

Chapter 2

Anelastic Wave Propagation (AWP)

Among all the numerical methods for solving the 3D elastodynamic equation, the finite-difference method is probably the most widely used. In this book, the numerical wave-equation solver adopted for solving F3DWI problems is the Anelastic Wave Propagation (AWP) code developed and maintained by Kim B. Olsen, Steven M. Day and Yifeng Cui (ODC) (Olsen 1994; Olsen et al. 1995, 2003, 2006, Cui et al. 2009). It implements the staggered-grid finite-difference scheme to solve the three-dimensional velocity-stress elastodynamic equation. The scheme is fourth-order accurate in space and second-order accurate in time. The AWP-ODC code implements the traction-free boundary condition for the free surface and can use either the Absorbing Boundary Conditions (ABC) of Cerjan (Cerjan et al. 1985) or the Perfectly Matched Layers (PML) of Berenger (Berenger 1994; Marcinkovich and Olsen 2003) for the four sides and the bottom of the computation domain. Representations of the seismic sources can be included in the staggered-grid system using the stress components for moment sources and/or the velocity components for body-force sources.

The code is written in Fortran 77/90. Message passing is implemented using MPI with domain decomposition. Each processor is responsible for updating the velocity and stress fields on its own portion of the mesh and also for handling the boundary conditions if some of its grid points are located on the external boundaries of the modeling volume. Two-grid-point-thick ghost layers containing the Latest velocity and stress fields are used for exchanging information with the neighboring sub-meshes. Some very large-scale simulations may take several days to execute therefore the AWP-ODC code includes checkpoint/restart capability. To ensure the integrity of the simulation output, the AWP-ODC code generates MD5 checksums in parallel for each mesh sub-array.

The accuracy of the AWP-ODC code has been extensively validated by researchers at the Southern California Earthquake Center (SCEC) in the SCEC 3D Numerical Simulation Code Validation Project (Day et al. 2001, 2003, 2005) using a hierarchy of test problems ranging from simple point-source problems in half-space or layered structural models (i.e., material properties are constant or vary only with depth) to finite-rupture sources in complex 3D structural models. Five different codes, including an earlier version of AWP-ODC, were compared in this validation

project. Four of them were finite-difference codes (Larsen and Schultz 1995; Graves 1996; Pitarka 1999) and one was a finite-element code (Akcelik et al. 2003). The validation was carried out mainly through cross-comparisons of the synthetic seismograms generated by the different numerical codes. For half-space and layered structural models, synthetic seismograms generated using a semi-analytic method were also included in the comparisons.

The AWP-ODC code has been extensively optimized for performance improvements on different types of HPC systems during the TeraShake project (Cui et al. 2009), an inter-disciplinary, collaborative project for simulating earthquake ruptures and seismic wave propagation in Southern California using the NSF-funded TeraGrid HPC resources¹. The result of those extensive optimization efforts is that the AWP-ODC code can now easily scale to tens of thousands of processors with very high parallel efficiency. In 2009 the AWP-ODC code was able to scale from 4096 to 40,960 cores with about 96% parallel efficiency on the Blue Gene/L system at the IBM TJ Watson Research Center in a strong scaling² experiment that used about 32 billion grid points. The weak scaling³ performance was also nearly perfect for up to 32,768 processors on the same system. In 2010, the AWP-ODC code was used for simulating seismic wavefields generated by a hypothetical magnitude-8 earthquake on southern San Andreas Fault⁴. The simulation was performed on 223,074 cores on the Jaguar Cray XT5 system at the National Center for Computational Sciences (NCCS) of the Oak Ridge National Laboratory in Tennessee and the AWP-ODC code achieved 220 TFLOPS of sustained performance. This study was selected as one of the year's Gordon Bell Prize⁵ finalists. Recently, optimization effort on the AWP-ODC code has been mainly focused on porting the code to GPUs and improving its efficiency on CPU-GPU hybrid heterogeneous HPC systems (Zhou et al. 2013).

¹ The TeraGrid was a grid-computing infrastructure combining HPC resources at 11 different partner sites. It was supported by the US National Science Foundation and operated from 2004 through 2011. The TeraGrid project was succeeded by the Extreme Science and Engineering Discovery Environment (XSEDE), a partnership involving HPC resources from 17 institutions, which has been supported by NSF since 2011.

² The overall problem size is fixed and the total number of processors is increased. Therefore the workload on each processor decreases with the number of processors.

³ The workload for each processor is constant and the overall problem size increases in proportion with the number of processors.

⁴ The "M8" earthquake simulation was the largest earthquake simulation coordinated by SCEC. It used a uniform mesh with 40-m grid spacing and the simulation volume was 810-km long, 405-km wide and 85-km deep. The total number of finite-difference grid points used in this simulation was about 436 billion. The full simulation took about 24 h and produced about 6 min of the 3D wavefields in the modeling volume.

⁵ The Gordon Bell prize is awarded by the Association of Computing Machinery (ACM) each year to recognize outstanding achievements in high-performance computing, especially in applying HPC technology to science and engineering applications.

2.1 Formulation

The AWP-ODC code solves the strong formulation⁶ of the velocity-stress equation of motion

$$\rho \dot{\mathbf{v}} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{f} \quad (2.1)$$

where ρ is the density, $\dot{\mathbf{v}}$ is the time derivative of the particle-velocity vector (i.e., particle acceleration), $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{f}$ is the body-force source. I use the over-dot (i.e., Newton's notation) to represent time derivatives in the following equations. I have dropped the dependence on the spatial and temporal coordinates $\mathbf{x}$ and t of the variables to clarify the basic physics. Written in index notation, Eq. 2.1 can be expressed as

$$\rho \dot{v}_i = \sigma_{ij,j} + f_i \quad (2.2)$$

where $i, j, k = 1, 2, 3$, repeated indices imply summation (i.e., Einstein convention), and $\sigma_{ij,j}$ is the divergence of the stress tensor $\sum_{j=1}^3 \partial \sigma_{ij} / \partial x_j$.

2.1.1 Elastic Media

For three-dimensional isotropic elastic media, the stress-strain relation used in the code is

$$\dot{\sigma}_{ij} = \lambda \dot{\epsilon}_{kk} \delta_{ij} + 2\mu \dot{\epsilon}_{ij} - \dot{m}_{ij} \quad (2.3)$$

where λ and μ are the time-independent, but spatially-varying, Lamé parameters,

$$\dot{\epsilon}_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i}) \quad (2.4)$$

is the strain rate tensor, δ_{ij} is the Kronecker delta, m_{ij} is the moment-density tensor of any moment source and $\dot{m}_{ij}$ is its time derivative (i.e., moment-rate-density tensor). The strain rate tensor $\dot{\epsilon}_{ij}$ is defined in terms of the spatial gradient of the particle velocity vector $\partial v_i / \partial x_j$ and its transpose $\partial v_j / \partial x_i$. Denoting the particle displacement vector as $\mathbf{u}$, the symmetric strain tensor $\boldsymbol{\epsilon}$ can then be expressed as

$$\begin{aligned} \boldsymbol{\epsilon} &= \frac{1}{2} \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right] \\ &= \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) & \frac{1}{2} \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) \\ \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2} \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) \\ \frac{1}{2} \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) & \frac{1}{2} \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}. \end{aligned} \quad (2.5)$$

⁶ The solution we are seeking must be sufficiently smooth and have regular second-order spatial derivatives. In the weak formulation, the solution we are seeking may have only first-order derivatives (Evans 2010).

This strain tensor will be playing an important role in constructing the Fréchet kernels in Chap. 4.

2.1.2 Viscoelastic Media

For three-dimensional isotropic viscoelastic media, the stress-strain relation can be defined as (Christensen 2003)

$$\sigma_{ij}(t) = \delta_{ij} \int_{-\infty}^t \lambda(t - \tau) \dot{\epsilon}_{kk}(\tau) d\tau + 2 \int_{-\infty}^t \mu(t - \tau) \dot{\epsilon}_{ij}(\tau) d\tau \quad (2.6)$$

where the Lamé parameters, $\lambda(t)$ and $\mu(t)$, are time-dependent relaxation functions. Here I have dropped the dependence on spatial coordinates $\mathbf{x}$ of the functions to reduce clutter. A direct integration of Eq. 2.6 is numerically intractable due to unrealistic requirements on computer memory and computing time, approximations are therefore needed. The standard approach is to convert the convolutive stress-strain relation Eq. 2.6 into a differential form, which can be numerically solved more easily using the finite-difference method.

2.1.2.1 The Complex Modulus and its Approximation

To demonstrate the basic physics, I will use a simplified scalar notation and write a scalar stress-strain relation analogous to Eq. 2.6

$$\sigma(t) = \int_{-\infty}^t \psi(t - \tau) \dot{\epsilon}(\tau) d\tau = \psi(t) * \dot{\epsilon}(t) \quad (2.7)$$

where $*$ represents the temporal convolution. I will deal with the conversion back to index notations for stress and strain tensors afterwards. Here the stress relaxation function $\psi(t)$ is the stress response to a unit Heaviside function in strain. In the frequency domain, Eq. 2.7 can be expressed as

$$\hat{\sigma}(\omega) = \hat{\psi}(\omega)(i\omega)\hat{\epsilon}(\omega) = \hat{M}(\omega)\hat{\epsilon}(\omega) \quad (2.8)$$

where $\hat{\sigma}$, $\hat{\psi}$ and $\hat{\epsilon}$ are Fourier transforms of the stress, the stress relaxation function and the strain and ω is the angular frequency. Here I have introduced the frequency-domain complex modulus

$$\hat{M}(\omega) = (i\omega)\hat{\psi}(\omega) \quad (2.9)$$

which, in the time domain, can be expressed as

$$M(t) = \dot{\psi}(t) \quad (2.10)$$

and corresponds to the stress response to the Dirac function in strain. The stress-strain relation 2.8 can then be expressed as

$$\sigma(t) = M(t) * \epsilon(t) \quad (2.11)$$

in the time domain. The anelastic attenuation factor or the seismic quality factor $Q(\omega)$ is defined in terms of the complex modulus as

$$Q(\omega) = \frac{\Re[\hat{M}(\omega)]}{\Im[\hat{M}(\omega)]} = \frac{1}{\tan \angle \hat{M}(\omega)} \quad (2.12)$$

where $\angle \hat{M}(\omega)$ is the phase of the complex modulus. For Eq. 2.8 to be causal, $\hat{M}(\omega)$ must satisfy the Kramers-Kronig relation⁷, therefore $\hat{M}(\omega)$ is uniquely determined by a given quality factor $Q(\omega)$ and vice versa. Observations have shown that $Q(\omega)$ is nearly constant over the seismic frequency range (i.e., periods from about 0.01 s to 1 h) (McDonal et al. 1958; Liu et al. 1976; Spencer 1981; Murphy III 1982). Our goal is to find an approximation to the complex modulus $\hat{M}(\omega)$ such that the quality factor $Q(\omega)$ fits a given target frequency-dependence over a specified frequency range and also the stress-strain relation Eq. 2.11 can be easily solved through a time-domain finite-differencing scheme. An approximation that has been widely used in the literature has the form

$$\hat{M}(\omega) = M_U \left(1 - \sum_{l=1}^N \frac{w_l}{1 + i\omega\tau_l} \right) \quad (2.13)$$

where M_U is the unrelaxed modulus, i.e.,

$$M_U = \lim_{\omega \rightarrow \infty} \hat{M}(\omega) = \lim_{t \rightarrow 0} \psi(t) \quad (2.14)$$

and the two sets of parameters w_l and τ_l ($l = 1, 2, \dots, N$), as well as N , can be chosen in such a way that the corresponding quality factor (considering Eq. 2.12),

$$Q(\omega) = \frac{1 - \sum_{l=1}^N \frac{w_l}{1 + (\omega\tau_l)^2}}{\sum_{l=1}^N \frac{w_l\omega\tau_l}{1 + (\omega\tau_l)^2}} \quad (2.15)$$

can adequately fit a target frequency-dependent model (Day and Minster 1984; Emmerich and Korn 1987; Blanch et al. 1995; Liu and Archuleta 2006). The time-domain expression for the approximate modulus 2.13 can be obtained through inverse Fourier transform,

$$M(t) = M_U \delta(t) - M_U \sum_{l=1}^N \frac{w_l}{\tau_l} e^{-t/\tau_l} H(t), \quad (2.16)$$

⁷ Causality implies that the complex function is analytic. The Kramers-Kronig relation indicates that the real and imaginary parts of any analytic function are related through the Hilbert transform, therefore the full function can be reconstructed by knowing just one of its parts. A corollary is that an analytic function can be reconstructed by knowing only its phase.

where $\delta(t)$ is the Dirac function and $H(t)$ is the Heaviside function. Bring Eq. 2.16 into the stress-strain relation Eq. 2.11, we obtain

$$\sigma(t) = M_U \epsilon(t) - M_U \sum_{l=1}^N \frac{w_l}{\tau_l} \int_{-\infty}^t \epsilon(\tau) e^{-(t-\tau)/\tau_l} d\tau. \quad (2.17)$$

Introduce the memory variables⁸,

$$\zeta_l(t) = \frac{w_l}{\tau_l} \int_{-\infty}^t \epsilon(\tau) e^{-(t-\tau)/\tau_l} d\tau, \quad (2.18)$$

the stress can be expressed as

$$\sigma(t) = M_U \left[\epsilon(t) - \sum_{l=1}^N \zeta_l(t) \right]. \quad (2.19)$$

Taking the time derivative on both sides of Eq. 2.18, we obtain

$$\begin{aligned} \dot{\zeta}_l(t) &= -\frac{w_l}{\tau_l^2} \int_{-\infty}^t \epsilon(\tau) e^{-(t-\tau)/\tau_l} d\tau + \frac{w_l}{\tau_l} \int_{-\infty}^{+\infty} \dot{H}(t-\tau) \epsilon(\tau) e^{-(t-\tau)/\tau_l} d\tau \\ &= -\frac{1}{\tau_l} \zeta_l(t) + \frac{w_l}{\tau_l} \int_{-\infty}^{+\infty} \delta(t-\tau) \epsilon(\tau) e^{-(t-\tau)/\tau_l} d\tau \\ &= -\frac{1}{\tau_l} \zeta_l(t) + \frac{w_l}{\tau_l} \epsilon(t). \end{aligned}$$

Therefore the memory variables $\zeta_l(t)$ follow simple first-order ordinary differential equations (ODEs)

$$\tau_l \dot{\zeta}_l(t) + \zeta_l(t) = w_l \epsilon(t) \quad (2.20)$$

which has a time-stepping solution (Day 1998; Liu and Archuleta 2006) given by the first-order exponential integrator method⁹ (Certaine 1960; Pope 1963)

$$\zeta_l \left(t_n + \frac{\Delta t}{2} \right) = e^{-\frac{\Delta t}{\tau_l}} \zeta_l \left(t_n - \frac{\Delta t}{2} \right) + w_l \left[1 - e^{-\frac{\Delta t}{\tau_l}} \right] \epsilon(t_n) \quad (2.21)$$

and also a finite-difference solution

$$\zeta_l \left(t_n + \frac{\Delta t}{2} \right) = \frac{\left(\frac{\tau_l}{\Delta t} - \frac{1}{2} \right) \zeta_l \left(t_n - \frac{\Delta t}{2} \right) + w_l \epsilon(t_n)}{\frac{\tau_l}{\Delta t} + \frac{1}{2}} \quad (2.22)$$

⁸ In the literature these variables are also known as relaxation functions, anelastic functions, internal variables, additional functions, etc.

⁹ To obtain Eq. 2.21, multiply both sides of Eq. 2.20 with e^{t/τ_l} and integrate over t from $t_n - \Delta t/2$ to $t_n + \Delta t/2$. The strain is approximated as a constant $\epsilon(t) \approx \epsilon(t_n)$ over the integration interval.

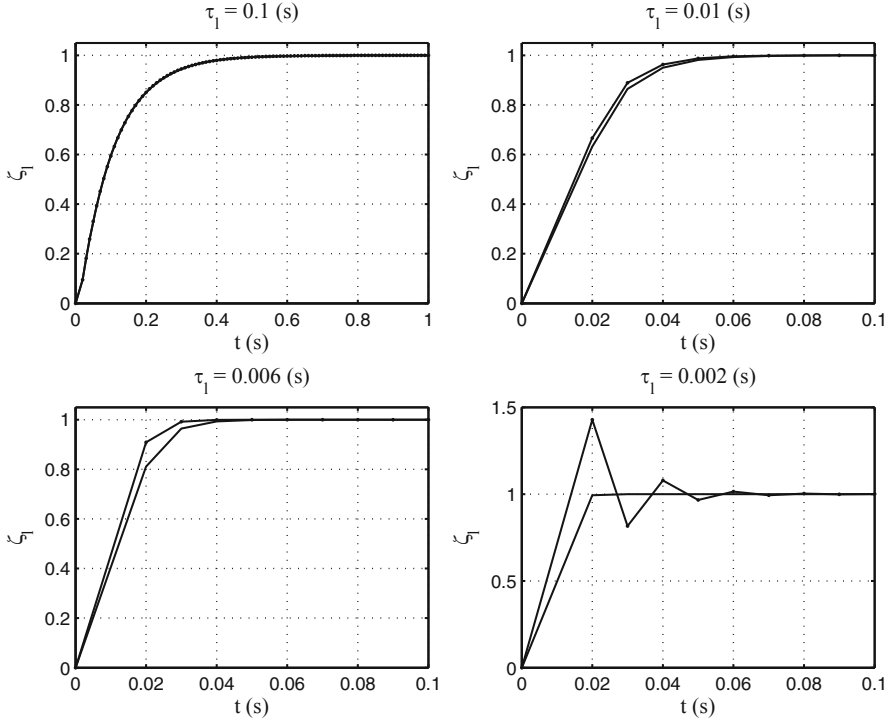


Fig. 2.1 Solution of the memory variable obtained using the exponential integrator method Eq. 2.21 (*solid lines*) and that obtained using the finite-difference method Eq. 2.22 (*dotted solid lines*) for different relaxation time τ_l . The initial condition is set as $z_l(0) = 0$, the weight coefficient $w_l = 1$, the strain is set to a constant $\epsilon = 1$, the time-step length $\Delta t = 0.01$ s.

where Δt is the time-step length and $t_n = (n - 1)\Delta t$ is the time of the n^{th} time step. In general, both solutions are consistent with each other. If we approximate $\exp(-\Delta t/\tau_l)$ in Eq. 2.21 using its Padé approximant¹⁰

$$(1 - 0.5\Delta t/\tau_l) / (1 + 0.5\Delta t/\tau_l), \quad (2.23)$$

we arrive at Eq. 2.22. For very small τ_l (i.e., the ODE is stiff¹¹), the exponential integrator solution 2.21 can alleviate some of the stiffness because part of the ODE is integrated exactly (Cox and Matthews 2002). Comparisons of the two solutions for a fixed Δt and different τ_l are shown in Fig. 2.1. When τ_l is reduced to 0.002 s,

¹⁰ The Padé approximant provides an approximation of a function using rational functions and often gives better accuracy than a truncated Taylor series (Baker and Graves-Morris 1996).

¹¹ The solutions of some differential equations may contain both rapid-varying, transient modes and longer-period slow modes. The numerical algorithms for solving such equations need to use very small time steps to ensure numerical stability even after the rapid-varying, transient modes are no longer visible in the solutions. Such differential equations are said to be stiff (Heath 2005).

the finite-difference solution shows strong oscillation, while the exponential integrator solution increases monotonically to 1. The oscillation in the finite-difference solution can be reduced by decreasing Δt , at the expense of higher computational cost. In each time step, after the memory variables are updated using Eq. 2.21 or 2.22, the stress can then be updated using Eq. 2.19.

2.1.2.2 Physical Interpretation of the Approximate Modulus

The approximation in Eq. 2.13 can be understood in multiple ways. For example, the $2N$ values of τ_l and ω_l can be related to the poles and residues of the rational approximation for $\hat{M}(\omega)$ (Day and Minster 1984). A more physical interpretation is in terms of the Generalized Zener Body (GZB, i.e., the Standard Linear Solid model) or the Generalized Maxwell Body (GMB-EK) rheological models. The GZB and the GMB-EK models are equivalent to each other and one can be derived from the other (Moczo and Kristek 2005). These generalized rheological models are based on serial and parallel compositions of two simplest linear rheological models: the Hooke body (i.e., the stress is proportional to the strain), which has the stress-strain relation

$$\hat{\sigma}_H(\omega) = M_H \hat{\epsilon}_H(\omega) \quad (2.24)$$

and is often depicted as a spring, and the Stokes body (i.e., the stress is proportional to the strain rate), which in the frequency domain has the stress-strain relation

$$\hat{\sigma}_S(\omega) = i\omega\eta_S \hat{\epsilon}_S(\omega) \quad (2.25)$$

and is often depicted as a dash pot. Here η_S is the time-independent viscosity. A Maxwell body is a Hooke body connected in series with a Stokes body. The stress in the Maxwell body $\hat{\sigma}_M$ should be equal to that in its Hooke and Stokes components, i.e.,

$$\hat{\sigma}_M = \hat{\sigma}_H = \hat{\sigma}_S \quad (2.26)$$

and the strain $\hat{\epsilon}_M$ should be equal to the summation of the strain in its Hooke and Stokes components, i.e.,

$$\hat{\epsilon}_M = \hat{\epsilon}_H + \hat{\epsilon}_S = \frac{\hat{\sigma}_H}{M_H} + \frac{\hat{\sigma}_S}{i\omega\eta_S} = \frac{M_H + i\omega\eta_S}{i\omega\eta_S M_H} \hat{\sigma}_M \quad (2.27)$$

therefore the effective modulus of a single Maxwell body in the frequency domain $\hat{M}_M$ can be expressed using the modulus of its Hooke component and the viscosity of its Stokes component as

$$\hat{M}_M(\omega) = \frac{i\omega\eta_S M_H}{M_H + i\omega\eta_S}. \quad (2.28)$$

A Generalized Maxwell Body (GMB) is N Maxwell bodies connected in parallel. A GMB-EK body is N Maxwell bodies and a single Hooke body connected in parallel

(Emmerich and Korn 1987). The stress in a GMB-EK body should be equal to the summation of the stresses in all of its parallel components and the strain should be equal to that in each of its individual parallel components. The effective modulus for such a GMB-EK body is therefore

$$\hat{M}(\omega) = M_H + \sum_{l=1}^N \frac{i\omega\eta_l M_l}{M_l + i\omega\eta_l} \quad (2.29)$$

where M_l and η_l are the modulus and the viscosity of the Hooke body and the Stokes body in the l^{th} Maxwell body and M_H is the modulus of the single Hooke body that is connected in parallel with the N Maxwell bodies. The unrelaxed and relaxed moduli for the GMB-EK body can therefore be derived using Eq. 2.29

$$M_U = \lim_{\omega \rightarrow \infty} \hat{M}(\omega) = M_H + \sum_{l=1}^N M_l, \quad (2.30)$$

$$M_R = \lim_{\omega \rightarrow 0} \hat{M}(\omega) = M_H. \quad (2.31)$$

If we introduce

$$\Delta M = M_U - M_R = \sum_{l=1}^N M_l \quad (2.32)$$

we can express the effective modulus of the GMB-EK body Eq. 2.29 using M_U and ΔM as

$$\begin{aligned} \hat{M}(\omega) &= M_U - \Delta M + \sum_{l=1}^N \frac{i\omega\eta_l M_l}{M_l + i\omega\eta_l} \\ &= M_U - \Delta M \left(\sum_{l=1}^N \frac{M_l}{\Delta M} - \sum_{l=1}^N \frac{M_l}{\Delta M} \frac{i\omega\eta_l}{M_l + i\omega\eta_l} \right) \\ &= M_U - \Delta M \sum_{l=1}^N \frac{M_l}{\Delta M} \left(1 - \frac{i\omega\eta_l}{M_l + i\omega\eta_l} \right) \\ &= M_U \left[1 - \frac{\Delta M}{M_U} \sum_{l=1}^N \frac{M_l}{\Delta M} \left(\frac{1}{1 + i\omega\eta_l/M_l} \right) \right] \\ &= M_U \left(1 - \sum_{l=1}^N \frac{M_l/M_U}{1 + i\omega\eta_l/M_l} \right). \end{aligned}$$

Compare with Eq. 2.13, the parameters w_l and τ_l in Eq. 2.13 can be understood as

$$w_l = M_l/M_U, \quad (2.33)$$

$$\tau_l = \eta_l / M_l, \quad (2.34)$$

and in the literature τ_l and w_l are often called the relaxation time and the weight coefficient, respectively. The inverse of the relaxation time

$$\omega_l = 1/\tau_l \quad (2.35)$$

is called the relaxation frequency. Another widely used notation for the weight coefficient is a_l , which is related to w_l through

$$w_l = a_l \frac{\Delta M}{M_U}. \quad (2.36)$$

Considering Eqs. 2.32 and 2.33, we have

$$\sum_{l=1}^N a_l = 1. \quad (2.37)$$

Bring Eq. 2.36 into the approximate modulus Eq. 2.13 and the ODEs for the memory variables Eq. 2.20, we obtain another set of equations also widely used in the literature

$$\hat{M}(\omega) = M_U - \Delta M \sum_{l=1}^N \frac{a_l \omega_l}{\omega_l + i\omega}, \quad (2.38)$$

$$\dot{\zeta}_l(t) + \omega_l \zeta_l(t) = \omega_l a_l \frac{\Delta M}{M_U} \epsilon(t). \quad (2.39)$$

Here I have used the relaxation frequency ω_l instead of the relaxation time τ_l in Eqs. 2.38 and 2.39. The approximate modulus can also be expressed in terms of the relaxed modulus M_R . Since $M_U = M_R + \Delta M$, considering Eq. 2.38, we have

$$\hat{M}(\omega) = M_R + \Delta M \left(1 - \sum_{l=1}^N \frac{a_l \omega_l}{\omega_l + i\omega} \right).$$

Considering Eq. 2.37, we obtain (Emmerich and Korn 1987)

$$\begin{aligned} \hat{M}(\omega) &= M_R + \Delta M \left(\sum_{l=1}^N a_l - \sum_{l=1}^N \frac{a_l \omega_l}{\omega_l + i\omega} \right) \\ &= M_R + \Delta M \sum_{l=1}^N \frac{ia_l \omega}{\omega_l + i\omega}. \end{aligned} \quad (2.40)$$

Based on this notation, the inverse of the quality factor can be expressed as (Emmerich and Korn 1987)

$$Q^{-1}(\omega) = \frac{\sum_{l=1}^N Y_l \frac{\omega/\omega_l}{1+(\omega/\omega_l)^2}}{1 + \sum_{l=1}^N Y_l \frac{(\omega/\omega_l)^2}{1+(\omega/\omega_l)^2}}, \quad (2.41)$$

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