

# Preface

The theory of finite-dimensional division algebras witnessed several breakthroughs in the latter decades of the twentieth century. Important advances, such as Amitsur's construction of noncrossed product division algebras and Platonov's solution of the Tannaka–Artin problem, relied on an inventive use of valuation theory, applied in the context of noncommutative rings. The subsequent development of valuation theory for finite-dimensional division algebras led to significant simplifications of the initial results and to a host of new constructions of division algebras satisfying various conditions, which shed much light on the structure of these algebras. In this research area, valuation theory has become a standard tool, for which this book is intended to provide a useful reference.

The theory of valuations and valuation rings has been extended to division rings in several different ways. We treat here only the most stringent of these extensions, which is the one that has turned out to be most useful in applications. Thus, our valuations on division algebras are defined by the same axioms as the (Krull) valuations on fields; hence they restrict to a valuation in the classical sense on the center of the division ring. Yet, noncommutative valuation theory has some significant features that give it a different flavor from the commutative theory. Notably, there are many fewer valuations on division algebras than on fields: A valuation always extends from a field  $F$  to any field containing  $F$ ; often there are many such extensions. But if  $D$  is a division algebra with center  $F$  and finite-dimensional over  $F$ , a valuation on  $F$  extends to  $D$  if and only if it has a *unique* extension to every field between  $F$  and  $D$ . Thus, very often it has no extension to  $D$  at all. But if it does extend, then the extension is unique. Consequently, the presence of a valuation on a division algebra  $D$  is a rather special phenomenon. When this occurs, it often gives a great deal of information about  $D$  and its subalgebras that can be virtually inaccessible for most division algebras. For this reason, valuation theory has had some of its greatest success in the construction of examples, such as noncrossed product algebras and division algebras with nontrivial reduced Whitehead group  $\text{SK}_1$ .

Henselian valuations on fields and Henselizations play a role for general valuations analogous to that of complete valuations and completions for rank 1 valuations. Henselian valuations on the center are even more important in the noncommutative theory because a Henselian valuation on a field  $F$  has a unique extension to each field algebraic over  $F$ . Consequently, it extends (uniquely) to each division algebra finite-dimensional over  $F$ . Much of the work on valued division algebra has thus focussed on algebras over Henselian fields. Also, notable results on arbitrary valued division algebras have been obtained by first proving the Henselian case (e.g., “Ostrowski’s Theorem” on the defect of valued division algebras).

Another distinctive feature of valuation theory on division algebras is a greater complexity of the residue structure, and some notable interaction between the residue algebra and the value group: There is a canonical action of the value group of a valued division algebra on the center of its residue division algebra. This provides an important piece of information even in the most classical cases studied by Hasse in the 1930s, as it is related to the local invariant of division algebras over local fields.

When division algebras are being investigated, simple algebras with zero divisors frequently arise, e.g., as tensor products or scalar extensions of division algebras. Therefore, it has been a drawback for noncommutative valuation theory that valuations make sense only for division algebras: The basic axiom that

$$v(ab) = v(a) + v(b) \quad (*)$$

breaks down if  $ab = 0$  for nonzero  $a$  and  $b$ . A few years ago the authors found a way to address this difficulty by defining a more general notion of value function that we call a *gauge*, which can exist on a (finite-dimensional) semisimple algebra  $A$  over a field  $F$ , with respect to a valuation on  $F$ . For a function  $\alpha$  on  $A$  to be a gauge, we replace the multiplicative condition  $(*)$  for a valuation with the following *surmultiplicativity* condition:

$$\alpha(ab) \geq \alpha(a) + \alpha(b) \quad \text{for all } a, b \in A.$$

The filtration of  $A$  induced by  $\alpha$  yields an associated graded algebra  $\text{gr}(A)$ , and gauges are distinguished among surmultiplicative value functions by a condition on  $\text{gr}(A)$ : This graded algebra must be graded semisimple, which means that it has no nilpotent homogeneous ideals.

Gauges work remarkably well. They show good behavior with respect to tensor products of algebras and scalar extensions. Moreover, there are natural constructions of gauges on many symbol algebras, cyclic algebras, and crossed product algebras. Additionally, over a Henselian base field, the gauges on the endomorphism algebra of a vector space are exactly the operator norms that are familiar in functional analysis.

Even for valuations on division algebras associated graded structures prove particularly useful. They encapsulate all the information about the residue algebra, the value group, and the canonical action of the value group on the

center of the residue algebra. They should be regarded as a substantially enhanced analogue of the residue algebra. Usually, when one passes from a ring to an associated graded ring one obtains a simplified structure, but at the price of significant loss of information about the original ring. With valued division algebras, the graded ring is definitely simpler and easier to work with than the original valued algebra, with surprisingly little lost in the transition to the graded setting. Indeed, if the valuation on the center is Henselian, we will see that under mild tameness conditions (which hold automatically whenever the characteristic of the residue field is 0 or prime to the degree of the division algebra) the graded algebra  $\text{gr}(D)$  associated to a division algebra  $D$  determines  $D$  up to isomorphism; moreover, the graded subalgebras of  $\text{gr}(D)$  then classify the subalgebras of  $D$ .

Graded structures are thus central to our approach of valuation theory. Our general strategy is to prove results first in the graded setting, where the arguments are often easier and more transparent. With gauges at our disposal, the passage to the corresponding results for valued division algebras is often very quick. To take full advantage of this method, we build a solid foundation on graded algebras with grade set lying in a torsion-free abelian group. It is worth pointing out that, in contrast with the classical theory, which mostly deals with valuations with value group  $\mathbb{Z}$ , our valuations take their values in arbitrary totally ordered abelian groups. Valuations of higher rank (i.e., with value group not embeddable in  $\mathbb{R}$ ) allow a greater richness in the possible structure of the value group and of the residue algebra. Moreover, new phenomena occur, such as totally ramified division algebras and algebras with noncyclic center of the residue—these have been particularly important in the construction of significant examples.

The material in this book can be roughly divided into three parts, which we briefly outline below, referring to the introduction of each chapter for additional information.

The first part consists of Chapters 1–4. They lay the groundwork for the theory of valuations on finite-dimensional division algebras and its extension to the theory of gauges on finite-dimensional semisimple algebras. The first chapter introduces the fundamental notions associated with valuations on division algebras and provides assorted examples. We view a valuation on the algebra as an extension of a known valuation on its center. In Chapter 2, the focus shifts to graded structures with a torsion-free abelian grade group. Graded rings in which the nonzero homogeneous elements are invertible are called graded division rings, because they display properties that are strikingly similar to those of the usual division rings. We are thus led to introduce graded vector spaces, and we develop a graded analogue of the Wedderburn and Noether theory of simple algebras. In Chapter 3, we return to the theme of valuations, which we extend to vector spaces and algebras over valued fields in order to define gauges on semisimple algebras. This first part of the book culminates in Chapter 4 with a determination of the necessary and sufficient

condition for the existence of gauges. This condition involves the division algebras Brauer-equivalent to the simple components of the semisimple algebra after scalar extension to a Henselization of the base field: These division algebras must each be defectless, which means that their dimension over their center must be the product of the residue degree and the ramification index. In particular, gauges always exist when the residue characteristic is zero.

The second part, comprising Chapters 5–7, addresses various topics related to the Brauer group of valued fields. We first discuss graded field extensions in Chapter 5, and review properties of valued field extensions from the perspective of their associated graded field extensions. Brauer groups of graded fields and of valued fields form the subject of Chapter 6. Valuation-theoretic properties define an ascending sequence of three subgroups of the Brauer group  $Br(F)$  of a valued field: the inertial part  $Br_{in}(F)$ , the inertially split part  $Br_{is}(F)$ , and the tamely ramified part  $Br_{tr}(F)$ . We use gauges to relate these subgroups to corresponding subgroups of the Brauer group  $Br(\text{gr}(F))$  of the associated graded field  $\text{gr}(F)$ . The main result of this part of the book yields for a Henselian field  $F$  a canonical index-preserving isomorphism  $Br_{tr}(F) \xrightarrow{\sim} Br(\text{gr}(F))$  mapping the Brauer class of a tame division algebra  $D$  to the Brauer class of  $\text{gr}(D)$ . We can then easily read off information about the pieces of  $Br_{tr}(F)$  from the corresponding data about  $Br(\text{gr}(F))$ . The inertial, or unramified part of the Brauer group is canonically isomorphic to the Brauer group of the residue field:  $Br_{in}(F) \cong Br(\overline{F})$ . The inertially split part  $Br_{is}(F)$  consists of the classes of division algebras split by the maximal inertial (= unramified) extension field of  $F$ . We give a generalization of Witt’s classical description of the Brauer group of a complete discretely-valued field, in the form of a “ramification” isomorphism from the quotient  $Br_{is}(F)/Br_{in}(F)$  to a group of characters of the absolute Galois group of the residue field  $\overline{F}$ . The next quotient  $Br_{tr}(F)/Br_{is}(F)$  is described in Chapter 7, where division algebras totally ramified over their centers are thoroughly investigated. When the base field is Henselian, the properties of such algebras can be read off from the extension of value groups, with the help of a canonical alternating pairing with values in the group of roots of unity of the residue field. Since totally ramified division algebras arise only when the value group has rank at least 2, such algebras have been relatively less studied in the literature; yet their structure is very simple and explicit.

In the third part of the book, Chapters 8–12, we apply the preceding results to investigate the structure of division algebras over Henselian fields, and we present several applications. Following the same methodology as in previous chapters, in Chapter 8 we first consider the structure of graded division algebras; we then derive corresponding structure theorems for division algebras over Henselian fields by relating the algebra to its associated graded algebra. We thereby recover easily several results that have been previously established by much more complicated methods. Historically, a primary application of valuation theory has been in the construction of significant examples. Our last four chapters are devoted to the presentation of such examples. In Chap-

ter 9 we obtain information on the maximal subfields and splitting fields of valued division algebras, and construct noncyclic division algebras with pure maximal subfields, noncyclic  $p$ -algebras, and noncrossed product algebras. Examples of division algebras that do not decompose into tensor products of proper subalgebras are given in Chapter 10, and Chapter 11 discusses reduced Whitehead group computations: We show that if  $D$  is a division algebra tamely ramified over a Henselian field then  $SK_1(D) \cong SK_1(\text{gr}(D))$ . This leads to quick proofs of many formulas for  $SK_1(D)$ . Finally, we give in Chapter 12 a modified version of recent results of Merkurjev and Baek–Merkurjev using valuation theory to obtain lower bounds on the essential dimension of central simple algebras of given degree and exponent.

The assumed background for this book is acquaintance with the classical theory of central simple algebras, together with a basic knowledge of the valuation theory of fields, as given for example in Bourbaki, *Algèbre Commutative*, Ch. VI. For the convenience of the reader, we have included an appendix covering some of the more technical facts we need in commutative valuation theory, especially concerning Henselian valuations and Henselizations. The theoretical aspects developed throughout the book are illustrated by many examples, which are listed by chapter in another appendix.

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## A note on notation

As pointed out above, we compare throughout most of the book algebras over valued fields and graded algebras. As a visual aid to help the reader determine whether a given statement lies in the context of graded algebras, we use sans serif letters ( $A, F, V, \dots$ ) to designate graded structures and associated constructions. Thus, for instance  $\text{End}_D(V)$  denotes the graded algebra of endomorphisms of the graded vector space  $V$  over the graded division algebra  $D$ , and we write  $\text{gr}(D)$  for the graded algebra associated to the valued division algebra  $D$ , and  $\text{Br}(F)$  for the Brauer group of a graded field  $F$ .

The blackboard bold symbols  $\mathbb{C}, \mathbb{F}_q, \mathbb{N}, \mathbb{Q}, \mathbb{Q}_p, \mathbb{R}$ , and  $\mathbb{Z}$  have their customary meanings: the complex numbers, the finite field of cardinality  $q$ , the

nonnegative integers, the rational numbers, the  $p$ -adic completion of  $\mathbb{Q}$ , the real numbers, and the integers.

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