Chapter 2
Kinematics

In this chapter we discuss the admissible kinematics of a continuous body in the physical space from a differential geometric point of view, as it is proposed by Epstein and Segev [1, 2]. A major part of the chapter deals with the introduction of the necessary differential geometric concepts. These geometric concepts are then directly applied to the description of a first gradient continuum as a model of a deformable body.

Section 2.1 introduces the objects of continuum mechanics, the body and the physical space as manifolds. The idea to regard a body as a smooth manifold originates from Noll [3] and is applied explicitly in [1]. In Sect. 2.2, tangent bundles, vector fields and global flows are defined to formulate the idea of a smooth spatial virtual displacement field. In Sect. 2.3, we introduce the configuration as a mapping between manifolds and discuss the infinite dimensional manifold structure of the set of all differentiable mappings. Furthermore, we introduce pullback tangent bundles which are required to represent elements of the tangent space of the configuration manifold, i.e. virtual displacement fields. In Sect. 2.4, we give a brief introduction to affine connections.

2.1 Body and Space

Many definitions of differential geometric concepts require notions from point set topology. We refer to textbooks like [4] for a detailed treatise on that topic. For the sake of completeness, we briefly introduce the necessary terminology of topology.

A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X having the three properties that (i) the empty set $\varnothing$ and the set X itself are elements of $\mathcal{T}$, (ii) the union of the elements of any subcollection of $\mathcal{T}$ is contained in $\mathcal{T}$, and (iii) the intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$. A *topological space* is the ordered pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X. Elements of $\mathcal{T}$ are called *open sets*, their complements *closed sets*. An open set $U \in \mathcal{T}$ containing $P \in X$ is

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called an *open neighborhood* of P. A topological space $(X, \mathcal{T})$ is called a *Hausdorff space* if for each pair of distinct points of X, there exist open neighborhoods of these points, that are disjoint. A space X is said to be *compact* if any open covering of X contains a finite subcollection that also covers X. A function $x \colon X_1 \to X_2$ between two topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ is said to be continuous if for each open subset V of X_2, its preimage under x, i.e. $x^{-1}(V)$, is an open subset of X_1. If a function $x \colon X_1 \to X_2$ is continuous and bijective with continuous inverse, then x is called a *homeomorphism*. The function x is said to be *proper* if for every compact set $K \subset X_2$, the preimage $x^{-1}(K)$ is compact.

We define the *closed n-dimensional upper half-space* $\mathbb{H}^n \subset \mathbb{R}^n$ as the set

$$\mathbb{H}^n := \{(a^1, \dots, a^n) \in \mathbb{R}^n \mid a^n \geq 0\} .$$

For $n > 0$, we denote the *interior* and the *boundary* of $\mathbb{H}^n$ by $\operatorname{Int} \mathbb{H}^n$ and $\partial \mathbb{H}^n$, respectively, which are defined as

$$\begin{aligned} \operatorname{Int} \mathbb{H}^n &:= \{(a^1, \dots, a^n) \in \mathbb{R}^n \mid a^n > 0\} , \\ \partial \mathbb{H}^n &:= \{(a^1, \dots, a^n) \in \mathbb{R}^n \mid a^n = 0\} . \end{aligned}$$

For the case $n = 0$, $\mathbb{H}^0 := \mathbb{R}^0 = \{0\}$, so $\operatorname{Int} \mathbb{H}^0 = \mathbb{R}^0$ and $\partial \mathbb{H}^0 = \varnothing$.

Definition 2.1 (*Topological Manifold with Boundary*) An *n-dimensional topological manifold with boundary* $\mathcal{M}$ is a Hausdorff space $(X, \mathcal{T})$ with a countable basis and the property, that every point P of X has an open neighborhood $U(P) \subset \mathcal{M}$, which is homeomorphic to an open set of $\mathbb{H}^n$.

The pair (U, x) consisting of an open neighborhood $U \subset \mathcal{M}$ and a homeomorphism x, which maps the open neighborhood U to an open set of $\mathbb{H}^n$, is called a *coordinate chart* on $\mathcal{M}$. We call (U, x) an *interior chart* if $x(U)$ is an open subset of $\mathbb{H}^n$ such that $x(U) \cap \partial \mathbb{H}^n = \varnothing$, and we call it a *boundary chart* if $x(U)$ is an open subset of $\mathbb{H}^n$ such that $x(U) \cap \partial \mathbb{H}^n \neq \varnothing$. A point $P \in \mathcal{M}$ is called an *interior point of* $\mathcal{M}$ if it is in the domain of some interior chart. It is a *boundary point of* $\mathcal{M}$ if it is in the domain of a boundary chart that maps P to $\partial \mathbb{H}^n$. The *boundary of* $\mathcal{M}$, denoted by $\partial \mathcal{M}$, is the set of all boundary points. The *interior of* $\mathcal{M}$ is the set of all interior points, denoted by $\operatorname{Int} \mathcal{M}$. For an interior chart (U, x), the canonical projection $\pi^i \colon \mathbb{R}^n \to \mathbb{R}$, $(a^1, \dots, a^n) \mapsto a^i$ induces the function $x^i \colon U(P) \to V \subset \mathbb{R}$, $x^i := \pi^i \circ x$, which extracts the i-th component of the homeomorphism x and is called the *component function of* x. The n-tuple $(x^1(P), \dots, x^n(P)) \in \mathbb{H}^n$ is called the *coordinate description of* P. For a boundary point $Q \in \partial \mathcal{M}$ the coordinate description is the n-tuple $(x^1(Q), \dots, x^{n-1}(Q), 0) \in \mathbb{H}^n$ where the nth component is zero.

If (U, x) and $(\tilde{U}, \tilde{x})$ are two charts such that $U \cap \tilde{U} \neq \varnothing$, the composite map $\tilde{x} \circ x^{-1} \colon x(U \cap \tilde{U}) \to \tilde{x}(U \cap \tilde{U})$ is called the *transition map from x to $\tilde{x}$*. The transition map relates two different coordinate descriptions of the same point on the manifold which is referred to as *change of coordinates*. Many of the discussed concepts are

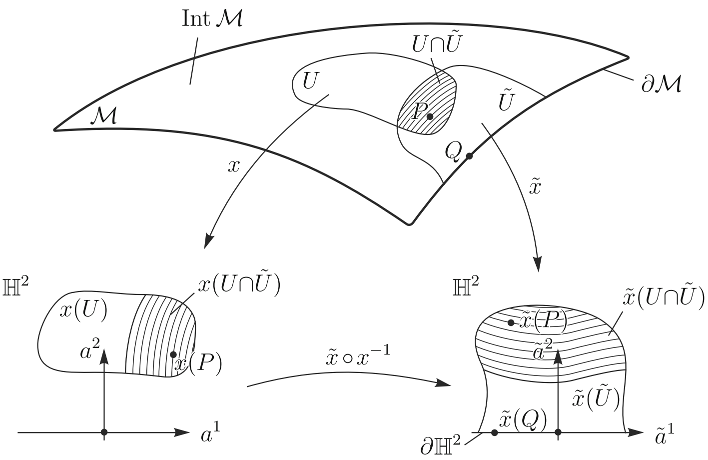

Fig. 2.1 Illustration of a two-dimensional topological manifold with boundary. The chart (U, x) and $(\tilde{U}, \tilde{x})$ are interior and boundary charts, respectively. The point P is an interior point, the point Q is a boundary point

depicted in Fig. 2.1 at the example of a 2-dimensional topological manifold with boundary.

If U and V are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, a function $\hat{\gamma}\colon U \to V$ is said to be C^k*-continuous* or in short C^k if each of its component functions is k-times continuously differentiable. The function $\hat{\gamma}$ is called *smooth* or C^∞ if all its component functions have continuous partial derivatives of all orders. If a C^k-continuous function is also bijective and has a C^k-continuous inverse map, it is called a C^k*-diffeomorphism*. For the case, that a bijective function and its inverse map are smooth, the function is called a *diffeomorphism*. Let $\hat{\rho}$ be a map from a subset, possibly closed, $D \subset \mathbb{R}^n$ to $\mathbb{R}^n$. The function $\hat{\rho}$ is called a (C^k-)diffeomorphism if at each point $x \in D$, it admits an extension to a (C^k-)diffeomorphism, defined on an open neighborhood of x in $\mathbb{R}^n$, cf. [5], Appendix C.

Two charts (U, x) and $(\tilde{U}, \tilde{x})$ are said to be *smoothly compatible* if either $U \cap \tilde{U} = \varnothing$ or the transition map $\tilde{x} \circ x^{-1}$ is a diffeomorphism. We define an *atlas for* $\mathcal{M}$ to be a collection of charts whose domains cover $\mathcal{M}$. An atlas $\mathcal{A}$ is called a *smooth atlas* if any two charts in $\mathcal{A}$ are smoothly compatible with each other. A smooth atlas $\mathcal{A}$ on $\mathcal{M}$ is *maximal* when any chart that is smoothly compatible with every chart in $\mathcal{A}$, is already contained in $\mathcal{A}$.

Definition 2.2 (*Smooth Manifold with Boundary*) An *n-dimensional smooth manifold with boundary* (or in short *smooth manifold*) is an n-dimensional topological manifold with boundary with a maximal smooth atlas $\mathcal{A}$.

One possibility to define an *n-dimensional smooth manifold without boundary* is to exchange the upper half space $\mathbb{H}^n$ by $\mathbb{R}^n$ in the previous definitions about smooth manifolds with boundary. Another possibility, which we choose here, is to define an *n-dimensional smooth manifold without boundary* as a smooth manifold with boundary, whose boundary $\partial\mathcal{M}$ is the empty set $\emptyset$.

Definition 2.3 (*Body*) A *body* is a compact m-dimensional smooth manifold with boundary. Typically, a body will be denoted by $\mathcal{B}$ and its dimension by m. A point P of the body $\mathcal{B}$ is called a *material point* of the body.

Neither $\mathbb{R}^m$ nor the upper half space $\mathbb{H}^m$ are compact sets with respect to the standard topology. Hence, these cannot be bodies by Definition 2.3. Nevertheless, non-compact bodies are often used in linear elasticity, cf. for instance [6]. In the following, we rely on some important mathematical results which do not allow relaxing the compactness assumption. Furthermore, it is worth noticing that the geometric definition of a body does not require metric concepts, such as length or angles. These are information of the body which are obtained by an embedding of the body into the physical space, which is defined in the following way.

Definition 2.4 (*Physical Space*) Let $n \geq m$. The *physical space* is an n-dimensional smooth manifold $\mathcal{S}$ without boundary. A point Q of the physical space $\mathcal{S}$ is called a *space point*.

2.2 Spatial Virtual Displacement Field

When not stated differently, $\mathcal{M}$ and $\mathcal{N}$ are henceforth smooth manifolds of dimensions m and n, respectively. Let $\gamma\colon \mathcal{N} \to \mathcal{M}$ be a map, $P \in \mathcal{N}$ and (V, x) be a chart on $\mathcal{M}$ such that $\gamma(P) \in V$. Furthermore, let (U, θ) be a chart on $\mathcal{N}$ with $P \in U$ and $\gamma(U) \subset V$. Then γ has, as depicted in Fig. 2.2, a *local representation around* P by the composition map $\hat{\gamma} := x \circ \gamma \circ \theta^{-1}\colon \mathbb{H}^n \to \mathbb{H}^m$. The function γ is said to be C^k*-continuous* or in short C^k if for each $P \in \mathcal{N}$ the local representation $\hat{\gamma}$ is

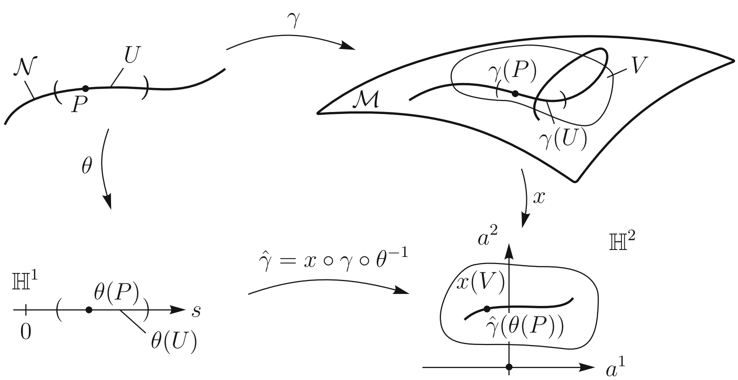

Fig. 2.2 Illustration of a function between a one- and a two-dimensional manifold

C^k. The function γ is called *smooth* or C^∞, if the local representation for each point $P \in \mathcal{N}$ is smooth. The set of all C^k and C^∞ functions between $\mathcal{N}$ and $\mathcal{M}$ are denoted by $C^k(\mathcal{N}, \mathcal{M})$ and $C^\infty(\mathcal{N}, \mathcal{M})$, respectively. If $\gamma \in C^k(\mathcal{N}, \mathcal{M})$ is bijective with a C^k-continuous inverse map, the function is called a *C^k-diffeomorphism*. In the case of a smooth function with a smooth inverse, the function is called a *diffeomorphism*. We denote the set of all smooth real-valued functions by $C^\infty(\mathcal{M}) := C^\infty(\mathcal{M}, \mathbb{R})$.

Definition 2.5 (*Germ*) Let U, V and $W \subset U \cap V$ be open neighborhoods of a point $P \in \mathcal{M}$. Given real-valued smooth functions $f: U \to \mathbb{R}$ and $g: V \to \mathbb{R}$, we define an equivalence relation $\sim_P$ as follows:

$$f \sim_P g \Leftrightarrow \exists\, W \text{ open neighborhood of } P: \ f \equiv g \text{ on } W \ .$$

A *germ of f at P* is the equivalence class

$$[f]_P := \{g: V \to \mathbb{R} \mid g \text{ smooth function in } P, \ (g, V) \sim_P (f, U)\} \ .$$

The set of all germs at P is denoted by $C_P^\infty(\mathcal{M})$.

Let $[f]_P$ and $[g]_P$ be germs at P and $\lambda \in \mathbb{R}$. With the operations

$$\begin{aligned} \lambda[f]_P + [g]_P &= [\lambda f + g]_P \ , \\ [f]_P[g]_P &= [fg]_P \ , \\ [f]_P(P) &= f(P) \ , \end{aligned}$$

the set of all germs $C_P^\infty(\mathcal{M})$ constitute a real vector space.

Definition 2.6 (*Tangent Space*) A linear map $\mathbf{v}: C_P^\infty(\mathcal{M}) \to \mathbb{R}$ is called a *derivation on* $C_P^\infty(\mathcal{M})$, if for all $[f]_P, [g]_P \in C_P^\infty(\mathcal{M})$ the Leibniz rule

$$\mathbf{v}([fg]_P) = f(P)\mathbf{v}([g]_P) + \mathbf{v}([f]_P)g(P) \tag{2.1}$$

holds. The set $T_P\mathcal{M}$ of all derivations on $C_P^\infty(\mathcal{M})$ is called the *tangent space of* $\mathcal{M}$ at P.

Proposition 2.7 *Let* $\mathbf{u}, \mathbf{v} \in T_P\mathcal{M}$, $[f]_P \in C_P^\infty(\mathcal{M})$ *and* $\lambda \in \mathbb{R}$. *Defining addition and scalar multiplication as*

$$\begin{aligned} (\mathbf{u} + \mathbf{v})([f]_P) &:= \mathbf{u}([f]_P) + \mathbf{v}([f]_P) \ , \\ (\lambda\mathbf{u})([f]_P) &:= \lambda\mathbf{u}([f]_P) \ , \end{aligned} \tag{2.2}$$

the tangent space at P is a vector space.

Proof Let $\mathbf{u}, \mathbf{v} \in T_P\mathcal{M}$, $[f]_P, [g]_P \in C_P^\infty(\mathcal{M})$ and $\lambda \in \mathbb{R}$. We need to show that an arbitrary linear combination $\lambda\mathbf{u} + \mathbf{v}$ is linear and satisfies the Leibniz rule (2.1). Linearity of $\lambda\mathbf{u} + \mathbf{v}$ follows directly from the definitions of addition and scalar multiplication (2.2). The Leibniz rule for the linear combination follows by linearity and straight forward computation:

$$
\begin{aligned}
(\lambda\mathbf{u}+\mathbf{v})([f]_P[g]_P) &= \lambda\mathbf{u}([f]_P[g]_P)+\mathbf{v}([f]_P[g]_P) \\
&\overset{(2.1)}{=} \lambda f(P)\mathbf{u}([g]_P)+\lambda\mathbf{u}([f]_P)g(P)+f(P)\mathbf{v}([g]_P)+\mathbf{v}([f]_P)g(P) \\
&= f(P)(\lambda\mathbf{u}+\mathbf{v})([g]_P)+(\lambda\mathbf{u}+\mathbf{v})([f]_P)g(P)\ .
\end{aligned}
$$

Definition 2.8 (*Induced Partial Derivative*) Let (U, x) be a chart on $\mathcal{M}$, $P \in U$ and $f : U \to \mathbb{R}$ a smooth function. We define an *induced partial derivative at P on $\mathcal{M}$* for $i \in \{1, \ldots, m\}$ as

$$\partial_{x^i}|_P([f]_P) \coloneqq \partial_i(f \circ x^{-1})|_{x(P)}\ , \tag{2.3}$$

where ∂_i denotes the ith partial derivative on $\mathbb{R}^m$.

Using the definition of the induced partial derivative together with the product rule of $\mathbb{R}^m$, it can easily be shown that the induced partial derivative at P is a linear map which satisfies the Leibniz rule (2.1) and consequently is a derivation on $C_P^\infty(\mathcal{M})$.

Theorem 2.9 *Let (U, x) be a chart on $\mathcal{M}$ and $P \in U$. The derivations $(\partial_{x^1}|_P, \ldots, \partial_{x^m}|_P)$ form a basis of the tangent space $T_P\mathcal{M}$. Consequently, applying a vector $\mathbf{v} \in T_P\mathcal{M}$ on a germ $[f]_P \in C_P^\infty(\mathcal{M})$, the vector can be represented as a linear combination*

$$\mathbf{v}([f]_P) = \mathbf{v}([x^i]_P)\partial_{x^i}|_P([f]_P) = v^i\partial_{x^i}|_P([f]_P)\ , \tag{2.4}$$

where summation over repeated indices is applied and the components v^i are defined as $\mathbf{v}([x^i]_P)$.

Proof For the proof we refer to [7], Sect. 1.8 or to [8], Sect. 5.6.

Excluded analytic functions, each germ of a smooth function has a representative which is defined on the whole $\mathcal{M}$, cf. [7]. Thus, we henceforth omit the brackets designating the equivalence class, defining a germ of a smooth function at a point on a manifold.

The definition of tangent vectors of $\mathcal{M}$ at a point P as the set of all derivations on $C_P^\infty(\mathcal{M})$ is a coordinate free and consequently chart independent definition. Nevertheless, in applications, charts have to be chosen and it is of major interest how objects transform under a change of coordinates. In the following, we show how the basis and the components of a tangent vector transform. Let (U, x) and $(\tilde{U}, \tilde{x})$ be charts of $\mathcal{M}$ and let $P \in U \cap \tilde{U}$. The definition of the induced partial derivative (2.3) together with the chain rule from higher dimensional calculus implies a transformation rule for a change of coordinates. Let $f \in C^\infty(\mathcal{M})$, then by a telescopic expansion it follows

$$
\begin{aligned}
\partial_{\tilde{x}^i}|_P(f) &\overset{(2.3)}{=} \partial_i(f \circ \tilde{x}^{-1})|_{\tilde{x}(P)} = \partial_i(f \circ x^{-1} \circ x \circ \tilde{x}^{-1})|_{\tilde{x}(P)} \\
&= \partial_j(f \circ x^{-1})|_{x(P)} \partial_i(x^j \circ \tilde{x}^{-1})|_{\tilde{x}(P)} = \Lambda_i^j \partial_{x^j}|_P(f) \ ,
\end{aligned}
$$

where we have recognized the transformation matrix $\Lambda_i^j := \partial_i(x^j \circ \tilde{x}^{-1})|_{\tilde{x}(P)}$. By an abuse of notation, where a point in $\mathbb{H}^m$ is named by the coordinate function $\tilde{x}^i$, the transformation matrix is often introduced as $\Lambda_i^j = \frac{\partial x^j}{\partial \tilde{x}^i}$, cf. for instance [9]. The transformation is independent of the choice of the smooth function f and we summarize the important result as follows:

$$
\partial_{\tilde{x}^i}|_P = \Lambda_i^j \partial_{x^j}|_P \ , \quad \Lambda_i^j := \partial_i(x^j \circ \tilde{x}^{-1})|_{\tilde{x}(P)} \ . \tag{2.5}
$$

Let $\mathbf{v} \in T_P\mathcal{M}$. The components $\tilde{v}^i = \mathbf{v}(\tilde{x}^i)$ of the coordinate representation in the chart $(\tilde{U}, \tilde{x})$ can be transformed further using the coordinate representation of $\mathbf{v}$ in the chart (U, x), i.e.

$$
\tilde{v}^i = \mathbf{v}(\tilde{x}^i) \overset{(2.4)}{=} v^j \partial_{x^j}|_P(\tilde{x}^i) \overset{(2.3)}{=} \partial_j(\tilde{x}^i \circ x^{-1})|_{x(P)} v^j = \tilde{\Lambda}^i_j v^j \ ,
$$

with the transformation matrix $\tilde{\Lambda}^i_j := \partial_j(\tilde{x}^i \circ x^{-1})|_{x(P)}$. Hence, the transformation rule for the components of a tangent vector is

$$
\tilde{v}^i = \tilde{\Lambda}^i_j v^j \ , \quad \tilde{\Lambda}^i_j := \partial_j(\tilde{x}^i \circ x^{-1})|_{x(P)} \ .
$$

Definition 2.10 (*Cotangent Space*) For each $P \in \mathcal{M}$, the *cotangent space at P*, denoted by $T_P^*\mathcal{M}$, is the dual space to $T_P\mathcal{M}$. An element of the cotangent space is called a *covector*.

Let $\mathrm{d}x^i|_P \in T_P^*\mathcal{M}$ denote a dual basis to $\partial_{x^j}|_P$ which satisfies $\mathrm{d}x^i|_P(\partial_{x^j}|_P) = \delta^i_j$. According to (A.5), a covector $\boldsymbol{\omega} \in T_P^*\mathcal{M}$ can be represented as a linear combination

$$
\boldsymbol{\omega} = \omega_i \mathrm{d}x^i|_P \ ,
$$

with the components $\omega_i = \boldsymbol{\omega}\left(\partial_{x^i}|_P\right)$. Let (U, x) and $(\tilde{U}, \tilde{x})$ be charts of $\mathcal{M}$ and let $P \in U \cap \tilde{U}$. Using (2.5), the transformation rule of the component $\tilde{\omega}_i$ follows by linearity and duality of the base vectors

$$
\tilde{\omega}_i = \boldsymbol{\omega}(\partial_{\tilde{x}^i})|_P \overset{(2.5)}{=} \omega_k \mathrm{d}x^k|_P(\Lambda_i^j \partial_{x^j}|_P) = \Lambda_i^j \omega_j \ .
$$

Thus, the transformation rule for the components of a covector is

$$
\tilde{\omega}_i = \Lambda_i^j \omega_j \ , \quad \Lambda_i^j = \partial_i(x^j \circ \tilde{x}^{-1})|_{\tilde{x}(P)} \ , \tag{2.6}
$$

which is the same as for the base vectors of a tangent vector. Since the transformation (2.6) is performed by Λ_i^j, i.e. the 'inverse' of $\tilde{\Lambda}_i^j$, it is classically called contravariant transformation. A covector $\boldsymbol{\omega}$ has its representation as a linear combination for any chart. Hence, the transformation of the components of a covector (2.6) immediately implies the transformation rule of the dual base vectors $\mathrm{d}x^i|_P$ by

$$\boldsymbol{\omega} = \omega_j \mathrm{d}x^j|_P = \tilde{\Lambda}^i_j \tilde{\omega}_i \mathrm{d}x^j|_P = \tilde{\omega}_i \mathrm{d}\tilde{x}^i|_P \ .$$

The transformation rule for the dual base vectors is

$$\mathrm{d}\tilde{x}^i|_P = \tilde{\Lambda}^i_j \mathrm{d}x^j|_P \ , \quad \tilde{\Lambda}^i_j = \partial_j(\tilde{x}^i \circ x^{-1})|_{x(P)} \ ,$$

which is the same transformation rule as for the components of a tangent vector, i.e. a covariant transformation.

Definition 2.11 (*Tangent Bundle*) The *tangent bundle of* $\mathcal{M}$ is the triple $(T\mathcal{M}, \pi_{\mathcal{M}}, \mathcal{M})$, where $T\mathcal{M}$ denotes the disjoint union of the tangent spaces at all points of $\mathcal{M}$

$$T\mathcal{M} := \bigcup_{P\in\mathcal{M}} \{P\} \times T_P\mathcal{M} \ .$$

The manifold $\mathcal{M}$ is *the base space* and $\pi_{\mathcal{M}}$ denotes the natural projection $\pi_{\mathcal{M}}\colon T\mathcal{M} \to \mathcal{M}$. The natural projection maps $\mathbf{v} \in T\mathcal{M}$ to its *base point* $P \in \mathcal{M}$.

Definition 2.12 (*Cotangent Bundle*) The *cotangent bundle of* $\mathcal{M}$ is the triple $(T^*\mathcal{M}, \pi_{\mathcal{M}}, \mathcal{M})$, where $T^*\mathcal{M}$ denotes the disjoint union of the cotangent spaces at all points of $\mathcal{M}$

$$T^*\mathcal{M} := \bigcup_{P\in\mathcal{M}} \{P\} \times T^*_P\mathcal{M} \ .$$

The manifold $\mathcal{M}$ is *the base space* and $\pi_{\mathcal{M}}$ denotes the natural projection $\pi_{\mathcal{M}}\colon T^*\mathcal{M} \to \mathcal{M}$. The natural projection maps $\boldsymbol{\omega} \in T\mathcal{M}$ to its *base point* $P \in \mathcal{M}$.

The tangent and cotangent bundle have again the structure of a manifold, cf. [5] or [7]. All upcoming operations on elements of the tangent bundle $T\mathcal{M}$ do not act on the base points. Hence, we often use the slight abuse of notation by referring to the vectorial part of $\mathbf{v} \in T\mathcal{M}$ by the same symbol, i.e. "$\mathbf{v} = (P, \mathbf{v})$". For any other bundle structure we do the same. From the context, however, it will be clear which object is meant.

Definition 2.13 (*Vector Field*) A *vector field on* $\mathcal{M}$ is a section of the map $\pi_{\mathcal{M}}\colon T\mathcal{M} \to \mathcal{M}$. That means, it is a continuous map $\mathbf{v}\colon \mathcal{M} \to T\mathcal{M}$ with the property that

$$\pi_{\mathcal{M}} \circ \mathbf{v} = \mathrm{Id}_{\mathcal{M}} \ .$$

The set of C^k-continuous sections on $T\mathcal{M}$ is denoted by $C^k(T\mathcal{M})$. The set of smooth sections is denoted by $\Gamma(T\mathcal{M})$.

Let (U, x) be a chart on $\mathcal{M}$ and $\mathbf{v} \in \Gamma(T\mathcal{M})$, then the value of $\mathbf{v}$ can be represented at any point $P \in U$ in coordinates as

$$\mathbf{v}(P) = (x(P), v^i(P)\partial_{x^i}|_P) \ .$$

This defines m functions $v^i : U \to \mathbb{R}$, called the *component functions of* $\mathbf{v}$ in the given chart.

Definition 2.14 (*Smooth Global Flow*) A *smooth global flow* on $\mathcal{M}$ is a smooth map $\varphi\colon \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ satisfying the following properties for all $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ and $P \in \mathcal{M}$:

$$\varphi(\varepsilon_1, \varphi(\varepsilon_2, P)) = \varphi(\varepsilon_1 + \varepsilon_2, P) \ , \quad \varphi(0, P) = P \ . \tag{2.7}$$

Let $f \in C^\infty(\mathcal{M})$ and $P \in \mathcal{M}$, then a smooth global flow $\varphi\colon \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ induces a smooth vector field $\delta\boldsymbol{\varphi} \in \Gamma(T\mathcal{M})$ defined by

$$\delta\boldsymbol{\varphi}(P)(f) = (\varphi(0, P), \delta\boldsymbol{\varphi}(P)(f)) := \big(P, \partial_1(f \circ \varphi)|_{(0,P)}\big) \ . \tag{2.8}$$

The smooth vector field $\delta\boldsymbol{\varphi}$ is called the *infinitesimal generator of* φ. We want to emphasize, that the δ-sign does not act as an operator and remains mainly as a decoration due to historical reasons.

Let (U, x) be a chart on $\mathcal{M}$ and $P \in U$, then the infinitesimal generator is represented at P as

$$\begin{aligned} \delta\boldsymbol{\varphi}(P)(f) &= \partial_1(f \circ x^{-1} \circ x \circ \varphi)|_{(0,P)} = \partial_i(f \circ x^{-1})|_{x(P)}\partial_1(x^i \circ \varphi)|_{(0,P)} \\ &= \partial_1(x^i \circ \varphi)|_{(0,P)}\partial_{x^i}|_P(f) = \delta\varphi^i(P)\partial_{x^i}|_P(f) \ , \end{aligned} \tag{2.9}$$

where the component functions of the infinitesimal generator evaluated at P are identified as $\delta\varphi^i(P) := \partial_1(x^i \circ \varphi)\big|_{(0,P)}$.

Definition 2.15 (*Spatial Virtual Displacement Field*) Let $\varphi\colon \mathbb{R} \times \mathcal{S} \to \mathcal{S}$ be a smooth global flow on the physical space $\mathcal{S}$ with an associated infinitesimal generator $\delta\boldsymbol{\varphi} \in \Gamma(T\mathcal{S})$. The infinitesimal generator of φ is called the *spatial virtual displacement field*.

2.3 Configuration Space

The following definition of the pullback bundle is illustrated in Fig. 2.3.

Definition 2.16 (*Pullback Tangent Bundle*) Let $(T\mathcal{M}, \pi_\mathcal{M}, \mathcal{M})$ be the tangent bundle and $\gamma\colon \mathcal{N} \to \mathcal{M}$ be a map. The *pullback tangent bundle by* γ is the bundle $(\gamma^*T\mathcal{M}, \gamma^*\pi_\mathcal{M}, \mathcal{N})$, where the total space is defined as

$$\gamma^*T\mathcal{M} := \{(P, \mathbf{v}) \in \mathcal{N} \times T\mathcal{M} : \pi_\mathcal{M}(\mathbf{v}) = \gamma(P)\}$$

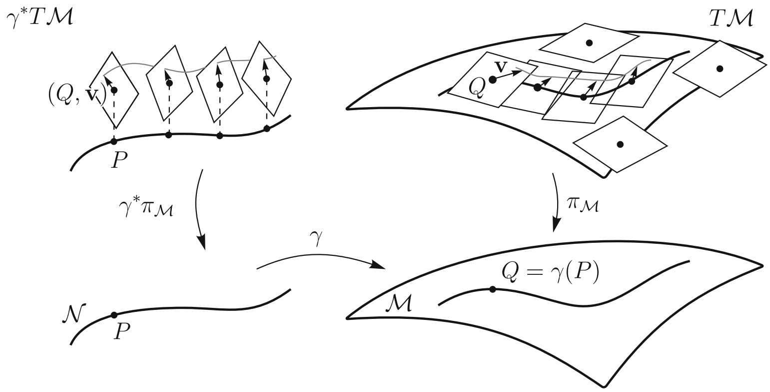

Fig. 2.3 Illustration of a pullback tangent bundle $\gamma^*T\mathcal{M}$ over a one-dimensional base manifold $\mathcal{N}$. Loosely, the pullback bundle can be thought of as a base manifold $\mathcal{N}$, in which at all points P on $\mathcal{N}$ and for $Q = \gamma(P)$, the tangent space $T_Q\mathcal{M}$ is attached

and the projection $\gamma^*\pi_{\mathcal{M}}$ of the pullback tangent bundle is defined as

$$(\gamma^*\pi_{\mathcal{M}})\,(P, \mathbf{v}) = P\ .$$

It can be shown that the pullback tangent bundle is a fiber bundle. For the definition of a fiber bundle we refer to textbooks like [10] or [11]. Let $\gamma \in C^k(\mathcal{N}, \mathcal{M})$ and $\mathbf{v} \in \Gamma(T\mathcal{M})$. Then the *pullback section* $\gamma^*\mathbf{v}$ is a C^k-section of $\gamma^*T\mathcal{M}$. The evaluation of the section at P is

$$\gamma^*\mathbf{v}(P) = (P, \mathbf{v}(\gamma(P)))\ .$$

Let $P \in \mathcal{N}$ and (V, x) be a chart on $\mathcal{M}$ with $\gamma(P) \in V$. Then for each $P \in \mathcal{N}$ the evaluation of $\gamma^*\mathbf{v}$ at P can be represented as

$$\gamma^*\mathbf{v}(P) = \Big(P, \Big((x \circ \gamma)(P), (v^i \circ \gamma)(P)(\partial_{x^i} \circ \gamma)|_P\Big)\Big)\ .$$

Let $\tilde{\mathbf{v}} \colon \mathcal{N} \to T\mathcal{M}$ be a C^k-continuous function such that $\pi_{\mathcal{M}}(\tilde{\mathbf{v}}) = \gamma$, then $\tilde{\mathbf{v}}$ is called a *vector field along* γ. For an appropriate chart (U, x) on $\mathcal{M}$ and for each $P \in \mathcal{N}$ the vector field along γ is represented in coordinates as

$$\tilde{\mathbf{v}}(P) = \Big((x \circ \gamma)(P), \tilde{v}^i(P)(\partial_{x^i} \circ \gamma)|_P\Big)\ .$$

The pullback section $\gamma^*\mathbf{v}$ and the vector field $\tilde{\mathbf{v}}$ along γ differ only in the additional base point in the pullback section. Hence, the isomorphism between the set of pullback sections $C^k(\gamma^*T\mathcal{M})$ and the set of vector fields along γ is obvious. Since a pullback section contains more geometric structure than a vector field along γ, we prefer in the following the pullback section.

Definition 2.17 (*Differential*) Let $k > 0$ and $\gamma\colon \mathcal{N} \to \mathcal{M}$ be a C^k-continuous map. The *differential* $D\gamma(P)$ *of* γ at P is a linear map

$$D\gamma(P)\colon T_P\mathcal{N} \to T_{\gamma(P)}\mathcal{M}$$

such that for $\mathbf{v} \in T_P\mathcal{N}$ and $f \in C^\infty(\mathcal{M})$

$$D\gamma(P)\mathbf{v}(f) = \mathbf{v}(f \circ \gamma)\ . \tag{2.10}$$

Let $P \in \mathcal{N}$, (V, x) be a chart on $\mathcal{M}$ with $\gamma(P) \in V$ and let (U, θ) be a chart on $\mathcal{N}$ with $P \in U$ and $\gamma(U) \subset V$. Then the coordinate representation of the differential $D\gamma(P)$ applied to a tangent vector $\mathbf{v} \in T_P\mathcal{N}$ is derived using the local representation $\hat{\gamma} := x \circ \gamma \circ \theta^{-1}$ as follows:

$$\begin{aligned} D\gamma(P)\mathbf{v}(f) &\overset{(2.10)}{=} \mathbf{v}(f \circ \gamma) = \mathbf{v}(f \circ x^{-1} \circ \hat{\gamma} \circ \theta) \overset{(2.4)}{=} v^i \partial_{\theta^i}|_P(f \circ x^{-1} \circ \hat{\gamma} \circ \theta) \\ &\overset{(2.3)}{=} v^i \partial_i(f \circ x^{-1} \circ \hat{\gamma})|_{\theta(P)} = v^i \partial_j(f \circ x^{-1})|_{x(\gamma(P))} \partial_i \hat{\gamma}^j|_{\theta(P)} \\ &\overset{(2.3)}{=} \partial_i \hat{\gamma}^j|_{\theta(P)} v^i \partial_{x^i}|_{\gamma(P)}(f) = F_i^j(P) v^i \partial_{x^i}|_{\gamma(P)}(f)\ , \end{aligned} \tag{2.11}$$

where in the last line we have made use of the component functions $F_i^j := \partial_i \hat{\gamma}^j \circ \theta$.

Definition 2.18 (*Tangent Map*) Let $k > 0$ and $\gamma\colon \mathcal{N} \to \mathcal{M}$ be a C^k-continuous map inducing the pullback tangent bundle $(\gamma^* T\mathcal{M}, \gamma^*\pi_{\mathcal{M}}, \mathcal{N})$. Then the *tangent map* $T\gamma$ is defined as the bundle homomorphism over $\mathcal{N}$

$$\begin{aligned} T\gamma\colon T\mathcal{N} &\to \gamma^* T\mathcal{M} \\ (P, \mathbf{v}) &\mapsto (P, (\gamma(P), D\gamma(P)\mathbf{v}(P)))\ , \end{aligned} \tag{2.12}$$

satisfying the commutative diagram:

$$\begin{tikzcd} T\mathcal{N} \arrow[rr, "T\gamma"] \arrow[dr, "\pi_{\mathcal{N}}"'] & & \gamma^* T\mathcal{M} \arrow[dl, "\gamma^*\pi_{\mathcal{M}}"] \\ & \mathcal{N} & \end{tikzcd}$$

Definition 2.19 (*Embedding*) A C^k-continuous and proper map $\gamma\colon \mathcal{N} \to \mathcal{M}$ is called a C^k*-embedding* if its tangent map $T\gamma$ is injective. The set of all C^k-embeddings is denoted by $\mathrm{Emb}^k(\mathcal{N}, \mathcal{M})$.

The analysis of mappings between manifolds is an important part of the theory of global analysis, cf. [12–14]. For a short historical overview of the theory of manifolds of mappings, which started in the late fifties with Eells [15], we refer to Marsden [16].

The beginning of global analysis was strongly influenced by the works [12, 17, 18]. The special case of embeddings is treated in [19]. For the application of global analysis in physics, we refer to [16, 20].

Theorem 2.20 (*Manifold Structure of $C^k(\mathcal{N}, \mathcal{M})$, [20], Theorem 5.4.1*) *Given two smooth manifolds $\mathcal{N}$ and $\mathcal{M}$ of which $\mathcal{N}$ is compact and $\mathcal{M}$ without boundary. Then for each integer $k < \infty$ the set $C^k(\mathcal{N}, \mathcal{M})$ is a smooth manifold modeled over Banach spaces, i.e. $C^k(\mathcal{N}, \mathcal{M})$ is a Banach manifold.*

Proof For a proof and a discussion about the topology of $C^k(\mathcal{N}, \mathcal{M})$, we refer to [20].

Definition 2.21 (*Configuration*) Let $\mathcal{B}$ be a body and $\mathcal{S}$ the physical space. We define the *configuration of a first gradient continuum* (or *continuous body*) to be a C^1-embedding κ of the body $\mathcal{B}$ into the physical space $\mathcal{S}$. The set of all C^1-embeddings, i.e. $Emb^1(\mathcal{B}, \mathcal{S})$, is called the *configuration manifold* $\mathcal{Q}$.

As recognized by Segev [2], the requirement that a configuration of a body into physical space is an embedding, is based upon two classical principles, cf. [21], Sect. 16. These are, the *permanence of matter* and the *principle of impenetrability*. The former states that no region of positive finite volume is deformed into one of zero or infinite volume. The latter states that one portion of matter never penetrates within another. In order that the set of configurations admits the structure of a manifold, Theorem 2.20 requires a body $\mathcal{B}$ to be a compact manifold.

Definition 2.22 (*Virtual Displacement Field*) Let $\delta\boldsymbol{\varphi} \in \Gamma(T\mathcal{S})$ be the spatial virtual displacement field and $\kappa \in \mathcal{Q}$. Then the *virtual displacement field* of a continuous body is defined as the pullback section $\delta\boldsymbol{\kappa} = \kappa^*\delta\boldsymbol{\varphi} \in C^1(\gamma^*T\mathcal{S})$.

Theorem 2.23 (*Tangent Space of $C^k(\mathcal{N}, \mathcal{M})$*) *Let $\mathcal{N}$ and $\mathcal{M}$ be manifolds of which $\mathcal{N}$ is compact and $\mathcal{M}$ without boundary. For any map $\gamma \in C^k(\mathcal{N}, \mathcal{M})$, the tangent space at γ $T_\gamma C^k(\mathcal{N}, \mathcal{M})$ is isomorphic to the set of pullback sections $C^k(\gamma^*T\mathcal{M})$.*

The identification of the tangent space at γ with C^k-sections of the pullback tangent bundle, is stated in [2]. For a proof it is referred to [12, 13, 18]. Also in [22] the same identification without a proof is stated with reference to [23, 24]. In [25] the isomorphism is mentioned merely as a note of Theorem 11.1 without proof. Nevertheless, a complete proof for the above stated assumptions could neither be found nor can be given in this book by the author. Strongly related results with proof can be found in [20], Theorem. 5.4.3, for the case of smooth mappings γ. Using the assumption of a Riemannian manifold $\mathcal{N}$ [18], "Corollaries for C^k", serves as a reference. Inspired by Binz et al. [20], we prove one direction which should support the reasonability of the theorem.

Proof (*Idea of Proof*) Let $\varphi\colon \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ be a global flow on $\mathcal{M}$. Then the composition function

$$\tilde{\varphi}\colon \mathbb{R} \times \mathcal{N} \to \mathcal{M}\ , \quad (\varepsilon, P) \mapsto \tilde{\varphi}(\varepsilon, P) = \varphi(\varepsilon, \gamma(P))$$

defines a smooth curve through the $C^k(\mathcal{N}, \mathcal{M})$ manifold. The properties of a global flow (2.7) imply that

$$\tilde{\varphi}(0, \cdot) = \gamma \ .$$

Let $f \in C^\infty(\mathcal{M})$ and $P \in \mathcal{N}$. Then the composition function $\tilde{\varphi}$ induces the section $\gamma^* \delta\tilde{\boldsymbol{\varphi}} \in C^k(\gamma^* T\mathcal{M})$ defined by

$$\gamma^* \delta\tilde{\boldsymbol{\varphi}}(P)(f) = (P, (\tilde{\varphi}(0, P), \delta\tilde{\boldsymbol{\varphi}}(P)(f))) = \big(P, \big(\gamma(P), \partial_1(f \circ \tilde{\varphi})|_{(0,P)}\big)\big) \ .$$

Let (U, x) be a chart on $\mathcal{M}$ and $\gamma(P) \in U$. Then by (2.9), the section through the pullback tangent bundle can locally be represented as

$$\gamma^* \delta\tilde{\boldsymbol{\varphi}}(P) = \Big(P, \Big((x \circ \gamma)(P), (\delta\varphi^i \circ \gamma)(P)(\partial_{x^i} \circ \gamma)|_P\Big)\Big) \ .$$

A tangent vector can alternatively be defined, cf. [26], by an equivalence class of curves which pass with the same velocity through the same point on the manifold. The composition function $\tilde{\varphi}$ is such a curve through $C^k(\mathcal{N}, \mathcal{M})$. Since the section $\gamma^* \delta\tilde{\boldsymbol{\varphi}}$ is obtained by taking the velocity of the smooth curve $\tilde{\varphi}$ at γ, a tangent vector of $C^k(\mathcal{N}, \mathcal{M})$ induces a section through the pullback tangent bundle $\gamma^* T\mathcal{M}$. The inverse, to show that a section $C^k(\gamma^* T\mathcal{M})$ induces a smooth curve through $C^k(\mathcal{N}, \mathcal{M})$ and that the involved mappings are bijective are necessary to finish the proof of the isomorphism rigorously.

Corollary 2.24 *The tangent space to* $\mathrm{Emb}^k(\mathcal{N}, \mathcal{M})$ *at* γ *is isomorphic to* $T_\gamma C^k(\mathcal{N}, \mathcal{M})$.

Proof According to [19] the set $\mathrm{Emb}^k(\mathcal{N}, \mathcal{M})$ is open in the set of $C^k(\mathcal{N}, \mathcal{M})$.

Due to Theorem 2.23, the virtual displacement field of a continuous body $\delta\boldsymbol{\kappa} \in C^1(\kappa^* T\mathcal{S})$ can be identified with an element of the tangent space $T_\kappa\mathcal{Q}$. This follows the tradition of analytical mechanics, where the virtual displacements are tangent vectors to the finite-dimensional configuration manifold, cf. [27].

2.4 Affine Connection

Definition 2.25 (*Affine Connection*) Let $\mathbf{u}, \mathbf{v} \in \Gamma(T\mathcal{M})$, $f \in C^\infty(\mathcal{M})$. An *(affine) connection* on $\mathcal{M}$ is a mapping ∇ which assigns to every pair $\mathbf{u}, \mathbf{v}$ another vector field $\nabla_\mathbf{u}\mathbf{v} \in \Gamma(T\mathcal{M})$ with the following properties:

$$\begin{aligned} &(a) \ \ \nabla_\mathbf{u}\mathbf{v} \text{ is bilinear in } \mathbf{u} \text{ and } \mathbf{v} \ , \\ &(b) \ \ \nabla_{f\mathbf{u}}\mathbf{v} = f\nabla_\mathbf{u}\mathbf{v} \ , \\ &(c) \ \ \nabla_\mathbf{u}(f\mathbf{v}) = f\nabla_\mathbf{u}\mathbf{v} + \mathbf{u}(f)\mathbf{v} \ . \end{aligned} \tag{2.13}$$

We call $\nabla_\mathbf{u}\mathbf{v}$ the *covariant derivative of* $\mathbf{v}$ *along* $\mathbf{u}$.

Let (U, x) be a chart on $\mathcal{M}$, then we define the m^3 functions Γ^k_{ij} by

$$\nabla_{\partial_{x^i}}\left(\partial_{x^j}\right) = \Gamma^k_{ij}\partial_{x^k} \ . \tag{2.14}$$

The Γ^k_{ij} are called the *Christoffel symbols* of the connection ∇.

Definition 2.26 (*Covariant Derivative*) Let $\boldsymbol{\omega} \in \Gamma(T^*\mathcal{M})$ and $\mathbf{u} \in \Gamma(T\mathcal{M})$. For every vector field $\mathbf{v} \in \Gamma(T\mathcal{M})$ we consider the tensor field $\nabla\mathbf{v} \in \Gamma(T\mathcal{M} \otimes T^*\mathcal{M})$ defined by

$$\nabla\mathbf{v}(\boldsymbol{\omega}, \mathbf{u}) := \boldsymbol{\omega}(\nabla_{\mathbf{u}}\mathbf{v}) \ . \tag{2.15}$$

The tensor field $\nabla\mathbf{v}$ is called the *covariant derivative of* $\mathbf{v}$.

Let (U, x) be a chart on $\mathcal{M}$, then $\mathbf{v} = v^i\partial_{x^i}$ and $\nabla\mathbf{v} = v^i_{;j}\partial_{x^i} \otimes \mathrm{d}x^j$. Notice the semicolon in the component of the covariant derivative. This has its origin from index notation, in which only components of the tensors are written. The semicolon distinguishes between partial derivative, i.e. application of the base vectors to the components of a vector, and covariant derivative of a vector field. According to the representation of a tensor as a linear combination (A.9) together with (2.13b) and (2.14), we obtain the component functions of the tensor field as

$$\begin{aligned} v^i_{;j} &= \nabla\mathbf{v}(\mathrm{d}x^i, \partial_{x^j}) \overset{(2.15)}{=} \mathrm{d}x^i(\nabla_{\partial_{x^j}}(v^k\partial_{x^k})) \\ &= \mathrm{d}x^i(\partial_{x^j}(v^k)\partial_{x^k} + v^k\Gamma^l_{jk}\partial_{x^l}) = \partial_{x^j}(v^i) + \Gamma^i_{jk}v^k \ . \end{aligned}$$

Definition 2.27 (*Covariant Derivative of Pullback Section*) Let $\gamma \in C^k(\mathcal{N}, \mathcal{M})$, $\mathbf{a} \in \Gamma(T\mathcal{N})$, $\mathbf{v} \in \Gamma(T\mathcal{M})$ with the associated pullback section $\gamma^*\mathbf{v} \in C^k(\gamma^*T\mathcal{M})$ and $\boldsymbol{\omega} \in C^k(\gamma^*T^*\mathcal{M})$. Let $\mathcal{M}$ be equipped with an affine connection ∇. Then, for every pullback section $\gamma^*\mathbf{v}$, the tensor field $(\gamma^*\nabla)(\gamma^*\mathbf{v}) \in C^k(\gamma^*T\mathcal{M} \otimes T^*\mathcal{N})$ over $\mathcal{N}$ is defined as

$$(\gamma^*\nabla)(\gamma^*\mathbf{v})(\boldsymbol{\omega}, \mathbf{a}) := \boldsymbol{\omega}(\gamma^*(\nabla_{T\gamma\mathbf{a}}\mathbf{v})) \ . \tag{2.16}$$

The tensor field $(\gamma^*\nabla)(\gamma^*\mathbf{v})$ is called *covariant derivative of* $\gamma^*\mathbf{v}$.

Let (U, θ) be a chart on $\mathcal{N}$ and let (V, x) be a chart on $\mathcal{M}$ such that $\gamma(U) \subset V$. Let $\mathbf{v} \in \Gamma(T\mathcal{M})$ be defined on the whole of V. Then the covariant derivative of the pullback section $\gamma^*\mathbf{v}$ corresponds to a tensor field $(\gamma^*\nabla)(\gamma^*\mathbf{v}) = (\gamma^*v^i)_{;j}(\partial_{x^i} \circ \gamma) \otimes \mathrm{d}\theta^j$. The computation of the component functions of the tensor field follows (A.9), i.e.

$$(\gamma^*v^i)_{;j} = (\gamma^*\nabla)(\gamma^*\mathbf{v})(\mathrm{d}x^i \circ \gamma, \partial_{\theta^j}) \overset{(2.16)}{=} (\mathrm{d}x^i \circ \gamma)(\gamma^*(\nabla_{T\gamma\partial_{\theta^j}}\mathbf{v})) \ .$$

Let $\hat{\gamma} = x \circ \gamma \circ \theta^{-1}$ be the local representation of γ around $P \in U$. Using (2.11), the vectorial part of the tangent map $T\gamma$ of a vector field $\partial_{\theta^j} \in \Gamma(T\mathcal{N})$ can locally be represented as

$$D\gamma\,\partial_{\theta^j} = (\partial_j \hat{\gamma}^i \circ \theta)\,\partial_{x^i}|_{\gamma(\cdot)} = F^i_j(\partial_{x^i} \circ \gamma)\;. \tag{2.17}$$

Let $P \in U$. Using a telescopic expansion and applying the chain rule, we show the following identity:

$$\begin{aligned}
\partial_{\theta^j}|_P(v^i \circ \gamma) &\overset{(2.3)}{=} \partial_j(v^i \circ x^{-1} \circ x \circ \gamma \circ \theta^{-1})|_{\theta(P)} \\
&= \partial_k(v^i \circ x^{-1})|_{(x(\gamma(P))}(\partial_j \hat{\gamma}^k \circ \theta)(P) \\
&= \partial_{x^k}|_{\gamma(P)}(v^i)\,F^k_j(P)\;.
\end{aligned} \tag{2.18}$$

Using property (2.13b) and the local representation by the Christoffel symbols (2.14) we compute:

$$\begin{aligned}
\gamma^*(\nabla_{T\gamma\partial_{\theta^j}}\mathbf{v}) &\overset{(2.17)}{=} F^i_j\gamma^*((\partial_{x^i}(v^k)\partial_{x^k} + v^k\Gamma^r_{ik}\partial_{x^r})|_{\gamma(\cdot)}) \\
&\overset{(2.18)}{=} (\partial_{\theta^j}(v^k \circ \gamma) + (v^r \circ \gamma)(\Gamma^k_{ir} \circ \gamma)F^i_j)(\partial_{x^k} \circ \gamma)\;.
\end{aligned}$$

Hence, the component functions of the covariant derivative of $\gamma^*\mathbf{v}$ are represented locally as

$$(\gamma^* v^i)_{;j} = \partial_{\theta^j}(v^k \circ \gamma) + (v^r \circ \gamma)(\Gamma^k_{ir} \circ \gamma)F^i_j\;. \tag{2.19}$$

Example 2.28 Let $\mathcal{N} = I$ be an interval of $\mathbb{R}$, $\gamma\colon I \to \mathcal{M}$ be a curve on $\mathcal{M}$ and $\mathbf{v} \in \Gamma(T\mathcal{M})$. An illustrative application of the covariant derivative of a pullback section is its correlation to the covariant derivative of $\mathbf{v}$ along a curve γ, denoted by $\nabla_{\dot\gamma}\mathbf{v}$. For the definition of a covariant derivative of $\mathbf{v}$ along a curve γ we refer to [28], Definition 2.7.3. Let $(I, \theta = \mathrm{Id}_\mathrm{I})$ and (U, x) be charts on I and $\mathcal{M}$, respectively, then $F^i_1 = \partial_1(x^i \circ \gamma)$. Using (2.16) and (2.19), we obtain a vector field $\mathbf{v}$ along γ when taking the covariant derivative of $\gamma^*\mathbf{v}$ along ∂_θ, i.e.

$$\begin{aligned}
&(\gamma^*\nabla)(\gamma^*\mathbf{v})(\cdot, \partial_\theta) = \\
&\quad\Big(\partial_\theta(v^k \circ \gamma) + (v^r \circ \gamma)(\Gamma^k_{ir} \circ \gamma)\partial_1(x^i \circ \gamma)\Big)\,(\partial_{x^k} \circ \gamma) = \nabla_{\dot\gamma}\mathbf{v}\;.
\end{aligned}$$

Since θ is the identity map, the induced partial derivative ∂_θ and the partial derivative ∂_1 coincide. For every $t \in I$, the covariant derivative of $\gamma^*\mathbf{v}$ along ∂_θ

$$\begin{aligned}
&(\gamma^*\nabla)(\gamma^*\mathbf{v})(\cdot, \partial_\theta)(t) = \\
&\quad\Big(\partial_1(v^k \circ \gamma)|_t + v^r(\gamma(t))\Gamma^k_{ir}(\gamma(t))\partial_1(x^i \circ \gamma)|_t\Big)\,(\partial_{x^k} \circ \gamma)|_t = \nabla_{\dot\gamma(t)}\mathbf{v}
\end{aligned}$$

corresponds to the covariant derivative of $\mathbf{v}$ along γ.

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