Chapter 2

Equations of the First Kind with a Difference Kernel

In this chapter we consider again the equation

$$Sf = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^{\omega} f(t)s(x-t)\,\mathrm{d}t = \varphi(x). \tag{2.0.1}$$

The operator S is now not assumed to be invertible. An important particular case of (2.0.1) is the equation of the first kind

$$Sf = \int_0^{\omega} f(t)k(x-t)\,\mathrm{d}t = \varphi(x), \quad k(x) \in L(-\omega,\omega). \tag{2.0.2}$$

Equations of the form (2.0.2) play an essential role in a number of applied problems [101]. There are, however, few rigorous results in the theory of such equations. The regularization method is not adaptable to a calculation of the difference structure of the kernel of (2.0.1) and (2.0.2). In this chapter we develop a method that takes into account the specific character of the kernels of (2.0.1) and (2.0.2). As in Chapter 1, a solution of (2.0.1) can be expressed in terms of the particular solutions

$$SN_1 = M, \quad SN_2 = \mathbb{1}. \tag{2.0.3}$$

The general results are illustrated and made more precise in the classical examples when the kernel $k(x)$ in (2.0.2) is one of the following functions:

$$\ln \frac{A}{2|x|}, \quad |x|^{-h}\ (-1 < h < 1), \quad \mathrm{e}^{-v|x|}\ (v > 0). \tag{2.0.4}$$

Operators with kernels (2.0.4), or similar ones, have been the object of study of a number of authors (Carleman [12], Kreĭn [38], Gakhov [17], Zadeh and Ragazzini [113]).

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2.1 Equations of the first kind with a special right-hand side

1. Let S be a bounded operator of the form

$$Sf = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^\omega f(t)s(x-t)\,\mathrm{d}t, \quad s(x) \in L^p(-\omega,\omega), \tag{2.1.1}$$

in the space $L^p(0,\omega)$ $(1 \le p \le 2)$, where $1/p + 1/q = 1$. The main equalities from Chapter 1 for S and S^* acting in $L^2(0,\omega)$ are also valid for S and S^* acting in $L^p(0,\omega)$. We note that the operators

$$Af = \mathrm{i}\int_0^x f(t)\,\mathrm{d}t, \quad A^*f = -\mathrm{i}\int_x^\omega f(t)\,\mathrm{d}t \tag{2.1.2}$$

are considered in $L^p(0,\omega)$ $(1 \le p \le 2)$ and, therefore, when $1 \le p < 2$ they are not adjoint to each other.

Proposition 2.1.1. *Let S be a bounded operator of the form* (2.1.1) *acting in $L^p(0,\omega)$. Then the operator identity* (1.1.10)*, where M and N are given in* (1.1.11)*, is valid. Formulas* (1.2.2) *and* (1.2.3) *hold for $\mathcal{L}_m$ given by* (1.2.1)*. The operator S is bounded in all the spaces $L^r(0,\omega)$ $(p \le r \le q)$ and the adjoint operator S^* acting in the spaces $L^{\widetilde{r}}(0,\omega)$ $(\widetilde{r} = r/(r-1))$ is given by the equality*

$$S^* = USU, \quad Uf = \overline{f(\omega - x)}. \tag{2.1.3}$$

The statements of the proposition above are proved similar to the corresponding statements in Chapter 1 and we prove here only the following lemma.

Lemma 2.1.2. *If $f \in L^p(0,\omega)$ and $Sf \in L^q(0,\omega)$, then*

$$\big\langle Sf, Uf\big\rangle = \big\langle f, S^*Uf\big\rangle. \tag{2.1.4}$$

Proof. Since $Sf \in L^q(0,\omega)$, it follows from (2.1.3) that $S^*Uf \in L^q(0,\omega)$. Hence, both sides of (2.1.4) are well defined. Now (2.1.4) is obtained directly from the following property of an involution:

$$\big\langle f_1, f_2\big\rangle = \big\langle Uf_2, Uf_1\big\rangle, \quad f_1 \in L^q(0,\omega), \quad f_2 \in L^p(0,\omega). \tag{2.1.5}$$

$\square$

2. Theorem 1.3.1 from Chapter 1 can be modified for the case of the equations of the first kind as follows.

Theorem 2.1.3. *Suppose that an operator S of the form* (2.1.1) *is bounded in $L^p(0,\omega)$ $(1 \le p \le 2)$ and that there are functions $N_1(x)$ and $N_2(x)$ in $L^p(0,\omega)$ satisfying* (2.0.3)*. Then*

$$SB_\gamma(x,\lambda) = \mathrm{e}^{\mathrm{i}x\lambda}, \quad \lambda \neq -\frac{1}{\mathrm{i}\gamma}, \tag{2.1.6}$$

where

$$\gamma = \langle SN_1, UN_2\rangle - \langle N_1, S^*UN_2\rangle, \quad B_\gamma(x,\lambda) = \frac{1}{\mathrm{i}\lambda\gamma+1}B(x,\lambda), \tag{2.1.7}$$

and $B(x,\lambda)$ belongs to $L^p(0,\omega)$ and is defined by (1.3.4)–(1.3.6)*.*

Proof. According to Proposition 2.1.1, the identity (1.1.10) and the recurrence formula (1.2.3) remain valid for S acting in $L^p(0,\omega)$. Since $N(x) \in L^q(0,\omega)$ and (1.2.3) holds, it follows that for a certain C we have

$$\|\mathcal{L}_{m+1}\|_p \le Cm\|\mathcal{L}_m\|_p \le C^{m+1}m! \tag{2.1.8}$$

Here $\|f\|_p$ is the norm in the space $L^p(0,\omega)$. By virtue of (2.1.8) the series

$$B_\gamma(x,\lambda) = \sum_{m=0}^{\infty} \frac{(\mathrm{i}\lambda)^m}{M!}\mathcal{L}_{m+1}$$

converges for $|\lambda| < C^{-1}$. Consequently,

$$SB_\gamma(x,\lambda) = \mathrm{e}^{\mathrm{i}\lambda x}, \quad |\lambda| < C^{-1}. \tag{2.1.9}$$

Using (1.2.2) we deduce that

$$B_\gamma(x,\lambda) = u_\gamma(x,\lambda) - \mathrm{i}\lambda\int_x^\omega B_\gamma(t,\lambda)\,\mathrm{d}t, \tag{2.1.10}$$

where

$$u_\gamma(x,\lambda) = a_\gamma(\lambda)N_1(x) + b_\gamma(\lambda)N_2(x), \tag{2.1.11}$$

$$a_\gamma(\lambda) = \mathrm{i}\lambda\int_0^\omega B_\gamma(t,\lambda)\,\mathrm{d}t, \quad b_\gamma(\lambda) = 1 + \mathrm{i}\lambda\int_0^\omega B_\gamma(t,\lambda)N(t)\,\mathrm{d}t. \tag{2.1.12}$$

Solving the integral equation (2.1.10) we derive

$$B_\gamma(x,\lambda) = u_\gamma(x,\lambda) - \mathrm{i}\lambda\int_x^\omega \mathrm{e}^{\mathrm{i}(x-t)\lambda}u_\gamma(t,\lambda)\,\mathrm{d}t. \tag{2.1.13}$$

Next we write the functions $a_\gamma(\lambda)$ and $a(\lambda)$ in the following form:

$$a_\gamma(\lambda) = \mathrm{i}\lambda\big\langle B_\gamma(x,\lambda), S^*UN_2\big\rangle, \quad a(\lambda) = \mathrm{i}\lambda\big\langle SB_\gamma(x,\lambda), UN_2\big\rangle. \tag{2.1.14}$$

Then from (2.1.4), (2.1.11), and (2.1.13), we obtain

$$a(\lambda) = a_\gamma(\lambda)\ \big(1 + \mathrm{i}\lambda\gamma\big), \tag{2.1.15}$$

where γ is defined by the first formula (2.1.7).

Writing $b_\gamma(\lambda)$ and $b(\lambda)$ in the form

$$b_\gamma(\lambda) = 1 + \mathrm{i}\lambda\Big\langle B_\gamma, S^*U(\mathbb{1} - N_1)\Big\rangle, \quad b(\lambda) = 1 + \mathrm{i}\lambda\Big\langle SB_\gamma, U(\mathbb{1} - N_1)\Big\rangle,$$

we derive a relation analogous to (2.1.15):

$$b(\lambda) = b_\gamma(\lambda)\ \big(1 + \mathrm{i}\lambda\gamma\big). \tag{2.1.16}$$

From (2.1.15) and (2.1.16) it follows that

$$u_\gamma(x,\lambda) = \frac{u(x,\lambda)}{1 + \mathrm{i}\lambda\gamma}, \quad B_\gamma(x,\lambda) = \frac{B(x,\lambda)}{1 + \mathrm{i}\lambda\gamma}, \quad |\lambda| < C^{-1},$$

that is, the theorem is true for $|\lambda| < C^{-1}$. Since $B(x,\lambda)$ and $\mathrm{e}^{\mathrm{i}\lambda x}$ are analytic in λ, the theorem follows. □

We introduce the function

$$\rho(\lambda,\mu) = \int_0^\omega B_\gamma(x,\lambda)\mathrm{e}^{\mathrm{i}\mu x}\,\mathrm{d}x. \tag{2.1.17}$$

A formula analogous to (1.3.12) can be deduced from Theorem 2.1.3:

$$\rho(\lambda,\mu) = -\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\omega\mu}}{\mathrm{i}\lambda\gamma + 1}\,\frac{a(\lambda)b(-\mu) - b(\lambda)a(-\mu)}{\lambda + \mu}. \tag{2.1.18}$$

2.2 Solutions of equations of the first kind

Let S be an operator with a difference kernel that is bounded in $L^p(0,\omega)$ and suppose that there are functions $N_1(x)$ and $N_2(x)$ satisfying (2.0.3). A solution of (2.0.1) for the particular right-hand side $\varphi(x) = \mathrm{e}^{i\lambda x}$ was given in Theorem 2.1.3. Using this result we construct a solution of (2.0.1) for the class $\varphi(x)$ from $W_p^{(2)}$.

1. We introduce the function

$$r(x,t) = N_2(\omega - t)N_1(x) - N_1(x-t)N_2(x). \tag{2.2.1}$$

From

$$r(x,t) = -r(\omega - t, \omega - x) \tag{2.2.2}$$

there follows the relation

$$\int\limits_x^{\omega} \int\limits_0^{\omega - x} f(x-t+s)r(t,s)\,\mathrm{d}s\,\mathrm{d}t = 0, \tag{2.2.3}$$

where $f(x)$ is an arbitrary function in $L^q(-\omega,\omega)$.

We denote by $W_p^{(l)}$ the set of functions $\varphi(x)$ such that $\varphi^{(l)}(x) \in L^p(0,\omega)$. We define an operator T on $W_p^{(2)}$ by

$$\begin{aligned} T\varphi = \int\limits_0^{\omega} \varphi'(t)r(x,t)\,\mathrm{d}t + \varphi(\omega)N_2(x) - \int\limits_x^{\omega} \varphi'(x-t+\omega)N_2(t)\,\mathrm{d}t \\ - \int\limits_x^{\omega} \int\limits_{\omega-x}^{\omega} \varphi''(x-t+s)r(t,s)\,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{2.2.4}$$

It is easy to see that T maps $W_p^{(2)}$ into $L^p(0,\omega)$. Taking into account (2.2.3), from (1.3.4)–(1.3.6) we obtain

$$B(x,\lambda) = T\mathrm{e}^{\mathrm{i}\lambda x}. \tag{2.2.5}$$

If $\gamma = 0$ then (2.1.6) and (2.2.5) imply that

$$ST\mathrm{e}^{\mathrm{i}\lambda x} = \mathrm{e}^{\mathrm{i}x\lambda}, \quad TSB(x,\lambda) = B(x,\lambda). \tag{2.2.6}$$

Using the first of the relations (2.2.6) we prove the following theorem.

Theorem 2.2.1. *Suppose that the conditions of Theorem* 2.1.3 *are fulfilled and that* $\gamma = 0$. *Then the operator T defined by* (2.2.4) *is a right inverse of S, that is,*

$$ST\varphi = \varphi, \quad \varphi \in W_p^{(2)}. \tag{2.2.7}$$

Thus, for $\gamma = 0$ and $\varphi \in W_p^{(2)}$ the function $f(x) = T\varphi$ is a solution of (2.0.1).

2. We introduce a simpler formula for T. To do this we need the identity

$$\int\limits_x^{\omega} \int\limits_{t-x}^{\omega} f(x-t+s)r(t,s)\,\mathrm{d}s\,\mathrm{d}t = \int\limits_x^{\omega} \int\limits_{\omega-x}^{\omega} f(x-t+s)r(t,s)\,\mathrm{d}s\,\mathrm{d}t, \tag{2.2.8}$$

which follows from (2.2.3) if we put $f(u) = 0$ for $-\omega < u < 0$. Further, in view of (2.2.2) we get

$$\int_x^\omega r(t, t-x)\,\mathrm{d}t = 0. \tag{2.2.9}$$

Taking (2.2.8) and (2.2.9) into account we rewrite (2.2.4) in the following form:

$$T\varphi = -\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\omega \int_{t-x}^\omega \varphi'(x-t+s)r(t,s)\,\mathrm{d}s\,\mathrm{d}t - \frac{\mathrm{d}}{\mathrm{d}x}\int_x^\omega \varphi(x-t+\omega)N_2(t)\,\mathrm{d}t. \tag{2.2.10}$$

Substituting the variables $s = u + t - x$ and $t = \dfrac{v+x-u}{2}$ we obtain the form

$$\begin{aligned} T\varphi = &-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\omega \left(\int_{x+u}^{2\omega-|x-u|} r\left(\frac{v+x-u}{2}, \frac{v+u-x}{2}\right)\mathrm{d}v \right)\varphi'(u)\,\mathrm{d}u \\ &-\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\omega N_2(t)\varphi(x-t+\omega)\,\mathrm{d}t. \end{aligned} \tag{2.2.11}$$

As in Chapter 1 we put

$$\begin{aligned} Q(x,t) &= N_2(\omega-t)N_1(x) + \Big(1-N_1(\omega-t)\Big)N_2(x), \\ \Phi(x,t) &= \frac{1}{2}\int_{x+t}^{2\omega-|x-t|} Q\left(\frac{s+x-t}{2}, \frac{s-x+t}{2}\right)\mathrm{d}s. \end{aligned} \tag{2.2.12}$$

Then the equality

$$Q(x,t) = r(x,t) + N_2(x) \tag{2.2.13}$$

holds. A direct calculation shows that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\omega \left(\int_{x+u}^{2\omega-|x-u|} N_2\left(\frac{v+u-x}{2}\right)\mathrm{d}v \right)\varphi'(u)\,\mathrm{d}u - \frac{\mathrm{d}}{\mathrm{d}x}\int_x^\omega N_2(t)\varphi(x-t+\omega)\,\mathrm{d}t \\ = \varphi(0)N_2(x). \end{aligned} \tag{2.2.14}$$

From (2.2.11), (2.2.13)–(2.2.14) we deduce the final formula for T:

$$T\varphi = -\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\omega \Phi(x,t)\varphi'(t)\,\mathrm{d}t + \varphi(0)N_2(x). \tag{2.2.15}$$

3. Now, let us consider the case where

$$\gamma = \big\langle SN_1, UN_2\big\rangle - \big\langle N_1, S^*UN_2\big\rangle \neq 0. \tag{2.2.16}$$

We put $\lambda_0 = -1/\mathrm{i}\gamma$. Comparing the residues of both sides of (2.1.6) at λ_0, we see that

$$SB(x, \lambda_0) = 0. \tag{2.2.17}$$

We rewrite (2.1.6) as follows:

$$S\Big(B(x,\lambda) - B(x,\lambda_0)\Big)\Big/\Big(\mathrm{i}\lambda\gamma + 1\Big) = \mathrm{e}^{\mathrm{i}\lambda x}. \tag{2.2.18}$$

We introduce the operators

$$C_\gamma\varphi = \frac{1}{\gamma}\int\limits_0^x \varphi(t)\mathrm{e}^{\mathrm{i}\lambda_0(x-t)}\,\mathrm{d}t, \quad T_\gamma = TC_\gamma. \tag{2.2.19}$$

It is easy to see that

$$C_\gamma \mathrm{e}^{\mathrm{i}\lambda x} = \frac{\mathrm{e}^{\mathrm{i}\lambda x} - \mathrm{e}^{\mathrm{i}\lambda_0 x}}{\mathrm{i}\lambda\gamma + 1}. \tag{2.2.20}$$

By (2.2.5), (2.2.19), and (2.2.20),

$$\frac{B(x,\lambda) - B(x,\lambda_0)}{\mathrm{i}\lambda\gamma + 1} = T_\gamma \mathrm{e}^{\mathrm{i}\lambda x}. \tag{2.2.21}$$

From (2.2.15) and (2.2.19) we see that T maps $W_p^{(1)}$ into $L^p(0,\omega)$ and can be written in the form

$$T_\gamma\varphi = -\frac{1}{\gamma}\,\frac{\mathrm{d}}{\mathrm{d}x}\int\limits_0^\omega \left(\varphi(t) + \mathrm{i}\lambda_0\int\limits_0^t \varphi(u)\mathrm{e}^{\mathrm{i}\lambda_0(t-u)}\,\mathrm{d}u\right)\Phi(x,t)\,\mathrm{d}t. \tag{2.2.22}$$

The relations (2.2.18) and (2.2.21) show that $ST_\gamma \mathrm{e}^{\mathrm{i}\lambda x} = \mathrm{e}^{\mathrm{i}\lambda x}$. Thus, we have the following theorem.

Theorem 2.2.2. *Suppose that the conditions of Theorem* 2.1.3 *are fulfilled and that* $\gamma \neq 0$*. Then the operator* T_γ *defined by* (2.2.22) *is a right inverse of* S*, that is,*

$$ST_\gamma\varphi = \varphi, \quad \varphi(x) \in W_p^{(1)}. \tag{2.2.23}$$

4. The problem of constructing the operators T and T_γ, cf. (2.2.19), is simplified if

$$R = \int\limits_0^\omega N_2(t)\,\mathrm{d}t \neq 0. \tag{2.2.24}$$

Then it follows from (1.2.2) for $m = 1$ that

$$N_1(x) = \frac{1}{R}\left(\int_x^\omega N_2(t)\,\mathrm{d}t + \mathcal{L}_2(x) + \alpha N_2(x)\right), \quad \alpha = -\int_0^\omega N_2(t)N(t)\,\mathrm{d}t. \tag{2.2.25}$$

When we now put

$$Q_1(x,t) = \frac{1}{R}\Big(N_2\big(\omega - t\big)\mathcal{L}_2(x) - \mathcal{L}_2\big(\omega - t\big)N_2(x)\Big), \tag{2.2.26}$$

$$Q_2(x,t) = \frac{1}{R}\left(N_2(\omega - t)\int_x^\omega N_2(s)\,\mathrm{d}s + \int_0^{\omega - t} N_2(s)\,\mathrm{d}s N_2(x)\right), \tag{2.2.27}$$

we deduce from (2.2.12) and (2.2.25)–(2.2.27) that

$$Q(x,t) = Q_1(x,t) + Q_2(x,t). \tag{2.2.28}$$

The corresponding function $\Phi(x,t)$ has the form

$$\Phi(x,t) = \Phi_1(x,t) + \Phi_2(x,t), \tag{2.2.29}$$

where

$$\Phi_k(x,t) = \frac{1}{2}\int_{x+t}^{2\omega - |x-t|} Q_k\left(\frac{s+x-t}{2}, \frac{s-x+t}{2}\right)\mathrm{d}s \quad (k = 1,2). \tag{2.2.30}$$

Using (2.2.15), (2.2.29), and (2.2.30) we represent T as follows:

$$T\varphi = -\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\omega \varphi'(t)\Phi_1(x,t)\,\mathrm{d}t + \frac{1}{R}\int_0^\omega \varphi(t)N_2(\omega - t)\,\mathrm{d}t\,N_2(x). \tag{2.2.31}$$

Thus, under the condition (2.2.24), it is enough to know the two functions $\mathcal{L}_1(x)$ and $\mathcal{L}_2(x)$ in order to construct the operator T.

It follows from (2.2.24) and (2.2.25) that formula (2.1.7) for γ can be written in the form

$$\gamma = \frac{1}{R}\int_0^\omega \Big(t\mathcal{L}_1(\omega - t) - \mathcal{L}_2(t)\Big)\,\mathrm{d}t. \tag{2.2.32}$$

5. We note that (2.2.26)–(2.2.31) also hold when S has a bounded inverse T in $L^2(0,\omega)$. In this case T admits the representation

$$T\varphi = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^\omega \left(\frac{\partial}{\partial t}\Phi_1(x,t)\right)\varphi(t)\,\mathrm{d}t + \frac{1}{R}\int_0^\omega N_2(\omega - t)\varphi(t)\,\mathrm{d}t\,N_2(x). \tag{2.2.33}$$

6. Under certain conditions, Theorems 2.2.1 and 2.2.2 provide a procedure to construct solutions of (2.0.1). The description of the whole set of solutions and also the question of uniqueness of the solution leads, as usual, to the study of the equation

$$Sf = 0. \tag{2.2.34}$$

Here H_S stands for the set of solutions of (2.2.34) belonging to $L^p(0,\omega)$. If $\dim H_S > 1$, then the following theorem is useful.

Theorem 2.2.3. *Let S be an operator with a difference kernel that is bounded in $L^p(0,\omega)$ $(1 \le p \le 2)$ and $1 < \dim H_S \le n < \infty$. Then H_S has a basis f_k $(0 \le k \le n-1)$ such that*

$$f_{k+1} = A^* f_k, \quad 0 \le k \le n-2, \tag{2.2.35}$$

where A and A^* are defined by (2.1.2).

Proof. Since the operator A^* has no finite-dimensional invariant subspaces, the inequality $A^*H_S \ne H_S$ is valid. According to (1.1.10), the operator S maps A^*H_S into the subspace spanned on $M(x)$ and $\mathbb{1}$. This subspace cannot be two-dimensional, because in that case there would exist absolutely continuous functions N_1 and N_2, and by virtue of Theorem 1.4.3, S would be invertible. Thus, $\dim SA^*H_S = 1$. Hence, the subspace A^*H_S has a common part H_S^1 of dimension $n-1$ with H_S. Similarly we derive that

$$A^*H_S \ne H_S^1, \quad \dim SA^*H_S^1 \le 1. \tag{2.2.36}$$

Putting $H_S^2 = A^*H_S^1 \cap H_S$, we deduce from (2.2.36) that

$$\dim H_S^2 \ge n-2. \tag{2.2.37}$$

It is easy to see that $H_S^2 \subset H_S^1$, and so we have the equality in (2.2.37). Repeating this process we obtain subspaces $H_S^k = A^*H_S^{k-1} \cap H_S$ $(2 \le k \le n-1)$, and

$$H_S \supset H_S^1 \supset \cdots \supset H_S^{n-1}, \quad \dim H_S^k = n-k. \tag{2.2.38}$$

Thus, there exists a function $f_0 \in H_S$ such that

$$f_k = A^{*k} f_0 \in H_S^k \quad (1 \le k \le n-1), \quad \|f_0\|_p \ne 0. \tag{2.2.39}$$

The system of functions $f_1, f_2, \ldots, f_{n-1}$ is linearly independent, because from

$$\left(\alpha_0 + \alpha_1 A^* + \cdots + \alpha_{n-1} A^{*n-1}\right) f_0 = 0, \tag{2.2.40}$$

it follows that $\alpha_0 = \alpha_1 = \cdots = \alpha_{n-1} = 0$. Thus, the functions f_k $(0 \le k \le n-1)$ form a basis of H_S and satisfy (2.2.35). This proves the theorem. □

A similar fact was proved by M.G. Kreĭn (verbal communication) for operators of the form

$$Sf = f(x) + \int_0^\omega f(t)k\big(\omega - t\big)\,\mathrm{d}t, \quad k(x) \in L(-\omega,\omega). \tag{2.2.41}$$

7. Under the assumption that $N_1(x)$ and $N_2(x)$ exist, the following theorem is true.

Theorem 2.2.4. *Suppose that an operator S of the form* (2.1.1) *is bounded in $L^p(0,\omega)$ and that there are functions $N_1(x)$ and $N_2(x)$ in $L^p(0,\omega)$ satisfying* (2.0.3).

I. *If $f \in H_S$ and $\|f\|_p \neq 0$, then $f \notin L^q(0,\omega)$.*

II. *If $p = 2$, then* $\dim H_S = 0$.

III. *The inequality* $\dim H_S \leq 1$ *is valid.*

IV. *If $\gamma \neq 1$ then* $\dim H_S = 1$.

Proof. Assume that $f \in L^q(0,\omega)$. By Theorem 2.1.3,

$$\left\langle \mathrm{e}^{\mathrm{i}x\lambda}, Uf \right\rangle = \left\langle SB_\gamma(x,\lambda), Uf \right\rangle. \tag{2.2.42}$$

Using (2.1.3) we deduce from (2.2.42) that

$$\left\langle \mathrm{e}^{\mathrm{i}x\lambda}, Uf \right\rangle = 0,$$

that is, $f(x) = 0$, which contradicts the condition $\|f\|_p \neq 0$. This proves the first assertion. Assertion II is immediate from the first assertion. It also follows from the first assertion that A^*H_S and H_S do not have common non-trivial elements. On the other hand, similar to the proof of Theorem 2.2.3, we deduce that

$$\dim SA^*H_S \leq 1.$$

Hence, we have $\dim A^*H_S \leq 1$, from which we derive III.

Now we consider the case, where $\gamma \neq 0$. We prove that

$$\|B(x,\lambda_0)\|_p \neq 0. \tag{2.2.43}$$

Let us assume that

$$\|B(x,\lambda_0)\|_p = 0. \tag{2.2.44}$$

From (1.3.4), (1.3.5) and (2.2.44) it follows that

$$a(\lambda_0)N_1(x) + b(\lambda_0)N_2(x) = 0. \tag{2.2.45}$$

From (1.3.6) and (2.2.45) we deduce, that

$$a(\lambda_0) = b(\lambda_0) = 0. \tag{2.2.46}$$

Putting

$$u_0(x) = a'(\lambda_0)N_1(x) + b'(\lambda_0)N_2(x), \tag{2.2.47}$$

$$B_0(x) = u_0(x) - \mathrm{i}\lambda_0 \int_x^\omega \mathrm{e}^{\mathrm{i}\lambda_0(x-t)} u_0(t)\,\mathrm{d}t, \tag{2.2.48}$$

we deduce from (2.1.6) that

$$\frac{1}{\mathrm{i}\gamma} SB_0(x) = \mathrm{e}^{\mathrm{i}\lambda_0 x}. \tag{2.2.49}$$

Using the notation (2.1.17) and formula (2.2.49), we have

$$\frac{1}{\mathrm{i}\gamma}\Big\langle SB_0(x), UB_\gamma(x,\lambda)\Big\rangle = \mathrm{e}^{\mathrm{i}\omega\lambda}\rho\big(\lambda, -\lambda_0\big). \tag{2.2.50}$$

Now, in view of (2.2.46), (2.2.50), and (2.1.18), we obtain

$$\Big\langle SB_0(x), UB_\gamma(x,\lambda)\Big\rangle = 0.$$

Hence, taking into account (2.1.4), we see that

$$\Big(a'(\lambda_0)b(\lambda) - b'(\lambda_0)a(\lambda)\Big)\gamma + \Big\langle B_0(x), S^*UB_\gamma(x,\lambda)\Big\rangle = 0. \tag{2.2.51}$$

Formulas (2.1.3) and (2.1.6) yield

$$\Big\langle B_0(x), S^*UB_\gamma(x,\lambda)\Big\rangle = \int_0^\omega B_0(x)\mathrm{e}^{\mathrm{i}\big(\omega - x\big)\lambda}\,\mathrm{d}x. \tag{2.2.52}$$

The formulas (2.2.48) and (2.2.52) lead to

$$\Big\langle B_0(x), S^*UB_\gamma(x,\lambda)\Big\rangle = \frac{a'(\lambda_0)\Big(b(\lambda) + \mathrm{e}^{\mathrm{i}\lambda_0\omega} - \mathrm{e}^{\mathrm{i}\lambda\omega}\Big) - b'(\lambda_0)a(\lambda)}{\mathrm{i}(\lambda_0 - \lambda)}. \tag{2.2.53}$$

We substitute the right-hand side of (2.2.53) in (2.2.51):

$$\Big(a'(\lambda_0)b(\lambda) - b'(\lambda_0)a(\lambda)\Big)\ \left(\gamma + \frac{1}{\mathrm{i}(\lambda_0 - \lambda)}\right) + a'(\lambda_0)\,\frac{\mathrm{e}^{\mathrm{i}\lambda_0\omega} - \mathrm{e}^{\mathrm{i}\lambda\omega}}{\mathrm{i}\big(\lambda_0 - \lambda\big)} = 0. \tag{2.2.54}$$

Taking (2.2.46) into account we pass in (2.2.54) to the limit as $\lambda \to \lambda_0$:

$$a'(\lambda_0) = 0. \tag{2.2.55}$$

It follows from (2.2.54) and (2.2.55) that

$$b'(\lambda_0)a(\lambda)\left(\gamma\frac{1}{\mathrm{i}(\lambda_0 - \lambda)}\right) = 0.$$

Hence,

$$b'(\lambda_0) = 0. \tag{2.2.56}$$

The equations (2.2.55) and (2.2.56) indicate that $\|u_0(x)\|_p = 0$. Then, by (2.2.47), $\|B_0(x)\|_p = 0$, which contradicts (2.2.49). Hence, (2.2.44) does not hold and $\|B(x,\lambda_0)\|_p \neq 0$. Now IV follows from III and (2.2.17). This proves the theorem. $\square$

8. The following result is useful for the study of various concrete examples.

Theorem 2.2.5. *Suppose that an operator S is defined by* (2.0.2) *and that there are functions $N_1(x)$ and $N_2(x)$ in $L^p(0,\omega)$ $(1 \le p \le 2)$ satisfying* (2.0.3). *If the functions*

$$g_m(x) = \int_0^\omega |N_m(t)| \cdot |k(x-t)| \, \mathrm{d}t \quad (m = 1, 2)$$

belong to $L^q(0,\omega)$, then $\gamma = 0$ and $\dim H_S = 0$.

Proof. The operator S^* has the form

$$S^* f = \int_0^\omega f(t) \overline{k(t-x)} \, \mathrm{d}t.$$

Then

$$\left\langle N_1, S^* U N_2 \right\rangle = \int_0^\omega N_1(x) \int_0^\omega N_2(\omega - t) k(t-x) \, \mathrm{d}t \, \mathrm{d}x. \tag{2.2.57}$$

Under the conditions of the theorem, the integral on the right-hand side of (2.2.57) converges absolutely. Hence, by Fubini's theorem we can change the order of integration:

$$\left\langle N_1, S^* U N_2 \right\rangle = \int_0^\omega \int_0^\omega N_1(x) k(t-x) \, \mathrm{d}x \, N_2(\omega - t) \, \mathrm{d}t = \left\langle S N_1, U N_2 \right\rangle. \tag{2.2.58}$$

By (2.1.7) and (2.2.58),

$$\gamma = 0. \tag{2.2.59}$$

Similarly,

$$\left\langle S N_m, U f \right\rangle = \left\langle N_m, S^* U f \right\rangle \tag{2.2.60}$$

if

$$f \in L^p(0,\omega), \quad S^* U f \in L^q(0,\omega).$$

From Theorem 2.1.3 we obtain

$$S B(x,\lambda) = \mathrm{e}^{\mathrm{i}\lambda x}. \tag{2.2.61}$$

Using (2.2.59)–(2.2.61) we deduce that

$$\left\langle \mathrm{e}^{\mathrm{i}\lambda x}, U f \right\rangle = \left\langle S B(x,\lambda), U f \right\rangle = \left\langle B(x,\lambda), S^* U f \right\rangle. \tag{2.2.62}$$

If $f \in H_S$ then $\left\langle \mathrm{e}^{\mathrm{i}\lambda x}, U f \right\rangle = 0$, by (2.1.3) and (2.2.62), that is, $\|f\|_p = 0$. This proves the theorem. $\square$

2.3 Generalized solutions

In many cases, equation (2.0.2) has only generalized functions as solutions. This situation is typical for equations connected with optimal problems of automatic control [101]. Under certain assumptions the results of Sections 2.1, 2.2 can be extended to this case.

1. We denote by $\mathfrak{D}$ the set of generalized functions of the form

$$f(x) = \alpha\delta(x) + \beta\delta(\omega - x) + f_1(x), \tag{2.3.1}$$

where $f_1(x) \in L(0, \omega)$ and $\delta(x)$ is the delta function.

We say that $g(x)$ belongs to the basic space K, if $g(x)$ is bounded on $[0, \omega]$ and is continuous at 0 and ω.

As usual (see [18, Chap. 1]) a generalized function f is a linear functional on K:

$$\langle g, f \rangle = \overline{\langle f, g \rangle} = \int_0^\omega g(x)\overline{f(x)}\,\mathrm{d}x. \tag{2.3.2}$$

By definition,

$$\langle g, f \rangle = \overline{\alpha}g(0) + \overline{\beta}g(\omega) + \int_0^\omega g(x)\overline{f}_1(x)\,\mathrm{d}x. \tag{2.3.3}$$

We introduce the operator

$$Sf = \int_0^\omega f(t)k(x - t)\,\mathrm{d}t, \tag{2.3.4}$$

where $k(x)$ is a continuous function on $[-\omega, \omega]$. The operator S maps functions from $\mathfrak{D}$ into continuous functions on $[0, \omega]$. Operators A and A^* on $\mathfrak{D}$ are defined by

$$Af = \mathrm{i}\int_0^x f(t)\,\mathrm{d}t = \mathrm{i}\alpha + \mathrm{i}\int_0^x f_1(t)\,\mathrm{d}t, \tag{2.3.5}$$

$$A^*f = -\mathrm{i}\int_x^\omega f(t)\,\mathrm{d}t = -\mathrm{i}\beta - \mathrm{i}\int_x^\omega f_1(t)\,\mathrm{d}t. \tag{2.3.6}$$

Theorem 2.3.1. *For any function f in $\mathfrak{D}$*

$$\big(AS - SA^*\big)f = \mathrm{i}\int_0^\omega f(t)\Big(M(x) + N(t)\Big)\,\mathrm{d}t, \tag{2.3.7}$$

where

$$M(x) = \int_0^x k(t)\,\mathrm{d}t, \quad N(x) = -\int_0^{-x} k(t)\,\mathrm{d}t. \tag{2.3.8}$$

Proof. From (2.3.4)–(2.3.6) and (2.3.8) it follows that

$$\big(AS - SA^*\big)\,\delta(x) = \mathrm{i}\int_0^x k(t)\,\mathrm{d}t = \mathrm{i}\int_0^\omega \Big(M(x) + N(t)\Big)\delta(t)\,\mathrm{d}t, \tag{2.3.9}$$

$$\big(AS - SA^*\big)\,\delta\big(\omega - x\big) = \mathrm{i}\int_0^x k\big(t-\omega\big)\,\mathrm{d}t + \mathrm{i}\int_0^\omega k\big(x-t\big)\,\mathrm{d}t. \tag{2.3.10}$$

Comparing (2.3.8) and (2.3.10) we have

$$\big(AS{-}SA^*\big)\,\delta\big(\omega{-}x\big) = \mathrm{i}\Big(M(x){+}N(\omega)\Big) = \mathrm{i}\int_0^\omega \Big(M(x){+}N(t)\Big)\,\delta\big(\omega{-}t\big)\,\mathrm{d}t. \tag{2.3.11}$$

In view of (2.3.9) and (2.3.11), the equality (2.3.7) holds for $f(x) = \delta(x)$ and $f(x) = \delta(\omega - x)$. The fact that (2.3.7) holds for $f(x) = f_1(x) \in L(0,\omega)$ is proved in a way which is similar to the proof of Theorem 1.1.3. Thus, (2.3.7) is valid for $f \in \mathfrak{D}$. This proves the theorem. □

Now, together with the operator S we consider the operator

$$S^* f = \int_0^\omega \overline{k(x-t)} f(t)\,\mathrm{d}t \tag{2.3.12}$$

which maps the functions from $\mathfrak{D}$ into the functions which are continuous on the segment $[0,\omega]$. Let us prove that for any pair of the functions $f(x)$ and $g(x)$ from $\mathfrak{D}$ the following equality holds:

$$\big\langle Sf, g\big\rangle = \big\langle f, S^* g\big\rangle. \tag{2.3.13}$$

Indeed, we put

$$f(x) = \alpha\,\delta(x) + \beta\,\delta(\omega - x) + f_1(x), \quad g(x) = \gamma\,\delta(x) + \nu\,\delta(\omega - x) + g_1(x),$$

where $f_1(x), g_1(x) \in L(0,\omega)$. By (2.3.4) and (2.3.12) we obtain

$$Sf = \alpha k(x) + \beta k(x-\omega) + \int_0^\omega k(x-t) f_1(t)\,\mathrm{d}t,$$

$$S^* g = \gamma\overline{k(-x)} + \nu\overline{k(\omega - x)} + \int_0^\omega \overline{k(t-x)} g_1(t)\,\mathrm{d}t.$$

After some calculations we see that the left-hand side of (2.3.13) equals the right-hand side.

2. Theorem 2.3.1 and relations (2.3.13) allow us to modify the results from Section 2.1 for the present case.

Theorem 2.3.2. *Suppose that S has the form* (2.3.4) *and that there exist functions N_1 and N_2 satisfying* (2.0.3)*. Then the function $B(x,\lambda)$ defined by* (1.3.4)–(1.3.6) *belongs to $\mathfrak{D}$ and*

$$SB(x,\lambda) = \mathrm{e}^{\mathrm{i}x\lambda}. \tag{2.3.14}$$

Proof. For a function f of the form (2.3.1) we introduce the norm

$$\|f\|_{\mathfrak{D}} = |\alpha| + |\beta| + \int_0^\omega |f_1(t)|\,\mathrm{d}t.$$

Since $N(x)$ is bounded on $[0,\omega]$, it follows from (1.2.2) that for some c

$$\|\mathcal{L}_{m+1}\|_{\mathfrak{D}} \le cm\|\mathcal{L}_m\|_{\mathfrak{D}} \le c^{m+1}m!.$$

Hence, the series

$$B(x,\lambda) = \sum_{m=0}^{\infty} \frac{(\mathrm{i}\lambda)^m}{m!}\mathcal{L}_{m+1}$$

converges for $|\lambda| < c^{-1}$. We also see that $B(x,\lambda) \in \mathfrak{D}$ and

$$SB(x,\lambda) = \mathrm{e}^{\mathrm{i}\lambda x}. \tag{2.3.15}$$

As in Theorem 1.3.1 we pass to the integral equation

$$B(x,\lambda) = u(x,\lambda) - \mathrm{i}\lambda\int_x^\omega B(t,\lambda)\,\mathrm{d}t, \tag{2.3.16}$$

where $u(x,\lambda)$ is defined by (1.3.5) and (1.3.6).

To solve (2.3.16) we use the rule for changing the order of integration:

$$\int_x^\omega\int_t^\omega f(v)g(v,t)\,\mathrm{d}v\,\mathrm{d}t = \int_x^\omega f(v)\int_x^v g(v,t)\,\mathrm{d}t\,\mathrm{d}v, \tag{2.3.17}$$

where $f(x) \in \mathfrak{D}$, and $g(x,t)$ is a continuous function of x and t $(0 \le x,t \le \omega)$.

We easily check that (2.3.17) holds for $f(x) = \delta(x)$ and $f(x) = \delta(\omega - x)$. It is known that formula (2.3.17) holds also for $f(x) \in L(0,\omega)$. Taking into account (2.3.17) we solve (2.3.16) by the method of successive approximations. This completes the proof of our assertion. □

Now, we introduce the function

$$\rho(\lambda,\mu)=\int_0^\omega B(x,\lambda)\mathrm{e}^{\mathrm{i}\mu x}\,\mathrm{d}x. \tag{2.3.18}$$

From Theorem 2.3.2 and (2.3.17) we derive that

$$\rho(\lambda,\mu)=-\mathrm{i}\mathrm{e}^{\mathrm{i}\omega\mu}\,\frac{a(\lambda)b(-\mu)-b(\lambda)a(-\mu)}{\lambda+\mu}. \tag{2.3.19}$$

3. The set of functions $\varphi(x)$, which have continuous second derivatives on $[0,\omega]$ is denoted by $C^{(2)}$. Like in Section 2.2, the operator T is introduced by formulas (2.2.1) and (2.2.4). This operator maps functions $\varphi(x)$ from $C^{(2)}$ into functions from $\mathfrak{D}$.

Theorem 2.3.3. *Suppose that the conditions of Theorem* 2.3.2 *are satisfied. If* $\varphi\in C^{(2)}$, *then the equation* (*of the first kind*)

$$Sf=\varphi \tag{2.3.20}$$

has a unique solution $f(x)=T\varphi$ *in* $\mathfrak{D}$.

Proof. The relation (2.2.3) remains valid for $N_1,N_2\in\mathfrak{D}$. Using Theorem 2.3.2 and formulas (2.2.1), (2.2.3), and (2.2.4) we deduce that $T\mathrm{e}^{\mathrm{i}\lambda x}=B(x,\lambda)$. Hence,

$$ST\mathrm{e}^{\mathrm{i}\lambda x}=\mathrm{e}^{\mathrm{i}\lambda x},\quad\text{that is,}\quad ST\varphi=\varphi,\quad \varphi\in C^{(2)}.$$

Thus, the generalized function $f(x)=T\varphi$ is a solution of (2.3.20). Suppose that this equation has more than one solution. Then there is a non-trivial solution of

$$Sf=0. \tag{2.3.21}$$

It follows from (2.3.12) and (2.3.21) that

$$S^*Uf=0,\quad\text{where}\quad Uf=\overline{f(\omega-x)}.$$

Using Theorem 2.3.2, we have

$$\langle \mathrm{e}^{\mathrm{i}x\lambda},Uf\rangle=\langle SB(x,\lambda),Uf\rangle.$$

When we now take into account (2.3.10), we obtain

$$\langle \mathrm{e}^{\mathrm{i}x\lambda},Uf\rangle=0,\quad\text{that is,}\quad f=0.$$

This proves the theorem. □

4. We consider the equation

$$Sf = \int_0^{\omega} \mathrm{e}^{-\nu|x-t|} f(t)\,\mathrm{d}t = \varphi(x), \quad \nu > 0. \tag{2.3.22}$$

After direct substitution we see that

$$N_1(x) = -\frac{1}{\nu}\,\delta(x), \tag{2.3.23}$$

$$N_2(x) = \frac{1}{2}\Big(\nu + \delta(x) + \delta(\omega - x)\Big). \tag{2.3.24}$$

Thus, according to (2.2.1), (2.2.4), (2.3.23), (2.3.24), for $\varphi \in C^{(2)}$, the equation (2.3.22) has one and only one solution in $\mathfrak{D}$.

2.4 On the behavior of solutions

Let us consider a bounded operator

$$Sf = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^{\omega} s(x-t) f(x)\,\mathrm{d}t, \quad s(x) \in L^q(-\omega,\omega), \quad \frac{1}{p} + \frac{1}{q} = 1 \tag{2.4.1}$$

acting in the space $L^p(0,\omega)$ $(1 \le p \le 2)$.

We study the equation

$$Sf = \varphi \tag{2.4.2}$$

and obtain conditions of the boundedness of $f(x)$ We also describe the behavior of $f(x)$ at the ends of the segment $[0,\omega]$. Such questions arise in a number of applied problems [1, 37, 101]. Separately we study the equations of the form (2.4.2), the solutions of which are in the class of generalized functions.

1. Let the conditions of Theorem 2.1.3 be fulfilled and the equation $Sf = 0$ has in $L^p(0,\omega)$ only one solution.

Then the equation (2.4.2) has one and only one solution

$$f(x) = T\varphi, \tag{2.4.3}$$

where the operator T is defined by (2.2.4).

We put

$$P_1\varphi = \int_0^{\omega} N_2(\omega - t)\varphi'(t)\,\mathrm{d}t, \quad P_2(\varphi) = \varphi(\omega) - \int_0^{\omega} N_1(\omega - t)\varphi'(t)\,\mathrm{d}t.$$

From (2.2.4) and (2.4.3) we have the assertions [60]:

Proposition 2.4.1. *The solution $f(x)$ of the equation* (2.4.2) *has the form*

$$f(x) = P_1(\varphi)N_1(x) + P_2(\varphi)N_2(x) + O(1), \quad 0 \le x \le \omega. \tag{2.4.4}$$

Proposition 2.4.2. *If any non-trivial linear combination $N_1(x)$ and $N_2(x)$ is an unbounded function, then for the boundedness of the solution of* (2.4.2) *it is necessary and sufficient that the equations*

$$P_1(\varphi) = 0, \quad P_2(\varphi) = 0 \tag{2.4.5}$$

be fulfilled.

Proposition 2.4.3. *If the relations* (2.4.5) *are fulfilled, then the solution of the equation* (2.4.2) *is a function continuous on the segment $[0, \omega]$ where*

$$f(0) = f(\omega) = 0. \tag{2.4.6}$$

2. The length ω of the segment $[0, \omega]$, on which the integral equation (2.4.2) is considered, is physically meaningful in a number of problems. For instance, in the contact theory of elasticity [1] the value ω coincides with the length of the region of the contact. In the theory of optimum synthesis, ω is the value of the apparatus memory [113]. Thus, the choice of ω is determined by the corresponding physical requirements.

Problem 2.4.4. *Let the kernel $s(x)$ be defined on the segment $-\Omega \le x \le \Omega$ and let the set of the functions $\varphi_\omega(x)$ $(0 \le x \le \omega \le \Omega)$ be given. Find such a value ω that the solution of the equation*

$$Sf_\omega = \varphi_\omega(x), \quad 0 \le x \le \omega \tag{2.4.7}$$

is continuous on the segment $[0, \omega]$, and

$$f_\omega(0) = f_\omega(\omega) = 0.$$

By Proposition 2.4.3 the solutions of Problem 2.4.4 coincide with the roots of the system of the transcendental equations

$$P_1(\varphi_\omega) = 0, \quad P_2(\varphi_\omega) = 0. \tag{2.4.8}$$

Problem 2.4.4 is not always solvable. For example, for $\varphi_\omega(x) = \mathbb{1}$, the system (2.4.8) has no solution.

3. Let the invertible operator

$$Sf = f(x) + \int_0^\omega k(x-t)f(t)\,\mathrm{d}t \tag{2.4.9}$$

act in $L(0,\omega)$. Then the functions $N_1(x)$, $N_2(x)$ are absolutely continuous and (see Section 1.4)

$$N_1(\omega)N_2(0) - N_2(\omega)\Big(N_1(0) - 1\Big) = 1. \tag{2.4.10}$$

From (2.4.10) it follows that $N_2(x)$ does not map into zero simultaneously at both ends of the segment $[0,\omega]$.

Using the absolute continuity of $N_1(x)$ and $N_2(x)$ we write the functionals $P_1(\varphi)$ and $P_2(\varphi)$ in the form

$$P_1(\varphi) = N_2(0)\varphi(\omega) - N_2(\omega)\varphi(0) + \int_0^\omega N_2'(t)\varphi(\omega - t)\,\mathrm{d}t, \tag{2.4.11}$$

$$P_2(\varphi) = \Big(1 - N_1(0)\Big)\varphi(\omega) + N_1(\omega)\varphi(0) - \int_0^\omega N_1'(t)\varphi(\omega - t)\,\mathrm{d}t. \tag{2.4.12}$$

In the case of the operator S in the form (2.4.9), it is possible to weaken the condition $\varphi'' \in L^p(0,\omega)$ replacing it by the demand of the continuity of $\varphi(x)$. In this case the following assertion holds (see (2.4.11), (2.4.12)).

Proposition 2.4.5. *Relation* (2.4.5) *is a necessary and sufficient condition under which* (2.4.6) *holds.*

4. Now we consider the equations

$$Sf = \int_0^\omega k(x-t)f(t)\,\mathrm{d}t = \varphi(x), \quad \varphi(x) \in C^{(2)} \tag{2.4.13}$$

where $k(x)$ is continuous on the segment $[-\omega,\omega]$. Let $N_1(x)$, $N_2(x) \in \mathfrak{D}$. Then by Theorem 2.3.3 the equation (2.4.13) has one and only one solution $f(x) = T\varphi$, where the operator T is defined by (2.2.4). Using (2.2.4) we obtain the following assertion.

Proposition 2.4.6. *Let $N_1, N_2 \in \mathfrak{D}$ and the equation* (2.4.5) *hold. Then the solution $f(x)$ of the equation* (2.4.13) *is a continuous function.*

5. In the theory of optimal synthesis the following problem is essential.

Problem 2.4.7. *Let the continuous kernel $k(x)$ be defined on the segment $-\Omega \le t \le \Omega$ and a set of the function $\varphi_\omega(x)$ be given. Find such a value ω that the solution of the equation*

$$Sf_\omega = \int_0^\omega k(x-t)f_\omega(t)\,\mathrm{d}t = \varphi_\omega(x), \quad \varphi_\omega(x) \in C^{(2)}$$

is continuous on the segment $[0,\omega]$.

According to Proposition 2.4.6 the solutions of Problem 2.4.7 coincide with the roots of the system of equations (2.4.8). If $\varphi_\omega(x) = \mathbb{1}$, then Problem 2.4.7 has no solution.

2.5 On a class of integro-differential equations

1. Let the operator S act in $L(-a,a)$ and let it be defined by the equality

$$Sf = \mu f(x) + \int_{-a}^{a} k(x-t)f(t)\,\mathrm{d}t \tag{2.5.1}$$

and $k(x) \in L(-a,a)$. We introduce the operator

$$\mathcal{A}f = -\frac{\mathrm{d}}{\mathrm{d}x} S \frac{\mathrm{d}}{\mathrm{d}x} f, \tag{2.5.2}$$

where $\mathcal{A}$ is acting in the domain $\mathfrak{D}_{\mathcal{A}}$ of the functions f satisfying the conditions

$$f''(x) \in L(-a,a), \quad f(-a) = f(a) = 0. \tag{2.5.3}$$

A number of problems of the Lévy processes theory lead to the equations of the form (see [30, 33, 97])

$$\mathcal{A}f = \varphi, \quad \varphi \in L(-a,a). \tag{2.5.4}$$

The integro-differential equations of this kind arise in some problems of diffraction theory [24]. This section is dedicated to the investigation of the equation (2.5.4), the solution of which, in fact, reduces to the inversion of the operator S with a difference kernel.

2. Let us show that the following equality (see [21, 100]) holds:

$$S\left(\frac{\mathrm{d}}{\mathrm{d}x} f\right) - \frac{\mathrm{d}}{\mathrm{d}x}(Sf) = k(x-a)f(a) - k(x+a)f(-a), \tag{2.5.5}$$

where $f'(x) \in L(-a,a)$. Integrating by parts, we have

$$S\left(\frac{\mathrm{d}}{\mathrm{d}x} f\right) = \mu f'(x) + \int_{-a}^{a} k'(x-t)f(t)\,\mathrm{d}t + k(x-a)f(a) - k(x-a)f(-a). \tag{2.5.6}$$

From (2.5.6) and the equality

$$\frac{\mathrm{d}}{\mathrm{d}x} Sf = \mu f'(x) + \int_{-a}^{a} k'(x-t)f(t)\,\mathrm{d}t$$

it follows that (2.5.5) holds in the case of smooth $k(x)$.

We can prove that (2.5.5) is true in the general case approximating non-smooth kernels by smooth ones. If $f''(x) \in L(-a,a)$, then from (2.5.5) we have

$$S\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - \frac{\mathrm{d}}{\mathrm{d}x}S\frac{\mathrm{d}}{\mathrm{d}x}f = k(x-a)f'(a) - f'(-a)k(x+a). \tag{2.5.7}$$

From (2.5.7) we deduce that $\mathcal{A}f \in L(-a,a)$ when $f'' \in L(-a,a)$.

We say that the operator S of the form (2.5.1) is a regular one, if the following conditions are fulfilled:

1) The equation

$$Sf = 0 \tag{2.5.8}$$

has in $L(-a,a)$ only a trivial solution.

2) In $L(-a,a)$ there exist functions $\mathcal{L}_k(x)$ $(k = 1,2)$ such that

$$S\mathcal{L}_k = x^{k-1}, \quad k = 1,2 \tag{2.5.9}$$

and

$$R = \int_{-a}^{a} \mathcal{L}_1(x)\,\mathrm{d}x \neq 0. \tag{2.5.10}$$

We remark that when $\mu \neq 0$, the invertibility of the operator S follows from the condition of regularity. We introduce the operator

$$T\varphi = -\frac{\mathrm{d}}{\mathrm{d}x}\int_{-a}^{a} \Phi(x,y)\varphi'(y)\,\mathrm{d}y + \frac{1}{R}\int_{-a}^{a} \mathcal{L}_1(-y)\varphi(y)\,\mathrm{d}y\mathcal{L}_1(x), \tag{2.5.11}$$

where

$$\Phi(x,y) = \frac{1}{2}\int_{x+y}^{2a-|x-y|} Q\left(\frac{s+x-y}{2}, \frac{s-x+y}{2}\right)\mathrm{d}s, \tag{2.5.12}$$

$$Q(x,y) = \frac{1}{R}\Big(\mathcal{L}_1(-y)\mathcal{L}_2(x) - \mathcal{L}_2(-y)\mathcal{L}_1(x)\Big). \tag{2.5.13}$$

Taking into account the transition from the segment $[0,\omega]$ to the segment $[-a,a]$, we deduce from Theorem 2.2.1 the following assertion.

Proposition 2.5.1. *If the operator S is regular and if $\varphi(x) \in W_p^{(2)}$ $(1 \leq p \leq 2)$, then $T\varphi \in L^p(-a,a)$ and the equality $ST\varphi = \varphi$ is true.*

3. In order to estimate the kernel $\Phi(x, y)$ we introduce the function

$$h(x) = \begin{cases} \int\limits_{v-a}^{a} \left|Q(u, u-v)\right| \mathrm{d}u, & 0 \leq v \leq 2a, \\ \int\limits_{-v-a}^{a} \left|Q(u+v, u)\right| \mathrm{d}u, & -2a \leq v \leq 0, \end{cases} \qquad v = x - y.$$

From (2.5.12), (2.5.13) we have

$$\left|\Phi(x, y)\right| \leq h(x-y), \qquad \int\limits_{-2a}^{2a} h(v)\,\mathrm{d}v < \infty. \tag{2.5.14}$$

It means that the operator

$$\mathcal{B}\varphi = \int\limits_{-a}^{a} \Phi(x, y)\varphi(y)\,\mathrm{d}y \tag{2.5.15}$$

is bounded in $L(-a, a)$.

Theorem 2.5.2. *If the operator S of the form* (2.5.1) *is regular and $\varphi(x) \in L(-a,a)$, then equation* (2.5.4) *has one and only one solution $f = \mathcal{B}\varphi$ which satisfies condition* (2.5.3).

Proof. The equation (2.5.4) is equivalent to the relation

$$-Sf' = \int\limits_{-a}^{x} \varphi(y)\,\mathrm{d}y + C,$$

where C is a constant. Using (2.5.1) and (2.5.11) we obtain:

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(\mathcal{B}\varphi) - \frac{1}{R} \int\limits_{-a}^{a} \left(\int\limits_{-a}^{y} \varphi(u)\,\mathrm{d}u + C \right) \mathcal{L}_1(-y)\,\mathrm{d}y \mathcal{L}_1(x). \tag{2.5.16}$$

Since $\Phi(\pm a, y) = 0$, from (2.5.16) and the requirement $f(\pm a) = 0$, we derive $f(x) = \mathcal{B}\varphi$. Hence, the assertion of the theorem follows. □

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