

Chapter 2

Graded Algebra

Since our approach to structures over valued fields relies in a fundamental way on filtrations and associated graded structures, our arguments often require information that is specific to graded modules. We collect in this chapter the basic definitions and results on graded algebras and modules that will be of constant use in subsequent chapters.

Because of the intended applications to valuation theory, we are interested only in graded structures where the grade group is abelian and torsion-free. With this restriction, the structure theory of graded algebras closely parallels the classical Wedderburn theory of algebras, provided that homogeneity conditions are imposed. We emphasize the analogy by using suggestive terminology: A commutative graded ring is said to be a *graded field* if its nonzero homogeneous elements are invertible (even when the ring is not a field), and graded modules over graded fields are called *graded vector spaces*. In §2.1 we lay the groundwork for linear algebra over graded fields, discussing graded vector spaces over graded division rings, their homomorphisms, and their tensor products. We next develop in §2.2 the graded analogue of the Wedderburn structure theory of semisimple algebras, showing that semisimple graded algebras are direct products of endomorphism algebras of graded vector spaces over graded division rings. We also establish in that section graded versions of the Double Centralizer Theorem (Th. 2.35) and of the Skolem–Noether Theorem (Th. 2.37). Finally, in §2.3 special attention is directed toward the degree zero component of simple graded algebras, which is a key part of their structure. This component is semisimple but usually not simple, and it carries a canonical action of the grade group. In the structures associated to algebras over valued fields, it plays the role of the residue algebra.

None of the results in this chapter is particularly deep. A major tool, introduced in §2.2.1, is the central quotients construction, which often allows us to reduce statements about graded algebras to corresponding statements on ungraded algebras. A deeper study of graded structures will unfold in subsequent chapters, to match our needs for the investigation of algebras

over valued fields: See Ch. 5 for the ramification theory and Galois theory of graded field extensions, Ch. 6 for the Brauer group of graded fields, Ch. 7 for the description of totally ramified graded division algebras, Ch. 8 for further discussion of graded division algebras, and Ch. 11 for the calculation of their SK_1 groups.

2.1 Graded linear algebra

This section gives the definitions and basic results concerning linear algebra over graded rings. We successively discuss graded rings (§2.1.1), graded modules (§2.1.2), homomorphisms (§2.1.3), and tensor products (§2.1.4). Throughout this section (and actually throughout the chapter), we let Γ denote a torsion-free abelian group, written additively, which will contain the set of degrees of all the graded objects we consider. The restriction to torsion-free abelian grade groups entails significant simplifications, which make the graded linear algebra a close analogue of the theory of vector spaces over division rings.

2.1.1 Graded rings

Let R be a Γ -graded ring, i.e., a ring with direct sum decomposition $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ where each R_γ is an additive abelian group and $R_\gamma \cdot R_\delta \subseteq R_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. Thus, every $r \in R$ is uniquely expressible as $r = \sum_{\gamma \in \Gamma} r_\gamma$ with each $r_\gamma \in R_\gamma$ and at most finitely many r_γ nonzero. The r_γ are called the *homogeneous components* of r . The *grade set* of R is the subset of Γ defined by

$$\Gamma_R = \{\gamma \in \Gamma \mid R_\gamma \neq \{0\}\}.$$

Note that Γ_R need not be a subgroup of the torsion-free abelian group Γ . The *homogeneous elements* of R are the elements of $\bigcup_{\gamma \in \Gamma} R_\gamma$. If $a \in R_\gamma$ and $a \neq 0$, we say that γ is the *degree* of a and write $\gamma = \deg a$. We assume always that a ring has a 1. For the graded ring R we have R_0 is a subring of R , as $R_0 \cdot R_0 \subseteq R_0$, and it is easy to check that $1 \in R_0$. Let R^\times denote the group of units of R . Define

$$\Gamma_R^\times = \{\gamma \in \Gamma_R \mid R_\gamma \cap R^\times \neq \emptyset\}. \quad (2.1)$$

Thus, Γ_R^\times is the set of degrees of homogeneous units of R . Clearly, Γ_R^\times is a subgroup of Γ (but it may not be the largest subgroup of Γ lying in Γ_R ; see Ex. 2.46 below).

The ring R with the grading forgotten is denoted by R^\natural . In other words, R^\natural is the underlying ungraded ring of R .

A graded ring D is called a *graded division ring* if $1 \neq 0$ in D and every nonzero homogeneous element of D is a unit. Commutative graded division

rings are called *graded fields*. For example, if D is a graded division ring, then its center $Z(D)$ is a graded field. The ungraded ring D^\natural of a graded division ring D is never a division algebra unless $\Gamma_D = \{0\}$, by Prop. 2.3(iii) below.

Example 2.1. Let D_0 be an arbitrary division ring and let $\varepsilon: \Gamma \rightarrow \text{Aut}(D_0)$ be a group homomorphism. The twisted group ring $D_0(\Gamma; \varepsilon)$ consists of the finite formal sums $\sum_{\gamma \in \Gamma} a_\gamma \gamma$ with $a_\gamma \in D_0$ for all γ (and $a_\gamma \neq 0$ for finitely many γ 's only). Multiplication in $D_0(\Gamma; \varepsilon)$ is defined by

$$a\gamma \cdot b\delta = a\varepsilon(\gamma)(b)(\gamma + \delta) \quad \text{for } a, b \in D_0 \text{ and } \gamma, \delta \in \Gamma,$$

and a Γ -grading is defined by $D_0(\Gamma; \varepsilon)_\gamma = D_0\gamma$ for $\gamma \in \Gamma$. More general examples can be obtained by considering factor sets of Γ in D_0 . See Exercise 5.1 for an example of a graded division ring that is *not* obtained by a group ring construction.

Remark 2.2. Although we do not need to fix an ordering on Γ in this chapter, note that the torsion-free abelian group Γ carries total orderings making it into an ordered abelian group: Let $\mathbb{H}(\Gamma)$ denote the divisible hull of Γ ,

$$\mathbb{H}(\Gamma) = \Gamma \otimes_{\mathbb{Z}} \mathbb{Q} = \varinjlim_n \frac{1}{n}\Gamma.$$

We may choose a total ordering on a \mathbb{Q} -base of $\mathbb{H}(\Gamma)$ and use it to build a lexicographical ordering on $\mathbb{H}(\Gamma)$. The group Γ then inherits an ordering from its embedding into $\mathbb{H}(\Gamma)$.

Basic properties of graded division rings are collected in the following proposition:

Proposition 2.3. *Let $D = \bigoplus_{\gamma \in \Gamma} D_\gamma$ be a graded division ring.*

- (i) *D has no zero divisors.*
- (ii) *For $a, b \in D \setminus \{0\}$, if ab is homogeneous, then a and b are homogeneous.*
- (iii) *D^\times consists of the nonzero homogeneous elements of D , and the degree map*

$$\deg: D^\times \rightarrow \Gamma$$

is a group homomorphism.

- (iv) *$\Gamma_D = \Gamma_D^\times$, which is a group.*
- (v) *D_0 is a division ring.*
- (vi) *Each nonzero D_γ is a 1-dimensional left and right D_0 -vector space.*

Proof. (i) and (ii): Choose a total ordering on Γ compatible with the addition (see Remark 2.2), and take nonzero $a, b \in D$. Write $a = a_1 + \dots + a_k$ with a_1, \dots, a_k homogeneous of increasing degrees; likewise write $b = b_1 + \dots + b_\ell$. Then $a_1 b_1$ is the homogeneous component of degree $\deg(a_1) + \deg(b_1)$ of ab , and $a_1 b_1 \neq 0$ since a_1 and b_1 are homogeneous, hence invertible. Therefore, $ab \neq 0$, showing that D has no zero divisors. Likewise, the highest

degree homogeneous component of ab is $a_k b_\ell$. If ab is homogeneous, we must have $\deg(a_1) + \deg(b_1) = \deg(a_k) + \deg(b_\ell)$, hence $\deg(a_1) = \deg(a_k)$ and $\deg(b_1) = \deg(b_\ell)$, which means that a and b are homogeneous.

(iii): This is immediate from (ii).

(iv): This is clear since every nonzero homogeneous element of D is a unit.

(v): We noted above that D_0 is a ring since D is a graded ring. The nonzero elements in D_0 form the kernel of $\deg: D^\times \rightarrow \Gamma$, which is a group. Therefore, D_0 is a division ring.

(vi): If $D_\gamma \neq \{0\}$, take any nonzero $a \in D_\gamma$. Then $D_\gamma a^{-1}$ is a nonzero left D_0 -subspace of D_0 , hence $D_\gamma a^{-1} = D_0$. Thus, $D_\gamma = D_0 a$, which is a 1-dimensional left D_0 -vector space. Likewise, $D_\gamma = a D_0$. \square

From this we obtain a convenient characterization of graded division rings:

Proposition 2.4. *Let R be a graded ring with $1 \neq 0$. Then, R is a graded division ring if and only if R_0 is a division ring and $\Gamma_R^\times = \Gamma_R$.*

Proof. Suppose R_0 is a division ring and $\Gamma_R^\times = \Gamma_R$. Take any $\delta \in \Gamma_R = \Gamma_R^\times$ and any $b \in R_\delta \cap R^\times$. Write $b^{-1} = \sum_{\gamma \in \Gamma_R} c_\gamma$ where each $c_\gamma \in R_\gamma$. The equation $1 = b^{-1}b = \sum_{\gamma \in \Gamma_R} c_\gamma b$ with each $c_\gamma b \in R_{\gamma+\delta}$ and $1 \in R_0$ implies that $c_\gamma b = 0$ for $\gamma \neq -\delta$. Hence, $c_{-\delta} b = 1$; likewise $b c_{-\delta} = 1$, showing that $b^{-1} = c_{-\delta} \in R_{-\delta}$. Therefore, $b^{-1} R_\delta \subseteq R_{-\delta} \cdot R_\delta \subseteq R_0$, so that $R_\delta = b b^{-1} R_\delta \subseteq b R_0$. Thus,

$$R_\delta \setminus \{0\} \subseteq b(R_0 \setminus \{0\}) \subseteq R^\times,$$

as R_0 is a division ring. Therefore, every nonzero homogeneous element of R is a unit; so, R is a graded division ring. This proves one implication of the proposition, and the converse is given in Prop. 2.3(iv) and (v). \square

A *graded subring* of a graded ring R is a subring $S \subseteq R$ such that $S = \bigoplus_{\gamma \in \Gamma} (R_\gamma \cap S)$. This decomposition defines a grading on S . As a special case, note that the center $Z(R)$ of R is a graded subring. Its 0-component $Z(R)_0$ satisfies

$$Z(R)_0 = Z(R) \cap R_0 \subseteq Z(R_0).$$

A graded subring that is also a graded division ring (resp. a graded field) is called a *graded sub-division ring* (resp. *graded subfield*).

2.1.2 Graded modules

Let R be a graded ring. A right module M over R^\natural equipped with a decomposition $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$, where each M_γ is an additive subgroup of M and $M_\gamma \cdot R_\delta \subseteq M_{\gamma+\delta}$ for $\gamma, \delta \in \Gamma$, is called a *right graded R -module*. Left graded modules are defined likewise. The *grade set* Γ_M is defined by

$$\Gamma_M = \{\gamma \in \Gamma \mid M_\gamma \neq \{0\}\}.$$

A *graded submodule* of a graded R -module M is an R^\natural -submodule $N \subseteq M$ such that $N = \bigoplus_{\gamma \in \Gamma} (M_\gamma \cap N)$. This decomposition defines a graded R -module structure on N . Just as for graded rings, we let M^\natural denote the underlying ungraded R^\natural -module of M .

For the rest of this subsection, we focus on the case where the base graded ring is a graded *division* ring D . Graded modules over a graded division ring are called *graded vector spaces*, because they are free modules: see Prop. 2.5 below. Note that if V is a graded (left or right) D -vector space, then each homogeneous component V_γ is a D_0 -vector space. The grade set Γ_V need not be a group, but it is a union of cosets of the group Γ_D . We denote by $|\Gamma_V : \Gamma_D|$ the number of these cosets (which may be infinite). Let $\Gamma_V = \coprod_{i \in I} \Gamma_i$ be the decomposition of Γ_V into disjoint cosets of Γ_D ; there is a corresponding direct sum decomposition of V into graded subspaces:

$$V = \bigoplus_{i \in I} V_i \quad \text{where} \quad V_i = \bigoplus_{\gamma \in \Gamma_i} V_\gamma \quad \text{for } i \in I. \quad (2.2)$$

We call (2.2) the *canonical decomposition* of V .

Proposition 2.5. *Every graded vector space over a graded division ring is a free module with a homogeneous base. More precisely, let V be a right graded vector space over the graded division ring D , with canonical decomposition (2.2). For $i \in I$, fix some $\gamma_i \in \Gamma_i$ and some D_0 -base $(e_{ij})_{j \in J_i}$ of V_{γ_i} . Then $(e_{ij})_{j \in J_i}$ is a homogeneous D -base of V_i , and $(e_{ij})_{i \in I, j \in J_i}$ is a homogeneous D -base of V . Moreover, every homogeneous D -base of V has cardinality $\sum_{i \in I} \dim_{D_0} V_{\gamma_i}$.*

Proof. The decomposition $V_{\gamma_i} = \bigoplus_{j \in J_i} e_{ij} D_0$ yields $V_{\gamma_i + \delta} = \bigoplus_{j \in J_i} e_{ij} D_\delta$ for all $\delta \in \Gamma$, hence

$$V_i = \bigoplus_{j \in J_i} e_{ij} D,$$

proving that $(e_{ij})_{j \in J_i}$ is a D -base of V_i . In view of the canonical decomposition, it follows that $(e_{ij})_{i \in I, j \in J_i}$ is a D -base of V .

Assume now that V is a right graded D -vector space. Let $(b_k)_{k \in K}$ be any homogeneous D -base of V . For $i \in I$, let $K_i = \{k \in K \mid \deg(b_k) \in \Gamma_i\}$. Since $\Gamma_i = \gamma_i + \Gamma_D$, for each $k \in K_i$ there is a nonzero $d_k \in D_{\gamma_i - \deg(b_k)}$; so, $b_k d_k \in V_{\gamma_i}$. Then, $(b_k d_k)_{k \in K_i}$ is a homogeneous D -base of V_{γ_i} . Hence, for each $i \in I$, $(b_k d_k)_{k \in K_i}$ must be a homogeneous D -base of V_i , and hence also a D_0 -base of V_{γ_i} . Thus, $|K| = \sum_{i \in I} |K_i| = \sum_{i \in I} \dim_{D_0} V_{\gamma_i}$. \square

Corollary 2.6. *Every graded subspace of a graded vector space has a complementary graded subspace.*

Proof. Let V be a right graded vector space over a graded division ring D and let $U \subseteq V$ be a graded subspace. We claim that there is a graded subspace $W \subseteq V$ such that $V = U \oplus W$. Consider the canonical decomposition of V as in (2.2) above: let $\Gamma_V = \coprod_{i \in I} \Gamma_i$ and $V = \bigoplus_{i \in I} V_i$ where $V_i = \bigoplus_{\gamma \in \Gamma_i} V_\gamma$

for $i \in I$. Since $U \subseteq V$ is a graded subspace, we also have a canonical decomposition of U : there is a subset $I_U \subseteq I$ such that $\Gamma_U = \coprod_{i \in I_U} \Gamma_i$ and, for each $i \in I_U$, there is a D_0 -subspace $U_\gamma \subseteq V_\gamma$ such that

$$U = \bigoplus_{i \in I_U} U_i \quad \text{where} \quad U_i = \bigoplus_{\gamma \in \Gamma_i} U_\gamma \quad \text{for } i \in I_U.$$

For $i \in I_U$, fix some $\gamma_i \in \Gamma_i$ and some D_0 -subspace $U'_{\gamma_i} \subseteq V_{\gamma_i}$ such that

$$V_{\gamma_i} = U_{\gamma_i} \oplus U'_{\gamma_i}.$$

By combining a D_0 -base \mathcal{B} of U_{γ_i} with a D_0 -base \mathcal{B}' of U'_{γ_i} , we obtain a D_0 -base of V_{γ_i} , which is a D -base of V_i by Prop. 2.5. Likewise, \mathcal{B} is a D -base of U_i . Therefore, letting U'_i be the D -span of \mathcal{B}' , we obtain

$$V_i = U_i \oplus U'_i \quad \text{for } i \in I_U.$$

Therefore, the following is a complementary subspace of U in V :

$$W = \left(\bigoplus_{i \in I_U} U'_i \right) \oplus \left(\bigoplus_{i \in I \setminus I_U} V_i \right). \quad \square$$

The rank of a graded vector space V over a graded division ring D , which is the number of elements in any homogeneous base, is also called its *dimension*, and is denoted by $\dim_D V$ or $[V:D]$. Proposition 2.5 shows that

$$\dim_D V = \sum_{i \in I} \dim_D V_i = \sum_{i \in I} \dim_{D_0} V_{\gamma_i}, \quad (2.3)$$

where I is a set with $|\Gamma_V : \Gamma_D|$ elements and $\{\gamma_i\}_{i \in I}$ is a set of coset representatives of Γ_V modulo Γ_D . In particular, if $\dim_D V = d < \infty$, then $|\Gamma_V : \Gamma_D| \leq d < \infty$ and $\dim_{D_0} V_\gamma \leq d < \infty$ for every $\gamma \in \Gamma$.

In one important case the dimensions $\dim_{D_0} V_{\gamma_i}$ are all equal, and the sum above can therefore be rewritten as a product: suppose D is a graded sub-division ring of another graded division ring $E = \bigoplus_{\gamma \in \Gamma} E_\gamma$. We may then consider E as a left (resp. right) graded D -vector space. We write $[E:D]_\ell$ (resp. $[E:D]_r$) for its dimension.¹ Clearly D_0 is a subring of E_0 and Γ_D is a subgroup of Γ_E .

Corollary 2.7. *With the notation above,*

$$[E:D]_\ell = [E_0:D_0]_\ell |\Gamma_E : \Gamma_D| \quad \text{and} \quad [E:D]_r = [E_0:D_0]_r |\Gamma_E : \Gamma_D|.$$

Proof. Consider E as a left graded D -vector space. For all $\gamma \in \Gamma_E$, the homogeneous component E_γ is a 1-dimensional E_0 -vector space; hence,

$$\dim_{D_0}(E_\gamma) = [E_0:D_0]_\ell.$$

By Prop. 2.5, it follows that $\dim_D(E) = [E_0:D_0]_\ell |\Gamma_E : \Gamma_D|$, proving the left equality. The proof of the right equation is analogous. \square

¹ When the dimensions as left and right vector spaces coincide—e.g., when D lies in the center of E —we write simply $[E:D]$.

2.1.3 Homomorphisms

Let $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ and $N = \bigoplus_{\gamma \in \Gamma} N_\gamma$ be right graded modules over a graded ring R , and let $\text{Hom}_{R^\natural}(M^\natural, N^\natural)$ be the group of R^\natural -linear maps (i.e., R^\natural -module homomorphisms) from the ungraded module M^\natural to N^\natural ; so $\text{Hom}_{R^\natural}(M^\natural, N^\natural)$ is a module over $Z(R^\natural)$. For $\gamma \in \Gamma$, we denote by $\text{Hom}_R(M, N)_\gamma$ the group of R -linear maps that shift the degree by γ ,

$$\text{Hom}_R(M, N)_\gamma = \{f \in \text{Hom}_{R^\natural}(M^\natural, N^\natural) \mid f(M_\delta) \subseteq N_{\delta+\gamma} \text{ for all } \delta \in \Gamma\}.$$

Thus, $\text{Hom}_R(M, N)_0$ consists of the (degree-preserving) graded homomorphisms from M to N , which are the morphisms in the category of graded R -modules. If there is an isomorphism in $\text{Hom}_R(M, N)_0$, we write $M \cong_g N$ to emphasize that M and N are isomorphic as graded modules (i.e., by a degree-preserving isomorphism), not simply as R^\natural -modules.

For $\gamma \neq 0$, homomorphisms in $\text{Hom}_R(M, N)_\gamma$ can also be viewed as degree-preserving maps after changing the grading on M (or N , or both). For any right graded R -module M and any $\gamma \in \Gamma$, we define the *shifted*² graded module $M(\gamma)$: it is the R^\natural -module M^\natural with a grading defined by shifting the grading of M , so that for $\delta \in \Gamma$,

$$M(\gamma)_\delta = M_{\gamma+\delta}.$$

Thus, $\Gamma_{M(\gamma)} = -\gamma + \Gamma_M$, and we may identify

$$\begin{aligned} \text{Hom}_R(M, N)_\gamma &= \text{Hom}_R(M(-\gamma), N)_0 = \text{Hom}_R(M, N(\gamma))_0 \\ &= \text{Hom}_R(M(\varepsilon), N(\gamma + \varepsilon))_0 \text{ for all } \varepsilon \in \Gamma. \end{aligned}$$

More generally, for all $\gamma, \delta, \varepsilon \in \Gamma$ we may identify

$$\text{Hom}_R(M(\gamma), N(\delta))_\varepsilon = \text{Hom}_R(M, N)_{\delta-\gamma+\varepsilon}. \quad (2.4)$$

Proposition 2.8. *If M^\natural is a finitely generated R^\natural -module, then*

$$\text{Hom}_{R^\natural}(M^\natural, N^\natural) = \bigoplus_{\gamma \in \Gamma} \text{Hom}_R(M, N)_\gamma.$$

With this decomposition, $\text{Hom}_{R^\natural}(M^\natural, N^\natural)$ is a graded $Z(R)$ -module denoted by $\text{Hom}_R(M, N)$. For all $\gamma, \delta \in \Gamma$ we have an identification of graded $Z(R)$ -modules

$$\text{Hom}_R(M(\gamma), N(\delta)) = \text{Hom}_R(M, N)(\delta - \gamma). \quad (2.5)$$

Proof. The sum $\sum_{\gamma \in \Gamma} \text{Hom}_R(M, N)_\gamma$ is clearly direct, so it suffices to show that this sum is all of $\text{Hom}_{R^\natural}(M^\natural, N^\natural)$ to establish the first part. By decomposing generators of M^\natural into homogeneous components, we may find a generating set $\{m_i\}_{i=1}^n$ consisting of homogeneous elements. Let $\delta_i = \deg m_i$

² The shift construction is also classically known as the *twist* construction. We will avoid this terminology because it may be confusing in a noncommutative context.

and let $f \in \text{Hom}_{\mathbb{R}^\natural}(\mathbb{M}^\natural, \mathbb{N}^\natural)$. For $i = 1, \dots, n$, let $y_{i,\gamma}$ be the homogeneous component of degree γ of $f(m_i)$, so

$$f(m_i) = \sum_{\gamma \in \Gamma} y_{i,\gamma}.$$

We claim that for each $\gamma \in \Gamma$ there exists an $f_\gamma \in \text{Hom}_{\mathbb{R}^\natural}(\mathbb{M}^\natural, \mathbb{N}^\natural)$ such that

$$f_\gamma(m_i) = y_{i,\gamma+\delta_i} \quad \text{for } i = 1, \dots, n.$$

It is clear that $f_\gamma \in \text{Hom}_{\mathbb{R}}(\mathbb{M}, \mathbb{N})_\gamma$ and $f = \sum_{\gamma \in \Gamma} f_\gamma$, hence the first part of the proposition follows.

To prove the claim, it suffices to show that if $a_1, \dots, a_n \in \mathbb{R}$ satisfy $\sum_{i=1}^n m_i a_i = 0$, then $\sum_{i=1}^n y_{i,\gamma+\delta_i} a_i = 0$ for all $\gamma \in \Gamma$; for then the map f_γ is well-defined by

$$f_\gamma \left(\sum_{i=1}^n m_i a_i \right) = \sum_{i=1}^n y_{i,\gamma+\delta_i} a_i \quad \text{for } a_1, \dots, a_n \in \mathbb{R}.$$

Suppose $\sum_{i=1}^n m_i a_i = 0$, and let $a_{i,\gamma}$ denote the homogeneous component of degree γ of a_i . For all $\delta \in \Gamma$, the homogeneous component of degree δ of $m_i a_i$ is $m_i a_{i,\delta-\delta_i}$, so we have $\sum_i m_i a_{i,\delta-\delta_i} = 0$. By applying f , we obtain $\sum_i f(m_i) a_{i,\delta-\delta_i} = 0$ for all $\delta \in \Gamma$. Now, for any $\gamma \in \Gamma$, the homogeneous component of degree $\gamma+\delta$ of $f(m_i) a_{i,\delta-\delta_i}$ is $y_{i,\gamma+\delta_i} a_{i,\delta-\delta_i}$, so $\sum_i y_{i,\gamma+\delta_i} a_{i,\delta-\delta_i} = 0$ for all $\gamma, \delta \in \Gamma$. Summing over δ , we obtain $\sum_i y_{i,\gamma+\delta_i} a_i = 0$ for all $\gamma \in \Gamma$, proving the claim.

If $f \in \text{Hom}_{\mathbb{R}}(\mathbb{M}, \mathbb{N})_\gamma$ and $z \in Z(\mathbb{R})_\delta$, then clearly $zf \in \text{Hom}_{\mathbb{R}}(\mathbb{M}, \mathbb{N})_{\gamma+\delta}$, hence $\text{Hom}_{\mathbb{R}}(\mathbb{M}, \mathbb{N})$ is a graded $Z(\mathbb{R})$ -module. Equation (2.5) follows immediately from (2.4). \square

If \mathbb{M}^\natural is a finitely generated \mathbb{R}^\natural -module, we set $\text{End}_{\mathbb{R}} \mathbb{M} = \text{Hom}_{\mathbb{R}}(\mathbb{M}, \mathbb{M})$. The grading on $\text{End}_{\mathbb{R}} \mathbb{M}$ is compatible with the composition of maps, so $\text{End}_{\mathbb{R}} \mathbb{M}$ has a natural graded ring structure. We next represent this graded ring as a graded matrix ring, assuming \mathbb{M} is a free graded module.

For any positive integer n , the $n \times n$ matrix ring $M_n(\mathbb{R})$ has an obvious grading, where for each $\gamma \in \Gamma$,

$$M_n(\mathbb{R})_\gamma = \begin{pmatrix} \mathbb{R}_\gamma & \dots & \mathbb{R}_\gamma \\ \vdots & \ddots & \vdots \\ \mathbb{R}_\gamma & \dots & \mathbb{R}_\gamma \end{pmatrix}. \quad (2.6)$$

But other gradings, as follows, arise naturally: Take any $\delta_1, \dots, \delta_n \in \Gamma$. Let $M_n(\mathbb{R})(\delta_1, \dots, \delta_n)$ denote $M_n(\mathbb{R}^\natural)$ as a ring, but graded so that

$$M_n(\mathbb{R})(\delta_1, \dots, \delta_n) = \begin{pmatrix} \mathbb{R} & \mathbb{R}(\delta_1 - \delta_2) & \dots & \mathbb{R}(\delta_1 - \delta_n) \\ \mathbb{R}(\delta_2 - \delta_1) & \mathbb{R} & \dots & \mathbb{R}(\delta_2 - \delta_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{R}(\delta_n - \delta_1) & \mathbb{R}(\delta_n - \delta_2) & \dots & \mathbb{R} \end{pmatrix}. \quad (2.7)$$

Thus, for any $\gamma \in \Gamma$ the component of degree γ is

$$M_n(R)(\delta_1, \dots, \delta_n)_\gamma = \begin{pmatrix} R_\gamma & R_{\gamma+\delta_1-\delta_2} & \dots & R_{\gamma+\delta_1-\delta_n} \\ R_{\gamma+\delta_2-\delta_1} & R_\gamma & \dots & R_{\gamma+\delta_2-\delta_n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{\gamma+\delta_n-\delta_1} & R_{\gamma+\delta_n-\delta_2} & \dots & R_\gamma \end{pmatrix}.$$

Let $T = M_n(R)(\delta_1, \dots, \delta_n)$. Since additively T is a direct sum of copies of R and for any $\delta \in \Gamma$ we have $R = \bigoplus_{\gamma \in \Gamma} R(\delta)_\gamma$, clearly $T = \bigoplus_{\gamma \in \Gamma} T_\gamma$. Moreover, if $A = (a_{ij}) \in T_\gamma$ and $B = (b_{ij}) \in T_\varepsilon$, then each $a_{ij} \in R_{\gamma+\delta_i-\delta_j}$ and $b_{jk} \in R_{\varepsilon+\delta_j-\delta_k}$, so $a_{ij}b_{jk} \in R_{\gamma+\delta_i+\varepsilon-\delta_k} = R(\delta_i - \delta_k)_{\gamma+\varepsilon}$. This yields that $AB \in T_{\gamma+\varepsilon}$, verifying that T is a graded ring.

Proposition 2.9. *Let R be a graded ring, let N be a finitely generated right graded R -module, and let $E = \text{End}_R N$. For any $\delta_1, \dots, \delta_n \in \Gamma$,*

$$\text{End}_R (N(\delta_1) \oplus \dots \oplus N(\delta_n)) \cong_g M_n(E)(\delta_1, \dots, \delta_n),$$

a graded ring isomorphism.

Proof. For $i = 1, \dots, n$, consider the canonical maps

$$q_i: N(\delta_i) \longrightarrow N(\delta_1) \oplus \dots \oplus N(\delta_n) \quad \text{and} \quad p_i: N(\delta_1) \oplus \dots \oplus N(\delta_n) \longrightarrow N(\delta_i).$$

To each endomorphism f of $N(\delta_1) \oplus \dots \oplus N(\delta_n)$ we associate the matrix $(f_{ij})_{i,j=1}^n$ where

$$f_{ij} = p_i \circ f \circ q_j: N(\delta_j) \longrightarrow N(\delta_i).$$

Thus, we have

$$\text{End}_R (N(\delta_1) \oplus \dots \oplus N(\delta_n)) \cong_g \begin{pmatrix} \text{Hom}_R (N(\delta_1), N(\delta_1)) & \text{Hom}_R (N(\delta_2), N(\delta_1)) & \dots & \text{Hom}_R (N(\delta_n), N(\delta_1)) \\ \text{Hom}_R (N(\delta_1), N(\delta_2)) & \text{Hom}_R (N(\delta_2), N(\delta_2)) & \dots & \text{Hom}_R (N(\delta_n), N(\delta_2)) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Hom}_R (N(\delta_1), N(\delta_n)) & \text{Hom}_R (N(\delta_2), N(\delta_n)) & \dots & \text{Hom}_R (N(\delta_n), N(\delta_n)) \end{pmatrix}.$$

This is a ring isomorphism, which is compatible with the gradings on each side. For the ij -entry on the right, Prop. 2.8 yields

$$\text{Hom}_R (N(\delta_j), N(\delta_i)) \cong_g (\text{End}_R N)(\delta_i - \delta_j),$$

which is the ij -entry of $M_n(E)(\delta_1, \dots, \delta_n)$. The result follows. \square

The “left” version of Prop. 2.9 says that if M is a finitely generated left graded R -module and $B = \text{End}_R M$, then

$$\text{End}_R (M(\gamma_1) \oplus \dots \oplus M(\gamma_n)) \cong_g M_n(B)(-\gamma_1, \dots, -\gamma_n).$$

Indeed, endomorphisms of left modules act on the right of their argument, hence their composition is in the reverse order as compared to the previous case. Thus, the matrix representation of an endomorphism f of $M(\gamma_1) \oplus \dots \oplus M(\gamma_n)$ is $(f_{ij})_{i,j=1}^n$ where

$$f_{ij} = q_i \circ f \circ p_j : M(\gamma_i) \longrightarrow M(\gamma_j).$$

To see that Prop. 2.9 applies in particular to endomorphism rings of graded vector spaces, we rephrase Prop. 2.5 as follows:

Corollary 2.10. *Every graded vector space over a graded division ring D is isomorphic to a direct sum of shifted 1-dimensional spaces $D(\gamma)$.*

Proof. In the notation of Prop. 2.5 and its proof, let $D^{(J_i)}$ for $i \in I$ denote the graded vector space of $|J_i|$ -tuples $(d_j)_{j \in J_i}$ such that $d_j \neq 0$ for only a finite number of $j \in J_i$. This graded vector space is a direct sum of copies of D and the map $(d_j)_{j \in J_i} \mapsto \sum_{j \in J_i} e_{ij} d_j$ is a graded vector space isomorphism $D^{(J_i)}(-\gamma_i) \cong_g V_i$. Therefore,

$$V \cong_g \bigoplus_{i \in I} D(-\gamma_i)^{(J_i)}. \quad \square$$

Let V, W be finite-dimensional right graded vector spaces over a graded division ring D . Consider decompositions of V and W as direct sums of shifted 1-dimensional graded vector spaces as in Cor. 2.10:

$$V \cong_g D(\gamma_1) \oplus \dots \oplus D(\gamma_n), \quad W \cong_g D(\delta_1) \oplus \dots \oplus D(\delta_m)$$

for some $\gamma_1, \dots, \delta_m \in \Gamma$.

We have $\text{End}_D D = D$. Therefore, arguing as in Prop. 2.9, we obtain

$$\begin{aligned} \text{Hom}_D(V, W) &\cong_g \\ &\begin{pmatrix} \text{Hom}_D(D(\gamma_1), D(\delta_1)) & \text{Hom}_D(D(\gamma_2), D(\delta_1)) & \dots & \text{Hom}_D(D(\gamma_n), D(\delta_1)) \\ \text{Hom}_D(D(\gamma_1), D(\delta_2)) & \text{Hom}_D(D(\gamma_2), D(\delta_2)) & \dots & \text{Hom}_D(D(\gamma_n), D(\delta_2)) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Hom}_D(D(\gamma_1), D(\delta_m)) & \text{Hom}_D(D(\gamma_2), D(\delta_m)) & \dots & \text{Hom}_D(D(\gamma_n), D(\delta_m)) \end{pmatrix} \\ &\cong_g \begin{pmatrix} D(\delta_1 - \gamma_1) & D(\delta_1 - \gamma_2) & \dots & D(\delta_1 - \gamma_n) \\ D(\delta_2 - \gamma_1) & D(\delta_2 - \gamma_2) & \dots & D(\delta_2 - \gamma_n) \\ \vdots & \vdots & \ddots & \vdots \\ D(\delta_m - \gamma_1) & D(\delta_m - \gamma_2) & \dots & D(\delta_m - \gamma_n) \end{pmatrix}. \end{aligned}$$

Consequently, we have

$$\Gamma_{\text{Hom}_D(V, W)} = \bigcup_{i,j} (\delta_i - \gamma_j + \Gamma_D) = \{\delta - \gamma \mid \delta \in \Gamma_W, \gamma \in \Gamma_V\}.$$

In particular, when $V = W$ we obtain:

Corollary 2.11. *Let V be a finite-dimensional graded vector space over a graded division ring D . If $V \cong_g D(\gamma_1) \oplus \dots \oplus D(\gamma_n)$ for some $\gamma_1, \dots, \gamma_n$, then*

$$\text{End}_D V \cong_g M_n(D)(\gamma_1, \dots, \gamma_n) \quad \text{and} \quad \Gamma_{\text{End}_D V} = \bigcup_{i,j} (\gamma_i - \gamma_j + \Gamma_D).$$

2.1.4 Tensor products

Let $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ be a right graded module and let $N = \bigoplus_{\gamma \in \Gamma} N_\gamma$ be a left graded module over the graded ring R . Letting $P_\gamma = \sum_{\delta+\varepsilon=\gamma} M_\delta \otimes_{\mathbb{Z}} N_\varepsilon$ for $\gamma \in \Gamma$, we have a grading on the \mathbb{Z} -module $M \otimes_{\mathbb{Z}} N$:

$$M \otimes_{\mathbb{Z}} N = \bigoplus_{\gamma \in \Gamma} P_\gamma.$$

The tensor product $M \otimes_R N$ is the quotient of $M \otimes_{\mathbb{Z}} N$ by the submodule Q spanned by the elements of the form $(mr) \otimes n - m \otimes (rn)$ for $m \in M$, $n \in N$ and $r \in R$. By decomposing m , n , and r into homogeneous components, we see that Q is also spanned by elements of this form with m , n , and r homogeneous. Therefore, Q is a graded submodule of $M \otimes_{\mathbb{Z}} N$, and it follows that $M \otimes_R N$ inherits the grading of $M \otimes_{\mathbb{Z}} N$. For $\gamma \in \Gamma$, the homogeneous component $(M \otimes_R N)_\gamma$ is the additive subgroup generated by the products $m_\delta \otimes n_\varepsilon$ with $m_\delta \in M_\delta$, $n_\varepsilon \in N_\varepsilon$, and $\delta + \varepsilon = \gamma$. Therefore, $\Gamma_{M \otimes_R N} \subseteq \Gamma_M + \Gamma_N$. Note that this grading provides $M \otimes_R N$ with a graded $Z(R)$ -module structure.

The graded tensor product construction will be used mostly in the case where R is a graded division ring D and M , N are graded D -vector spaces V , W . For $\delta, \varepsilon \in \Gamma$ we have a canonical map $V_\delta \otimes_{D_0} W_\varepsilon \rightarrow V \otimes_D W$. This map is injective since any D_0 -base of V_δ (resp. W_ε) is D -linearly independent in V (resp. W), see Prop. 2.5. Therefore, we may identify $V_\delta \otimes_{D_0} W_\varepsilon$ with a subgroup of $V \otimes_D W$, and we have for $\gamma \in \Gamma$

$$(V \otimes_D W)_\gamma = \sum_{\delta+\varepsilon=\gamma} (V_\delta \otimes_{D_0} W_\varepsilon). \quad (2.8)$$

Note however that for $v_\delta \in V_\delta$, $w_\varepsilon \in W_\varepsilon$, and $d_\lambda \in D_\lambda$, the elements

$$(v_\delta d_\lambda) \otimes w_\varepsilon \in V_{\delta+\lambda} \otimes_{D_0} W_\varepsilon \quad \text{and} \quad v_\delta \otimes (d_\lambda w_\varepsilon) \in V_\delta \otimes_{D_0} W_{\lambda+\varepsilon}$$

are identified in $V \otimes_D W$. Thus, the sum on the right in (2.8) is not a direct sum.

Proposition 2.12. *Let V be a right graded vector space and W a left graded vector space over a graded division ring D . Let $(\gamma_i)_{i \in I}$ be a collection of representatives in Γ of the cosets of $\Gamma_V \cap \Gamma_W$ modulo Γ_D , so that*

$$\Gamma_V \cap \Gamma_W = \coprod_{i \in I} (\gamma_i + \Gamma_D).$$

Then $\Gamma_{V \otimes_D W} = \Gamma_V + \Gamma_W$ and, for $\delta \in \Gamma_V$, $\varepsilon \in \Gamma_W$,

$$(V \otimes_D W)_{\delta+\varepsilon} = \bigoplus_{i \in I} (V_{\delta+\gamma_i} \otimes_{D_0} W_{\varepsilon-\gamma_i}).$$

Proof. The equality $\Gamma_{V \otimes W} = \Gamma_V + \Gamma_W$ is clear from (2.8). Now, fix $\delta \in \Gamma_V$ and $\varepsilon \in \Gamma_W$. If $\lambda \in \Gamma_V$ and $\mu \in \Gamma_W$ satisfy $\lambda + \mu = \delta + \varepsilon$, then $\lambda - \delta = \varepsilon - \mu \in \Gamma_V \cap \Gamma_W$; hence, we may find $i \in I$ and $d \in D^\times$ such that

$$\lambda - \delta = \varepsilon - \mu = \gamma_i + \deg(d).$$

For $v_\lambda \in V_\lambda$ and $w_\mu \in W_\mu$ we have in $V \otimes_D W$

$$v_\lambda \otimes w_\mu = v_\lambda d^{-1} \otimes dw_\mu \in V_{\delta+\gamma_i} \otimes_{D_0} W_{\varepsilon-\gamma_i}.$$

Thus, for all $\lambda, \mu \in \Gamma$ such that $\lambda + \mu = \delta + \varepsilon$, the image of $V_\lambda \otimes_{D_0} W_\mu$ in $V \otimes_D W$ lies in the image of $\sum_{i \in I} (V_{\delta+\gamma_i} \otimes_{D_0} W_{\varepsilon-\gamma_i})$.

The latter sum is direct because for $i \in I$ the $V_{\delta+\gamma_i}$'s lie in different summands of the canonical decomposition (2.2) of V . More explicitly, for every coset $\Lambda = \gamma + \Gamma_D$ in Γ_V consider the projection

$$p_\Lambda: V \longrightarrow \bigoplus_{\lambda \in \Lambda} V_\lambda.$$

This map is D -linear; it yields a $Z(D)$ -linear map

$$p_\Lambda \otimes id_W: V \otimes_D W \longrightarrow \left(\bigoplus_{\lambda \in \Lambda} V_\lambda \right) \otimes_D W.$$

The restriction of this map to $V_{\delta+\gamma_i} \otimes_{D_0} W_{\varepsilon-\gamma_i}$ is the identity if $\delta + \gamma_i \in \Lambda$; it is the 0 map otherwise. Therefore, if $\sum_{i \in I} x_i = 0$ with $x_i \in V_{\delta+\gamma_i} \otimes_{D_0} W_{\varepsilon-\gamma_i}$ for $i \in I$, then by applying $p_{\delta+\gamma_j+\Gamma_D} \otimes id_W$ we obtain $x_j = 0$. \square

2.2 Wedderburn structure theory

In this section, we develop the analogue for graded rings of the Wedderburn structure theory of semisimple algebras, with a view toward its application to the definition of gauges in the next chapter. It turns out that, under the appropriate restrictions to homogeneous ideals and graded modules, the whole theory can be carried out in the graded case. The arguments we use should make this point clear; they are slight variations on the most classical proofs (dealing with finite-dimensional algebras). The only special feature of the graded case is that the classification of simple graded algebras up to isomorphism involves a collection of elements in the grade group Γ ; see Prop. 2.27.

Throughout the section, we fix a base graded field F with torsion-free abelian grade group Γ . The group Γ contains the degrees of all the graded structures we consider in this section. We let F^\natural denote F with its grading forgotten, i.e., F^\natural is the underlying ungraded ring of F . Proposition 2.3 shows

that F^\natural is an integral domain; hence, we may consider its quotient field, for which we use the notation $q(F)$:

$$q(F) = \text{Quot}(F^\natural).$$

We start with the basic definitions, then we show in §2.2.2 that semisimple graded algebras are direct products of simple graded algebras. In §2.2.3, we show that simple graded algebras are algebras of endomorphisms of graded vector spaces over graded division rings, and we discuss tensor products of graded algebras and analogues of the Double Centralizer Theorem and the Skolem–Noether Theorem in §2.2.4.

2.2.1 Semisimple graded algebras and central quotients

Mimicking the classical (ungraded) theory, we define a *graded F-algebra* as a graded ring A with 1 that is also a graded F -vector space, in which the multiplication and the scalar multiplication are related by

$$(\lambda a)b = \lambda(ab) = a(\lambda b) \quad \text{for } \lambda \in F \text{ and } a, b \in A.$$

We identify F with a graded subring of A by mapping $\lambda \in F$ to $\lambda 1 \in A$ and we say that the graded F -algebra A is *central* if its center $Z(A)$ is F . If A is also a graded division ring, we call it a *graded division algebra*. If A and B are graded F -algebras, we define a grading on the direct product $A \times B$ by

$$A \times B = \bigoplus_{\gamma \in \Gamma} (A_\gamma \times B_\gamma).$$

Homogeneous left (resp. right) ideals in a graded algebra A are simply the graded submodules of A for its structure of left (resp. right) module. A finite-dimensional graded algebra A is said to be *semisimple*³ if $\{0\}$ is its only homogeneous two-sided nilpotent ideal. Then $\{0\}$ is also the only homogeneous left nilpotent ideal and the only homogeneous right nilpotent ideal because for any left (resp. right) ideal $J \subseteq A$ the two-sided ideal JA (resp. AJ) satisfies for every integer $n \geq 1$,

$$(JA)^n = J(AJ)^{n-1}A = J^nA \quad (\text{resp. } (AJ)^n = A(JA)^{n-1}J = AJ^n).$$

Clearly, a finite direct product of semisimple graded algebras is again graded semisimple.

When A has enough homogeneous units, we can verify the graded simplicity of A by examining A_0 . For this, let

$$\Gamma_A^\times = \{\deg(a) \mid a \in A^\times \text{ and } a \text{ is homogeneous}\} \subseteq \Gamma_A.$$

³ We sometimes call such a graded algebra *graded semisimple*, for emphasis.

Lemma 2.13. *Let A be a graded algebra over a graded field F . If A_0 is semi-simple and $\Gamma_A^\times = \Gamma_A$, then A is graded semisimple.*

Proof. Suppose A is not graded semisimple. Then, there is a nonzero nilpotent homogeneous ideal N of A . Take $\gamma \in \Gamma_A$ with $N_\gamma \neq \{0\}$. By hypothesis, there is a $u \in A^\times \cap A_\gamma$. Then, $\{0\} \subsetneq u^{-1}N_\gamma \subseteq N_0$. Hence, N_0 is a nonzero nilpotent ideal of A_0 . This cannot occur since A_0 is assumed semisimple. Hence, A must be graded semisimple. \square

The converse of Lemma 2.13 also holds, without the hypothesis on Γ_A^\times . See Cor. 2.42 below.

A *simple graded F -algebra* is a finite-dimensional graded F -algebra $A \neq \{0\}$ in which $\{0\}$ and A are the only homogeneous two-sided ideals. To prove that a graded algebra is simple, it suffices to extend scalars to the quotient field $q(F)$, as the next lemma shows:

Lemma 2.14. *Let A be a graded algebra over a graded field F and let*

$$q_F(A) = A^\natural \otimes_{F^\natural} q(F),$$

which we consider as an algebra over $q(F)$. We have

$$[q_F(A):q(F)] = [A:F] \quad \text{and} \quad Z(q_F(A)) = Z(A)^\natural \otimes_{F^\natural} q(F) \quad (=: q_F(Z(A))).$$

Moreover, if $q_F(A)$ is simple, then A is graded simple. If A is a finite-dimensional graded division algebra, then $q_F(A)$ is a division algebra.

Note that the converse of these last statements also holds: see Prop. 2.28 below.

Proof. Every element of $q_F(A)$ has the form $\sum_i \alpha_i \otimes \lambda_i^{-1}$ for some $\alpha_i \in A$ and some nonzero $\lambda_i \in F$. Reducing to the common denominator $\lambda = \prod_i \lambda_i$, we may rewrite the element in the form $\alpha \otimes \lambda^{-1}$, where $\alpha = \sum_i (\alpha_i \prod_{j \neq i} \lambda_j)$. Therefore, it is easy to see that any F -base of A is a $q(F)$ -base of $q_F(A)$, and that the center of $q_F(A)$ is $q_F(Z(A))$.

Now, let $I \subseteq A$ be a nonzero homogeneous two-sided ideal. By Cor. 2.6, there is a graded F -subspace $W \subseteq A$ such that $A = I \oplus W$. The tensor product $I^\natural \otimes_{F^\natural} q(F)$ is a nonzero two-sided ideal of $q_F(A)$, and we have

$$q_F(A) = (I^\natural \otimes_{F^\natural} q(F)) \oplus (W^\natural \otimes_{F^\natural} q(F)).$$

Therefore, if $q_F(A)$ is simple we must have $I^\natural \otimes_{F^\natural} q(F) = q_F(A)$, hence $W^\natural \otimes_{F^\natural} q(F) = \{0\}$, and therefore $W = \{0\}$. It follows that $I = A$. Thus, A is graded simple.

If A is a finite-dimensional graded division algebra, then Prop. 2.3 shows that A^\natural has no zero divisors. Then $q_F(A)$ is a finite-dimensional $q(F)$ -algebra without zero divisors; hence, it is a division algebra. \square

The $q(F)$ -algebra $q_F(A) = A^\natural \otimes_{F^\natural} q(F)$ is less dependent on F than its definition suggests, as we see by relating it to the ring of central quotients

of A^\natural . Recall that if R is a commutative ring and \mathcal{N} is its set of non-zero-divisors, then the *total quotient ring* of R is its localization

$$q(R) = R_{\mathcal{N}} = \{rs^{-1} \mid r \in R, s \in \mathcal{N}\}.$$

The set \mathcal{N} is the largest multiplicatively closed subset S of $R \setminus \{0\}$ such that R embeds in the localization R_S . If B is a noncommutative ring, then the *ring of central quotients* of B is

$$q(B) = B \otimes_{Z(B)} q(Z(B)),$$

where $Z(B)$ is the center of B . The canonical map $B \rightarrow q(B)$, $b \mapsto b \otimes 1$, need not be injective, however, since a non-zero-divisor of $Z(B)$ could become a zero-divisor in B . (See Exercise 2.3 below for an example.) For a graded F -algebra A , we write $q(A)$ for $q(A^\natural)$. The next lemma shows that for the graded F -algebras usually considered in this book, $q_F(A)$ and $q(A)$ coincide.

Lemma 2.15. *Let A be a graded algebra over a graded field F . Then, $A^\natural \subseteq q_F(A)$. If $Z(A)^\natural$ is integral over F^\natural (e.g., if $[A:F] < \infty$), then*

$$q_F(A) = q(A).$$

Proof. Let $\varphi: A^\natural \rightarrow q_F(A)$ be the canonical map given by $a \mapsto a \otimes 1$. If there is an $a \in \ker \varphi$ with $a \neq 0$, choose $s \in F \setminus \{0\}$ with $sa = 0$. Let \leq be some total ordering on the torsion-free abelian group Γ , and let s_γ (resp. a_δ) be the nonzero homogeneous component of s (resp. a) of lowest degree. Then $s_\gamma a_\delta$ must vanish, since it is a homogeneous component of sa . But $s_\gamma \in F^\times$, so $a_\delta = 0$, a contradiction. Hence, φ is injective, and we identify A^\natural with its image $\varphi(A^\natural)$ in $q_F(A)$.

Let \mathcal{N} be the set of non-zero-divisors of $Z(A)^\natural$. We have just shown that $F \setminus \{0\} \subseteq \mathcal{N}$. Hence, there is a canonical monomorphism

$$\psi: Z(A)^\natural \otimes_{F^\natural} q(F) \longrightarrow q(Z(A))$$

given by $z \otimes (bt^{-1}) \mapsto (zb)t^{-1}$ for all $z \in Z(A)$, $b \in F$, $t \in F \setminus \{0\}$. Suppose that $Z(A)^\natural$ is integral over F^\natural . To see that ψ is surjective, take any $t \in \mathcal{N}$; let $f = c_n X^n + \dots + c_0 \in F[X]$ be a nonzero polynomial of minimal degree such that $f(t) = 0$. Then, $c_0 \neq 0$ as $c_n t^{n-1} + \dots + c_1 \neq 0$ and t is not a zero divisor. Hence, $t^{-1} = -c_0^{-1}(c_n t^{n-1} + \dots + c_1) \in \text{im } \psi$. Since $\{t^{-1} \mid t \in \mathcal{N}\}$ generates the ring $q(Z(A))$ over $Z(A)$, the map ψ is an isomorphism. Thus, $Z(A)^\natural \otimes_{F^\natural} q(F) \cong q(Z(A))$, and hence,

$$\begin{aligned} q_F(A) &= A^\natural \otimes_{F^\natural} q(F) \cong A^\natural \otimes_{Z(A)^\natural} (Z(A)^\natural \otimes_{F^\natural} q(F)) \\ &\cong A^\natural \otimes_{Z(A)^\natural} q(Z(A)) = q(A). \end{aligned}$$

We use these canonical isomorphisms to identify $q_F(A)$ with $q(A)$. □

We conclude this subsection with examples of simple graded algebras. The easiest ones are obtained by scalar extension from F_0 . For any F_0 -algebra A ,

the F -algebra $A \otimes_{F_0} F$ has an obvious grading, for which $A \otimes_{F_0} F_\gamma$ is the homogeneous component of degree γ , for any $\gamma \in \Gamma$.

Proposition 2.16. *For any finite-dimensional simple algebra A over F_0 , the graded F -algebra $A \otimes_{F_0} F$ is graded simple, and $Z(A \otimes_{F_0} F) = Z(A) \otimes_{F_0} F$. If A is a division algebra, then $A \otimes_{F_0} F$ is a graded division algebra.*

Proof. Suppose first A is a division algebra. Let $\gamma \in \Gamma$. The homogeneous component of degree γ in $A \otimes_{F_0} F$ is $A \otimes \lambda$ for any $\lambda \in F^\times$ homogeneous of degree γ . It is clear that each nonzero element in this homogeneous component is invertible, hence $A \otimes_{F_0} F$ is a graded division algebra.

When A is assumed only to be simple (and finite-dimensional), the preceding argument applies to $Z(A)$, and shows that $Z(A) \otimes_{F_0} F$ is a graded field, hence $q(Z(A) \otimes_{F_0} F)$ is a field. The equality $Z(A \otimes_{F_0} F) = Z(A) \otimes_{F_0} F$ is clear. Since $[A \otimes_{F_0} F : F] = [A : F_0] < \infty$, we have $q_F(A \otimes_{F_0} F) = q(A \otimes_{F_0} F)$ by Lemma 2.15, hence

$$q_F(A \otimes_{F_0} F) = (A \otimes_{F_0} F) \otimes_{Z(A) \otimes_{F_0} F} q(Z(A) \otimes_{F_0} F) = A \otimes_{Z(A)} q(Z(A) \otimes_{F_0} F).$$

It follows that $q_F(A \otimes_{F_0} F)$ is simple (see Draxl [63, Cor. 3, p. 30] or Pierce [178, Prop. b, p. 226]). Therefore, Lemma 2.14 shows that $A \otimes_{F_0} F$ is graded simple. \square

The following examples turn out to be typical of all simple graded F -algebras by Th. 2.26 below.

Proposition 2.17. *Let D be a (finite-dimensional) graded division algebra over F and let V be a finite-dimensional right graded D -vector space. The graded F -algebra $\text{End}_D V$ is graded simple. Its center consists of scalar multiplications by elements in the center of D , so $Z(\text{End}_D V) = Z(D)$.*

Proof. Using a homogeneous base of V , we may identify $\text{End}_D V$ with a graded matrix ring (see Cor. 2.11), so $(\text{End}_D V)^\natural \cong M_n(D^\natural)$ for some integer n . It follows that $q_F(\text{End}_D V) \cong M_n(q_F(D))$. The latter is a simple $q(F)$ -algebra, since Lemma 2.14 shows that $q_F(D)$ is a division algebra over $q(F)$. Therefore, it follows from Lemma 2.14 that $\text{End}_D V$ is a simple graded algebra. The equality $Z(\text{End}_D V) = Z(D)$ is easily proved using a matrix representation of $\text{End}_D V$. \square

The graded symbol algebras that we define next are explicit examples of central simple graded algebras; such symbol algebras will occur frequently in the sequel (see for example §7.2.2).

Definition 2.18. Let F be a graded field and $n \geq 2$ an integer such that F_0 contains a primitive n -th root of unity ω (in particular, $\text{char } F_0 \nmid n$). For any homogeneous elements $a, b \in F^\times$, consider the $q(F)$ -algebra A generated by two elements i, j subject to the following relations:

$$i^n = a, \quad j^n = b, \quad ij = \omega ji.$$

This algebra is a *symbol algebra* as defined in Draxl⁴ [63, §11] or Gille–Szamuely [84, p. 36]; it is a central simple algebra of degree n (i.e., dimension n^2) over $q(F)$, and we have

$$A = \bigoplus_{k,\ell=0}^{n-1} q(F) i^k j^\ell.$$

For this algebra we use the symbol algebra notation

$$A = (a, b/q(F))_n \quad \text{or} \quad (a, b/q(F))_{\omega, n}.$$

Since $i^n, j^n \in F$, the F^\natural -submodule of A spanned by $(i^k j^\ell)_{k,\ell=0}^{n-1}$ is an F^\natural -algebra. We provide it with a grading extending the grading on F by declaring i and j to be homogeneous of degree $\frac{1}{n} \deg a$ and $\frac{1}{n} \deg b$ respectively. Thus, for any $\gamma \in \Gamma$, we let

$$S_\gamma = \bigoplus_{k,\ell=0}^{n-1} F_{\varepsilon(\gamma,k,\ell)} i^k j^\ell \quad \text{where } \varepsilon(\gamma, k, \ell) = \gamma - \frac{k}{n} \deg a - \frac{\ell}{n} \deg b,$$

and we define

$$S = \bigoplus_{k,\ell=0}^{n-1} F i^k j^\ell = \bigoplus_{\gamma \in \Gamma} S_\gamma.$$

Easy calculations show that $S_\gamma \cdot S_\delta \subseteq S_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$; hence, S is a graded F -algebra in which S_γ is the homogeneous component of degree γ . Note that since the powers of i and j are units of S , we have

$$\Gamma_S = \Gamma_S^\times = \langle \frac{1}{n} \deg a, \frac{1}{n} \deg b \rangle + \Gamma_F. \quad (2.9)$$

The graded algebra S is said to be a *graded symbol algebra* of degree n over F ; we use the following notation:

$$S = (a, b/F)_n \quad \text{or} \quad (a, b/F)_{\omega, n}. \quad (2.10)$$

Graded symbol algebras of degree 2 are also called *graded quaternion algebras*.

Proposition 2.19. *Every graded symbol algebra of degree n over F is a central simple graded F -algebra of dimension n^2 .*

Proof. It is clear from the construction of graded symbol algebras that extending scalars to the quotient field yields an (ungraded) symbol algebra over $q(F)$:

$$q_F((a, b/F)_n) = (a, b/q(F))_n.$$

Symbol algebras over $q(F)$ are known to be central simple: see Draxl [63, Th. 1, p. 78] or Gille–Szamuely [84, Cor. 2.5.5]. Therefore, the proposition follows from Lemma 2.14. \square

As further examples of graded simple algebras, graded analogues of cyclic and crossed product algebras will be defined in §6.1.2.

⁴ Draxl uses the term “power norm residue algebra.”

2.2.2 The structure of semisimple graded algebras

Our goal in this subsection is to show that every semisimple graded algebra is a direct product of simple graded algebras; see Th. 2.23. Our arguments also yield information on graded modules over semisimple graded algebras. Defining a *simple graded module* over a graded algebra to be a graded module $M \neq \{0\}$ in which $\{0\}$ and M are the only graded submodules, we show in Prop. 2.24 that every finitely generated graded module over a semisimple graded algebra is a direct sum of simple graded submodules.

Lemma 2.20. *Every minimal homogeneous left ideal J in a semisimple graded F -algebra A is generated by a homogeneous idempotent.*

Proof. Since $J^2 \neq \{0\}$, there exists an $a \in J$ such that $Ja \neq \{0\}$, and we may choose a to be homogeneous. Then Ja is a homogeneous left ideal contained in J ; hence $Ja = J$. Therefore, there exists an $e \in J$ such that $ea = a$. By substituting for e its homogeneous component of degree 0, we may assume that e is homogeneous of degree 0. By multiplying by e we obtain $e^2a = ea$; hence

$$e^2 - e \in \{x \in J \mid xa = 0\}.$$

The set on the right is a homogeneous left ideal contained in J ; it is not J since $Ja \neq \{0\}$, hence it is $\{0\}$ and therefore $e^2 - e = 0$; i.e., e is an idempotent. Since $e \in J$ we have $J = Ae$. \square

Note that in a graded algebra every nonzero homogeneous idempotent has degree 0.

Proposition 2.21. *Let J be a nonzero homogeneous left ideal in a semisimple graded F -algebra A . There are nonzero homogeneous idempotents e_1, \dots, e_r such that*

- e_1, \dots, e_r are pairwise orthogonal, i.e., $e_i e_j = 0$ for $i \neq j$;
- Ae_i is a minimal homogeneous left ideal for all $i = 1, \dots, r$, and
- $J = Ae_1 \oplus \dots \oplus Ae_r = A(e_1 + \dots + e_r)$.

In particular, $e_1 + \dots + e_r$ is an idempotent that generates J .

Proof. We argue by induction on $\dim_F J$. If J is minimal, the proposition readily follows from the lemma. Otherwise, let $J_1 \subset J$ be a minimal homogeneous left ideal. Lemma 2.20 yields a homogeneous idempotent e_1 such that $J_1 = Ae_1$. We have $e_1 = e_1^2 \in Je_1$; hence Je_1 is a nonzero homogeneous left ideal contained in J_1 . Therefore, $Je_1 = J_1$. We have

$$J = Je_1 \oplus J(1 - e_1) = Ae_1 \oplus J(1 - e_1) \quad (2.11)$$

and $J(1 - e_1)$ is a homogeneous left ideal since $1 - e_1$ is homogeneous. By induction, there are pairwise orthogonal nonzero homogeneous idempotents

e'_2, \dots, e'_r such that Ae'_i is a minimal homogeneous left ideal for $i = 2, \dots, r$, and

$$J(1 - e_1) = Ae'_2 \oplus \dots \oplus Ae'_r. \quad (2.12)$$

Since $e'_i \in J(1 - e_1)$, we have

$$e'_i e_1 = 0 \quad \text{for } i = 2, \dots, r. \quad (2.13)$$

For $i = 2, \dots, r$, let $e_i = (1 - e_1)e'_i$. Then $e_i \in Ae'_i$ and, by (2.13)

$$e'_i e_i = e'_i(1 - e_1)e'_i = e_i'^2 = e'_i \quad \text{for } i = 2, \dots, r.$$

Hence $e'_i \in Ae_i$ and it follows that $Ae_i = Ae'_i$ for $i = 2, \dots, r$. Therefore, by (2.11) and (2.12), we have

$$J = Ae_1 \oplus Ae_2 \oplus \dots \oplus Ae_r.$$

Moreover, for $i = 2, \dots, r$ we have $e_i e_1 = 0$ by (2.13), and

$$e_1 e_i = e_1(1 - e_1)e'_i = 0.$$

Using (2.13) again, we have also for $i, j \in \{2, \dots, r\}$ with $i \neq j$

$$e_i e_j = (1 - e_1)e'_i(1 - e_1)e'_j = (1 - e_1)e'_i e'_j = 0.$$

Finally, e_i is an idempotent for $i = 2, \dots, r$ since

$$e_i^2 = (1 - e_1)e'_i(1 - e_1)e'_i = (1 - e_1)e_i'^2 = (1 - e_1)e'_i.$$

Thus, e_1, \dots, e_r satisfy all the requirements, and it remains only to show that J is generated by $e_1 + \dots + e_r$.

To complete the proof, observe that

$$e_i = e_i(e_1 + \dots + e_r) \in A(e_1 + \dots + e_r) \quad \text{for } i = 1, \dots, r,$$

hence

$$Ae_1 \oplus \dots \oplus Ae_r \subseteq A(e_1 + \dots + e_r).$$

The reverse inclusion is obvious, and it follows that $J = A(e_1 + \dots + e_r)$. \square

Corollary 2.22. *Every two-sided homogeneous ideal J in a semisimple graded algebra A is generated by a homogeneous central idempotent.*

Proof. By Prop. 2.21 we have $J = Ae$ for some homogeneous idempotent $e \in J$. Since J is two-sided, we have $ex \in J$ for all $x \in A$, hence $ex = exe$. It follows that $(1 - e)Ae$ is a homogeneous right ideal of A . This ideal satisfies $((1 - e)Ae)^2 = \{0\}$ since e is idempotent, hence $(1 - e)Ae = \{0\}$ since A is graded semisimple. Therefore, we have $(1 - e)xe = 0$ for all $x \in A$, hence $xe = exe$. Thus, for all $x \in A$ we have

$$xe = exe = ex,$$

which shows that e is a central idempotent. \square

Theorem 2.23. *Every semisimple graded algebra A is a direct product of simple graded algebras, which are uniquely determined. Its center $Z(A)$ is a direct product of graded fields.*

Proof. The center of a simple graded algebra is a graded field, since any nonzero central homogeneous element that is not invertible generates a non-trivial two-sided ideal. Therefore, the theorem is clear if A is graded simple. If it is not, Cor. 2.22 yields a homogeneous central idempotent $e \neq 0, 1$. Then

$$A = Ae \oplus A(1 - e) \cong_g Ae \times A(1 - e),$$

and $Ae, A(1 - e)$ are semisimple graded algebras. Arguing by induction on dimension, we may assume that each of these algebras is a direct product of simple graded algebras, hence A also is such a product. If

$$A \cong_g A_1 \times \dots \times A_n$$

with A_1, \dots, A_n simple graded algebras, then the center $Z(A)$ satisfies

$$Z(A) \cong_g Z(A_1) \times \dots \times Z(A_n).$$

Since each $Z(A_i)$ is a graded field, it follows that $Z(A)$ is a direct product of graded fields. Moreover, the simple graded components A_1, \dots, A_n are isomorphic to Ae_1, \dots, Ae_n for e_1, \dots, e_n the primitive idempotents of $Z(A)$, hence they are uniquely determined. \square

By applying Prop. 2.21 to the left ideal $J = A$, we obtain a decomposition of A into a direct sum of minimal homogeneous left ideals

$$A = Ae_1 \oplus \dots \oplus Ae_r. \quad (2.14)$$

There is a similar decomposition for every finitely generated left graded A -module, as the next proposition shows.

Proposition 2.24. *Every finitely generated left graded module over a semisimple graded algebra A is a direct sum of simple left graded A -submodules.*

Proof. Let g_1, \dots, g_n be homogeneous generators of some left graded A -module M . From (2.14) it follows that

$$M = \sum_{i=1}^r \sum_{j=1}^n Ae_i g_j.$$

Each summand $Ae_i g_j$ is either $\{0\}$ or a simple graded submodule of M since for any graded submodule $N \subset Ae_i g_j$ the set $\{x \in Ae_i \mid xg_j \in N\}$ is a homogeneous left ideal contained in the minimal homogeneous left ideal Ae_i . Therefore, M is a sum of finitely many simple graded submodules $M = \sum_{i \in I} M_i$. Pick a maximal subset $J \subseteq I$ such that the sum $\sum_{j \in J} M_j$ is direct. For each $i \in I \setminus J$ we have

$$M_i \cap \left(\bigoplus_{j \in J} M_j \right) \neq \{0\},$$

otherwise J would not be maximal. But this intersection is a graded submodule of the simple graded module M_i , hence it equals M_i ; therefore, $M_i \subseteq \bigoplus_{j \in J} M_j$. Thus, it follows that $\sum_{i \in I} M_i = \bigoplus_{j \in J} M_j$, hence M is a direct sum of simple graded submodules. \square

2.2.3 The Wedderburn Theorem for simple graded algebras

We start our discussion of the graded analogue of Wedderburn's Theorem with the graded version of Schur's Lemma:

Lemma 2.25. *If M and N are simple graded modules over a graded algebra A , then every graded homomorphism $f: M \rightarrow N$ is either 0 or an isomorphism. In particular, $\text{End}_A M$ is a graded division ring for the grading induced by the grading on M (see Prop. 2.8).*

Proof. This readily follows from the observation that the kernel and the image of any graded A -module homomorphism $M \rightarrow N$ are graded submodules of M and N respectively. \square

Now, let A be a simple graded algebra over F . Since $[A:F]$ is finite, there is a minimal nonzero homogeneous left ideal $J \subset A$. Let $D = \text{End}_A J$ (acting on J on the right). Since J is a finite-dimensional graded F -vector space, the F -algebra D is graded and finite-dimensional. Moreover, J carries a right D -module structure, and multiplication on the left by elements of A defines a canonical graded F -algebra homomorphism

$$\rho: A \longrightarrow \text{End}_D J.$$

Theorem 2.26. *The F -algebra D is a graded division F -algebra and the map ρ is an isomorphism of graded F -algebras. So,*

$$A \cong_g \text{End}_D J.$$

Moreover, every simple left graded A -module S is isomorphic to some shift $J(\gamma)$ of J for some $\gamma \in \Gamma$, and $\text{End}_A S \cong_g D$ as graded F -algebras.

As a consequence, the graded division algebra D is uniquely determined by A up to graded isomorphism, independent of the choice of J . (This is also shown in Prop. 2.27 below.) This D is called the *graded division algebra associated to A* .

Proof. Since J is a simple graded module, D is a graded division algebra by the graded analogue of Schur's Lemma, see Lemma 2.25. Therefore, J is a graded D -vector space. If $a \in A$, $x \in J$, and $f \in D$ are homogeneous, then by the definition of $\deg f$ we have whenever $ax^f \neq 0$,

$$\deg(ax^f) = \deg a + \deg x + \deg f.$$

Thus, the map ρ is a homomorphism of graded algebras. It is therefore injective since A is graded simple. To show that ρ is surjective, it suffices to show that its image is a left ideal in $\text{End}_D J$, since this image contains the identity. Since A is graded simple and $J \cdot A$ is a two-sided homogeneous ideal in A , we have $J \cdot A = A$; hence $\rho(J) \cdot \rho(A) = \rho(A)$ and therefore

$$(\text{End}_D J) \cdot \rho(A) = (\text{End}_D J) \cdot \rho(J) \cdot \rho(A).$$

We claim that $(\text{End}_D J) \cdot \rho(J) \subset \rho(J)$; it then follows that

$$(\text{End}_D J) \cdot \rho(A) \subseteq \rho(J) \cdot \rho(A) \subseteq \rho(A),$$

which shows that $\rho(A)$ is a left ideal in $\text{End}_D J$. Thus, to prove the claim it suffices to show that ρ is an isomorphism. Note that for $x, y \in J$ we have $xy \in J$, hence multiplication on the right by y defines an element in $D = \text{End}_A J$. Therefore, for $g \in \text{End}_D J$ we have $g(xy) = g(x)y$, which means that

$$g \circ \rho(x) = \rho(g(x)) \in \rho(J).$$

The claim is thus proved.

Now, let S be a simple left graded A -module. The set $\{x \in A \mid xS = \{0\}\}$ is a two-sided homogeneous ideal of A , hence it is $\{0\}$. Therefore, $J \cdot S \neq \{0\}$ and we may find a homogeneous element $s \in S$ such that $J \cdot s \neq \{0\}$. The map $x \mapsto xs$ is a nonzero homomorphism of left graded A -modules $J \rightarrow S$. Since J and S are graded simple, this homomorphism is bijective, and it identifies S with the shift $J(-\deg s)$. Since $\text{End}_A(J(\gamma)) = \text{End}_A J$ for all $\gamma \in \Gamma$, we have $\text{End}_A S \cong_g D$. \square

Another way to view a central simple graded algebra A is as a matrix ring with a shifted grading. Using the notation of Th. 2.26, we fix a decomposition of the right graded D -vector space J into 1-dimensional vector spaces as in Cor. 2.10:

$$J \cong_g D(\delta_1) \oplus \dots \oplus D(\delta_n). \quad (2.15)$$

By Th. 2.26 and Prop. 2.9, we have a matrix representation of A as in Cor. 2.11:

$$A \cong_g \text{End}_D J \cong_g M_n(D)(\delta_1, \dots, \delta_n). \quad (2.16)$$

If we let A^\natural, D^\natural denote the F -algebras A and D with their grading forgotten, we thus have $A^\natural \cong M_n(D^\natural)$.

With the notation above, we have:

Proposition 2.27. *Let V be a finite-dimensional right graded vector space over some graded division F -algebra E and let A be as in (2.16). There is a graded algebra isomorphism $A \cong_g \text{End}_E V$ if and only if the following conditions hold:*

- (i) $E \cong_g D$, and
- (ii) there exists a $\gamma \in \Gamma$ such that $V \cong_g E(\gamma + \delta_1) \oplus \dots \oplus E(\gamma + \delta_n)$.

Proof. Suppose there is an isomorphism $A \cong_g \text{End}_E V$. We use it as an identification. The grading on V gives it a left graded A -module structure since by definition of the grading on $\text{End}_E V$ we have for all homogeneous elements $a \in A$, $v \in V$ such that $a(v) \neq 0$

$$\deg(a(v)) = \deg a + \deg v.$$

It is clear that V is a simple graded A -module since for every nonzero homogeneous element $v \in V$ we have $A \cdot v = V$. Therefore, Th. 2.26 shows that there is an isomorphism of graded A -modules $V \cong_g J(\gamma)$ for some $\gamma \in \Gamma$, and an isomorphism of graded F -algebras $D \cong_g \text{End}_A V$. On the other hand, we claim that $\text{End}_A V$ consists of scalar multiplications by elements in E , so $D \cong_g E$. To prove the claim, choose a homogeneous E -base $(v_i)_{i=1}^n$ of V , and for $i, j = 1, \dots, n$ let $\varepsilon_{ij} \in A$ be the linear transformation of V satisfying

$$\varepsilon_{ij}(v_k) = \begin{cases} v_i & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Suppose $g \in \text{End}_A V$ and let

$$v_j^g = \sum_{i=1}^n v_i e_{ij} \quad \text{for } j = 1, \dots, n, \text{ with } e_{ij} \in E.$$

For $k, \ell = 1, \dots, n$ we have

$$\varepsilon_{k\ell}(v_j^g) = v_k e_{\ell j} \quad \text{and} \quad \varepsilon_{k\ell}(v_j)^g = \begin{cases} v_k^g & \text{if } \ell = j, \\ 0 & \text{if } \ell \neq j. \end{cases}$$

But $\varepsilon_{k\ell}(v_j^g) = \varepsilon_{k\ell}(v_j)^g$ since g is A -linear, hence

$$e_{\ell j} = 0 \text{ if } \ell \neq j \quad \text{and} \quad v_k^g = v_k e_{jj} \text{ for all } j, k = 1, \dots, n.$$

Therefore, $e_{11} = \dots = e_{nn}$, and g is scalar multiplication by e_{11} , which proves the claim that $\text{End}_A V = E$, hence $D \cong_g E$.

By counting dimensions over F , we see that $\dim_E V = n$, and condition (ii) follows from the isomorphism $V \cong_g J(\gamma)$. Thus, conditions (i) and (ii) hold.

Conversely, suppose (i) and (ii) hold. Since $(\gamma + \delta_i) - (\gamma + \delta_j) = \delta_i - \delta_j$, by Prop. 2.9 the matrix representation of $\text{End}_E V$ is

$$\text{End}_E V \cong_g M_n(E)(\delta_1, \dots, \delta_n).$$

In view of (2.16), it follows that any isomorphism $D \cong_g E$ induces an isomorphism $A \cong_g \text{End}_E V$. \square

As a consequence of Th. 2.26, we can now prove the converse of the last statement of Lemma 2.14, relating properties of a finite-dimensional graded F -algebra A to properties of its associated $q(F)$ -algebra $q(A) = q_F(A) = A^{\natural} \otimes_{F^{\natural}} q(F)$ (see Lemma 2.15).

Proposition 2.28. *Let A be a finite-dimensional graded F -algebra.*

- (i) *A is a graded division algebra if and only if $q(A)$ is a division algebra.*
- (ii) *A is a simple graded algebra if and only if $q(A)$ is a simple algebra.*
- (iii) *A is a semisimple graded algebra if and only if $q(A)$ is a semisimple algebra.*

Proof. Lemma 2.14 shows that $q(A)$ is finite-dimensional; it also establishes the “only if” part of (i) and the “if” part of (ii). If $q(A)$ is a division algebra, the “if” part of (ii) shows that A is graded simple. Moreover, A^\natural has no zero divisors, so in a representation of A as in (2.16) above we must have $n = 1$, hence A is a graded division algebra. This completes the proof of (i).

For (ii), it remains only to prove the “only if” part. Assume A is a central simple graded algebra, and fix a representation as in (2.16) above:

$$A \cong_g M_n(D)(\delta_1, \dots, \delta_n)$$

where D is a graded division algebra. Then,

$$q(A) \cong M_n(q(D)).$$

By (i), we know that $q(D)$ is a division algebra; hence, $q(A)$ is a simple algebra.

For (iii), first assume that A is semisimple. Then, by Th. 2.23,

$$A = B_1 \times \dots \times B_n$$

for some simple graded algebras B_i . Hence,

$$q(A) \cong q(B_1) \times \dots \times q(B_n),$$

and each $q(B_i)$ is simple by (ii). Thus, $q(A)$ is semisimple. On the other hand, if A is not semisimple, then it has a nonzero nilpotent homogeneous ideal N . Then, $N^\natural \otimes_{F^\natural} q(F)$ is a nonzero nilpotent ideal of $q(A)$. Hence, $q(A)$ is not semisimple. \square

Corollary 2.29. *If A is a central simple graded F -algebra, then $[A:F] = d^2$ for some integer d .*

Proof. By Lemma 2.14 and Prop. 2.28(ii), the $q(F)$ -algebra $q(A) = q_F(A)$ is central simple, hence there is an integer d such that $[q(A):q(F)] = d^2$. (The integer d is the degree of the central simple $q(F)$ -algebra $q(A)$.) By Lemma 2.14, we also have $d^2 = [A:F]$. \square

The integer $d \geq 1$ as in Cor. 2.29 is called the *degree* of the central simple graded algebra A . We use the notation $\deg A$ for the degree of A ; thus

$$\deg A = \deg q(A).$$

Note that this definition is consistent with the definition of a graded symbol algebra of degree n , see Prop. 2.19. We also define the (*Schur*) *index* of a central simple graded algebra A as the degree of the division algebra D associated to A by Wedderburn’s Theorem. Thus (see Th. 2.26),

$$\text{ind } A = \deg D \quad (= \deg q(D) = \text{ind } q(A)).$$

2.2.4 Centralizers and simple subalgebras

For arbitrary graded F -algebras A, B the tensor product $A \otimes_F B$ carries a grading defined in §2.1.4, which is compatible with the multiplication. Thus, $A \otimes_F B$ is a graded F -algebra. It contains as graded subalgebras $A \otimes 1$ and $1 \otimes B$, which we identify with A and B . In this subsection, we determine when a tensor product of semisimple graded algebras is graded semisimple. We then prove graded analogues of two essential tools for the study of subalgebras of central simple algebras: the Double Centralizer Theorem and the Skolem–Noether Theorem. We start with some general remarks on centralizers.

If $A' \subseteq A$ is a graded subalgebra, we let $C_A(A')$ denote the centralizer of A' in A , i.e.,

$$C_A(A') = \{a \in A \mid aa' = a'a \text{ for all } a' \in A'\}.$$

It is a graded subalgebra of A .

Proposition 2.30. *If $A' \subseteq A$ and $B' \subseteq B$ are graded subalgebras of arbitrary graded F -algebras A, B , then*

$$C_{A \otimes_F B}(A' \otimes_F B') = C_A(A') \otimes_F C_B(B').$$

Proof. Let $x \in C_{A \otimes_F B}(A' \otimes_F B')$. For some F -base $(a_i)_{i \in I}$ of A , write $x = \sum_{i \in I} a_i \otimes x_i$ with each $x_i \in B$. Since x centralizes $1 \otimes B'$, we must have $x_i \in C_B(B')$ for all $i \in I$. Rewriting $x = \sum_{j \in J} u_j \otimes b_j$ where $(b_j)_{j \in J}$ is an F -base of $C_B(B')$, we have $u_j \in C_A(A')$ for all $j \in J$ since x centralizes $A' \otimes 1$. Therefore, x lies in $C_A(A') \otimes_F C_B(B')$, and we have proved $C_{A \otimes_F B}(A' \otimes_F B') \subseteq C_A(A') \otimes_F C_B(B')$. The reverse inclusion is obvious. \square

The particular case where $A' = A$ and $B' = B$ determines the center of $A \otimes_F B$:

Corollary 2.31. $Z(A \otimes_F B) = Z(A) \otimes_F Z(B)$.

We now study the semisimplicity of tensor products, in the special case where one of the factors is graded simple with center F .

Proposition 2.32. *Let A be a central simple graded F -algebra, and let B be a (possibly infinite-dimensional) graded F -algebra with $Z(B)$ integral over F . The graded $Z(B)$ -algebra $A \otimes_F B$ is graded simple (resp. graded semisimple) if and only if B is a graded simple (resp. graded semisimple) $Z(B)$ -algebra.*

Proof. If B is not graded simple, then any nontrivial two-sided homogeneous ideal $J \subseteq B$ yields a nontrivial two-sided homogeneous ideal $A \otimes_F J \subseteq A \otimes_F B$. Similarly, if B is not graded semisimple, we may take for J a nontrivial homogeneous nilpotent ideal in B and obtain a nontrivial homogeneous nilpotent ideal in $A \otimes_F B$. The “only if” part of the proposition is thus clear.

Now, suppose B is graded simple, so finite-dimensional over the graded field $Z(B)$. Then $q(B)$ is simple by Prop. 2.28; similarly, $q(A)$ is central simple

over $q(F)$. Hence, $q(A) \otimes_{q(F)} q(B)$ is a simple algebra (see Draxl [63, Cor. 3, p. 30] or Pierce [178, Prop. b, p. 226]). Now, $Z(A \otimes_F B) = F \otimes_F Z(B) = Z(B)$, which is integral over F by hypothesis. Hence, by Lemma 2.15,

$$q(A \otimes_F B) = q_F(A \otimes_F B) = q_F(A) \otimes_{q(F)} q_F(B) = q(A) \otimes_{q(F)} q(B).$$

It therefore follows from Prop. 2.28 (or Lemma 2.14) that $A \otimes_F B$ is graded simple.

If B is graded semisimple, we have $B \cong_g B_1 \times \dots \times B_r$ for some simple graded F -algebras B_1, \dots, B_r by Th. 2.23. Then,

$$A \otimes_F B \cong_g (A \otimes_F B_1) \times \dots \times (A \otimes_F B_r).$$

Each direct factor is graded simple, hence $A \otimes_F B$ is graded semisimple. \square

As a corollary, we obtain a criterion for the graded semisimplicity of the tensor product of semisimple graded algebras:

Corollary 2.33. *Let A, B be semisimple graded F -algebras. The graded algebra $A \otimes_F B$ is graded semisimple if and only if $Z(A) \otimes_F Z(B)$ is graded semisimple.*

Proof. Suppose first A and B are graded simple. Since

$$A \otimes_F B = A \otimes_{Z(A)} (Z(A) \otimes_F B),$$

Prop. 2.32 shows that $A \otimes_F B$ is graded semisimple if and only if $Z(A) \otimes_F B$ is semisimple. But

$$Z(A) \otimes_F B = (Z(A) \otimes_F Z(B)) \otimes_{Z(B)} B;$$

so, one more application of Prop. 2.32 shows that this condition holds if and only if $Z(A) \otimes_F Z(B)$ is graded semisimple. The corollary is thus proved if A and B are simple.

In the general case of semisimple graded algebras A, B , we use Th. 2.23 to decompose A and B into direct products of simple algebras. Let

$$A = A_1 \times \dots \times A_n \quad \text{and} \quad B = B_1 \times \dots \times B_m$$

where A_1, \dots, A_n and B_1, \dots, B_m are simple graded F -algebras. We have

$$A \otimes_F B = \prod_{i,j} A_i \otimes_F B_j \quad \text{and} \quad Z(A) \otimes_F Z(B) = \prod_{i,j} Z(A_i) \otimes_F Z(B_j).$$

Since a direct product of graded algebras is graded semisimple if and only if each factor is graded semisimple, $A \otimes_F B$ is graded semisimple if and only if $A_i \otimes_F B_j$ is graded semisimple for all i, j . Likewise, $Z(A) \otimes_F Z(B)$ is graded semisimple if and only if $Z(A_i) \otimes_F Z(B_j)$ is graded semisimple for all i, j . The special case of simple graded algebras considered above shows that $A_i \otimes_F B_j$ is graded semisimple if and only if $Z(A_i) \otimes_F Z(B_j)$ is graded semisimple. The corollary follows. \square

Since a semisimple graded F -algebra is a direct product of simple graded F -algebras and the center of a simple graded F -algebra is a graded field, Cor. 2.33 reduces the question of semisimplicity of $A \otimes_F B$ to consideration of tensor products of graded fields. We will see below in Prop. 5.21 that if L is a finite-degree separable (= étale) graded field extension of F and K is any graded field extension of F , then $L \otimes_F K$ is a semisimple K -algebra.

Proposition 2.34. *Let A be a simple graded F -algebra with center Z . Let N be a finitely generated left graded A -module. Let $E = \text{End}_A N$ (acting on N on the right). Then, E is graded simple with $Z(E) = Z$ and*

$$[A:F] \cdot [E:F] = (\dim_F N)^2. \quad (2.17)$$

Moreover, $\text{End}_E N \cong_g A$ and $A \otimes_Z E^{\text{op}} \cong_g \text{End}_Z N$.

Proof. Let J be a minimal nonzero homogeneous left ideal of A , and let $D = \text{End}_A J$. As in §2.2.3, we write $J \cong_g D(\delta_1) \oplus \dots \oplus D(\delta_n)$; then, as in (2.16),

$$A \cong_g M_n(D)(\delta_1, \dots, \delta_n),$$

so $Z(A) = Z(D)$. By Prop. 2.24 and Th. 2.26 we have $N \cong_g \bigoplus_{i=1}^k J(\lambda_i)$ for suitable λ_i in Γ . Hence, by the “left version” of Prop. 2.9,

$$E \cong_g M_k(D)(-\lambda_1, \dots, -\lambda_k).$$

Let $P = \bigoplus_{i=1}^k D(-\lambda_i)$ viewed as a right graded D -vector space. By Prop. 2.9,

$$\text{End}_D P \cong_g M_k(D)(-\lambda_1, \dots, -\lambda_k) \cong_g E.$$

Hence, by Prop. 2.17, E is graded simple and $Z(E) = Z(D) = Z(A)$. Let $d = [D:F]$. Then, $[A:F] = n^2 d$, $[E:F] = k^2 d$, and $\dim_F N = k \dim_F J = knd$. These equalities yield formula (2.17).

Let $B = \text{End}_E N$ (acting on N on the left). The map $\rho: A \rightarrow B$ given by $\rho(a)(x) = ax$ for $a \in A$, $x \in N$ is a graded F -algebra homomorphism, which is injective, as A is graded simple. Because E is graded simple, formula (2.17) applies with E (resp. B) replacing A (resp. E), showing that

$$[E:F] \cdot [B:F] = (\dim_F N)^2 = [A:F] \cdot [E:F].$$

Therefore, $[B:F] = [A:F]$, so ρ is a graded isomorphism. The last isomorphism of the proposition is proved similarly: the module actions of A and E on N give graded F -algebra homomorphisms $A \rightarrow \text{End}_Z N$ and $E^{\text{op}} \rightarrow \text{End}_Z N$ whose images commute. Moreover, the elements of Z in A have the same action on N as their images in E . Hence, there is a graded F -algebra homomorphism $\tau: A \otimes_Z E^{\text{op}} \rightarrow \text{End}_Z N$. This τ is injective, as $A \otimes_Z E^{\text{op}}$ is graded simple by Prop. 2.32. Then τ is also surjective, since by (2.17),

$$\begin{aligned} [A \otimes_Z E^{\text{op}} : Z] &= [A:Z] \cdot [E:Z] = [A:F] \cdot [E:F] / [Z:F]^2 \\ &= (\dim_F N)^2 / [Z:F]^2 = [\text{End}_Z N : Z]. \end{aligned} \quad \square$$

Theorem 2.35 (Double Centralizer Theorem). *Let A be a central simple graded F -algebra, let B be a simple graded F -subalgebra of A , and let $C = C_A(B)$. Then,*

- (i) C is a simple graded F -algebra with $Z(C) = Z(B)$, and $C_A(C) = B$.
- (ii) $[C:F] \cdot [B:F] = [A:F]$.
- (iii) $B \otimes_{Z(B)} C \cong_g C_A(Z(B))$. In particular, if $Z(B) = F$, then $B \otimes_F C \cong_g A$.

Proof. By using the multiplication in A , we may view A as a left graded B -module and a right graded A -module. Since the multiplication is associative, A is a graded B - A -bimodule, or, equivalently, a right graded $B^{\text{op}} \otimes_F A$ -module. The module action is given by

$$t \cdot \left(\sum b_i^{\text{op}} \otimes a_i \right) = \sum b_i t a_i \quad \text{for } t \in A, b_i \in B, a_i \in A.$$

Let $T = B^{\text{op}} \otimes_F A$. Since A is graded central simple over F and B^{op} is graded simple, Prop. 2.32 shows that T is graded simple with $Z(T) = Z(B^{\text{op}}) = Z(B)$. Let $\text{End}_A A$ denote the graded endomorphism ring of A as a right graded A -module. With $\text{End}_A A$ acting on A on the left, we have the graded ring isomorphism $\eta: \text{End}_A A \rightarrow A$ given by $\eta(f) = f(1)$. The inverse map is given by $\eta^{-1}(a)(y) = ay$ for all $a, y \in A$. Now, $\text{End}_T A$ is the graded subring of $\text{End}_A A$ consisting of those A -endomorphisms of A that also commute with the action of B on A . That is,

$$\eta(\text{End}_T A) = \{a \in A \mid aby = bay \text{ for all } b \in B, y \in A\} = C.$$

Thus, $C \cong_g \text{End}_T A$. By Prop. 2.34, $\text{End}_T A$, hence also C , is graded simple, with center $Z(C) = Z(\text{End}_T A) = Z(T) = Z(B)$. Furthermore, formula (2.17) gives

$$[A:F]^2 = [T:F] \cdot [\text{End}_T A:F] = [B:F] \cdot [A:F] \cdot [C:F],$$

which yields (ii). Clearly $B \subseteq C_A(C)$. But since C is graded simple, part (ii) applies with C replacing B . Thus,

$$[C_A(C):F] \cdot [C:F] = [A:F] = [C:F] \cdot [B:F],$$

which yields $C_A(C) = B$ by dimension count. This proves (i).

For (iii), the inclusions $B \hookrightarrow A$ and $C \hookrightarrow A$ with images that commute yield a graded ring homomorphism $\sigma: B \otimes_{Z(B)} C \rightarrow A$ given by $\sum b_i \otimes c_i \mapsto \sum b_i c_i$. Let $B \cdot C$ denote $\text{im}(\sigma)$, which is the graded subalgebra of A generated by B and C . So, using (i),

$$C_A(B \cdot C) = C_A(B) \cap C_A(C) = C_A(B) \cap B = Z(B).$$

Since $B \otimes_{Z(B)} C$ is graded simple by Prop. 2.32, σ is injective, so

$$B \cdot C \cong_g B \otimes_{Z(B)} C.$$

This isomorphism shows that $B \cdot C$ is graded simple, and hence, by (i),

$$B \cdot C = C_A(C_A(B \cdot C)) = C_A(Z(B)).$$

Thus, $B \otimes_{Z(B)} C \cong_g B \cdot C = C_A(Z(B))$, which is the first isomorphism of (iii). The second isomorphism of (iii) is a special case of the first, as $C_A(F) = A$. \square

The graded Double Centralizer Theorem gives an upper bound on the dimensions of graded subfields of a graded simple algebra:

Corollary 2.36. *Let A be a central simple graded F -algebra, and let K be a graded subfield of A with $F \subseteq K$. Then,*

- (i) $[K:F] \mid \deg A$;
- (ii) *If A is a graded division algebra, then K is a maximal graded subfield of A if and only if $[K:F] = \deg A$.*

Proof. Let $C = C_A(K) \supseteq K$. By Th. 2.35(ii),

$$[K:F]^2 \mid [K:F] \cdot [C:F] = [A:F] = (\deg A)^2. \quad (2.18)$$

This yields (i), and it shows that if $[K:F] = \deg A$, then K is a maximal subfield of A . (This could also be proved by passing to $q(F)$ and invoking Lemma 2.14.) For the other implication in (ii), assume A is a graded division algebra and suppose $[K:F] < \deg A$. Then, (2.18) shows that $[C:F] > \deg A > [K:F]$; hence, $C \not\supseteq K$. Therefore, there is a nonzero homogeneous $c \in C \setminus K$. The commutative graded subalgebra $K[c]$ of A has no zero divisors, since A has none, and $K[c]$ is finite-dimensional over F . Hence, $K[c]$ is a graded subfield of A strictly containing K , showing that K is not a maximal graded subfield of A . \square

The graded version of the Skolem–Noether Theorem is more delicate than its ungraded counterpart. Graded isomorphisms of simple graded subalgebras of a central simple graded algebra A will be shown always to be induced by an inner automorphism of A . But what is really desired is a *graded* inner automorphism of A , which is only assured if one conjugates by a homogeneous unit of A . This is not always possible, but we will clarify exactly when it can be done. Thus, part (i) of the next theorem is general, but of little value. Parts (ii) and (iii) are the useful analogues of the Skolem–Noether Theorem in the graded setting.

Theorem 2.37. *Let A be a central simple graded F -algebra. Let B be a simple graded subalgebra of A , let $C = C_A(B)$ and $Z = Z(B)$. Let $f: B \rightarrow A$ be a graded F -algebra homomorphism. Then,*

- (i) *There is a $t \in A^\times$ with $f(b) = tbt^{-1}$ for all $b \in B$.*
- (ii) *If C is a graded division ring, then the t of part (i) can be chosen to be homogeneous in A .*
- (iii) *The t in part (i) can be chosen to be homogeneous if and only if there is a graded homomorphism $g: C \rightarrow A$ such that $g|_Z = f|_Z$ and $g(C)$ centralizes $f(B)$ in A .*

Proof. Let $T = B \otimes_F A^{\text{op}}$, which is a simple graded algebra by Prop. 2.32. We have two ways to view A as a graded B - A -bimodule, hence a left graded T -module. The first action is given, as usual, by $(\sum b_i \otimes a_i^{\text{op}}) \cdot y = \sum b_i y a_i$. The second is given by $(\sum b_i \otimes a_i^{\text{op}}) \cdot y = \sum f(b_i) y a_i$. To distinguish these two module structures, we let A denote A with the first T -action and write A' for A with the second T -action. Let S be a simple left graded T -module. By Prop. 2.24 and Th. 2.26,

$$A \cong_g \bigoplus_{i=1}^k S(\gamma_i) \quad \text{and} \quad A' \cong_g \bigoplus_{j=1}^{\ell} S(\delta_j)$$

for some $\gamma_1, \dots, \gamma_k, \delta_1, \dots, \delta_{\ell} \in \Gamma$. Clearly $\ell = k$ by dimension count. As usual, we write T^{\natural} for T with its grading ignored, and likewise $A^{\natural}, A'^{\natural}, S^{\natural}$ for A, A', S with gradings suppressed. Then, as T^{\natural} -modules we have $A^{\natural} \cong \bigoplus_{i=1}^k S^{\natural} \cong A'^{\natural}$. Let $h: A^{\natural} \rightarrow A'^{\natural}$ be a T^{\natural} -module isomorphism. This means

$$h(bya) = f(b)h(y)a \quad \text{for all } b \in B, y, a \in A. \quad (2.19)$$

Let $t = h(1)$ and $s = h^{-1}(1)$. By setting $b = a = 1$ in (2.19), we obtain $h(a) = ta$ for all $a \in A$. Using this, then setting $a = y = 1$ in (2.19), we have

$$tb = h(b) = h(b \cdot 1 \cdot 1) = f(b)t \quad \text{for all } b \in B. \quad (2.20)$$

Now, $1 = h(s) = st$. Also, $h(ts - 1) = tst - t = t(st - 1) = 0$. Since h is injective, this shows $ts = 1$. Thus, $t \in A^{\times}$ and $s = t^{-1}$. Formula (2.20) yields $f(b) = tbt^{-1}$ for all $b \in B$, proving (i).

For (ii), suppose C is a graded division ring. Since $s = t^{-1}$, we have

$$f(b) = tbt^{-1} = s^{-1}bs$$

for $b \in B$. By combining this with (2.20), we see that

$$tb = f(b)t \quad \text{and} \quad bs = sf(b) \quad \text{for all } b \in B. \quad (2.21)$$

Since $st = 1 \neq 0$ there must be homogeneous components s_{δ} of s and t_{γ} of t with $s_{\delta}t_{\gamma} \neq 0$. Let $\delta = \deg s_{\delta}$ and $\gamma = \deg t_{\gamma}$. For any $\varepsilon \in \Gamma_B$, take any $b_{\varepsilon} \in B_{\varepsilon}$. Then $f(b_{\varepsilon}) \in B_{\varepsilon}$ as f is a graded homomorphism. The $(\gamma + \varepsilon)$ -component in the equation $tb_{\varepsilon} = f(b_{\varepsilon})t$ yields $t_{\gamma}b_{\varepsilon} = f(b_{\varepsilon})t_{\gamma}$. Since every $b \in B$ is the sum of its homogeneous components, we obtain

$$t_{\gamma}b = f(b)t_{\gamma} \quad \text{and, likewise,} \quad bs_{\delta} = s_{\delta}f(b) \quad \text{for all } b \in B. \quad (2.22)$$

Thus, $s_{\delta}t_{\gamma}b = s_{\delta}f(b)t_{\gamma} = bs_{\delta}t_{\gamma}$ for all $b \in B$; hence, $s_{\delta}t_{\gamma} \in C_A(B) = C$. Since $s_{\delta}t_{\gamma}$ is homogeneous in the graded division ring C , we have $s_{\delta}t_{\gamma} \in C^{\times} \subseteq A^{\times}$. Therefore, $t_{\gamma} \in A^{\times}$, and (2.22) shows $f(b) = t_{\gamma}bt_{\gamma}^{-1}$ for all $b \in B$, proving (ii).

(iii): Suppose there is a homogeneous unit $t \in A^{\times}$ with $f(b) = tbt^{-1}$ for all $b \in B$. Define $g: C \rightarrow A$ by $g(c) = tct^{-1}$. Because t is homogeneous, g is a graded homomorphism. For all $b \in B, c \in C$, and $z \in Z$, we have

$$f(b)g(c) = tbt^{-1} = tcbt^{-1} = g(c)f(b)$$

and $g(z) = f(z)$. Thus, g has the desired properties.

Conversely, suppose there is a $g: C \rightarrow A$ as described in (iii). Then, there is a well-defined graded homomorphism $\mu: B \otimes_Z C \rightarrow A$ given by $\mu(\sum b_i \otimes c_i) = \sum f(b_i)g(c_i)$. By the graded Double Centralizer Theorem 2.35 there is a graded isomorphism

$$\eta: B \otimes_Z C \longrightarrow C_A(Z), \quad \sum b_i \otimes c_i \mapsto \sum b_i c_i.$$

Let $f' = \mu \circ \eta^{-1}: C_A(Z) \rightarrow A$, a graded homomorphism. Since B is graded simple, its center Z is a graded field, so also graded simple. Therefore, the graded Double Centralizer Theorem 2.35 applied to Z shows that $C_A(Z)$ is graded simple and $C_A(C_A(Z)) = Z$. Therefore, by parts (i) and (ii), there is a homogeneous unit $t \in A^\times$ with $f'(y) = tyt^{-1}$ for all $y \in C_A(Z)$. Then, $f(b) = btb^{-1}$ for all $b \in B$, as $f'|_B = f$. \square

Corollary 2.38. *Let A be a central simple graded F -algebra. Then every graded F -automorphism of A has the form $\text{int}(t)$ for some homogeneous unit $t \in A^\times$.*

Proof. This is immediate from Th. 2.37(ii) since $C_A(A) = F$, which is a graded field. \square

The following example illustrates the need for the added conditions in (ii) and (iii) of Th. 2.37 in order to assure that one can conjugate by a homogeneous unit.

Example 2.39. Let F be a graded field with $\Gamma_F = \mathbb{Z}$. We have the graded F -vector spaces $V = F(0) \oplus F(\frac{1}{2})$, $W = V$, and $W' = F(0) \oplus F(0)$. It is immediate from the definitions that $F(\gamma) \otimes_F F(\delta) \cong_g F(\gamma + \delta)$ for any $\gamma, \delta \in \Gamma$. Hence,

$$V \otimes_F W \cong_g F(0) \oplus F(\frac{1}{2}) \oplus F(\frac{1}{2}) \oplus F(1) \cong_g F(0) \oplus F(\frac{1}{2}) \oplus F(0) \oplus F(\frac{1}{2}) \cong_g V \otimes_F W'.$$

Using Prop. 2.9, let

$$A = M_4(F)(0, \frac{1}{2}, 0, \frac{1}{2}) \cong_g \text{End}_F(V \otimes_F W) \cong_g \text{End}_F(V \otimes_F W'),$$

$$B = M_2(F)(0, \frac{1}{2}) \cong_g \text{End}_F V,$$

$$C = B \cong_g \text{End}_F W,$$

$$C' = M_2(F)(0, 0) \cong_g \text{End}_F W'.$$

Then, $C \not\cong_g C'$, since $\Gamma_C = \frac{1}{2}\mathbb{Z}$ while $\Gamma_{C'} = \mathbb{Z}$. Note that the F -bilinear map $\text{End}_F V \times \text{End}_F W \rightarrow \text{End}_F(V \otimes_F W)$ given by $(g, h) \mapsto g \otimes h$ induces a well-defined graded F -algebra homomorphism

$$\psi: \text{End}_F V \otimes_F \text{End}_F W \longrightarrow \text{End}_F(V \otimes_F W).$$

This map is injective since its domain is graded simple by Prop. 2.17 and Prop. 2.32, and it is surjective by dimension count; hence, ψ is a graded

isomorphism. Thus, there are graded isomorphisms $\mu: B \otimes_F C \xrightarrow{\sim} A$ and $\eta: B \otimes_F C' \xrightarrow{\sim} A$ which are the compositions

$$\begin{aligned} B \otimes_F C &\xrightarrow{\sim} \text{End}_F V \otimes_F \text{End}_F W \xrightarrow{\sim} \text{End}_F(V \otimes_F W) \xrightarrow{\sim} A, \quad \text{and} \\ B \otimes_F C' &\xrightarrow{\sim} \text{End}_F V \otimes_F \text{End}_F W' \xrightarrow{\sim} \text{End}_F(V \otimes_F W') \xrightarrow{\sim} A. \end{aligned}$$

Now, B is graded simple by Prop. 2.17, so $\mu(B)$ is a simple graded subalgebra of A . The graded homomorphism $f = \eta \circ \mu^{-1}|_{\mu(B)}: \mu(B) \rightarrow A$ has image $\eta(B)$. Since $C_{B \otimes_F C}(B) = C$ and $C_{B \otimes_F C'}(B) = C'$ by Prop. 2.30, the isomorphisms μ and η yield $C_A(\mu(B)) = \mu(C)$ and $C_A(\eta(B)) = \eta(C')$. By Th. 2.37(i) there is a $t \in A^\times$ with $f(y) = tyt^{-1}$ for all $y \in \mu(B)$. But there can be no such t that is homogeneous. For, if t were homogeneous, $\text{int}(t)$ would be a graded automorphism of A , which would yield a graded isomorphism of centralizers, $\mu(C) \cong_g \eta(C')$. This cannot occur, as $\mu(C) \cong_g C \not\cong_g C' \cong_g \eta(C')$.

2.3 Degree zero elements in simple graded algebras

The focus in this section is on the degree zero component of a central simple graded algebra, which is described explicitly in §2.3.1; see Prop. 2.41. We also give an explicit description of the group of degrees of invertible homogeneous elements, and of its action on the center of the degree zero component; see Prop. 2.44. As a result, we characterize in Prop. 2.45 the grade group of the associated graded division algebra. In §2.3.2, we consider in particular graded algebras obtained by scalar extension from their zero component. These algebras, which are said to be *inertial*, have a particularly simple behavior under tensor products.

2.3.1 The grade group action

For a central simple graded algebra A , the grade set Γ_A and the homogeneous component A_0 of degree zero have special connections, which are made explicit in this subsection. We start with the case of graded division rings, which need not be finite-dimensional over any subfield.

For a graded division ring D , any $x \in D^\times$ is homogeneous, hence the inner automorphism $\text{int}(x)$ preserves the grading and induces an automorphism $\text{int}(x)_0$ of D_0 . This automorphism restricts to an automorphism of $Z(D_0)$ fixing $Z(D)_0$. The restriction is the identity when $x \in D_0^\times$. Since D_0^\times is the kernel of the degree homomorphism $\deg: D^\times \rightarrow \Gamma_D$, we have a well-defined canonical homomorphism

$$\theta_D: \Gamma_D \longrightarrow \text{Aut}(Z(D_0)/Z(D)_0) \quad (2.23)$$

such that

$$\theta_D(\deg(x)) = \text{int}(x)_0|_{Z(D_0)} \quad \text{for } x \in D^\times.$$

The analogy between this canonical homomorphism and the canonical homomorphism (1.1) of a valuation is illustrated in the following proposition (cf. Prop. 1.5(v)).

Proposition 2.40. *Let D be a graded division ring with center F . The fixed ring of $Z(D_0)$ under the automorphisms in $\theta_D(\Gamma_D)$ is F_0 . Hence, if $[D_0:F_0] < \infty$, then $Z(D_0)$ is abelian Galois over F_0 , and $\text{im}(\theta_D)$ is the entire Galois group $\mathcal{G}(Z(D_0)/F_0)$.*

Proof. By the definition of θ_D , every element b of $Z(D_0)$ lying in the fixed ring of $\text{im}(\theta_D)$ commutes with all nonzero homogeneous elements of D . Since the homogeneous elements additively generate D , we have $b \in Z(D)_0 = F_0$. If $[D_0:F_0] < \infty$ then $[Z(D_0):F_0] < \infty$, and it follows by Galois theory that $Z(D_0)$ is Galois over F_0 and $\text{im}(\theta_D) = \mathcal{G}(Z(D_0)/F_0)$. This Galois group is abelian since it is a homomorphic image of the abelian group Γ_D . \square

Now, let A be a (finite-dimensional) simple graded F -algebra, let J be a minimal nonzero homogeneous left ideal of A , and let $D = \text{End}_A J$, the graded division algebra associated to A . In view of Th. 2.26, we identify A with $\text{End}_D J$. For convenience, assume A is central, i.e., $F = Z(A) = Z(D)$. We let $n = \dim_D J$.

While A is graded simple, A_0 need not be simple. The structure of A_0 is determined by the canonical decomposition of J as a right graded D -vector space, as follows: let $\Gamma_1, \dots, \Gamma_k$ be the distinct cosets of Γ_D in Γ_J , and for each Γ_i choose a coset representative γ_i , so

$$\Gamma_J = \Gamma_1 \cup \dots \cup \Gamma_k \quad \text{with} \quad \Gamma_i = \gamma_i + \Gamma_D.$$

Take the canonical decomposition of J as in (2.2),

$$J = J_1 \oplus \dots \oplus J_k, \quad \text{where} \quad J_i = \bigoplus_{\gamma \in \Gamma_i} J_\gamma.$$

For $i = 1, \dots, k$, let

$$r_i = \dim_D J_i, \quad \text{so} \quad J_i \cong_g D(\gamma_i)^{r_i} \quad \text{and} \quad \sum_{i=1}^k r_i = n. \quad (2.24)$$

The decomposition $J = \bigoplus_{i=1}^k J_i \cong_g \bigoplus_{i=1}^k D(\gamma_i)^{r_i}$ yields a matrix representation of $A = \text{End}_D J$:

$$A \cong_g M_n(D) \left(\underbrace{\gamma_1, \dots, \gamma_1}_{r_1}, \underbrace{\gamma_2, \dots, \gamma_2}_{r_2}, \dots, \underbrace{\gamma_k, \dots, \gamma_k}_{r_k} \right). \quad (2.25)$$

In $A = \text{End}_D J$, let e_i be the projection on J_i parallel to $\bigoplus_{j \neq i} J_j$, so

$$e_i|_{J_i} = \text{id}_{J_i} \quad \text{and} \quad e_i|_{J_j} = 0 \text{ for } j \neq i.$$

Each e_i is homogeneous of degree 0 and idempotent. Set

$$\begin{aligned} B_i &= e_i A_0 e_i = \{f \in A_0 \mid f(J_i) \subseteq J_i, \text{ and } f(J_j) = 0 \text{ for } j \neq i\} \\ &\cong (\text{End}_{D_0} J_i)_0. \end{aligned} \quad (2.26)$$

Proposition 2.41. *The idempotents e_1, \dots, e_k are the primitive central idempotents of A_0 , and*

$$A_0 = B_1 \oplus \dots \oplus B_k \cong M_{r_1}(D_0) \times \dots \times M_{r_k}(D_0).$$

Therefore,

$$Z(A_0) = Z(A_0)e_1 \oplus \dots \oplus Z(A_0)e_k \cong Z(D_0) \times \dots \times Z(D_0).$$

Proof. Note that A_0 consists of the grade-preserving endomorphisms of J . For $i \neq j$ we have $\Gamma_{J_i} \cap \Gamma_{J_j} = \Gamma_i \cap \Gamma_j = \emptyset$. So, any $f \in A_0$ must map each J_i to itself, and hence $A_0 = \bigoplus_{i=1}^k B_i$. To see that e_1, \dots, e_k are the primitive central idempotents in A_0 , it now suffices to prove that B_i is simple for each i . Let $(b_{ij})_{j=1}^{r_i}$ be a D_0 -base of J_{γ_i} ; it is also a D -base of J_i by Prop. 2.5. Thus, we may identify J_i with $J_{\gamma_i} \otimes_{D_0} D$. Any map in $\text{End}_D(J_i)_0$ is degree-preserving, and hence maps J_{γ_i} to itself. Thus, there is a ring homomorphism $(\text{End}_D J_i)_0 \rightarrow \text{End}_{D_0}(J_{\gamma_i})$, given by $g \mapsto g|_{J_{\gamma_i}}$. This has an inverse given by the map sending $h \in \text{End}_{D_0}(J_{\gamma_i})$ to $h \otimes id_D \in \text{End}_D(J_{\gamma_i} \otimes_{D_0} D)$. Thus,

$$(\text{End}_D J_i)_0 \cong \text{End}_{D_0}(J_{\gamma_i}) \cong M_{r_i}(D_0),$$

which shows that B_i is simple and yields the isomorphisms of the proposition. (In the matrix representation (2.25), A_0 is realized in block diagonal form, with i -th block $M_{r_i}(D_0)$, because whenever $i \neq j$ we have $D(\gamma_i - \gamma_j)_0 = D_{\gamma_i - \gamma_j} = \{0\}$. The idempotent e_i is identified with the identity matrix of the i -th block.) \square

Corollary 2.42. *Let S be a semisimple graded F -algebra. Then, S_0 is semisimple.*

Proof. Since S is graded semisimple, by Th. 2.23 $S = A_1 \times \dots \times A_k$, where each A_i is graded simple. Then S_0 is the direct product of the degree-0 parts of the A_i , each of which is semisimple, by Prop. 2.41. Hence, S_0 is semisimple. \square

Proposition 2.41 also yields a convenient criterion for when A is a graded division algebra:

Corollary 2.43. *A finite-dimensional simple graded F -algebra A is a graded division algebra if and only if A_0 is a division ring.*

Proof. If A is a graded division algebra then A_0 is a division ring, as noted in Prop. 2.3(v). Conversely, suppose A_0 is a division ring. Then, in the notation of Prop. 2.41, $k = 1$ and $r_1 = 1$, so $n = r_1 = 1$, and $A \cong_g M_1(D) \cong_g D$, where D is a graded division ring. \square

We have seen in Prop. 2.40 that there is an action of Γ_D on $Z(D_0)$ given by the epimorphism $\theta_D: \Gamma_D \rightarrow \mathcal{G}(Z(D_0)/F_0)$. We now describe the corresponding map for the simple graded algebra A . It is more complicated because the homogeneous elements of A need not all be units and A_0 need not be simple. We continue with the notation preceding Prop. 2.41. Since $\Gamma_J = \bigcup_{i=1}^k (\gamma_i + \Gamma_D)$, we have as in Cor. 2.11,

$$\Gamma_A = \bigcup_{i,j} (\gamma_i - \gamma_j + \Gamma_D), \quad (2.27)$$

which need not be a group. Recall from (2.1) that

$$\Gamma_A^\times = \{\deg(a) \mid a \in A^\times \text{ and } a \text{ is homogeneous}\}. \quad (2.28)$$

Clearly, Γ_A^\times is a subgroup of Γ with

$$\Gamma_D = \Gamma_D^\times \subseteq \Gamma_A^\times \subseteq \Gamma_A, \quad \text{and} \quad |\Gamma_A^\times / \Gamma_D| \leq k^2 < \infty.$$

For each homogeneous $a \in A^\times$, its inner automorphism $\text{int}(a)$ preserves the homogeneous components of A , so is a graded automorphism of A ; thus, $\text{int}(a)$ restricts to an automorphism of A_0 and of $Z(A_0)$. Define

$$\theta_A: \Gamma_A^\times \longrightarrow \text{Aut}(Z(A_0)/F_0) \quad (2.29)$$

by

$$\theta_A(\gamma) = \text{int}(a)|_{Z(A_0)} \quad \text{for any homogeneous } a \in A^\times \text{ with } \deg(a) = \gamma. \quad (2.30)$$

This θ_A is a well-defined group homomorphism since homogeneous units of degree 0 act trivially on $Z(A_0)$ by conjugation. To help understand Γ_A^\times and θ_A , we partition Γ_J according to the r_i of (2.24): let ℓ_1, \dots, ℓ_m be the distinct values in $\{r_1, \dots, r_k\}$. For $t = 1, \dots, m$, let

$$S_t = \{i \in \{1, \dots, k\} \mid r_i = \ell_t\} \quad \text{and} \quad \Sigma_t = \bigcup_{i \in S_t} \Gamma_i.$$

Thus,

$$\{1, \dots, k\} = S_1 \cup \dots \cup S_m \quad \text{and} \quad \Gamma_J = \Sigma_1 \cup \dots \cup \Sigma_m.$$

Proposition 2.44.

(i) *We have*

$$\Gamma_A^\times = \{\varepsilon \in \Gamma \mid \varepsilon + \Sigma_t = \Sigma_t \text{ for } 1 \leq t \leq m\}. \quad (2.31)$$

Hence, Γ_A^\times is the largest subgroup of Γ such that each Σ_t is a union of cosets of Γ_A^\times . Also, Γ_A is a union of cosets of Γ_A^\times (though Γ_A^\times need not be the largest group with this property; see Ex. 2.46 below).

(ii) *Γ_A^\times acts by translation on the set of cosets $\Gamma_J / \Gamma_D = \{\Gamma_1, \dots, \Gamma_k\}$, sending each Σ_t / Γ_D to itself. Each coset Γ_i has stabilizer Γ_D under this action, so its orbit has size $|\Gamma_A^\times / \Gamma_D|$. The action of Γ_A^\times on the central*

primitive idempotents e_1, \dots, e_k of $Z(\mathbf{A}_0)$ via $\theta_{\mathbf{A}}$ corresponds to its action on $\Gamma_1, \dots, \Gamma_k$: for $\varepsilon \in \Gamma_{\mathbf{A}}^{\times}$ and $i, j \in \{1, \dots, k\}$, we have

$$\varepsilon + \Gamma_i = \Gamma_j \quad \text{if and only if} \quad \theta_{\mathbf{A}}(\varepsilon)(e_i) = e_j.$$

Proof. Take any $\varepsilon \in \Gamma_{\mathbf{A}}^{\times}$ and any $f \in \mathbf{A}^{\times} \cap \mathbf{A}_{\varepsilon}$; recall that $\mathbf{A}_{\varepsilon} = \text{End}_{\mathbf{D}}(\mathbf{J})(\varepsilon)$. Take any canonical component \mathbf{J}_i of \mathbf{J} and any nonzero homogeneous $z \in \mathbf{J}_i$. Then $f(z)$ is homogeneous and nonzero as f is bijective, say $f(z) \in \mathbf{J}_j$. Since $\varepsilon + \deg z = \deg f(z) \in \Gamma_j$, we have $\varepsilon + \Gamma_i = \Gamma_j$. Thus f , which shifts all degrees by ε , maps every homogeneous element of \mathbf{J}_i to \mathbf{J}_j ; hence, $f(\mathbf{J}_i) \subseteq \mathbf{J}_j$. As f is injective, this yields

$$r_i = \dim_{\mathbf{D}} \mathbf{J}_i = \dim_{\mathbf{D}} f(\mathbf{J}_i) \leq \dim_{\mathbf{D}} \mathbf{J}_j = r_j.$$

Since $f^{-1} \in \mathbf{A}^{\times} \cap \mathbf{A}_{-\varepsilon}$ and $-\varepsilon + \Gamma_j = \Gamma_i$, the same argument shows that $f^{-1}(\mathbf{J}_j) \subseteq \mathbf{J}_i$ and $r_j \leq r_i$. Therefore, $r_j = r_i$, which shows that \mathbf{J}_i and \mathbf{J}_j lie in the same piece Σ_t of $\Gamma_{\mathbf{J}}$.

Let $\tau_{\varepsilon}: \Gamma/\Gamma_{\mathbf{D}} \rightarrow \Gamma/\Gamma_{\mathbf{D}}$ be the translation-by- ε map given by $\gamma + \Gamma_{\mathbf{D}} \mapsto \varepsilon + \gamma + \Gamma_{\mathbf{D}}$. Clearly τ_{ε} is bijective, and we have just shown that τ_{ε} maps each $\Sigma_t/\Gamma_{\mathbf{D}}$ into itself; necessarily $\tau_{\varepsilon}(\Sigma_t) = \Sigma_t$ as $|\Sigma_t/\Gamma_{\mathbf{D}}| < \infty$. Thus, for any $\varepsilon \in \Gamma_{\mathbf{A}}^{\times}$, we have $\varepsilon + \Sigma_t = \Sigma_t$ for $t = 1, \dots, m$.

Conversely, take any $\rho \in \Gamma$ with $\rho + \Sigma_t = \Sigma_t$ for all t . For each t and $i \in S_t$, there is some $\pi(i) \in S_t$ such that $\rho + \Gamma_i = \Gamma_{\pi(i)}$. Since i and $\pi(i)$ lie in the same S_t , we have $r_i = r_{\pi(i)}$, hence $\dim_{\mathbf{D}} \mathbf{J}_i = \dim_{\mathbf{D}} \mathbf{J}_{\pi(i)}$. For the ρ -shift $\mathbf{J}_i(\rho)$ we have

$$\Gamma_{\mathbf{J}_i(\rho)} = \rho + \Gamma_i = \Gamma_{\pi(i)} \quad \text{and} \quad \dim_{\mathbf{D}} \mathbf{J}_i(\rho) = \dim_{\mathbf{D}} \mathbf{J}_i = \dim_{\mathbf{D}} \mathbf{J}_{\pi(i)}.$$

Therefore, there is a graded \mathbf{D} -vector space isomorphism $g_i: \mathbf{J}_i(\rho) \rightarrow \mathbf{J}_{\pi(i)}$. Thus, $g_i \in \text{Hom}_{\mathbf{D}}(\mathbf{J}_i, \mathbf{J}_{\pi(i)})(\rho)$ and g_i is bijective. Choose such a g_i for each \mathbf{J}_i , and let

$$g = (g_1, \dots, g_k) \in \text{Hom}_{\mathbf{D}}\left(\bigoplus_{i=1}^k \mathbf{J}_i, \bigoplus_{i=1}^k \mathbf{J}_{\pi(i)}\right)(\rho).$$

Since the injective translation-by- ρ map τ_{ρ} sends $\Gamma_{\mathbf{J}}/\Gamma_{\mathbf{D}} = \bigcup_{t=1}^m \Sigma_t/\Gamma_{\mathbf{D}}$ to itself, π is a permutation of $\{1, \dots, k\}$; hence,

$$\bigoplus_{i=1}^k \mathbf{J}_{\pi(i)} = \mathbf{J} = \bigoplus_{i=1}^k \mathbf{J}_i.$$

Therefore, $g \in \text{End}_{\mathbf{D}}(\mathbf{J})(\rho) = \mathbf{A}_{\rho}$. Moreover, g is bijective since each g_i is an isomorphism, hence $g \in \mathbf{A}^{\times}$. Thus, $\rho \in \Gamma_{\mathbf{A}}^{\times}$. This proves (2.31), from which it is immediate that each Σ_t is a union of cosets of $\Gamma_{\mathbf{A}}^{\times}$, and $\Gamma_{\mathbf{A}}^{\times}$ is the largest group with this property.

For any $\gamma \in \Gamma_{\mathbf{A}}$ and $\varepsilon \in \Gamma_{\mathbf{A}}^{\times}$, choose a nonzero $h \in \mathbf{A}_{\gamma}$ and $g \in \mathbf{A}^{\times} \cap \mathbf{A}_{\varepsilon}$. Then $0 \neq hg \in \mathbf{A}_{\gamma+\varepsilon}$, so $\gamma + \varepsilon \in \Gamma_{\mathbf{A}}$. This shows that $\Gamma_{\mathbf{A}}$ is a union of cosets of $\Gamma_{\mathbf{A}}^{\times}$.

(ii) For each $\varepsilon \in \Gamma_{\mathbf{A}}^{\times}$ we have seen that the translation-by- ε map τ_{ε} sends each $\Sigma_t/\Gamma_{\mathbf{D}}$ to itself. Clearly, for any $\gamma \in \Gamma$, $\tau_{\varepsilon}(\gamma + \Gamma_{\mathbf{D}}) = \gamma + \Gamma_{\mathbf{D}}$ if and only

if $\varepsilon \in \Gamma_D$. Thus, for the group action of Γ_A^\times on Γ_J/Γ_D by translation, each coset Γ_i has stabilizer group Γ_D , so it has orbit size $|\Gamma_A^\times/\Gamma_D|$.

Take any $\varepsilon \in \Gamma_A^\times$ and any $h \in A^\times \cap A_\varepsilon$. We have seen that h permutes J_1, \dots, J_k : for $i = 1, \dots, k$ there is a $j \in \{1, \dots, k\}$ such that $h(J_i) = J_j$, where $\Gamma_j = \varepsilon + \Gamma_i$. It follows that $he_i = e_jh$, hence $\theta_A(\varepsilon)(e_i) = e_j$ when $\tau_\varepsilon(\Gamma_i) = \Gamma_j$, proving (ii). \square

By using the action of Γ_A^\times on the idempotents of $Z(A_0)$, we can also recover Γ_D as a subgroup of Γ_A^\times . For the following statement, fix an isomorphism of F_0 -algebras

$$\varphi: Z(A_0) \xrightarrow{\sim} Z(D_0) \times \dots \times Z(D_0).$$

We use φ to associate to every $\sigma \in \mathcal{G}(Z(D_0)/F_0)$ the automorphism $d(\sigma)$ of $Z(A_0)$ defined by

$$d(\sigma)(\varphi^{-1}(z_1, \dots, z_k)) = \varphi^{-1}(\sigma(z_1), \dots, \sigma(z_k)) \quad \text{for } z_1, \dots, z_k \in Z(A_0).$$

Since $\mathcal{G}(Z(D_0)/F_0)$ is abelian by Prop. 2.40, the automorphism $d(\sigma)$ does not depend on the choice of φ : this is because for any other isomorphism φ' we may find a permutation π of $\{1, \dots, k\}$ and $\tau_1, \dots, \tau_k \in \mathcal{G}(Z(D_0)/F_0)$ such that

$$\varphi'^{-1}(z_1, \dots, z_k) = \varphi^{-1}(\tau_1(z_{\pi(1)}), \dots, \tau_k(z_{\pi(k)})) \quad \text{for } z_1, \dots, z_k \in Z(A_0).$$

Then

$$\begin{aligned} d(\sigma)(\varphi'^{-1}(z_1, \dots, z_k)) &= \varphi^{-1}(\tau_1\sigma(z_{\pi(1)}), \dots, \tau_k\sigma(z_{\pi(k)})) \\ &= \varphi'^{-1}(\sigma(z_1), \dots, \sigma(z_k)). \end{aligned}$$

Thus, the map

$$d: \mathcal{G}(Z(D_0)/F_0) \longrightarrow \text{Aut}(Z(A_0)/F_0), \quad \sigma \mapsto d(\sigma) \quad (2.32)$$

is a group homomorphism that does not depend on the choice of φ .

Proposition 2.45. *With the same notation as in Prop. 2.41 and 2.44,*

(i) $\Gamma_D = \{\gamma \in \Gamma_A^\times \mid \theta_A(\gamma)(e_1) = e_1\} = \dots = \{\gamma \in \Gamma_A^\times \mid \theta_A(\gamma)(e_k) = e_k\}$.

(ii) *There is a commutative diagram*

$$\begin{array}{ccc} \Gamma_D & \xhookrightarrow{i} & \Gamma_A^\times \\ \theta_D \downarrow & & \downarrow \theta_A \\ \mathcal{G}(Z(D_0)/F_0) & \xhookrightarrow{d} & \text{Aut}(Z(A_0)/F_0), \end{array}$$

where $i: \Gamma_D \rightarrow \Gamma_A^\times$ is the inclusion map and d is the map (2.32).

Proof. Take any $\varepsilon \in \Gamma_A^\times$. From Prop. 2.44(ii), we know that for any $i \in \{1, \dots, k\}$ the equation $\theta_A(\varepsilon)(e_i) = e_i$ holds if and only if $\tau_\varepsilon(\Gamma_i) = \Gamma_i$, if and only if $\varepsilon \in \Gamma_D$. This proves (i).

(ii) Take any homogeneous D -vector space base (b_1, \dots, b_n) of J . For $c \in D$, define $f_c \in A$ by

$$f_c \left(\sum_{i=1}^n b_i d_i \right) = \sum_{i=1}^n b_i c d_i$$

for $d_i \in D$. The map $c \mapsto f_c$ is an injective graded ring homomorphism $D \rightarrow A$. We identify $Z(D_0)$ with $Z(B_i)$ by $z \mapsto e_i f_z$. For any $\delta \in \Gamma_D$, choose a nonzero $c \in D_\delta$; so $f_c \in A^\times \cap A_\delta$. Then, for $z \in Z(D_0)$, we have $\theta_D(\delta)(z) = c z c^{-1}$ while $\theta_A(\delta) = \text{int}(f_c)|_{Z(A_0)}$. Since $\delta \in \Gamma_D$, we have seen in (i) that $\theta_A(\delta)(e_i) = e_i$ for each i . So,

$$\theta_A(\delta)(e_i f_z) = e_i f_c f_z f_c^{-1} = e_i f_{c z c^{-1}} = e_i f_{\theta_D(\delta)(z)}.$$

Thus, the diagram in (ii) is commutative. \square

Example 2.46. Let D be a graded division algebra with $\Gamma_D = \mathbb{Z}$. Consider the graded D -vector space $J = D \oplus D(\frac{1}{3})$. We have $\Gamma_J = \mathbb{Z} \cup (\frac{1}{3} + \mathbb{Z})$ and J has canonical components $J_1 \cong_g D$ and $J_2 \cong_g D(\frac{1}{3})$; so, $\Gamma_1 = \mathbb{Z}$, $\Gamma_2 = \frac{1}{3} + \mathbb{Z}$, $r_1 = r_2 = 1$, and $\Sigma_1 = \Gamma_J$. For $A = \text{End}_D J$ we have $\Gamma_A^\times = \Gamma_D = \mathbb{Z}$, since no larger subgroup of Γ translates Γ_J to itself. But

$$\Gamma_A = \mathbb{Z} \cup (\frac{1}{3} + \mathbb{Z}) \cup (-\frac{1}{3} + \mathbb{Z}) = \frac{1}{3}\mathbb{Z},$$

which is a group strictly containing Γ_A^\times .

When A_0 is simple, the structure is less complicated:

Proposition 2.47. *For any central simple graded F -algebra A , the following conditions are equivalent:*

- (a) A_0 is simple.
- (b) $\Gamma_A = \Gamma_D$.
- (c) $\dim_{D_0} A_0 = \dim_D A$.
- (d) $Z(A_0)$ is a field.

When these conditions hold, $A \cong_g M_n(D)$ with the standard grading on $M_n(D)$ (as in (2.6)), and

$$A_0 \cong M_n(D_0), \quad \Gamma_A^\times = \Gamma_A = \Gamma_D, \quad \text{and} \quad \theta_A = \theta_D.$$

Proof. Suppose Γ_J is a single coset $\gamma + \Gamma_D$ of Γ_D . Then, the canonical decomposition of J has only one component, J itself, and conditions (a)–(d) hold by Prop. 2.41 and (2.27). Furthermore, by Prop. 2.9,

$$A \cong_g M_n(D)(\gamma, \dots, \gamma) \cong_g M_n(D)(0, \dots, 0),$$

which is $M_n(D)$ with its standard grading as in (2.6). Since $\Gamma_D \subseteq \Gamma_A^\times \subseteq \Gamma_A$, we must then have $\Gamma_A^\times = \Gamma_A$, and $\theta_A = \theta_D$ by Prop. 2.45. On the other hand, if $|\Gamma_J/\Gamma_D| > 1$, then by Prop. 2.41 and (2.27) none of (a)–(d) holds. \square

We can also identify when $\Gamma_A^\times = \Gamma_A$:

Proposition 2.48. *For any central simple graded F -algebra A , the following conditions are equivalent:*

- (a) $\Gamma_A^\times = \Gamma_A$.
- (b) Γ_A^\times acts transitively on the primitive central idempotents of A_0 .
- (c) $Z(A_0)$ is a Galois étale F_0 -algebra with group $\text{im}(\theta_A)$.
- (d) Γ_J is a coset of a group and all the r_i are equal. (The group is then Γ_A^\times .)

When these equivalent conditions hold,

$$[A:F] = [A_0:F_0] \cdot |\Gamma_A:\Gamma_F|. \quad (2.33)$$

Proof. We will repeatedly use Prop. 2.44 and 2.45 without specific mention.

(a) \Rightarrow (b) Suppose $\Gamma_A^\times = \Gamma_A$. For any two cosets $\gamma_i + \Gamma_D$ and $\gamma_j + \Gamma_D$ in Γ_J/Γ_D , let $\varepsilon = \gamma_j - \gamma_i \in \Gamma_A$ (see (2.27)). Then, $\varepsilon \in \Gamma_A^\times$ by hypothesis, and $\varepsilon + (\gamma_i + \Gamma_D) = \gamma_j + \Gamma_D$. Hence, Γ_A^\times acts transitively by translation on Γ_J/Γ_D , so its equivalent action on the simple components of $Z(A_0)$ is also transitive.

(b) \Rightarrow (c) By using a diagonal embedding of D in A as in the proof of Prop. 2.45(ii), we have

$$Z(A_0) = Z(A_0)e_1 \oplus \dots \oplus Z(A_0)e_k$$

with each $Z(A_0)e_i \cong Z(D_0)$. Recall from Prop. 2.40 that the field $Z(D_0)$ is Galois over F_0 with Galois group $\text{im}(\theta_D)$. Hence, $Z(A_0)$ is étale over F_0 . Let

$$G = \text{im}(\theta_A) \subseteq \text{Aut}_{F_0}(Z(A_0)).$$

For $Z(A_0)$ to be G -Galois over F_0 we need that $|G| = \dim_{F_0} Z(A_0)$ and the fixed ring $Z(A_0)^G$ is F_0 (cf. Knus et al. [115, Def. (18.15)]). We have $|\text{im}(\theta_D)| = [Z(D_0):F_0]$. Since Γ_A^\times acts transitively on $\{e_1, \dots, e_k\}$, and the stabilizer of each e_i is Γ_D , we have $|\Gamma_A^\times:\Gamma_D| = k$ and $\ker(\theta_A) \subseteq \Gamma_D$. The diagonal action of Γ_D on $Z(A_0)$ via θ_A then shows that $\ker(\theta_A) = \ker(\theta_D)$. Thus,

$$|G| = |\text{im}(\theta_A)| = |\Gamma_A^\times:\Gamma_D| \cdot |\text{im}(\theta_D)| = k [Z(D_0):F_0] = \dim_{F_0} Z(A_0).$$

Now take any $e_1 z_1 + \dots + e_k z_k \in Z(A_0)^G$ with each $z_i \in Z(D_0)$. From the diagonal action of Γ_D on $Z(A_0)$ via θ_A , each z_i lies in $Z(D_0)^{\text{im}(\theta_D)} = F_0$. Because of the transitive F_0 -linear action of G on $\{e_1, \dots, e_k\}$, we have $z_1 = \dots = z_k$, so

$$e_1 z_1 + \dots + e_k z_k = (e_1 + \dots + e_k) z_1 = z_1 \in F_0.$$

Thus, $Z(A_0)^G = F_0$ and $Z(A_0)$ is G -Galois over F_0 .

(c) \Rightarrow (b) Suppose $Z(A_0)$ is Galois over F_0 with group $\text{im}(\theta_A)$. If the orbit of e_1 under the action of Γ_A^\times is $\{e_{i_1}, \dots, e_{i_q}\}$, then the nonzero idempotent $e_{i_1} + \dots + e_{i_q}$ of $Z(A_0)$ is Γ_A^\times -stable so lies in the field $Z(A_0)^{\text{im}(\theta_A)}$. Hence, $e_{i_1} + \dots + e_{i_q} = 1$, so $\{i_1, \dots, i_q\} = \{1, \dots, k\}$. This shows that Γ_A^\times acts transitively on the e_i .

(b) \Rightarrow (d) If Γ_A^\times acts transitively on the e_i , then its corresponding translation action on the cosets $\Gamma_1, \dots, \Gamma_k$ of Γ_D is transitive. So, $\Gamma_J = \Gamma_1 \cup \dots \cup \Gamma_k$ must be a single coset of Γ_A^\times . Since $\Gamma_J = \coprod_{t=1}^m \Sigma_t$ and each Σ_t is a union of cosets of Γ_A^\times , there can be only one Σ_t . Hence $r_1 = \dots = r_k$.

(d) \Rightarrow (a) Suppose Γ_J is a coset of a group Ω and $r_1 = \dots = r_k$. The equality of the r_i implies that there is only one Σ_t , which must then be all of Γ_J . Hence,

$$\Gamma_A^\times = \{\varepsilon \in \Gamma \mid \varepsilon + \Gamma_J = \Gamma_J\} = \Omega.$$

So, Γ_J is a coset of Γ_A^\times , which shows that $\Gamma_A^\times = \{\gamma - \delta \mid \gamma, \delta \in \Gamma_J\} = \Gamma_A$ (see (2.27)).

Suppose conditions (a)–(d) hold. Then, for each $\gamma \in \Gamma_A = \Gamma_A^\times$ there is a $c_\gamma \in A^\times \cap A_\gamma$. Hence, the map $A_0 \rightarrow A_\gamma$ given by $a \mapsto c_\gamma a$ is an F_0 -vector space isomorphism. If $\Gamma_A = \bigcup_{i=1}^k (\gamma_i + \Gamma_F)$ (disjoint union), then by Prop. 2.5,

$$[A:F] = \sum_{i=1}^k \dim_{F_0} A_{\gamma_i} = k [A_0:F_0] = |\Gamma_A:\Gamma_F| \cdot [A_0:F_0]. \quad \square$$

Note that the equality (2.33) fails to hold in Ex. 2.46 where $\Gamma_A^\times \neq \Gamma_A$, $[A:F] = 4$, $[A_0:F_0] = 2$, and $|\Gamma_A:\Gamma_F| = 3$.

2.3.2 Inertial graded algebras

One easy way to build a graded algebra over a graded field F is by scalar extension from algebras over F_0 ; we already met this construction in Prop. 2.16 and consider here an important special case.

Lemma 2.49. *For a finite-dimensional graded F -algebra A , the following conditions are equivalent:*

- (a) $A = A_0 \otimes_{F_0} F$ (with each $A_\gamma = A_0 \cdot F_\gamma$).
- (b) $[A:F] = [A_0:F_0]$.
- (c) $\Gamma_A = \Gamma_F$.

When they hold, we have $\Gamma_A^\times = \Gamma_A = \Gamma_F$.

Proof. (a) \Rightarrow (b) is clear. (b) \Rightarrow (a) Let $B = A_0 \cdot F = A_0 \otimes_{F_0} F$, which is a graded F -subalgebra of A with each $B_\gamma = A_0 \otimes_{F_0} F_\gamma$ and $[B:F] = [A_0:F_0]$. If (b) holds, then $[B:F] = [A:F] < \infty$, which shows that $A = B = A_0 \otimes_{F_0} F$.

(b) \Leftrightarrow (c) Write Γ_A as a disjoint union of Γ_F -cosets, $\Gamma_A = \bigcup_{i \in I} (\gamma_i + \Gamma_F)$ with, say, $\gamma_j = 0$. By Prop. 2.5,

$$[A:F] = [A_0:F_0] + \sum_{i \neq j} \dim_{F_0} A_{\gamma_i},$$

with each $\dim_{F_0} A_{\gamma_i} \geq 1$. Hence, $[A:F] = [A_0:F_0]$ if and only if Γ_A contains no other cosets of Γ_F but Γ_F itself.

When (c) holds, we have $\Gamma_F \subseteq \Gamma_A^\times \subseteq \Gamma_A = \Gamma_F$, hence equality holds throughout. \square

Definition 2.50. Let F be a graded field. A (finite-dimensional) semisimple graded F -algebra A is said to be *inertial (over F)* if the equivalent conditions of Lemma 2.49 hold and $Z(A_0)$ is a separable (= étale) algebra over the field F_0 ; i.e., $Z(A_0)$ is a direct product of separable field extensions of F_0 . This is an analogue for graded algebras of the notion of an inertial extension in valuation theory, and it encompasses the finite-dimensional case of inertial graded field extensions that will be considered in §5.1.3. Corollary 2.42 shows that A_0 is a semisimple F_0 -algebra when A is inertial over F . Then $Z(A_0) \cong L_1 \times \dots \times L_k$, where each L_i is a field finite-dimensional over F_0 . The condition that $Z(A_0)$ be separable over F_0 is equivalent to: each field L_i is separable over F_0 .

Conversely, if A is a finite-dimensional graded F -algebra for which the conditions in Lemma 2.49 hold, then Lemma 2.13 shows that A is semisimple, hence inertial over F , if A_0 is semisimple and $Z(A_0)$ is separable over F_0 .

Remark 2.51. If A is an inertial graded F -algebra, then the graded subalgebras B of A all have the form $B = B_0 \otimes_{F_0} F$, where B_0 is an F_0 -subalgebra of A_0 . Note that $Z(A) = Z(A_0) \otimes_{F_0} F$. Also, A is graded simple if and only if A_0 is simple, if and only if $Z(A_0)$ is a field: see Prop. 2.16 and 2.47.

Proposition 2.52. *Let A be a simple graded F -algebra with associated graded division algebra D . Then A is inertial over F if and only if D is inertial over F and $A \cong_g M_n(D)$ for some integer n , with the standard grading. When this occurs, we have*

$$A_0 \cong M_n(D_0), \quad \Gamma_A = \Gamma_A^\times = \Gamma_D = \Gamma_F, \quad \text{and} \quad \theta_A = \theta_D.$$

Proof. By the graded version of Wedderburn's Theorem (Th. 2.26), we have $A \cong_g \text{End}_D(J)$ for some right graded D -vector space J . Let $n = \dim_D J$. If A is inertial over F , then

$$\Gamma_F \subseteq \Gamma_D \subseteq \Gamma_A^\times \subseteq \Gamma_A = \Gamma_F,$$

hence $\Gamma_A = \Gamma_A^\times = \Gamma_D = \Gamma_F$. Proposition 2.47 then shows that $A \cong_g M_n(D)$, $A_0 \cong M_n(D_0)$, and $\theta_A = \theta_D$. It follows that $Z(D_0) \cong Z(A_0)$, which is separable over F_0 , hence D is inertial over F . Conversely, if D is inertial over F , then for $A = M_n(D)$ we have $\Gamma_A = \Gamma_D = \Gamma_F$ and $Z(A_0) \cong Z(D_0)$, so A is inertial over F . \square

Example 2.53. Let n be a positive integer, and let F be a graded field such that F_0 contains a primitive n -th root of unity ω (so $\text{char } F_0$ is 0 or prime to n), and let $a, b \in F_0^\times$. Let S be the symbol algebra $(a, b/F_0)_n$, a central simple F_0 -algebra of degree n . Then, $S \otimes_{F_0} F$ is the graded symbol algebra $(a, b/F)_n$ as in Def. 2.18 above. Thus, $(a, b/F)_n$ is an inertial graded F -algebra with degree 0 component S .

Inertial graded algebras are well-behaved with respect to tensor products and scalar extensions:

Proposition 2.54. *If A and B are inertial graded F -algebras, then $A \otimes_F B$ is also inertial over F with $(A \otimes_F B)_0 = A_0 \otimes_{F_0} B_0$.*

Proof. We have

$$A \otimes_F B = (A_0 \otimes_{F_0} F) \otimes_F (B_0 \otimes_{F_0} F) = (A_0 \otimes_{F_0} B_0) \otimes_{F_0} F. \quad (2.34)$$

So, $(A \otimes_F B)_0 = A_0 \otimes_{F_0} B_0$. Let

$$Z = Z((A \otimes_F B)_0) = Z(A_0 \otimes_{F_0} B_0) = Z(A_0) \otimes_{F_0} Z(B_0).$$

Then, Z is a separable F_0 -algebra since $Z(A_0)$ and $Z(B_0)$ are each separable F_0 -algebras. Say $Z \cong L_1 \times \dots \times L_k$, for fields L_1, \dots, L_k , each separable over F_0 . Each L_i is an algebra over $Z(A_0)$ and over $Z(B_0)$, and $A_0 \otimes_{Z(A_0)} L_i \otimes_{Z(B_0)} B_0$ is a central simple L_i -algebra. So, as

$$\begin{aligned} A_0 \otimes_{F_0} B_0 &= A_0 \otimes_{Z(A_0)} (Z(A_0) \otimes_{F_0} Z(B_0)) \otimes_{Z(B_0)} B_0 \\ &\cong A_0 \otimes_{Z(A_0)} (L_1 \times \dots \times L_k) \otimes_{Z(B_0)} B_0 \\ &\cong \prod_{i=1}^k A_0 \otimes_{Z(A_0)} L_i \otimes_{Z(B_0)} B_0, \end{aligned}$$

$A_0 \otimes_{F_0} B_0$ is semisimple. Moreover, (2.34) yields $\Gamma_{A \otimes B} = \Gamma_F = \Gamma_{A \otimes B}^\times$. Hence, by (2.34) and Lemma 2.49, $A \otimes_F B$ is graded semisimple. Thus, $A \otimes_F B$ is inertial over F . \square

Proposition 2.55. *Let A be a semisimple graded F -algebra which is inertial over F , and let K be any graded field extension of F . Then, $A \otimes_F K$ is an inertial K -algebra with $(A \otimes_F K)_0 = A_0 \otimes_{F_0} K_0$.*

Proof. Since A is inertial, we have

$$A \otimes_F K = A_0 \otimes_{F_0} K = (A_0 \otimes_{F_0} K_0) \otimes_{K_0} K \quad \text{and} \quad \Gamma_{A \otimes_F K} = \Gamma_K.$$

Therefore,

$$(A \otimes_F K)_0 = A_0 \otimes_{F_0} K_0 \quad \text{and} \quad Z((A \otimes_F K)_0) = Z(A_0) \otimes_{F_0} K_0.$$

Since $Z(A_0)$ is separable over F_0 , it follows that $Z((A \otimes_F K)_0)$ is separable over K_0 . Moreover, A_0 is semisimple by Cor. 2.42, hence $A_0 \otimes_{F_0} K_0$ is semisimple. By Lemma 2.13, it follows that $A \otimes_F K$ is semisimple, because $\Gamma_{A \otimes_F K}^\times = \Gamma_{A \otimes_F K}$ as $\Gamma_K \subseteq \Gamma_{A \otimes_F K}^\times \subseteq \Gamma_{A \otimes_F K} = \Gamma_K$. Therefore, $A \otimes_F K$ is inertial over K . \square

Proposition 2.56. *Let A be a central simple graded F -algebra which is inertial over F , and let B be any simple graded F -algebra. Let $C = A \otimes_F B$, which is a simple graded F -algebra. Then,*

$$\begin{aligned} C_0 &= A_0 \otimes_{F_0} B_0, \quad \Gamma_C = \Gamma_B, \quad Z(C_0) = Z(B_0), \\ \Gamma_C^\times &= \Gamma_B^\times, \quad \text{and} \quad \theta_C = \theta_B. \end{aligned}$$

Proof. The graded simplicity of C is given by Prop. 2.32. Since

$$C = A \otimes_F B = A_0 \otimes_{F_0} F \otimes_F B = A_0 \otimes_{F_0} B,$$

we have $C_\gamma = A_0 \otimes_{F_0} B_\gamma$, for each $\gamma \in \Gamma$. Hence, $C_0 = A_0 \otimes_{F_0} B_0$ and $\Gamma_C = \Gamma_B$. Since $F = Z(A) = Z(A_0) \otimes_{F_0} F$, we must have $Z(A_0) = F_0$, and hence

$$Z(C_0) = Z(A_0) \otimes_{F_0} Z(B_0) = Z(B_0).$$

If $\gamma \in \Gamma_B^\times$, then there is some $b \in B_\gamma \cap B^\times$. Then $1 \otimes b \in C_\gamma \cap C^\times$, so $\gamma \in \Gamma_C^\times$ and

$$\theta_C(\gamma) = \text{int}(1 \otimes b)|_{Z(C_0)} = \text{int}(b)|_{Z(B_0)} = \theta_B(\gamma).$$

Hence, $\Gamma_B^\times \subseteq \Gamma_C^\times$ and $\theta_C|_{\Gamma_B^\times} = \theta_B$.

To prove that $\Gamma_B^\times = \Gamma_C^\times$, it suffices to show that $\Gamma_C^\times \subseteq \Gamma_B^\times$. Let $\gamma \in \Gamma_C^\times$. The homogeneous component $C_\gamma = A_0 \otimes_{F_0} B_\gamma$ then contains an element u that is invertible in C . Let $m = [A:F] = [A_0:F_0]$. Note that A_0 is central simple over F_0 , by Remark 2.51. Therefore, we have F_0 -algebra isomorphisms

$$A_0^{\text{op}} \otimes_{F_0} A_0 \cong \text{End}_{F_0} A_0^{\text{op}} \cong M_m(F_0).$$

(Compare Lemma 6.3 for the graded analogue.) By tensoring with A_0^{op} , we obtain isomorphisms

$$A_0^{\text{op}} \otimes_{F_0} C = A_0^{\text{op}} \otimes_{F_0} A_0 \otimes_{F_0} B \cong_g M_m(F_0) \otimes_{F_0} B = M_m(B),$$

for the standard grading on $M_m(B)$. Now, $1 \otimes u \in A_0^{\text{op}} \otimes_{F_0} C_\gamma \cong M_m(B)_\gamma$. This element is invertible in $M_m(B)$ since u is invertible in C , hence $\gamma \in \Gamma_{M_m(B)}^\times$. Suppose $B \cong_g \text{End}_E J$ for some graded division algebra E and some right graded E -vector space J . Then $M_m(B) \cong_g \text{End}_E(J^m)$. Clearly, $\Gamma_{J^m} = \Gamma_J$ and if $J = \bigoplus_{i=1}^k J_i$ is the canonical decomposition of J determined by the cosets of Γ_E in Γ_J as in (2.2), then the canonical decomposition of J^m is $\bigoplus_{i=1}^k J_i^m$. Since each $\Gamma_{J_i^m} = \Gamma_{J_i}$ and $\dim_E J_i^m = m \dim_E J_i$, Γ_{J^m} has the same pieces Σ_t as Γ_J in the partition of Γ_J by the dimensions of the J_i as in Prop. 2.44. Hence, $\Gamma_{M_m(B)}^\times = \Gamma_B^\times$ by the characterization in Prop. 2.44(i), and it follows that $\gamma \in \Gamma_B^\times$. Thus, $\Gamma_C^\times \subseteq \Gamma_B^\times$. When combined with the first paragraph results, this yields $\Gamma_C^\times = \Gamma_B^\times$ and $\theta_C = \theta_B$. \square

We now consider scalar extension of central simple graded algebras by inertial graded field extensions.

Proposition 2.57. *Let L be an inertial graded field extension of a graded field F , and let A be a central simple graded F -algebra. Then,*

$$(A \otimes_F L)_0 = A_0 \otimes_{F_0} L_0, \quad Z((A \otimes_F L)_0) = Z(A_0) \otimes_{F_0} L_0,$$

$$\Gamma_{A \otimes_F L} = \Gamma_A, \quad \Gamma_{A \otimes_F L}^\times = \Gamma_A^\times.$$

Moreover, the following diagram commutes:

$$\begin{array}{ccc} \Gamma_A^\times & \xlongequal{\quad} & \Gamma_{A \otimes L}^\times \\ \theta_A \downarrow & & \downarrow \theta_{A \otimes L} \\ \text{Aut}(Z(A_0)/F_0) & \xrightarrow{\iota} & \text{Aut}(Z(A \otimes_F L)_0/L_0) \end{array}$$

where

$$\iota(\tau) = \tau \otimes \text{id}_{L_0} \quad \text{for } \tau \in \text{Aut}(Z(A_0)/F_0).$$

Proof. Let $T = A \otimes_F L = A \otimes_{F_0} L_0$. It is clear that $T_0 = A_0 \otimes_{F_0} L_0$ and hence $Z(T_0) = Z(A_0) \otimes_{F_0} L_0$. Since A is a graded subalgebra of T , we have

$$\Gamma_A \subseteq \Gamma_T \subseteq \Gamma_A + \Gamma_L = \Gamma_A + \Gamma_F = \Gamma_A,$$

so $\Gamma_T = \Gamma_A$, and $\Gamma_A^\times \subseteq \Gamma_T^\times$. To get equality in the last inclusion, let $B = \text{End}_F(L)$, and view L as a graded F -subalgebra of B by the left regular representation. Since L is inertial over F , we have $L \cong_g F^n$ as graded F -vector spaces where $n = [L:F]$. Hence, B is an inertial central simple graded F -algebra. Therefore, Prop. 2.56 shows that $\Gamma_{A \otimes_F B}^\times = \Gamma_A^\times$. But the inclusions of graded algebras $A \subseteq T \subseteq B \otimes_F A$ yield $\Gamma_A^\times \subseteq \Gamma_T^\times \subseteq \Gamma_{A \otimes_F B}^\times$. Hence, $\Gamma_T^\times = \Gamma_A^\times$. So, for any $\gamma \in \Gamma_T^\times$ there exists an $a \in A_\gamma \cap A^\times$. Then $a \otimes 1 \in T_\gamma \cap T^\times$, so as $\theta_A(\gamma) = \text{int}_A(a)|_{Z(A_0)}$, we have

$$\theta_T(\gamma) = \text{int}_T(a \otimes 1)|_{Z(T_0)} = (\text{int}_A(a)|_{Z(A_0)}) \otimes \text{id}_{L_0} = \iota(\theta_A(\gamma)).$$

Thus, $\theta_T = \iota \circ \theta_A$. □

Our final result in this section demonstrates how the action of the grade group of a graded division algebra on the homogeneous component of degree zero behaves under an inertial graded field extension.

We let D be a central graded division algebra over a graded field F and consider an inertial graded field extension L of F (possibly of infinite degree). Recall from Prop. 2.40 that $Z(D_0)$ is a Galois field extension of F_0 with abelian Galois group. Let $G = \mathcal{G}(Z(D_0)/F_0)$. With G acting on the left factor, the étale L_0 -algebra $Z(D_0) \otimes_{F_0} L_0$ is G -Galois. Since G is abelian, any element that fixes any one of the primitive idempotents of $Z(D_0) \otimes_{F_0} L_0$ fixes each of the idempotents. Let $H \subseteq G$ be the subgroup of all such stabilizing elements. There is a field extension K of L_0 such that

$$Z(D_0) \otimes_{F_0} L_0 \cong K \times \dots \times K. \quad (2.35)$$

The extension K/L_0 is Galois with Galois group isomorphic to H . The isomorphism $H \xrightarrow{\sim} \mathcal{G}(K/L_0)$ is obtained via the inclusions

$$H \subseteq G \subseteq \text{Aut}((Z(D_0) \otimes_{F_0} L_0)/L_0))$$

and the isomorphism (2.35), but it does not depend on the choice of this isomorphism: see the discussion preceding Prop. 2.45. The isomorphism (2.35)

yields F_0 -embeddings of $Z(D_0)$ in K . Since $Z(D_0)$ is Galois over F_0 , all these embeddings have the same image, and K can be viewed as the field compositum of L_0 and $Z(D_0)$. Since moreover G is abelian, the intersection $Z(D_0) \cap L_0$ in K is independent of the choice of embedding $Z(D_0) \hookrightarrow K$, and we have by Galois theory

$$H \cong \mathcal{G}(K/L_0) \cong \mathcal{G}(Z(D_0)/(Z(D_0) \cap L_0)).$$

Proposition 2.58. *Use the notation above, and let E be the graded division algebra associated to $D \otimes_F L$. Then,*

- (i) E_0 is the associated division algebra of $D_0 \otimes_{Z(D_0)} K$.
- (ii) $\Gamma_E = \theta_D^{-1}(H) \subseteq \Gamma_D$, and there is a commutative diagram:

$$\begin{array}{ccc} \Gamma_E & \hookrightarrow & \Gamma_D \\ \theta_E \downarrow & & \downarrow \theta_D \\ \mathcal{G}(K/L_0) & \hookrightarrow & G \end{array}$$

where the upper horizontal map is the inclusion and the lower horizontal map is the composition $\mathcal{G}(K/L_0) \xrightarrow{\sim} H \hookrightarrow G$.

Proof. Proposition 2.41 shows that E_0 is the division algebra associated to each simple factor of the semisimple algebra $(D \otimes_F L)_0$. By Prop. 2.57, we have

$$(D \otimes_F L)_0 = D_0 \otimes_{F_0} L_0.$$

By decomposing $D_0 \otimes_{F_0} L_0 = D_0 \otimes_{Z(D_0)} (Z(D_0) \otimes_{F_0} L_0)$ and using (2.35), we obtain

$$(D \otimes_F L)_0 \cong (D_0 \otimes_{Z(D_0)} K) \times \dots \times (D_0 \otimes_{Z(D_0)} K).$$

Therefore, the simple factors of $(D \otimes_F L)_0$ are isomorphic to $D_0 \otimes_{Z(D_0)} K$. Assertion (i) follows.

To prove (ii), note that by Prop. 2.57 we have $Z((D \otimes_F L)_0) = Z(D_0) \otimes_{F_0} L_0$ and $\Gamma_{D \otimes L}^\times = \Gamma_D^\times = \Gamma_D$, and the following diagram commutes:

$$\begin{array}{ccc} \Gamma_D & \xlongequal{\quad} & \Gamma_{D \otimes L}^\times \\ \theta_D \downarrow & & \downarrow \theta_{D \otimes L} \\ G & \xrightarrow{\iota} & \text{Aut}((Z(D_0) \otimes_{F_0} L_0)/L_0) \end{array} \quad (2.36)$$

Now, Prop. 2.45 shows that Γ_E consists of the $\gamma \in \Gamma_{D \otimes L}^\times$ such that $\theta_{D \otimes L}(\gamma)$ fixes the primitive idempotents of $Z(D_0) \otimes_{F_0} L_0$. Since diagram (2.36) commutes, these γ lie in Γ_D and can also be characterized by the condition that $\theta_D(\gamma) \in H$. Moreover, Prop. 2.45 also yields a commutative diagram

$$\begin{array}{ccc}
\Gamma_E & \hookrightarrow & \Gamma_{D \otimes L}^\times \\
\theta_E \downarrow & & \downarrow \theta_{D \otimes L} \\
\mathcal{G}(K/L_0) & \xrightarrow{d} & \text{Aut}((Z(D_0) \otimes_{F_0} L_0)/L_0)
\end{array}$$

Then (ii) follows by combining this diagram with (2.36). \square

Exercises

Exercise 2.1. Let D be a graded division ring.

- (i) Let $\gamma, \delta \in \Gamma$. Show that the shifted right graded D -vector spaces $D(\gamma)$ and $D(\delta)$ are isomorphic as graded D -modules if and only if $\gamma \equiv \delta \pmod{\Gamma_D}$.
- (ii) Let V be a right graded D -vector space and W a left graded D -vector space. Establish a canonical isomorphism of graded $Z(D)$ -vector spaces $V(\gamma) \otimes_D W(\delta) \cong_g (V \otimes_D W)(\gamma + \delta)$.

Exercise 2.2. Let D be a graded division ring and let $\delta_1, \dots, \delta_n, \varepsilon_1, \dots, \varepsilon_n \in \Gamma$. Show that $M_n(D)(\delta_1, \dots, \delta_n) \cong_g M_n(D)(\varepsilon_1, \dots, \varepsilon_n)$ if and only if there is a permutation σ of $\{1, \dots, n\}$ and a $\gamma \in \Gamma$ such that $\delta_i - \varepsilon_{\sigma(i)} \in \gamma + \Gamma_D$ for all $i = 1, \dots, n$.

Exercise 2.3. Let F be a field, let t be a commuting indeterminate over F , and let $A = F[t]\{x, y\}$, the free algebra in noncommuting indeterminates x and y over the polynomial ring $F[t]$. Thus, A has a base as a free $F[t]$ -module consisting of all words of finite length in x and y . Clearly $Z(A) = F[t]$. Let I be the two-sided ideal of A generated by tx , and let $B = A/I$, which is generated as an F -algebra by the images $\bar{t}, \bar{x}, \bar{y}$ of t, x, y . Show that $Z(B) = F[\bar{t}] \cong F[t]$, and $\bar{x} \neq 0$, but $\bar{t}\bar{x} = 0$. Thus, $q(Z(B)) \cong F(t)$, a rational function field over F , but B does not embed in its ring of central quotients $q(B) = B \otimes_{Z(B)} q(Z(B)) \cong B \otimes_{F[t]} F(t)$. In fact, $q(B) \cong F(\bar{t})[\bar{y}]$, a commutative polynomial ring in \bar{y} over the field $F(t)$. (Hint: There is a $\mathbb{Z} \times \mathbb{Z}$ -grading on A given by degree in t and total degree in x and y . Since I is a homogeneous ideal of A with respect to this grading, there is an induced grading on B . From this one can determine an F -vector space base of B .)

Exercise 2.4. This exercise gives the graded version of a standard identity for symbol algebras, cf. Draxl [63, Lemma 7, p. 81]. Let $n = n_1 n_2$ for some relatively prime integers $n_1, n_2 \geq 2$, and suppose F is a graded field such that F_0 contains a primitive n -th root of unity ω . If $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ satisfy $m_1 n_1 + m_2 n_2 = 1$, show that for all homogeneous elements $a, b \in F^\times$

$$(a, b/F)_{\omega, n} \cong_g (a^{m_2}, b/F)_{\omega^{n_2}, n_1} \otimes_F (a^{m_1}, b/F)_{\omega^{n_1}, n_2}.$$

Exercise 2.5. Use the same notation as in the preceding exercise, and let $\omega_1, \omega_2 \in F_0$ be primitive roots of unity of order n_1 and n_2 respectively. Show that for all homogeneous elements $a_1, b_1, a_2, b_2 \in F^\times$

$$(a_1, b_1/F)_{\omega_1, n_1} \otimes_F (a_2, b_2/F)_{\omega_2, n_2} \cong_g (a_1^{n_2} a_2^{n_1}, b_1^{n_2} b_2^{n_1}/F)_{\omega_1 \omega_2, n}.$$

Exercise 2.6. Let F be a graded field, let A be a central simple graded F -algebra, and let B be an arbitrary graded F -algebra. Show that there is a bijection between two-sided homogeneous ideals of $A \otimes_F B$ and two-sided homogeneous ideals of B , which maps $J \subseteq A \otimes B$ to $J \cap (1 \otimes B) \subseteq B$ and $K \subseteq B$ to $A \otimes_F K \subseteq A \otimes_F B$. (This result yields an alternative proof of Prop. 2.32.)

Exercise 2.7. Show that if A and B are finite-dimensional graded algebras over a graded field F , then $A \times B$ is inertial over F if and only if A and B are each inertial over F .

Exercise 2.8. Let A and B be (finite-dimensional) central graded division algebras over a graded field F . Assume that $\Gamma_A \cap \Gamma_B = \Gamma_F$.

- (i) Prove that $(A \otimes_F B)_0 = A_0 \otimes_{F_0} B_0$, $\Gamma_{A \otimes_F B} = \Gamma_{A \otimes_F B}^\times = \Gamma_A + \Gamma_B$ and that $\theta_{A \otimes_F B}: \Gamma_A + \Gamma_B \rightarrow \text{Aut}(Z(A)_0 \otimes_{F_0} Z(B)_0 / F_0)$ is given by $\gamma + \delta \mapsto \theta_A(\gamma) \otimes \theta_B(\delta)$, for all $\gamma \in \Gamma_A, \delta \in \Gamma_B$.

Let S be a separable closure of F_0 . Since $Z(A)_0$ (resp. $Z(B)_0$) is Galois over F_0 , we may identify it with its unique F_0 -isomorphic copy in S . Then the compositum $Z(A)_0 \cdot Z(B)_0$ and the intersection $Z(A)_0 \cap Z(B)_0$ are well-defined subfields of S . Moreover, as $Z(A)_0$ is abelian Galois over F_0 , the identification of $\mathcal{G}(Z(A)_0/F_0)$ with the Galois group of the image of $Z(A)_0$ in S is independent of the choice of F_0 -homomorphism $Z(A)_0 \hookrightarrow S$. Likewise for $Z(B)_0$. See the remarks preceding Prop. 2.45.

- (ii) Let D be the graded division algebra associated to $A \otimes_F B$. Prove that $Z(D)_0 \cong Z(A)_0 \cdot Z(B)_0$ and that D_0 is the division algebra associated to

$$A_0 \otimes_{Z(A)_0} (Z(A)_0 \cdot Z(B)_0) \otimes_{Z(B)_0} B_0.$$

Let $Z = Z(A)_0 \cap Z(B)_0 \subseteq S$. Recall (see Pierce [178, Lemma b, p. 256]) that since Z is Galois over F_0 , we have

$$Z \otimes_{F_0} Z \cong \prod_{\sigma \in \mathcal{G}(Z/F_0)} e_\sigma Z,$$

where the primitive idempotents $\{e_\sigma \mid \sigma \in \mathcal{G}(Z/F_0)\}$ of $Z \otimes_{F_0} Z$ are characterized by the condition that $e_\sigma(c \otimes 1) = e_\sigma(1 \otimes \sigma(c))$ for all $c \in Z$.

- (iii) Prove that $\Gamma_D = \{\gamma + \delta \mid \gamma \in \Gamma_A, \delta \in \Gamma_B, \text{ and } \theta_A(\gamma)|_Z = \theta_B(\delta)|_Z\}$. Hence $|\Gamma_A + \Gamma_B : \Gamma_D| = [Z:F_0]$.

Notes

Graded rings and modules are a classical topic, which is well-documented in the literature; see for example Bourbaki [31, § II.11, § III.3]. The idea to consider a graded ring in which nonzero homogeneous elements are invertible as a “graded field” can be traced back to Năstăsescu [169] in the special case where the grade group is \mathbb{Z} . Elaborating on this idea, it is natural to develop for semisimple graded algebras the analogue of the Wedderburn theory of semisimple algebras; this was done by Năstăsescu–Van Oystaeyen [170, § II.9] (for $\Gamma = \mathbb{Z}$); see also Boulagouaz [25] and Hwang–Wadsworth [103, §1]. The graded version of the Skolem–Noether Theorem (Th. 2.37) is due to Hwang–Wadsworth [103, Prop. 1.6]. The description of the zero-component of a simple graded algebra in §2.3 comes from Tignol–Wadsworth [246, §2].

Exercise 2.8 is a graded version of Morandi–Wadsworth [163, Cor. 3.12], which is for valued division algebras.

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Tignol, J.-P.; Wadsworth, A.R.

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