, which depend on the time derivatives of the stocks, show coarser perturbations, in keeping with this large interest rate spread, as shown in Fig. 15. The most important feature is the initial drop in output (and therefore consumption) of good-2 (shown in magenta), which in the Utopia solution was unaffected by the innovation event in good-1. The output of good-1 also falls (shown in cyan in the figure) relative to its Utopia trajectory. Here we see the first feature of the small-parameter approximation indicating its incompleteness as a solution. In an initial post-shock interval, only failed-innovation type-1 firms offer, and they have measure $(1 - \xi)$. When prices have risen suitably, the successfully-innovating type-1 firms enter (as in Utopia and in the previous chapters), so that all type-1 firms are offering. Simply adding these two corrections to the Utopia solution produces a discontinuity that is an approximation error. In a full solution, adjustment of the matching conditions would absorb this correction (which

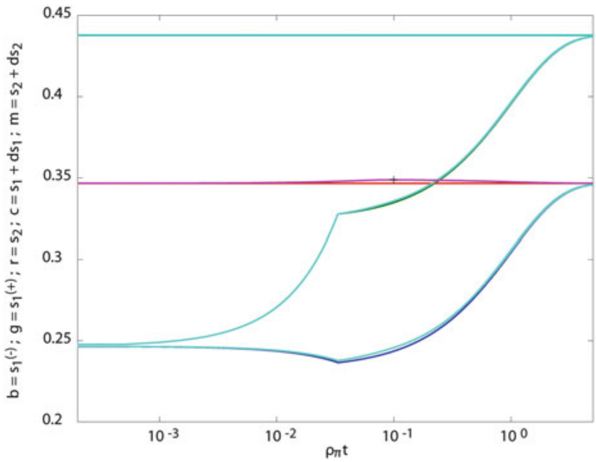

Fig. 14 Same recovery trajectories as in Fig. 13, with $\rho_\pi t$ plotted on logarithmic scale as in Fig. 2

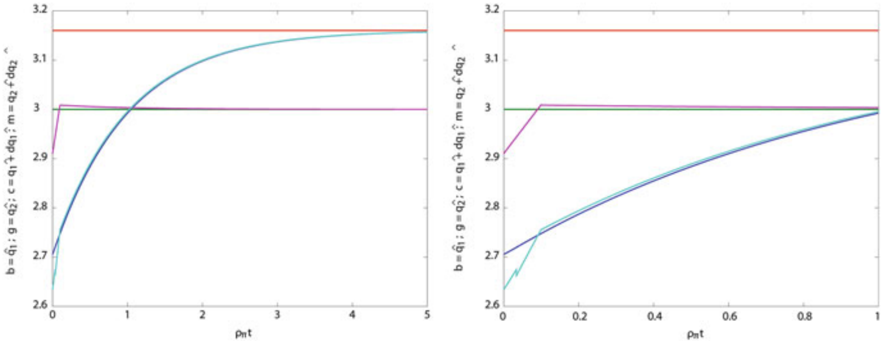

Fig. 15 Offer levels for the two goods in the Utopia solution and with nonzero interest rate spread under the parameters of Fig. 13. Utopia offer rates from Fig. 3 are $\hat{q}_1$ (*blue*) and $\hat{q}_2$ (*green*). Perturbed output due to the interest-rate discontinuity for $\hat{q}_1$ is *cyan* and for $\hat{q}_2$ is magenta. *Left panel* is the full relaxation trajectory (post-innovation asymptote for $\hat{q}_1$ shown in *red*); *right panel* expands the interval following the innovation shock. Unlike the stocks from Fig. 13, which show only a small perturbation, the consumption rates $\hat{q}$ show the larger effect that might have been expected for the large interest rate spread $\left(\rho_{B,L} - \rho_{B,D}\right)/2\rho = 0.25$ chosen to make the effects visible. The small discontinuity in $\hat{q}_1$ visible in the *right-hand panel* comes from the fact that the perturbations in output levels were not fed back—in this leading-order approximation—to the leading-order optimization problem; doing so would have lead to a correction in the matching conditions for the offer levels of the type-1 firms by a small fraction of its Utopia solution value, to absorb this discontinuity

is only $\approx 0.3\,\%$ even for a wide spread) and restore continuity to the offer rates.[25]

6.3.4 Further Properties

Solutions for bank balances, bid levels, and other properties, can be carried through, and are qualitatively like those in the Utopia model. The equations for both intra-economy and aggregate account balances have already been presented in Sect. 5.4.10 in a form compatible with this model. The change in total money supply is responsive only to the central bank rate ρ_C, and initial loans can be fully repaid to converge to a turnpike solution, as in Utopia. The structure of intra-economy lending differs from that in the Utopia solution because borrowers and lenders pay at different rates, while bank profits are distributed to both types of consumers in equal measure. These differences change the quantitative properties of account

[25] In a true small-parameter expansion with both $\rho_\pi \Delta t \ll 1$ and $\left(\rho_{B,L} - \rho_{B,D}\right)/2\rho \ll 1$, the value t_{split} would be shorter than the natural recovery time for stocks $s_1^{(\pm)}$, so that the output of the successfully-innovating firms would never even respond to the interest-rate spread. The resulting solution would be simpler in structure than the one presented here, as well as smaller in magnitude.

dynamics and their steady state values, but not their qualitative character. The terminal transient differs minimally from the Utopia solution, because the account balances do not change sign, so one type of consumers remains a lender with a fixed rate throughout the transient, while the other remains a borrower also with a fixed rate throughout the transient.

6.3.5 Concluding Comments Regarding Interest Rate Spreads

At small spreads, the introduction of bank profits creates small quantitative change but no qualitative change to the Utopia solution. This result demonstrates that the independent scaling between money supply and private debt (with respect to powers of $\rho_B \Delta t$) is not a fragile or fine-tuned property of Utopia, and can be retained in more institutionally complex models. The functions of varying the money supply and monetizing private debt are likewise robust. However, any active response by banks to consumer demand, which either limits the money supply or makes consumers' discounting of money time-dependent, in the sense of Eq. (47), propagates shocks from innovation across sectors and impairs optimal planning of production schedules.

6.3.6 A Note on Fractional-Reserve Lending

Although we do not present a formal model of banking with fractional reserves we conjecture that many of the basic qualitative aspects the models deliver the same message. In particular an official currency, a central bank and commercial banks are all artifacts to deal with evaluation, perception and substitutes for trust needed to promote and protect trade. The lack of natural physical laws for creating and destroying money call for the apparatus of sociopolitical laws to replace the laws for the creation and consumption of physical goods. The specifics of reserve ratio banking, reserves and excess reserves are discussed elsewhere [5].

Although we have presented an analysis on varying the money supply when there is, in essence no basic uncertainty in our models beyond one innovation decision, this is only the tip of an iceberg. We did not deal with the presence of a stream of random events that more closely characterizes ongoing innovation. The modeling considerations indicate that there is a welter of worthwhile case distinctions that depend on factors other than the mechanism of varying the money supply. In particular it is our belief that a key factor in the existence of a two tiered mechanism involving both a central bank and commercial banks is (as Bagehot observed) the importance of the banking system as a distributed perception device. Our efforts were devoted to variation of the money supply. In doing so we were able to illustrate how the failure to so adequately can cause considerable fluctuation that might be otherwise avoided.

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Appendix: Supporting Algebra for Non-Cooperative Equilibria of Game Models

Steady Post-Innovation Output and Stable Money Supply Lead to Stable Bid Levels

This section shows from the first-order conditions for consumption that, if output levels converge to steady late-time values, and if the money supply converges to a steady value, then bid levels by both type-1 and type-2 agents also converge to steady values. This condition is not an accounting identity, but part of the optimization problem that agents must solve. It requires only one strategic degree of freedom to be met, which is the overall consumption asymmetry $\hat{\epsilon}$ that governs agents' bid levels throughout the post-innovation consumption schedule.

From the notation of Eq. (30) in the main text, for the consumption asymmetries ϵ_1 and ϵ_2, and the fact that consumption rates c_i and $\tilde{c}_{\tilde{\imath}}$ are related to bid rates b_i and $\tilde{b}_{\tilde{\imath}}$ through the same prices p_i, the ratios of consumption levels of the same good by the two types of agents may be written in terms of the $\hat{q}_i$ and $\hat{\epsilon}$ as

$$\begin{aligned} \frac{c_1}{\tilde{c}_2} &= \frac{b_1}{\tilde{b}_2} = \frac{1 + 2\hat{\epsilon}/\hat{q}_1}{1 - 2\hat{\epsilon}/\hat{q}_1} \\ \frac{\tilde{c}_1}{c_2} &= \frac{\tilde{b}_1}{b_2} = \frac{1 + 2\hat{\epsilon}/\hat{q}_2}{1 - 2\hat{\epsilon}/\hat{q}_2}. \end{aligned} \tag{56}$$

Introducing two further notational abbreviations

$$x_1 \equiv \frac{2\hat{\epsilon}}{\hat{q}_1} \qquad\qquad x_2 \equiv \frac{2\hat{\epsilon}}{\hat{q}_2}, \tag{57}$$

the bid rates by either agent type are written in terms of the total bid rates B_1 and B_2 as

$$\begin{aligned} b_1 &= \frac{1 + x_1}{2} B_1 & \tilde{b}_2 &= \frac{1 - x_1}{2} B_1 \\ \tilde{b}_1 &= \frac{1 + x_2}{2} B_2 & b_2 &= \frac{1 - x_2}{2} B_2 \end{aligned} \tag{58}$$

The bid rates B_1 and B_2 are then related to the total money supply by Eq. (34) in the main text.

As long as the values of the late-time interest rates are well-defined, $\hat{\epsilon}_t$ at late t has a fixed value, by Eq. (40). Then, as long as production levels q_i converge to steady values, the ratios of both B_i to the total money supply converge to steady values by Eq. (34). Finally, under these two conditions, the relations of all b_i and $\tilde{b}_i$ to the total money supply also converge, by Eq. (58).

This completes the result, and shows that steady credit and debt balances for the two agents $a_{1,T}$ and $a_{2,T}$ can be attained with a suitably chosen $\hat{\epsilon}$ by Eq. (23).

Solutions for the Utopia Economy

This section provides solutions for the non-cooperative equilibria of the Utopia model of Sect. 6.2. We begin with the output equations for good-2, which does not undergo an innovation shock.

The Unshocked Good Remains at Steady State Unperturbed

The main Eq. (51) for the response of output decisions to prices, under the condition (40) on shadow prices, becomes

$$\left[\frac{d}{dt}\left(\frac{d}{dt} - \rho_\pi\right) + 2\gamma_2 \bar{f}_2''\right](s_2 - \bar{s}_2) = 0. \tag{59}$$

Since the initial condition from the pre-shock equilibrium was $s_{2,0} = \bar{s}_2$, the unique bounded solution is $s_{2,t} = \bar{s}_2$ for all t.

Recovery of the Shocked Good

$s_{1,t}^{(-)}$ denotes the stock of the type-1 firms that tried to innovate and failed, and $s_{1,t}^{(+)}$ denotes the stock of the type-1 firms that succeeded. The initial conditions for both stocks in the periods immediately following the innovation are $s_{1,0+}^{(\pm)} \to \bar{s}_1 - j$. The steady-state stock for failed-innovation firms is $\bar{s}_1 \equiv (1/2)\log 2$, and the steady-state stock for successfully-innovating firms is $\tilde{s}_1 = \bar{s}_1 + (1/2)\log(1+\theta)$.

In an initial interval following the shock, only a measure $(1-\xi)$ of firms offer in markets. The recovery equation (51) for these firms becomes

$$\frac{(1-\xi)}{2\gamma_1}\frac{d}{dt}\left(\frac{d}{dt} - \rho_\pi\right)\left(s_1^{(-)} - \bar{s}_1\right) \approx -\bar{f}_1''\left(s_1^{(-)} - \bar{s}_1\right). \tag{60}$$

This solution will govern offers $q_{1,t}$ until the shadow prices of successfully-innovating firms fall to intersect market prices. Thereafter the successfully-innovating firms also begin to offer.

Once both firms have entered, both relax to the new steady states with the converging solution to the equation

$$\frac{1}{2\gamma_1}\frac{d}{dt}\left(\frac{d}{dt}-\rho_\pi\right)\left(s_1^{(-)}-\bar{s}_1\right)\approx -\bar{f}_1''\left(s_1^{(-)}-\bar{s}_1\right). \tag{61}$$

These are both second-order linear equations, which possess growing and decaying solutions. We first introduce notations for characteristic rates in the two regimes:

$$\begin{aligned}\omega_+^2 &\equiv -2\gamma_1\bar{f}_1'' \quad \text{evaluates on Eq. (1) to} \quad 4\gamma_1\rho_\pi\\ \omega_-^2 &\equiv -\frac{2\gamma_1}{(1-\xi)}\bar{f}_1''=\frac{\rho_+^2}{(1-\xi)}\end{aligned} \tag{62}$$

In terms of these, the solutions for the relaxation time constants are

$$\begin{aligned}\frac{1}{\tau}&=\pm\sqrt{\omega_-^2+\frac{\rho_\pi^2}{4}}-\frac{\rho_\pi}{2} && t\leqslant t_1\\ \frac{1}{\tau}&=\pm\sqrt{\omega_+^2+\frac{\rho_\pi^2}{4}}-\frac{\rho_\pi}{2} && t>t_1.\end{aligned} \tag{63}$$

Both the positive and negative roots are needed in the initial transient for $t \leqslant t_1$. Only the positive root is required for relaxation toward the turnpike solution in the initial transient for $t > t_1$. The negative root in the second line of Eq. (63) will become important again, however, for the growing solution in the terminal transient.

Equivalent expressions exist for production by type-2 firms. In the numerical example, where the production and consumption parameters are set to equal values for the two types, the type-2 dynamics will depend on the same time constants as the dynamics for type-1 firms in the interval $t > t_1$.

Relaxation and Matching Conditions

The timescale for relaxation shared among models is the discount rate in the profit criterion ρ_π. Therefore introduce a dimensionless coordinate

$$z \equiv \rho_\pi t. \tag{64}$$

Two scale factors that define local timescales relative to z are given shorthand notations $\sqrt{\pm}$, which denote

$$\sqrt{+}\equiv\sqrt{1+\frac{4\rho_+^2}{\rho_\pi^2}}=\sqrt{1+\frac{8\gamma_1\bar{f}_1''}{\rho_\pi^2}}=\sqrt{1+\frac{16\gamma_1}{\rho_\pi}}$$

$$\sqrt{-} \equiv \sqrt{1 + \frac{4\rho_-^2}{\rho_\pi^2}} = \sqrt{1 + \frac{16\gamma_1}{(1-\xi)\,\rho_\pi}}. \tag{65}$$

The two trajectories in the initial interval after the innovation event are

$$\begin{aligned} s_{1,z}^{(+)} - \tilde{s}_1 &= -j - \frac{1}{2}\log(1+\theta) + \frac{f_{1,\infty}(1+\theta)}{\rho_\pi}\left(e^z - 1\right) \\ s_{1,z}^{(-)} - \bar{s}_1 &= e^{z/2}\left[-j\mathrm{ch}\left(\frac{z}{2}\sqrt{-}\right) + \sigma\mathrm{sh}\left(\frac{z}{2}\sqrt{-}\right)\right]. \end{aligned} \tag{66}$$

The trajectory for $s_{1,z}^{(+)}$ is fully determined by the production function because these firms are not responsive to markets. The trajectory for $s_{1,z}^{(-)}$ is determined by its initial conditions up to a single parameter σ which will be determined by matching conditions when successful innovators enter the markets.

The market prices and the shadow prices of successful type-1 firms become equal at some time z_1, which we will identify numerically. (The existence of a unique intersection is assured because the shadow prices of successful firms are falling while the market prices that can be maintained by the unsuccessful firms are rising, during the initial post-innovation interval.)

When the successful type-1 firms have entered the markets, their stocks relax with a fixed offset equal to the difference of late-time steady-state stocks, according to the functions

$$s_{1,z}^{(+)} - \tilde{s}_1 = s_{1,z}^{(-)} - \bar{s}_1 = \left(s_{1,z_1}^{(-)} - \bar{s}_1\right) e^{(z-z_1)\left(\sqrt{+}-1\right)/2} \tag{67}$$

The undetermined parameter σ in Eq. (66) is set by the requirement that the total offering $q_{1,t}$ be continuous through the transition at $z = z_1$, because continuity of q_1 is required for continuity of the price against which firms perform their discounting.

In the numerical solutions of Sect. 6.2.2, the radicals determining the relaxation time constants (65) evaluate to $\sqrt{+} = 3$ and $\sqrt{-} = \sqrt{11} \approx 3.3166$. The resulting time constants (63) are given by $1/\rho_\pi\tau = \left(\pm\sqrt{11} - 1\right)/2$ for $t \leqslant t_1$; $1/\rho_\pi\tau = 1$ for $t > t_1$. The matching parameter that makes both prices and quantities continuous is $\sigma \approx -0.14536$. The remaining features of these solutions are presented as plots in the main text.

Terminal Transient

A terminal transient is solved in terms of the divergences of the three working stocks from their steady-state turnpike values. The functional forms (using properties of non-cooperative equilibria previously derived for stocks when all firms optimize against a shared price system) are given by

$$s_{1,t}^{(+)} - \tilde{s}_1 = s_{1,t}^{(-)} - \bar{s}_1 = \left(s_{1,T}^{(-)} - \bar{s}_1\right) e^{(t-T)/\tau}$$

$$s_{2,t} - \bar{s}_2 = (s_{2,T} - \bar{s}_2)\, e^{(t-T)/\tau}. \tag{68}$$

When (as is the case in the numerical example) $\gamma_1 = \gamma_2 = \rho_\pi/2$, the time constant in both divergences is given by the negative root in the second line of Eq. (63), which evaluates to

$$\frac{1}{\tau} = -\frac{\left(\sqrt{+} + 1\right)}{2}\rho_\pi = -2\rho_\pi. \tag{69}$$

The two parameters in the solution (68), $s^{(-)}_{1,T}$ and $s_{2,T}$, are determined by the requirements that $(a_{1,T} - a_{2,T}) = 0$ and $(a_{1,T} + a_{2,T}) = 0$. Initial conditions are $(a_{1,t} + a_{2,t}) = 0$ as $t \to -\infty$, and $\rho_C\,(a_{1,t} - a_{2,t}) = \tilde{b}_1 - \tilde{b}_2$ of the turnpike solution for $t \to -\infty$. Results of numerical solution are shown in the figures of Sect. 6.2.3.

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