Parametric Estimation Problem for a Time Periodic Signal in a Periodic Noise

Khalil El Waled

1 Introduction

Consider the following periodic signal plus periodic noise model

$$d\zeta_t = f(t, \theta)dt + \sigma(t)dW_t, \tag{1}$$

where $f(\cdot, \cdot) : \mathbb{R} \times \Theta \mapsto \mathbb{R}$, $\sigma(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ are continuous functions and periodic in t with the same period P, $\sigma(\cdot)$ is positive, $\theta \in \Theta$ is an unknown parameter; Θ is a compact of $\mathbb{R}$, $\{W_t\}$ is a Brownian motion defined over a complete probability space $(\Omega, \mathscr{F}, \mathrm{P})$.

As an application, let $\{\xi_t, t \geq 0\}$ be the time-dependent geometric Brownian motion which verifies the following linear stochastic differential equation

$$d\xi_t = f(t)\xi_t dt + \sigma(t)\xi_t dW_t. \tag{2}$$

The connection between these processes is given by

$$d\zeta_t = \frac{d\xi_t}{\xi_t}.$$

Such types of processes appear in many domains, for instance, in finance [13] (Black–Scholes–Merton model), mechanics ([8, 12]) and in biology [1, 9].

The problem of parameter estimation when the process is continuously observed has been studied by many others (see for instance [15, 16]...). Ibragimov and Has'minskil ([11], Sect. 5 in Chap. 3) have considered the general model

$$dX_\varepsilon(t) = S_\varepsilon(t, \theta) + db(t).$$

K. El Waled (✉)
IRMAR, CNRS Umr 6625, Université Rennes 2, Rennes, France
e-mail: khalil.elwaled@gmail.com; khalil.elwaled@uhb.fr

F. Chaari et al. (eds.), *Cyclostationarity: Theory and Methods - II*,
Applied Condition Monitoring 3, DOI 10.1007/978-3-319-16330-7_2

Although model (1) can be reduced to the well-known model $X_\varepsilon(t)$, we prefer to deal with the first case and we are interested in its properties that are driven by the periodicity of $f(\cdot,\cdot)$ and $\sigma(\cdot)$. These properties will enable us to establish the convergence almost sure, to give an explicit expression of the Fisher information and to prove simply the efficiency of the MLE.

Since in practice it is difficult to obtain a complete continuous time observation of the sample path of a process. So the problem of discretization has to be considered.

Kasonga [14] has used the least square method to show the consistency of the estimators based on the discrete schemas. The case of stationary diffusion models is studied in [3, 5].

Using maximum contrast, and for small variance diffusion models [6], has shown under some classical assumptions, asymptotic results. Harison [7] has used this method to estimate the drift parameter for some one-dimensional nonstationary Gaussian diffusion models.

This paper is organized as follows: First, in Sect. 2 we study the parametric estimation problem of θ from a continuous time observation. We prove the existence of an estimator that converges in probability. When $f(t,\theta)=\theta f(t)$ we get the mean square convergence, the asymptotic normality of this estimator and the asymptotic efficiency.

In the Sect. 3 we assume that the observations have been done at discrete moments. We compute the likelihood function and we propose an estimator of the unknown parameter θ then we establish its convergence in probability. When $f(t,\theta)=\theta f(t)$ we show the mean square convergence and the asymptotic normality property as well as the asymptotic efficiency.

For an easier reading of the paper proofs which are long are put in the appendix.

2 Drift Estimation from a Continuous Time Observation

2.1 *Likelihood*

In the continuous time observation context to estimate the unknown parameter θ we can use the method of the maximum of likelihood. To compute the likelihood function of this model we apply Theorem 7.18 in [16] and Corollary which follows it to the next two processes (see also Theorem 6.10 in [10])

$$d\zeta_t^\theta = f(t,\theta)dt + \sigma(t)dW_t, \tag{3}$$

$$d\eta_t = \sigma(t)dW_t. \tag{4}$$

These two processes satisfy the four conditions of Theorem 7.18 in [16]. So $\mu_\theta^T \sim \nu^T$, where

$$\mu_\theta^T := \mathscr{L}(\zeta_t^\theta,\ 0\le t\le T),\ \nu^T := \mathscr{L}(\eta_t,\ 0\le t\le T).$$

In addition, the conditions of the corollary which follows this Theorem 7.18 are satisfied and we have P_θ-a.s.

$$\frac{d\mu_\theta^T}{d\nu^T}(\zeta^\theta) = \exp\left(\int_0^T \frac{\rho(s,\theta)}{\sigma(s)} d\zeta_s^\theta - \frac{1}{2}\int_0^T \rho^2(s,\theta)ds\right) \tag{5}$$

where $\rho(s,\theta) := \frac{f(s,\theta)}{\sigma(s)}$. Remark that $\sigma(\cdot)$ is positive continuous so $\inf_{t\in[0,P]}\sigma(t) > 0$ and $\rho(\cdot)$ is well defined and continuous.

To simplify the computation we take $T = nP$. Assuming that the real value of the unknown parameter θ is θ_0, the likelihood function is given by

$$L_n(\theta) := \exp\left(\int_0^{nP} \frac{\rho(s,\theta)}{\sigma(s)} d\zeta_s^{\theta_0} - \frac{1}{2}\int_0^{nP} \rho^2(s,\theta)ds\right) \tag{6}$$

Thanks to the continuity of $f(\cdot,\cdot)$ and $\sigma(\cdot)$ and the compactness of Θ there exists $\hat{\theta}_n$ such that

$$\hat{\theta}_n = \arg\sup_{\theta\in\Theta} L_n(\theta).$$

Maximizing $L_n(\theta)$ in θ is identical to maximizing the function $U_n(\theta)$ given by

$$U_n(\theta) := \frac{1}{nP}\int_0^{nP} \frac{\rho(s,\theta)}{\sigma(s)} d\zeta_s^{\theta_0} - \frac{1}{2nP}\int_0^{nP} \rho^2(s,\theta)ds. \tag{7}$$

In the following we need the next separability condition:

(S) Assume that for each $\theta \neq \theta_0$ there exists an s such that $f(s,\theta) \neq f(s,\theta_0)$.

2.2 *Consistency of the Estimator*

Here we prove the consistency of the MLE $\hat{\theta}_n$. To establish this result we first check that $U_n(\cdot)$ is a contrast process in the sense of [2] (Definition 3.2.7).

Lemma 1 *Under the continuity and the periodicity of $f(\cdot,\cdot)$ and $\sigma(\cdot)$ in addition to the separability condition (S), $U_n(\cdot)$ is a contrast process.*

Proof We show that $U_n(\theta)$ converges in P_{θ_0} to the nonrandom contrast function $K(\theta,\theta_0)$ where

$$K(\theta,\theta_0) := -\frac{1}{2P}\int_0^P (\rho(s,\theta_0) - \rho(s,\theta))^2\, ds + \frac{1}{2P}\int_0^P \rho^2(s,\theta_0)ds. \tag{8}$$

Recall that

$$d\zeta_s^{\theta_0} = f(s,\theta_0)ds + \sigma(s)dW_s.$$

So from the periodicity of $f(\cdot,\cdot)$ and $\sigma(\cdot)$ we have

$$\begin{aligned} U_n(\theta) &= \frac{1}{P}\int_0^P \rho(s,\theta_0)\rho(s,\theta)ds - \frac{1}{2P}\int_0^P \rho^2(s,\theta)ds \\ &\quad + \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{P}\int_0^P \rho(s,\theta)dW_s^{(kP)} \\ &= -\frac{1}{2P}\int_0^P (\rho(s,\theta_0)-\rho(s,\theta))^2 ds + \frac{1}{2P}\int_0^P \rho^2(s,\theta_0)ds \\ &\quad + \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{P}\int_0^P \rho(s,\theta)dW_s^{(kP)} \end{aligned}$$

where $W_s^{(kP)} = W_{s+kP} - W_{kP}$.

Since the random variables $\frac{1}{P}\int_0^P \rho(s,\theta)dW_s^{(kP)}$, $\{k=0,\ldots,n-1\}$ are iid then thanks to the Strong Law of Large Numbers

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{P}\int_0^P \rho(s,\theta)dW_s^{(kP)} = 0 \;\; \mathrm{P}_{\theta_0} - \text{a.s.}$$

Therefore $U_n(\theta)$ converges almost surely to $K(\theta,\theta_0)$, so in probability.

Thanks to the fact that for each $\theta \neq \theta_0$ there exists an s such that $f(s,\theta) \neq f(s,\theta_0)$ the function $\theta \mapsto K(\theta,\theta_0)$ has a strict maximum when $\theta = \theta_0$ and

$$K(\theta_0,\theta_0) = \frac{1}{2P}\int_0^P \rho^2(s,\theta_0)ds.$$

This achieves the proof of the lemma.

Theorem 1 *Assume that $f(\cdot,\cdot)$ and $\sigma(\cdot)$ are continuous and periodic in t, $f(\cdot,\cdot)$ is continuously differentiable in θ and the separability condition (S) is satisfied. Then the MLE $\hat{\theta}_n$ is consistent.*

$$\hat{\theta}_n \overset{\mathrm{P}_{\theta_0}}{\rightarrow} \theta_0.$$

2.3 Consistency and Efficiency When $f(t,\theta) = \theta f(t)$

The choice of the case $f(t,\theta) = \theta f(t)$ is justified by its linearity which makes it relevant. The modèle (1), in this case, is given as $d\zeta_t = \theta f(t)dt + \sigma(t)dW_t$. In the continuous case, the non-parametric estimation of the function $f(\cdot)$ was studied [4, 11]. For the parametric estimation, thanks to (6), the likelihood is defined as

$$L_n(\theta) = \exp\left(\theta \int_0^{nP} \frac{\rho(s)}{\sigma(s)} d\zeta_s^{\theta_0} - \frac{\theta^2}{2} \int_0^{nP} \rho^2(s)ds\right).$$

The score function is equal to

$$S(\theta) := \frac{\partial \ln L_n(\theta)}{\partial \theta} = \int_0^{nP} \frac{\rho(s)}{\sigma(s)} d\zeta_s^{\theta_0} - \theta \int_0^{nP} \rho^2(s)ds.$$

This implies that the MLE $\hat{\theta}_n$ is equal to

$$\hat{\theta}_n := \frac{\int_0^{nP} \frac{\rho(s)}{\sigma(s)} d\zeta_s^{\theta_0}}{\int_0^{nP} \rho^2(s)ds} = \frac{\int_0^{nP} \frac{\rho(s)}{\sigma(s)} (\theta_0 f(s)ds + \sigma(s)dW_s)}{\int_0^{nP} \rho^2(s)ds}.$$

So we can write $\hat{\theta}_n$ as:

$$\hat{\theta}_n = \theta_0 + \frac{V_n}{I_n},$$

where

$$V_n := \int_0^{nP} \rho(s)dW_s, \;\; I_n := \int_0^{nP} \rho^2(s)ds.$$

Then $\hat{\theta}_n$ is unbiased. Moreover, we are going to state the almost sure convergence, mean square convergence, the asymptotic normality and the asymptotic efficiency.

2.3.1 Almost Sure Convergence, Mean Square Convergence, Asymptotic Normality

Theorem 2 *$\hat{\theta}_n$ converges almost surely to θ_0.*

Remark 1 Noticing that I_n is the quadratic variation of the martingale V_n we can directly deduce from the martingale convergence the strong convergence of $\hat{\theta}_n$.

Since

$$\mathscr{L}\left(\int_0^{nP} \rho(s)dW_s\right) = \mathscr{N}\left(0, \int_0^{nP} \rho^2(s)ds\right) \text{ and } \hat{\theta}_n - \theta_0 = \frac{\int_0^{nP} \rho(s)dW_s}{\int_0^{nP} \rho^2(s)ds},$$

we deduce that

$$\mathscr{L}\left(\hat{\theta}_n - \theta_0\right) = \mathscr{N}\left(0, I_n^{-1}\right). \tag{9}$$

So we obtain the mean square convergence as well as the asymptotic normality.

Theorem 3 *$\hat{\theta}_n$ converges in mean square to θ_0, and $\bar{\theta}_n = \sqrt{n}(\hat{\theta}_n - \theta_0)$ is asymptotically normal: $\bar{\theta}_n \sim \mathcal{N}\left(0, I^{-1}\right)$, where $I = \int_0^P \rho^2(s)ds$ is the Fisher information.*

2.3.2 Asymptotic Efficiency of $\hat{\theta}_n$

To justify the relevance of this estimator we are going to establish that it is asymptotically efficient. For this purpose we use the Hájek-Le Cam inequality (see [15, 17] for further details).

We show first that the family $(\mathrm{P}_\theta^{(n)})_{\theta\in\Theta}$ is locally asymptotically normal (see Definition 1.2.1 in [15]).

Proposition 1 *Using the notations of Definition 1.2.1 in [15] $(\mathrm{P}_\theta^{(n)})_{\theta\in\Theta}$ is locally asymptotically normal at θ_0 for any θ_0.*

Theorem 4 *The estimator $\hat{\theta}_n$ is asymptotically efficient for the square error (see Definition 1.2.2 in [15]).*

Proof From (9)

$$\mathrm{E}\left[I_n(\hat{\theta}_n - \theta_0)^2\right] = 1.$$

Hence $\hat{\theta}_n$ is asymptotically efficient.

3 Drift Estimation from a Discrete Time Observation

In the second section we have assumed that the process is observed continuously throughout the interval $[0, T]$, however it is difficult to record numerically a continuous time process, and the observations generally take place at discrete moments. Hence the motivation of this section.

Recall the model of signal plus noise type that we consider

$$d\zeta_t = f(t,\theta)dt + \sigma(t)dW_t \quad t \in [0, T]. \tag{10}$$

The goal of this section is the estimation of the parameter θ.

Thanks to the simplicity of this model we are able to compute its likelihood.

Assume that the observations take place at the instants $t_i := i\Delta_n$, $i \in 0 \ldots n-1$ of the interval $[0, T]$, where $\Delta_n = \frac{T}{n}$. Assume also that $T = n\Delta_n = N_n P$ and that $T = n\Delta_n \to \infty$ and $\Delta_n \to 0$ when $n \to \infty$. So $N_n \to \infty$ when $n \to \infty$.

To simplify the representation denote

$$Y_i^\theta := \int_{t_i}^{t_{i+1}} \rho(s,\theta)ds + W_{t_{i+1}} - W_{t_i}. \tag{11}$$

Hence

$$Y_i^\theta \sim \mathcal{N}\left(\int_{t_i}^{t_{i+1}} \rho(s,\theta)ds,\, \Delta_n\right). \tag{12}$$

Therefore, the log-likelihood is given by

$$\begin{aligned}\Lambda_n(\theta) &:= \sum_{i=0}^{n-1}\left(-\ln(\sqrt{2\pi\Delta_n}) - \frac{1}{2\Delta_n}\left(Y_i^{\theta_0} - \int_{t_i}^{t_{i+1}} \rho(s,\theta)ds\right)^2\right) \qquad (13)\\ &= -\frac{n}{2}\ln(2\pi\Delta_n) - \frac{1}{2\Delta_n}\sum_{i=0}^{n-1}\left({Y_i^{\theta_0}}^2 - 2Y_i^{\theta_0}\int_{t_i}^{t_{i+1}} \rho(s,\theta)ds\right)\\ &\quad -\frac{1}{2\Delta_n}\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}} \rho(s,\theta)ds\right)^2\end{aligned}$$

where θ_0 is the true value of the unknown parameter θ. Thanks to the continuity of $f(\cdot,\cdot)$ and $\sigma(\cdot)$ and the compactness of Θ there exists $\hat{\theta}_n$ such that

$$\hat{\theta}_n = \arg\sup_{\theta\in\Theta} \Lambda_n(\theta).$$

Since θ does not depend on the observation $Y_i^{\theta_0}$ then

$$\hat{\theta}_n = \arg\sup_{\theta\in\Theta} U_n(\theta),$$

where

$$U_n(\theta) := \frac{1}{n\Delta_n^2}\sum_{i=0}^{n-1} Y_i^{\theta_0}\int_{t_i}^{t_{i+1}} \rho(s,\theta)ds - \frac{1}{2n\Delta_n^2}\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}} \rho(s,\theta)ds\right)^2. \tag{14}$$

3.1 Consistency of the Estimator

As in the continuous case, to prove the convergence in P_{θ_0} of $\hat{\theta}_n$ to the true value of the parameter θ_0 we check that $U_n(\theta)$ is a contrast process in the sense of [2] (Definition 3.2.7) and then we apply a version of (Theorem 3.2.4, [2], see also Theorem 5.7 of van der Vaart [17]).

Proposition 2 *Under the continuity and the periodicity of $f(\cdot,\cdot)$ and $\sigma(\cdot)$ and under the separability condition (S), $U_n(\theta)$ is a contrast process.*

Theorem 5 *In addition to the conditions of Proposition 2 assume that $\rho(s,\theta)$ is continuously differentiable, then $\hat{\theta}_n$ converges in probability to the real parameter θ_0.*

3.2 Study of the Case $f(t,\theta)=\theta f(t)$

Now we consider the case $f(t,\theta)=\theta f(t)$ where $f(\cdot)$ is a continuous and periodic function with same period as $\sigma(\cdot)$ and θ is unknown parameter. In this case the expression of the estimator $\hat{\theta}_n$ is explicit and we show that its estimator converges in mean square and it is asymptotically normal.

From (13) the log-likelihood is given by

$$\Lambda_n(\theta) := -\frac{n}{2}\ln(2\pi\Delta_n) - \frac{1}{2\Delta_n}\sum_{i=0}^{n-1}\left(Y_i^{\theta_0} - \theta\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2, \tag{15}$$

where $\rho(s)=\frac{f(s)}{\sigma(s)}$, Y_i^{θ} is given by (11). Hence the score function $S(\theta)$ is

$$S(\theta) = \frac{1}{\Delta_n}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\rho(s)ds\left(Y_i^{\theta_0} - \theta\int_{t_i}^{t_{i+1}}\rho(s)ds\right).$$

So we get the expression of the MLE

$$\hat{\theta}_n = \frac{\sum_{i=0}^{n-1} Y_i^{\theta_0}\int_{t_i}^{t_{i+1}}\rho(s)ds}{\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2} = \theta_0 + \frac{\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\rho(s)ds\left(W_{t_{i+1}} - W_{t_i}\right)}{\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2}.$$

Then $\hat{\theta}_n$ is unbiased estimator. Furthermore, we are going to show that it converges in mean square and it is asymptotically normal.

3.2.1 Mean Square Convergence, Asymptotic Normality

Theorem 6 *$\hat{\theta}_n$ converges in mean square to θ_0, furthermore we have the rate of this convergence*

$$\lim_{n\to\infty} \mathrm{E}_{\theta_0}\left[\left|\sqrt{n\Delta_n}(\hat{\theta}_n - \theta_0)\right|^2\right] = \left(\frac{1}{P}\int_0^P \rho^2(s)ds\right)^{-1}.$$

Since

$$\sqrt{n\Delta_n}(\hat{\theta}_n - \theta_0) \sim \mathcal{N}\left(0, \frac{n\Delta_n^2}{\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2}\right).$$

We deduce that

$$\lim_{n\to\infty} \mathscr{L}\left(\sqrt{n\Delta_n}(\hat{\theta}_n - \theta_0)\right) = \mathscr{N}\left(0, \left(\frac{1}{P}\int_0^P \rho^2(s)ds\right)^{-1}\right).$$

Therefore

Theorem 7 $\sqrt{n\Delta_n}(\hat{\theta}_n - \theta_0)$ *is asymptotically normal.*

3.2.2 Asymptotic Efficiency of $\hat{\theta}_n$

Here as in the continuous case we show that the estimator $\hat{\theta}_n$ is asymptotically efficient. We establish first that the family $(\mathrm{P}_\theta^{(n)})_{\theta\in\Theta}$ is locally asymptotically normal (see Definition 1.2.1 in [15]).

Proposition 3 $(\mathrm{P}_\theta^{(n)})_{\theta\in\Theta}$ *is locally asymptotically normal at θ_0 for any θ_0.*

Theorem 8 *The estimator $\hat{\theta}_n$ is also asymptotically efficient for the square error (see Definition 1.2.2 in [15]).*

Proof According to (26)

$$\begin{aligned}
\mathrm{E}_{\theta_0}\left[\left|\Phi_n^{-1}(\hat{\theta}_n - \theta_0)\right|^2\right] &= \Phi_n^{-2}\mathrm{E}_{\theta_0}\left[\left|\frac{\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\rho(s)ds(W_{t_{i+1}} - W_{t_i})}{\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2}\right|^2\right] \\
&= \Phi_n^{-2}\frac{\Delta_n}{\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2} \\
&= \Phi_n^{-2}\Phi_n^2 \\
&= 1.
\end{aligned}$$

Hence $\hat{\theta}_n$ is asymptotically efficient.

4 Simulation

As an example we illustrate this estimator $\hat{\theta}_n$ with different values of θ_0, $\theta_0 = 0$, $\theta_1 = 1$ and $\theta = -4$.

Assume that the sample size $T = nP = 1000$, the period $P = 1$. First, we consider the next model

$$d\zeta_t = \theta\cos(2\pi t) + dW_t, \text{ i.e. } f(t) = \cos(2\pi t),\ \sigma(t) = 1,$$

and the discretization step δ .

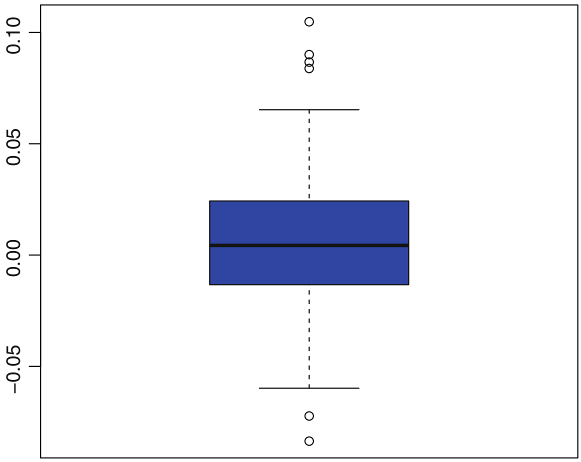

Fig. 1 Boxplot of the values of the estimator $\hat{\theta}_n$ from 100 repetitions

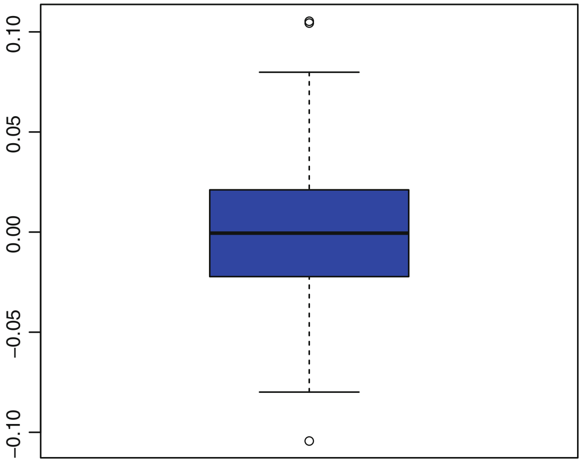

Fig. 2 Boxplot of the values of the estimator $\hat{\theta}_n$ from 1000 repetitions

For $\theta_0 = 0$, $\delta = 10^{-2}$ in Figs. 1 and 2 we see that all the values are around 0 and the interquartile range is less than 0.2.

For $\theta_0 = 1$, $\delta = 10^{-3}$ in Fig. 3 we see the importance of the step of discretization δ, the median is equal to 1 and the interquartile range is equal to 0.1.

Now we illustrate this estimator with another model (Fig. 4):

$$f(t) = \cos(2\pi t),\ \sigma(t) = 2 + \sin(2\pi t),\ \theta_0 = -4,\ \delta = 10^{-3}.$$

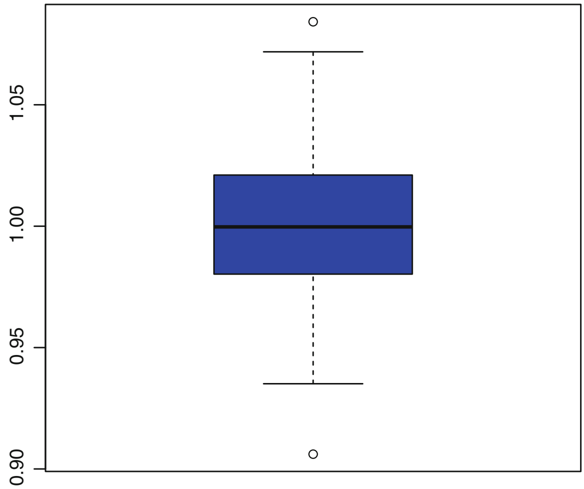

Fig. 3 Boxplot of the values of the estimator $\hat{\theta}_n$ from 1000 repetitions

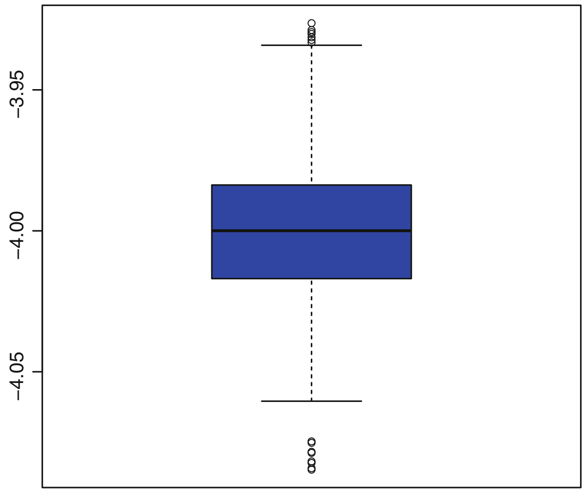

Fig. 4 Boxplot of the values of the estimator $\hat{\theta}_n$ from 1000 repetitions

As we can see the median of the 1000 values is -4 and the interquartile range is around 0.1.

When $f(t) = \sigma(t) = 1, \theta = 1$, Fig. 5 shows that all the values are next to 1. In the previous cases although the dispersion around the median is not very big, the degradation with respect to the case $f(t) = \sigma(t) = 1$ is due to the variations of $f(t)$ and $\sigma(t)$.

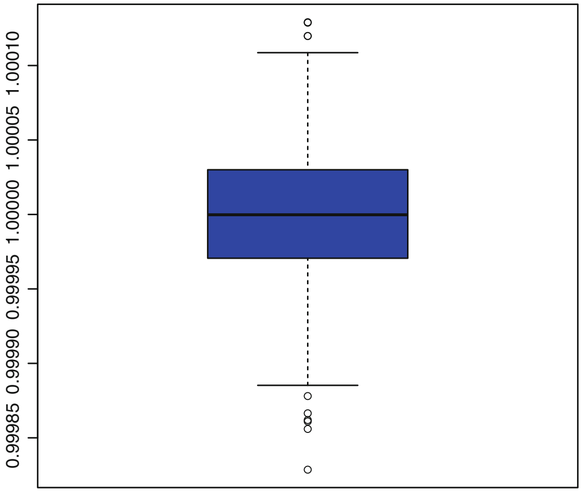

Fig. 5 Boxplot of the values of the estimator $\hat{\theta}_n$ from 1000 repetitions

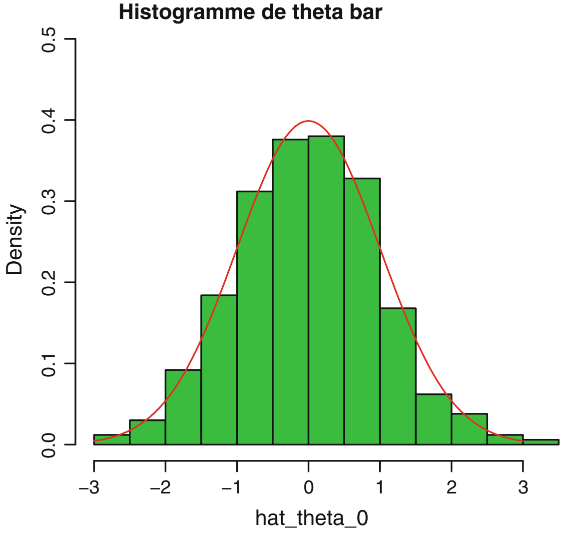

Fig. 6 Histogram $\bar{\theta}_n$, $\theta_0 = 0$ from 1000 repetitions

Now we check the asymptotic normality property. From the Theorem 3

$$\bar{\theta}_n := \sqrt{n}(\hat{\theta}_n - \theta_0), \lim_{n\to\infty} \mathscr{L}\left(\sqrt{n}(\hat{\theta}_n - \theta_0)\right) = \mathscr{N}\left(0, \frac{1}{\int_0^P \rho^2(s)ds}\right).$$

In Figs. 6, 7 and 8 we note that the histograms obtained by the 1000 repetitions of the simulations fit to the Gaussian distribution.

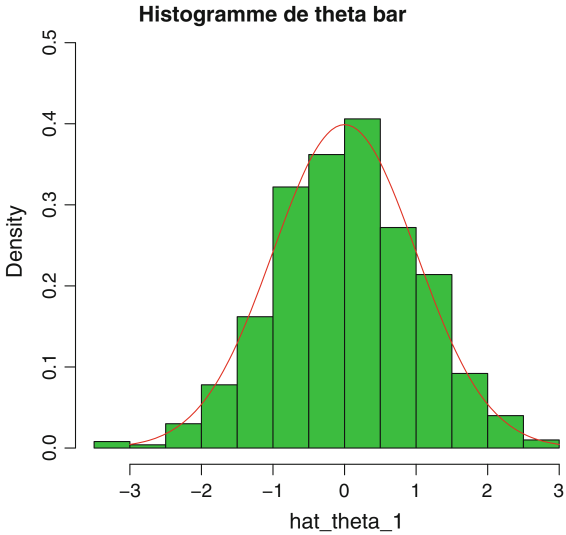

Fig. 7 Histogram of $\bar{\theta}_n$, $\theta_0 = 1$ from 1000 repetitions

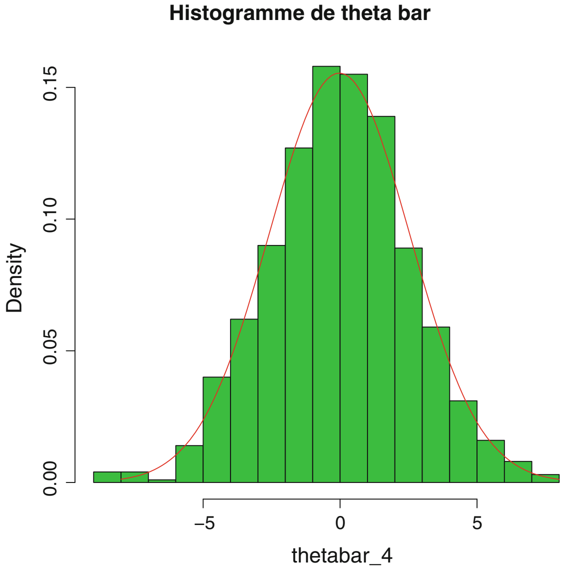

Fig. 8 Histogram of $\bar{\theta}_n$, $\theta_0 = -4$ from 1000 repetitions

5 Conclusion

We have considered a model of type signal plus noise given by

$$d\zeta_t = f(t,\theta)dt + \sigma(t)dW_t.$$

From a continuous time observation throughout the interval $[0, T]$ we have built an estimator $\hat{\theta}_n$ of the unknown parameter θ, using the method of maximum likelihood. We have established that this estimator converges in probability. When $f(t,\theta) = \theta f(t)$, we have given the expression of $\hat{\theta}_n$ and we have established its mean square convergence, almost sure convergence, asymptotic normality and asymptotic efficiency. From a discrete time observation on $[0, T]$ we have computed the likelihood function and we have shown the same convergences as in the continuous case except the strong consistency convergence.

As a prospect I am going to consider model (1) but with $\sigma(\cdot)$ which can take the value 0. I plan also to study the estimation of the parameters θ_i from the observations of ζ_t following the model

$$d\xi_t = (\theta_1 f_1(t) + \theta_2 f_2(t) \ldots \theta_n f_n(t))\, dt + \sigma(t)dW_t.$$

Acknowledgments I would like to thank the region of Brittany for funding my Ph.D. scholarship.

Appendix

Proof **Theorem 1**

We apply (Theorem 3.2.4, [2]) which gives the convergence in probability of the maximum of the contrast process $U_n(\theta)$. See also (Theorem 5.7 of van der Vaart 2005). For this purpose we need to show the two following conditions:

(i) Θ is a compact of $\mathbb{R}$, the functions $\theta \mapsto U_n(\theta)$, $\theta \mapsto K(\theta,\theta_0)$ are continuous.

(ii) for all $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\lim_{n\to\infty} \mathrm{P}_\theta \left(\sup_{|\theta-\theta'|<\eta} \left| U_n(\theta) - U_n(\theta') \right| > \varepsilon \right) = 0.$$

The first condition is readily obtained from the hypothesis on $f(\cdot,\cdot)$, $\sigma(\cdot)$ and Θ, and from the definitions of $U_n(\cdot)$ and $K(\cdot,\cdot)$.

Below we prove the second condition

$$\begin{aligned}
U_n(\theta) - U_n(\theta') &= \frac{1}{2nP}\int_0^{nP} \left(\rho(s,\theta) - \rho(s,\theta')\right)\left(2\rho(s,\theta_0) - \rho(s,\theta) - \rho(s,\theta')\right) ds \\
&\quad + \frac{1}{nP}\int_0^{nP} (\rho(s,\theta) - \rho(s,\theta'))dW_s \\
&= \frac{1}{2P}\int_0^{P} \left(\rho(s,\theta) - \rho(s,\theta')\right)\left(2\rho(s,\theta_0) - \rho(s,\theta) - \rho(s,\theta')\right) ds \\
&\quad + \frac{1}{nP}\int_0^{P} (\rho(s,\theta) - \rho(s,\theta'))dW_s \\
&:= A_1(\theta,\theta') + A_2(\theta,\theta')
\end{aligned} \tag{16}$$

where $A_1(\theta,\theta')$ is the nonrandom term of this equality and $A_2(\theta,\theta')$ is the random term.

The absolute value of the nonrandom term of this equality is bounded by a multiple of η where $|\theta - \theta'| \le \eta$ and the second term converges in mean to 0 when $n \to \infty$. Indeed

As $\rho(\cdot,\cdot)$ is continuously differentiable then

$$\rho(s,\theta) - \rho(s,\theta') = (\theta - \theta')\rho'(s,\theta_1),\ \theta_1 \in [\theta,\theta'],$$

where $\rho'(\cdot,\cdot)$ is the derivative of $\rho(\cdot,\cdot)$ with respect to θ. Denote

$$C_0 := \sup_{\theta\in\Theta,\ s\in[0,P]} \left|\rho'(s,\theta)\right|.$$

Thus

$$\begin{aligned}
\sup_{\theta,\theta'\in\Theta,\ s\in[0,P]} \left|2\rho(s,\theta_0) - \rho(s,\theta) - \rho(s,\theta')\right| &= 2\sup_{\theta\in\Theta,\ s\in[0,P]} |\rho(s,\theta_0) - \rho(s,\theta)| \\
&\le C_1 \sup_{\theta\in\Theta} |\theta_0 - \theta| \\
&\le C
\end{aligned}$$

where C is positive constant because Θ is a compact. So for the nonrandom part

$$\sup_{|\theta-\theta'|<\eta} \left|A_1(\theta,\theta')\right| \le \sup_{|\theta-\theta'|<\eta} \frac{|\theta-\theta'|}{2P}\int_0^P C ds \le \frac{C}{2}\eta \tag{17}$$

Now show that the random part converges in mean to 0. First

$$A_2(\theta,\theta') = \frac{1}{nP}\int_0^P (\rho(s,\theta) - \rho(s,\theta'))dW_s = \int_\theta^{\theta'} \frac{1}{nP}\int_0^P \rho'(s,u)dW_s\, du.$$

The set Θ being compact

$$\begin{aligned}
\mathrm{E}_{\theta_0}\left[\sup_{|\theta-\theta'|<\eta}\left|A_2(\theta,\theta')\right|\right] &\leq \mathrm{E}_{\theta_0}\left[\sup_{|\theta-\theta'|<\eta}\int_{\theta}^{\theta'}\left|\frac{1}{nP}\int_0^P \rho'(s,u)dW_s\right| du\right]\\
&\leq \mathrm{E}_{\theta_0}\left[\int_{\Theta}\left|\frac{1}{nP}\int_0^P \rho'(s,u)dW_s\right| du\right]\\
&\leq \int_{\Theta}\mathrm{E}_{\theta_0}\left[\left|\frac{1}{nP}\int_0^P \rho'(s,u)dW_s\right|\right] du\\
&\leq \int_{\Theta}\left(\mathrm{E}_{\theta_0}\left[\left|\frac{1}{nP}\int_0^P \rho'(s,u)dW_s\right|^2\right]\right)^{\frac{1}{2}} du\\
&= \int_{\Theta}\left(\frac{1}{n^2P^2}\int_0^{nP} \rho'^2(s,u)ds\right)^{\frac{1}{2}} du\\
&= \frac{1}{\sqrt{nP}}\int_{\Theta}\left(\frac{1}{P}\int_0^{P} \rho'^2(s,u)ds\right)^{\frac{1}{2}} du.
\end{aligned}$$

Then

$$\lim_{n\to\infty}\mathrm{E}_{\theta_0}\left[\sup_{|\theta-\theta'|<\eta}\left|A_2(\theta,\theta')\right|\right] = 0. \tag{18}$$

According to the equality (16) we have for all $\varepsilon > 0$

$$\begin{aligned}
\mathrm{P}_{\theta_0}\left(\sup_{|\theta-\theta'|<\eta}\left|U_n(\theta)-U_n(\theta')\right| > \varepsilon\right) &\leq \mathrm{P}_{\theta_0}\left(\sup_{|\theta-\theta'|<\eta}\left|A_1(\theta,\theta')\right| > \frac{\varepsilon}{2}\right)\\
&\quad + \mathrm{P}_{\theta_0}\left(\sup_{|\theta-\theta'|<\eta}\left|A_2(\theta,\theta')\right| > \frac{\varepsilon}{2}\right)
\end{aligned}$$

Since

$$\lim_{n\to\infty}\mathrm{E}_{\theta_0}\left[\sup_{|\theta-\theta'|<\eta}\left|A_2(\theta,\theta')\right|\right] = 0$$

then

$$\lim_{n\to\infty}\mathrm{P}_{\theta_0}\left(\sup_{|\theta-\theta'|<\eta}\left|A_2(\theta,\theta')\right| > \frac{\varepsilon}{2}\right) = 0.$$

Thanks to (17) is suffice, for all $\varepsilon > 0$, to take $\eta = \frac{\varepsilon}{2C}$ to have

$$\sup_{|\theta-\theta'|<\eta} \left|A_1(\theta,\theta')\right| < \frac{\varepsilon}{2}.$$

So for all $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\lim_{n\to\infty} \mathrm{P}_{\theta_0}\left(\sup_{|\theta-\theta'|<\eta} \left|U_n(\theta) - U_n(\theta')\right| > \varepsilon\right) = 0.$$

Therefore, the second condition is fulfilled. So the MLE $\hat{\theta}_n$ is consistent.

Proof **Theorem 2**

Recall that $f(\cdot)$ and $\sigma(\cdot)$ are continuous and periodic with the same period $P > 0$. On one hand

$$I_n = \int_0^{nP} \rho^2(s)ds = n\int_0^P \rho^2(s)ds.$$

So

$$\lim_{n\to\infty} \frac{I_n}{n} = \int_0^P \rho^2(s)ds.$$

On the other hand

$$V_n = \sum_{k=0}^{n-1} \int_{kP}^{(k+1)P} \rho(s)dW_s = \sum_{k=0}^{n-1} \int_0^P \rho(s)dW_s^{(kP)}.$$

As

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \int_0^P \rho(s)dW_s^{(kP)} = \mathrm{E}_{\theta_0}\left[\int_0^P \rho(s)dW_s^{(kP)}\right] = 0 \;\; \mathrm{P}_{\theta_0} - a.s.$$

we deduce that

$$\lim_{n\mapsto\infty} \hat{\theta}_n - \theta_0 = \lim_{n\mapsto\infty} \hat{\theta}_n - \theta_0 = \lim_{n\mapsto\infty} \frac{\frac{1}{n}V_n}{\frac{1}{n}I_n} = 0 \;\; \mathrm{P}_{\theta_0} - a.s.$$

Proof **Proposition 1**

Denote

$$\Phi_n := I_n^{-\frac{1}{2}} = \left(\int_0^{nP} \rho^2(s)ds\right)^{-\frac{1}{2}},$$

$$\Delta_n(\zeta^\theta) := I_n^{-\frac{1}{2}} \int_0^{nP} \rho(s) dWs.$$

According to (6) we have

$$\begin{aligned} Z_n(u) &:= \frac{L_n(\theta_0 + \Phi_n u)}{L_n(\theta_0)} \\ &= \exp\left(\Phi_n u \int_0^{nP} \frac{\rho(s)}{\sigma(s)} d\zeta_s^{\theta_0} - \frac{(\theta_0 + \Phi_n u)^2 - \theta_0^2}{2} I_n \right) \\ &= \exp\left(\Phi_n u \int_0^{nP} \frac{\rho(s)}{\sigma(s)} d\zeta_s^{\theta_0} - \frac{\Phi_n^2 u^2}{2} I_n - \theta_0 \Phi_n u I_n \right). \end{aligned}$$

Since

$$d\zeta_t^{\theta_0} = \theta_0 f(t) dt + \sigma(t) dW_t$$

then we have

$$\begin{aligned} Z_n(u) &= \exp\left(\theta_0 u I_n^{-\frac{1}{2}} I_n + u I_n^{-\frac{1}{2}} \int_0^{nP} \rho(s) dW_s + \frac{1}{2} u^2 - \theta_0 u I_n^{\frac{1}{2}} \right) \\ &= \exp\left(u I_n^{-\frac{1}{2}} \int_0^{nP} \rho(s) dW_s - \frac{1}{2} u^2 \right) \\ &= \exp\left(u \Delta_n(\zeta^\theta) - \frac{1}{2} u^2 \right). \end{aligned} \tag{19}$$

As

$$\mathscr{L}\left(\int_0^{nP} \rho(s) dW_s \right) = \mathscr{N}(0, I_n)$$

then we have

$$\mathscr{L}\left\{ \Delta_n(\zeta^\theta) | \mathrm{P}_{\theta_0} \right\} = \mathscr{N}(0, 1).$$

So from (19) the family $(\mathrm{P}_\theta^{(n)})_{\theta \in \Theta}$ is locally asymptotically normal at θ_0 for every θ_0.

Proof **Proposition 2** We show that $U_n(\theta)$ converges in P_{θ_0} to the contrast function $K(\theta, \theta_0)$ defined by (8). To do this we use the next lemma

Lemma 2 *For the continuous and periodic function $\rho(\cdot, \cdot)$ and $\Delta_n \to 0$ when $n \to \infty$ we have*

$$\lim_{n\to\infty} \frac{1}{2n\Delta_n^2} \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} \rho(s, \theta) - \int_{t_i}^{t_{i+1}} \rho(s, \theta_0) \right)^2 ds = \frac{1}{2P} \int_0^P (\rho(s, \theta_0) - \rho(s, \theta))^2 ds.$$

$$\lim_{n\to\infty} \frac{1}{2n\Delta_n^2} \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} \rho(s,\theta_0)ds \right)^2 = \frac{1}{2P} \int_0^P \rho^2(s,\theta_0)ds.$$

We continue the proof of Proposition 2, recall that

$$Y_i^{\theta_0} = \int_{t_i}^{t_{i+1}} \rho(s,\theta_0)ds + W_{t_{i+1}} - W_{t_i}.$$

So

$$\begin{aligned}
U_n(\theta) &:= \frac{1}{n\Delta_n^2} \sum_{i=0}^{n-1} Y_i^{\theta_0} \int_{t_i}^{t_{i+1}} \rho(s,\theta)ds - \frac{1}{2n\Delta_n^2} \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} \rho(s,\theta)ds \right)^2 \\
&= \frac{1}{n\Delta_n^2} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \rho(s,\theta)ds \int_{t_i}^{t_{i+1}} \rho(s,\theta_0)ds - \frac{1}{2n\Delta_n^2} \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} \rho(s,\theta)ds \right)^2 \\
&\quad + \frac{1}{n\Delta_n^2} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \rho(s,\theta)ds (W_{t_{i+1}} - W_{t_i}) \\
&= -\frac{1}{2n\Delta_n^2} \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} \rho(s,\theta) - \int_{t_i}^{t_{i+1}} \rho(s,\theta_0) \right)^2 ds \qquad (20) \\
&\quad + \frac{1}{2n\Delta_n^2} \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} \rho(s,\theta_0)ds \right)^2 + \frac{1}{n\Delta_n^2} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \rho(s,\theta)ds (W_{t_{i+1}} - W_{t_i}).
\end{aligned}$$

To complete the proof and thanks to Lemma 2 it remains to establish the convergence in P_{θ_0} probability of the last part of (20). For this purpose we show the convergence in mean square.

From the independence of the increment of the Brownian motion $W_{t_{i+1}} - W_{t_i}$ we can write

$$\begin{aligned}
&\mathrm{E}_{\theta_0} \left[\left| \frac{1}{n\Delta_n^2} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \rho(s,\theta)ds (W_{t_{i+1}} - W_{t_i}) \right|^2 \right] \\
&= \frac{1}{n^2\Delta_n^4} \sum_{i=0}^{n-1} \mathrm{E}_{\theta_0} \left[\left| \int_{t_i}^{t_{i+1}} \rho(s,\theta)ds (W_{t_{i+1}} - W_{t_i}) \right|^2 \right] \\
&= \frac{1}{n^2\Delta_n^3} \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} \rho(s,\theta)ds)^2 \right) \\
&= \frac{1}{n\Delta_n} \frac{1}{n\Delta_n^2} \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} \rho(s,\theta)ds \right)^2 .
\end{aligned}$$

Thanks to (20) we deduce that

$$\lim_{n\to\infty} \mathrm{E}_{\theta_0}\left[\left|\frac{1}{n\Delta_n^2}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \rho(s,\theta)ds(W_{t_{i+1}} - W_{t_i})\right|^2\right] = 0.$$

So the contrast process $U_n(\theta)$ converges in mean square to the contrast function $K(\theta, \theta_0)$ which has a strict maximum when $\theta = \theta_0$

$$K(\theta_0, \theta_0) = \frac{1}{2P}\int_0^P \rho^2(s, \theta_0)ds.$$

This achieves the proof of Proposition 2.

Proof **Theorem 5**

As in Theorem 1 we apply (Theorem 3.2.4 in [2]). By their definitions, $U_n(\theta)$ and $K(\theta, \theta_0)$ satisfy the first condition

For the second condition

$$\begin{aligned}
&U_n(\theta) - U_n(\theta') && (21)\\
&= \frac{1}{n\Delta_n^2}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \rho(s,\theta_0)ds\int_{t_i}^{t_{i+1}}(\rho(s,\theta) - \rho(s,\theta'))ds - \frac{1}{2n\Delta_n^2}\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\rho(s,\theta)ds\right)^2\\
&\quad + \frac{1}{2n\Delta_n^2}\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\rho(s,\theta')ds\right)^2 + \frac{1}{n\Delta_n^2}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(\rho(s,\theta) - \rho(s,\theta'))ds(W_{t_{i+1}} - W_{t_i})\\
&= \frac{1}{2n\Delta_n^2}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(\rho(s,\theta) - \rho(s,\theta'))ds\int_{t_i}^{t_{i+1}}(2\rho(s,\theta_0) - \rho(s,\theta) - \rho(s,\theta'))ds\\
&\quad + \frac{1}{n\Delta_n^2}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(\rho(s,\theta) - \rho(s,\theta'))ds(W_{t_{i+1}} - W_{t_i})\\
&:= B_1(\theta, \theta') + B_2(\theta, \theta') && (22)
\end{aligned}$$

As in the continuous case we can establish that the absolute value of the nonrandom part $B_1(\theta, \theta')$ is bounded by a multiple of η and the random part $B_2(\theta, \theta')$ converges in mean to 0. Hence we have the second condition.

So

$$\hat{\theta}_n \overset{\mathrm{P}_{\theta_0}}{\rightarrow} \theta_0.$$

Proof **Theorem 6**

$$
\begin{aligned}
\mathrm{E}_{\theta_0}\left[\left|\sqrt{n\Delta_n}(\hat{\theta}_n-\theta_0)\right|^2\right] &= n\Delta_n \mathrm{E}_{\theta_0}\left[\left|\frac{\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\rho(s)ds\left(W_{t_{i+1}}-W_{t_i}\right)}{\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2}\right|^2\right] \\
&= \frac{n\Delta_n \mathrm{E}_{\theta_0}\left[\left|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\rho(s)ds\left(W_{t_{i+1}}-W_{t_i}\right)\right|^2\right]}{\left(\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2\right)^2} \\
&= \frac{n\Delta_n^2\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2}{\left(\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2\right)^2} \\
&= \frac{n\Delta_n^2}{\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2}.
\end{aligned}
$$

From equality (20) with $f(s,\theta)=\theta f(s)$ we have

$$
\lim_{n\to\infty}\frac{1}{n\Delta_n^2}\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2=\frac{1}{P}\int_0^P\rho^2(s)ds.
$$

So

$$
\lim_{n\to\infty}\mathrm{E}_{\theta_0}\left[\left|\sqrt{n\Delta_n}(\hat{\theta}_n-\theta_0)\right|^2\right]=\left(\frac{1}{P}\int_0^P\rho^2(s)ds\right)^{-1}.
$$

Proof **Proposition 3**

Denote

$$
\Phi_n := \sqrt{\Delta_n}\left(\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2\right)^{-\frac{1}{2}}, \tag{23}
$$

$$
\Delta_n(\zeta^\theta) := \frac{\Phi_n}{\Delta_n}\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\rho(s)ds(W_{t_{i+1}}-W_{t_i})\right). \tag{24}
$$

According to (12) and (13) we have

$$
\begin{aligned}
Z_n(u) &:= \frac{L_n(\theta_0+\Phi_n u)}{L_n(\theta_0)} \\
&= \frac{\prod_{i=0}^{n-1} \frac{1}{\sqrt{2\pi\Delta_n}} \exp\left(-\frac{1}{2\Delta_n}\left(Y_i^{\theta_0} - (\theta_0+\Phi_n u)\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2\right)}{\prod_{i=0}^{n-1} \frac{1}{\sqrt{2\pi\Delta_n}} \exp\left(-\frac{1}{2\Delta_n}\left(Y_i^{\theta_0} - \theta_0\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2\right)} \\
&= \prod_{i=0}^{n-1} \exp\left(-\frac{1}{\Delta_n}\left(Y_i^{\theta_0}\Phi_n u\int_{t_i}^{t_{i+1}}\rho(s)ds - \theta_0\Phi_n u\left(\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2\right)\right) \\
&\quad\times \prod_{i=0}^{n-1} \exp\left(\frac{1}{2\Delta_n}\Phi_n^2 u^2\left(\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2\right)
\end{aligned}
$$

Since

$$
Y_i^{\theta_0} = \theta_0\int_{t_i}^{t_{i+1}}\rho(s)ds + W_{t_{i+1}} - W_{t_i},
$$

then thanks to (23) and (24) we have

$$
\begin{aligned}
Z_n(u) &= \prod_{i=0}^{n-1} \exp\left(\frac{u\Phi_n}{\Delta_n}\int_{t_i}^{t_{i+1}}\rho(s)ds(W_{t_{i+1}}-W_{t_i}) - \frac{u^2\Phi_n^2}{2\Delta_n}\left(\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2\right) \\
&= \exp\left(\frac{u\Phi_n}{\Delta_n}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\rho(s)ds(W_{t_{i+1}}-W_{t_i}) - \frac{u^2\Phi_n^2}{2\Delta_n}\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2\right) \\
&= \exp\left(u\Delta_n(\zeta^\theta) - \frac{1}{2}u^2\right) \qquad (25)
\end{aligned}
$$

From the independence of the increments $W_{t_{i+1}} - W_{t_i}$, $i = 0, \ldots, n-1$ and (23)

$$
\begin{aligned}
\mathscr{L}\left(\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\rho(s)ds(W_{t_{i+1}}-W_{t_i})\right) &= \mathscr{N}\left(0, \Delta_n\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\rho(s)ds\right)^2\right) \\
&= \mathscr{N}\left(0, \Phi_n^{-2}\Delta_n^2\right). \qquad (26)
\end{aligned}
$$

So from (24)

$$
\mathscr{L}\left\{\Delta_n(\zeta^\theta)|\mathrm{P}_{\theta_0}\right\} = \mathscr{N}(0,1).
$$

Hence from (25) the family $(\mathrm{P}_\theta^{(n)})_{\theta\in\Theta}$ is locally asymptotically normal at θ_0 for every θ_0.

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http://www.springer.com/978-3-319-16329-1

Cyclostationarity: Theory and Methods - II
Contributions to the 7th Workshop on Cyclostationary Systems And Their Applications, Grodek, Poland, 2014
Chaari, F.; Leskow, J.; Napolitano, A.; Zimroz, R.; Wyłomańska, A.; Dudek, A. (Eds.)
2015, VIII, 206 p. 93 illus., 16 illus. in color., Hardcover
ISBN: 978-3-319-16329-1