

Contents

1	Valuations on Division Rings	1
1.1	Basic definitions and examples	1
1.1.1	Valuations and associated structures	2
1.1.2	Examples: twisted Laurent series	3
1.1.3	Examples: iterated Laurent series	4
1.1.4	Examples: Mal'cev–Neumann series	7
1.2	Valuations on finite-dimensional division algebras	9
1.2.1	The fundamental inequality	9
1.2.2	Extension of a valuation from the center	10
1.2.3	Composite valuations	13
1.2.4	Examples: rational functions over a division algebra	15
1.2.5	Examples: Laurent series	16
1.2.6	Examples: iterated Laurent series	17
1.2.7	Examples: discrete rank 1 valuations	20
1.2.8	Examples: local fields	23
1.2.9	Example: the field of rational numbers	24
1.3	The graded ring associated to a valued division algebra	25
1.3.1	Definition	25
1.3.2	Examples	27
	Exercises	28
	Notes	30
2	Graded Algebra	33
2.1	Graded linear algebra	34
2.1.1	Graded rings	34
2.1.2	Graded modules	36
2.1.3	Homomorphisms	39
2.1.4	Tensor products	43
2.2	Wedderburn structure theory	44
2.2.1	Semisimple graded algebras and central quotients	45

2.2.2	The structure of semisimple graded algebras	50
2.2.3	The Wedderburn Theorem for simple graded algebras	53
2.2.4	Centralizers and simple subalgebras	57
2.3	Degree zero elements in simple graded algebras	64
2.3.1	The grade group action	64
2.3.2	Inertial graded algebras	72
	Exercises	78
	Notes	80
3	Value Functions	81
3.1	Value functions on vector spaces	81
3.1.1	Value functions and associated graded vector spaces	82
3.1.2	Value functions over maximally complete fields	85
3.1.3	Subspaces	88
3.1.4	Homomorphisms	90
3.1.5	Tensor products	94
3.2	Value functions on algebras	97
3.2.1	Surmultiplicativity	97
3.2.2	Gauges	103
3.2.3	Gauges on symbol algebras	107
3.2.4	Gauges on tensor products	113
3.2.5	Gauges from graded algebras	115
3.2.6	Residually simple gauges	120
3.2.7	Composition of value functions and gauges	124
	Exercises	130
	Notes	132
4	Existence and Fundamental Properties of Gauges	135
4.1	The Henselization of a division algebra	135
4.2	Defect	140
4.2.1	Defect of valued division algebras	140
4.2.2	Defectlessness of valuations in semisimple algebras	152
4.3	The structure of gauges	155
4.3.1	Gauges on algebras over Henselian fields	155
4.3.2	Gauges on central simple algebras	160
4.3.3	Gauges on commutative algebras	162
4.3.4	Gauges on simple and semisimple algebras	167
4.4	Existence of gauges on semisimple algebras	177
4.4.1	Descent of norms	178
4.4.2	Existence of gauges	183
	Exercises	189
	Notes	189

5	Graded and Valued Field Extensions	193
5.1	Graded field extensions	193
5.1.1	The quotient field	194
5.1.2	Algebraic extensions	195
5.1.3	Ramification in graded field extensions	206
5.1.4	Galois theory of graded fields	212
5.2	Valued fields from a graded perspective	221
5.2.1	Uniform polynomials	221
5.2.2	Tamely ramified extensions of valued fields	227
	Exercises	234
	Notes	236
6	Brauer Groups	239
6.1	The Brauer group of a graded field	239
6.1.1	Definition of $\text{Br}(F)$	240
6.1.2	Homological interpretation of $\text{Br}(F)$	245
6.1.3	The canonical Brauer group filtration (graded case)	252
6.2	The Brauer group of a valued field	261
6.2.1	Inertial gauges	262
6.2.2	Tame gauges	268
6.2.3	The canonical Brauer group filtration (valued case)	277
	Exercises	289
	Notes	293
7	Total Ramification	297
7.1	Symplectic modules	298
7.1.1	Alternating pairings on torsion abelian groups	298
7.1.2	The group $\text{Symp}(\Omega)$	302
7.1.3	Symplectic modules and second exterior powers	309
7.2	Tensor products of symbol algebras	317
7.2.1	Armatures	318
7.2.2	Armatures in graded algebras	330
7.2.3	Armature gauges	336
7.3	Total ramification in graded algebras	340
7.3.1	Totally ramified graded division algebras	341
7.3.2	The Brauer group of an inertially closed graded field	350
7.3.3	The top level of the Brauer group filtration	355
7.4	Total ramification in algebras over valued fields	360
7.4.1	Totally ramified division algebras	360
7.4.2	The tame Brauer group of a Henselian field	367
	Exercises	371
	Notes	373

8	Division Algebras over Henselian Fields	377
8.1	Tame and inertial lifts in valued division algebras	378
8.2	Canonical subalgebras	388
8.2.1	Canonical subalgebras of a graded division algebra	388
8.2.2	Canonical subalgebras of a tame division algebra	391
8.3	Semiramified algebras	394
8.3.1	The <i>Dec</i> subgroup of the Brauer group	395
8.3.2	Bicyclic algebras	397
8.3.3	Semiramified graded division algebras	405
8.3.4	Semiramified division algebras over Henselian fields	413
8.4	Brauer class factorizations	419
8.4.1	The $S \otimes T$ decomposition (graded case)	420
8.4.2	Inertially split graded algebras	425
8.4.3	The $S \otimes T$ decomposition (valued case)	433
8.4.4	Inertially split algebras over a Henselian field	435
8.4.5	Example: the \mathcal{S} construction	439
8.4.6	The case of discrete rank 1 valuations	441
	Exercises	444
	Notes	449
9	Subfields and Splitting Fields of Division Algebras	455
9.1	Splitting fields of tame division algebras	456
9.1.1	Inertial and inertially split algebras	456
9.1.2	Tame algebras	458
9.1.3	Example: the \mathcal{T} construction	461
9.2	Pure subfields in semiramified division algebras	463
9.2.1	Noncyclic algebras with pure maximal subfields	464
9.2.2	Pure totally ramified subfields	469
9.3	Galois subfields in p -algebras	472
9.4	Noncrossed product algebras	476
9.4.1	Noncrossed products of degree p^2	480
9.4.2	Noncrossed products of degree 8	482
9.4.3	Iterative construction of noncrossed products	484
	Exercises	486
	Notes	486
10	Indecomposable Division Algebras	491
10.1	Indecomposable algebras of nonprime exponent	492
10.2	Indecomposable algebras of prime exponent	501
10.2.1	Jacob's indecomposability criterion	501
10.2.2	Indecomposable algebras of exponent 2 and degree 8	507
10.2.3	Characteristic p calculations	510
10.2.4	Indecomposable algebras of odd prime exponent	515
10.3	Decompositions into tensor products of symbol algebras	521
	Exercises	529

Notes	531
11 Computation of $SK_1(D)$	535
11.1 SK_1 of graded division algebras	536
11.2 SK_1 of division algebras over Henselian fields	546
Exercise	554
Notes	555
12 The Essential Dimension of Central Simple Algebras	561
12.1 A brief introduction to essential dimensions	562
12.2 Descent of semiramified division algebras	566
12.3 Lower bounds	575
12.4 Application to decomposability	581
Exercises	583
Notes	583
Appendices	
A Commutative Valuation Theory	585
A.1 Extensions of valuations on fields	585
A.2 Henselian valuations	595
A.3 Properties of the Henselization	604
A.4 Ranks of value groups and their factor groups	608
B List of Examples	613
References	621
Subject Index	635
Index of Notation	641

Value Functions on Simple Algebras, and Associated
Graded Rings

Tignol, J.-P.; Wadsworth, A.R.

2015, XV, 643 p., Hardcover

ISBN: 978-3-319-16359-8