Mean-Variance Portfolio Analysis: The Markowitz Model

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2.1 Basic Notions

The Markowitz model[1] describes a market with N assets characterized by a random vector of returns

$$R = (R_1, \ldots, R_N).$$

The following data are assumed to be given:

- The expected value (mean) $m_i = ER_i$ of each random variable R_i, $i = 1, 2, \ldots, N$;
- The covariances $\sigma_{ij} = Cov(R_i, R_j)$ for all pairs of random variables R_i and R_j.

The covariance of two random variables, X and Y, is defined by

$$Cov(X, Y) = E[X - EX][Y - EY] = E(XY) - (EX)(EY).$$

We will denote by m the vector of the expected returns

$$m = (m_1, \ldots, m_N)$$

and by V the covariance matrix

$$V = (\sigma_{ij}),\ \sigma_{ij} = Cov(R_i, R_j)$$

[1]Markowitz, H., Portfolio Selection, Journal of Finance 7, 77–91, 1952. Markowitz was awarded a Nobel Prize in Economics in 1990, jointly with W. Sharpe and M. Miller.

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of the random vector $R = (R_1, \ldots, R_N)$. (The expectations and the covariances are assumed to be well-defined and finite.) The matrix V has N rows and N columns. The element at the intersection of ith row and jth column is σ_{ij}.

Expectations and Covariances of Returns Consider a portfolio $x = (x_1, \ldots, x_N)$, where x_i is the amount of money invested in asset i. Recall that the return on the portfolio x is computed according to the formula

$$R_x = \sum_{i=1}^{N} x_i R_i .$$

Consequently, the expected return $m_x = ER_x$ on the portfolio x is given by

$$m_x = \sum_{i=1}^{N} x_i m_i = \langle m, x \rangle$$

where

$$m_i = ER_i$$

and

$$m = (m_1, \ldots, m_N).$$

The variance $VarR_x$ of the portfolio return R_x can be computed as follows:

$$\begin{aligned}
\sigma_x^2 &= Var(R_x) = E(R_x - m_x)^2 \\
&= E\left(\sum_{i=1}^{N} x_i R_i - \sum_{i=1}^{N} x_i m_i\right)^2 = E\left[\sum_{i=1}^{N} x_i (R_i - m_i)\right]^2 \\
&= E\left[\sum_{i=1}^{N} x_i (R_i - m_i)\right]\left[\sum_{j=1}^{N} x_j (R_j - m_j)\right] \\
&= E\left[\sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j (R_i - ER_i)(R_j - ER_j)\right] \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} x_i Cov(R_i, R_j) x_j \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} x_i \sigma_{ij} x_j = \langle x, Vx \rangle .
\end{aligned}$$

Thus we have the following formulas for the expectation and the variance of the return R_x on the portfolio x:

$$m_x = ER_x = \langle m, x \rangle, \tag{2.1}$$

$$\sigma_x^2 = Var(R_x) = \langle x, Vx \rangle. \tag{2.2}$$

Markowitz's Approach to Portfolio Selection This approach is often used in practical decisions. Given the constraint $\sum x_i = 1$ on the portfolio weights, investors choose a portfolio x, having two objectives:

- Maximization of the expected value $m_x = ER_x$ of the portfolio return;
- Minimization of the portfolio *risk*, which is measured by $\sigma_x^2 = VarR_x$ or σ_x.

We denote by σ_x the *standard deviation* of the random variable R_x:

$$\sigma_x = \sqrt{VarR_x} = \sqrt{E(R_x - m_x)^2}.$$

It is the fundamental assumption of the Markowitz approach that only two numbers characterize the portfolio: the expectation and the variance of the portfolio return. The variance is used as a very simple measure of risk: the more "variable" the random return R_x on the portfolio x, the higher the variance of R_x. If the return R_x is certain, its variance is equal to zero, and so such a portfolio is *risk-free*.

2.2 Optimization Problem: Formulation and Discussion

The Markowitz Optimization Problem According to individual preferences, an investor puts weights on the conflicting objectives m_x and σ_x^2 and maximizes

$$\tau m_x - \sigma_x^2$$

given the parameter $\tau \geq 0$. This parameter is called *risk tolerance*. Hence, according to Markowitz, the optimization problem to be solved is as follows:

$$\max_{x \in R^N} \{\tau m_x - \sigma_x^2\}$$

subject to

$$x_1 + \ldots + x_N = 1.$$

More explicitly, the above problem can be written

$$\max_{x \in R^N} \left\{ \tau \sum_{i=1}^{N} m_i x_i - \sum_{i=1}^{N} \sum_{j=1}^{N} x_i \sigma_{ij} x_j \right\}$$

subject to

$$x_1 + \ldots + x_N = 1.$$

Using the notation

$$e = (1, 1, \ldots, 1)$$

for the vector whose all coordinates are equal to one and writing $\langle \cdot, \cdot \rangle$ for the scalar product, we can represent the Markowitz optimization problem as follows:

$$\max_{x \in R^N} \{ \tau \langle m, x \rangle - \langle x, Vx \rangle \}$$

subject to

$$\langle e, x \rangle = 1.$$

Advantages and Disadvantages of the Markowitz Approach The Markowitz approach has the following important *advantages*:

- The preferences of the investor are described in a most simple way. Only one positive number, the risk tolerance τ, has to be determined.
- Only the expectations $m_i = ER_i$ and the covariances $\sigma_{ij} = Cov(R_i, R_j)$ of asset returns are needed.
- The optimization problem is quadratic concave, and powerful numerical algorithms exist for finding its solutions.
- Most importantly, the Markowitz optimization problem admits an explicit analytic solution, which makes it possible to examine its quantitative and qualitative properties in much detail.

The main *drawback* of the Markowitz approach is its inability to cover situations in which the distribution of the portfolio return cannot be fully characterized by such a scarce set of data as m_i and σ_{ij}.

Efficient Portfolios Portfolios obtained by using the Markowitz approach are termed *efficient*.

Definition A portfolio x^* is called (*mean-variance*) *efficient* if it solves the optimization problem

$$(\mathbf{M}_\tau) \qquad \max_{x \in R^N} \{\tau m_x - \sigma_x^2\}$$
$$\text{subject to: } x_1 + \ldots + x_N = 1$$

for some $\tau \geq 0$.

2.3 Assumptions

Basic Assumptions We will start the analysis of the Markowitz model under the following assumptions (later, an alternative set of assumptions will be considered).

Assumption 1 The covariance matrix V is *positive definite*.

This assumption means that

$$\langle x, Vx \rangle \left(= \sum_{i,j=1}^{N} x_i \sigma_{ij} x_j \right) > 0 \text{ for each } x \neq 0.$$

Since $\langle x, Vx \rangle = Var(R_x)$, we always have $\langle x, Vx \rangle \geq 0$. The above assumption requires that $\langle x, Vx \rangle = 0$ *only if* $x = 0$. As a consequence of Assumption 1, we obtain $VarR_i > 0$, i.e., *all the assets* $i = 1, 2, \ldots, N$ *are risky*.

If Assumption 1 is satisfied, then the objective function

$$\tau m_x - \sigma_x^2 = \tau \langle m, x \rangle - \langle x, Vx \rangle$$

in the Markowitz problem $(\mathbf{M}_\tau)$ is strictly concave and the solution to $(\mathbf{M}_\tau)$ exists and is unique.[2]

The set of efficient portfolios is a one-parameter family with parameter τ ranging through the set $[0, \infty)$ of all non-negative numbers.

The efficient portfolio x^{MIN} corresponding to $\tau = 0$ is termed the *minimum variance portfolio*. It minimizes $VarR_x = \langle x, Vx \rangle$ over all normalized portfolios x.

What Happens If Assumption 1 Fails to Hold? Then there is a portfolio $y \neq 0$ with $\langle y, Vy \rangle = 0$. Hence

$$Var(R_y) = Var(y_1 R_1 + \ldots + y_N R_N) = 0.$$

[2] For details see Mathematical Appendix A.

Thus R_y is equal to a constant, c, with probability one. If $c \neq 0$, we can assume without loss of generality that $c > 0$ (replace y by $-y$ if needed!). The property

$$y_1 R_1 + \ldots + y_N R_N = c > 0 \text{ with probability } 1$$

means the existence of a *risk-free investment strategy with strictly positive return* (which is ruled out in the present context).

If $c = 0$, then the equality $y_1 R_1 + \ldots + y_N R_N = 0$, holding for some $(y_1, \ldots, y_N) \neq 0$, means that the random variables $R_1, \ldots, R_N$ are *linearly dependent*. Then at least one of them (any one for which $y_i \neq 0$) can be expressed as a linear combination of the others, which means the existence of a *redundant asset*.

In addition to Assumption 1, we will need the following

Assumption 2 There are at least two assets i and j with expected returns $m_i \neq m_j$.

What If Assumption 2 Does Not Hold? If Assumption 2 is not satisfied, then there is only one efficient portfolio, x^{MIN}. Indeed, if Assumption 2 does not hold, then all the numbers $m_1, \ldots, m_N$ are the same and are equal, say, to some number θ. Then we have $m = \theta e$, i.e., the vectors m and $e = (1, 1, \ldots, 1)$ are collinear. In the Markowitz problem ($\mathbf{M}_\tau$), we have to maximize

$$\tau \langle m, x \rangle - \langle x, Vx \rangle$$

under the constraint

$$\langle e, x \rangle = 1.$$

If $m = \theta e$, then for every x satisfying the constraint $\langle e, x \rangle = 1$, the value of the objective function is equal to

$$\tau \langle m, x \rangle - \langle x, Vx \rangle = \tau\theta \langle e, x \rangle - \langle x, Vx \rangle = \tau\theta - \langle x, Vx \rangle.$$

For each τ, the maximum value of this function is attained at $x = x^{MIN}$ because x^{MIN} minimizes $\langle x, Vx \rangle$ on the set of all normalized portfolios.

2.4 Efficient Portfolios and Efficient Frontier

Efficient Frontier We can draw a diagram depicting the set of all points (σ_x^2, m_x) in the plane corresponding to all efficient portfolios x. This set is called the *efficient frontier*. The efficient frontier is a curve of the following typical form (Fig. 2.1):

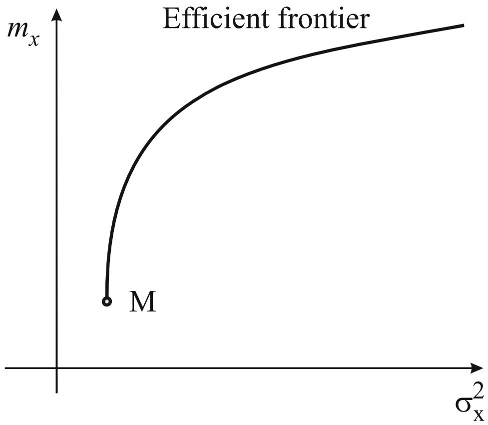

Fig. 2.1 Efficient frontier

The point M of the curve in the above diagram corresponds to the minimum variance efficient portfolio (for which $\tau = 0$). All the other points (σ_x^2, m_x) of the curve represent the variances and the expectations of the returns on efficient portfolios x with $\tau > 0$.

Efficient Portfolios: An Equivalent Definition We give an equivalent definition of an efficient portfolio (which is often used in the literature).

Proposition 2.1 *A normalized portfolio $x^* \in R^N$ is efficient if and only if there exists no normalized portfolio $x \in R^N$ such that*

$$m_x \geq m_{x^*} \text{ and } \sigma_x^2 < \sigma_{x^*}^2.$$

The last two inequalities mean that x^* solves the optimization problem

$$(\mathbf{M}^\mu) \qquad \min_{x \in R^N} \sigma_x^2$$

subject to

$$m_x \geq \mu \text{ and } \sum x_i = 1,$$

where $\mu = m_{x^*}$ and $x = (x_1, \ldots, x_N)$.

Proof "Only if": We have to show that if x^* is a solution to $(\mathbf{M}_\tau)$, then x^* is a solution to $(\mathbf{M}^\mu)$ with $\mu = m_{x^*}$. Suppose the contrary: x^* is a solution to $(\mathbf{M}_\tau)$, but not to $(\mathbf{M}^\mu)$, i.e., there is a normalized portfolio x for which $m_x \geq \mu = m_{x^*}$ and $\sigma_x^2 < \sigma_{x^*}^2$. Then $\tau m_x - \sigma_x^2 > \tau m_{x^*} - \sigma_{x^*}^2$, which means that x^* is *not* a solution to $(\mathbf{M}_\tau)$. A contradiction.

"If": We have to show that if x^* is a solution to $(\mathbf{M}^\mu)$ with $\mu = m_{x^*}$, then x^* is a solution to $(\mathbf{M}_\tau)$ for some $\tau \geq 0$. It can be shown that there exists a Lagrange multiplier $\gamma \geq 0$ relaxing the constraint $m_x \geq \mu$ in $(\mathbf{M}^\mu)$:

$$-\sigma_x^2 + \gamma(m_x - \mu) \leq -\sigma_{x^*}^2 + \gamma(m_{x^*} - \mu)$$

for each normalized portfolio x. This implies

$$\gamma m_x - \sigma_x^2 \leq \gamma m_{x^*} - \sigma_{x^*}^2.$$

By setting $\tau = \gamma$, we obtain that x^* is a solution to $(\mathbf{M}_\tau)$, which completes the proof. $\square$

Remark The above proof is based on a general result on the existence of Lagrange multipliers for convex optimization problems—the Kuhn–Tucker theorem. This theorem is presented in Mathematical Appendix B.

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