Chapter 2
Continuous Fourier Transformation

Abstract This chapter deals with the mapping of arbitrary functions in the time domain to Fourier transformed functions in the frequency domain. The so-called δ-function is introduced and forward and inverse transformations are defined and illustrated with examples. The polar representation of the Fourier transform is given and shifting rules are discussed. Convolution, cross-correlation, and autocorrelation with Parseval's theorem are illustrated with examples. It concludes with a discussion of pitfalls and truncation errors.

In this chapter we relax the requirement of periodicity of the function $f(t)$. Hence, instead of discrete Fourier coefficients we end up with the continuous function $F(\omega)$. The integration interval is the entire real axis $(-\infty, +\infty)$.

Mapping of an *Arbitrary* Function $f(t)$ to the Fourier-Transformed Function $F(\omega)$

2.1 Continuous Fourier Transformation

We'll look at what happens at the transition from a series- to an integral-representation:

$$\text{Series:} \qquad C_k = \frac{1}{T}\int_{-T/2}^{+T/2} f(t)\mathrm{e}^{-2\pi \mathrm{i}kt/T}\,\mathrm{d}t. \tag{2.1}$$

$$\text{Now:} \quad T \to \infty \qquad \underset{\text{discrete}}{\omega_k = \frac{2\pi k}{T}} \quad \to \quad \underset{\text{continuous}}{\omega,}$$

$$\lim_{T\to\infty}(TC_k) \to \int_{-\infty}^{+\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t. \tag{2.2}$$

Before we get into the definition of the Fourier transformation, we have to do some homework.

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2.1.1 Even and Odd Functions

A function is called even, if:

$$f(-t) = f(t). \tag{2.3}$$

A function is called odd, if:

$$f(-t) = -f(t). \tag{2.4}$$

Any general function may be split into an even and an odd part. We've heard that before, at the beginning of Chap. 1, and of course it's true whether the function $f(t)$ is periodic or not.

2.1.2 The δ-Function

The δ-function is a distribution,[1] not a function. In spite of that, it's always called δ-function. Its value is zero anywhere except when its argument is equal to 0. In this case it is ∞. If you think that's too steep or pointed for you, you may prefer a different definition:

$$\begin{aligned} \delta(t) &= \lim_{a\to\infty} f_a(t) \\ \text{with } f_a(t) &= \begin{cases} a \text{ for } -\dfrac{1}{2a} \le t \le \dfrac{1}{2a} \\ 0 \text{ else} \end{cases} . \end{aligned} \tag{2.5}$$

Now we have a pulse for the duration of $-1/2a \le t \le 1/2a$ with height a and keep diminishing the width of the pulse while keeping the area unchanged (normalised to 1), viz. the height goes up while the width gets smaller. That's the reason why the δ-function often is also called impulse. At the end of the previous chapter we already had heard about a representation of the δ-function: Dirichlet's kernel for $N \to \infty$. If we restrict things to the basis interval $-\pi \le t \le +\pi$, we get:

$$\int_{-\pi}^{+\pi} D_N(x)\mathrm{d}x = \pi, \text{ independent of } N, \tag{2.6}$$

[1] Generalised function. The theory of distributions is an important basis of modern analysis, and impossible to understand without additional reading. A more in-depth treatment of its theory, however, is not required for the applications in this book.

and thus:

$$\frac{1}{\pi}\lim_{N\to\infty}\int_{-\pi}^{+\pi} f(t)D_N(t)\mathrm{d}t = f(0). \tag{2.7}$$

In the same way, the δ-function "picks" the integrand where the latter's argument is 0 during integration (we always have to integrate over the δ-function!):

$$\int_{-\infty}^{+\infty} f(t)\delta(t)\mathrm{d}t = f(0). \tag{2.8}$$

Another representation for the δ-function, which we'll frequently use, is:

$$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t. \tag{2.9}$$

Purists may multiply the integrand with a damping-factor, for example $\mathrm{e}^{-\alpha|t|}$, and then introduce $\lim_{\alpha\to 0}$. This won't change the fact that everything gets "oscillated" or averaged away for all frequencies $\omega \neq 0$ (venial sin: let's think in whole periods for once!), whereas for $\omega = 0$ integration will be over the integrand 1 from $-\infty$ to $+\infty$, i.e. the result will have to be ∞.

2.1.3 Forward and Inverse Transformation

Let's define:

Definition 2.1 (*Forward transformation*)

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t. \tag{2.10}$$

Definition 2.2 (*Inverse transformation*)

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\mathrm{e}^{+\mathrm{i}\omega t}\mathrm{d}\omega. \tag{2.11}$$

Caution:

i. In the case of the forward transformation, there is a minus sign in the exponent (cf. (1.27)), in the case of the inverse transformation, this is a plus sign.
ii. In the case of the inverse transformation, $1/2\pi$ is in front of the integral, contrary to the forward transformation.

The asymmetric aspect of the formulas has tempted many scientists to introduce other definitions, for example to write a factor $1/\sqrt{(2\pi)}$ for forward as well as inverse transformation. That's no good, as the definition of the average $F(0) = \int_{-\infty}^{+\infty} f(t)\mathrm{d}t$ would be affected. Weaver's representation is correct, though not widely used:

$$\text{Forward transformation:} \qquad F(\nu) = \int\limits_{-\infty}^{+\infty} f(t)\mathrm{e}^{-2\pi\mathrm{i}\nu t}\mathrm{d}t,$$

$$\text{Inverse transformation:} \qquad f(t) = \int\limits_{-\infty}^{+\infty} F(\nu)\mathrm{e}^{2\pi\mathrm{i}\nu t}\mathrm{d}\nu.$$

Weaver, as can be seen, doesn't use the angular frequency ω, but rather the frequency ν. This effectively made the formulas look symmetrical, though it saddles us with many factors 2π in the exponent. We'll stick to the definitions (2.10) and (2.11).

We now want to demonstrate, that the inverse transformation returns us to the original function. For the forward transformation, we often will use $\mathrm{FT}(f(t))$, and for the inverse transformation we'll use $\mathrm{FT}^{-1}(F(\omega))$. We'll start with the inverse transformation and insert:

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} F(\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\mathrm{d}\omega\int\limits_{-\infty}^{+\infty} f(t')\mathrm{e}^{-\mathrm{i}\omega t'}\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t' \\ &= \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} f(t')\mathrm{d}t'\int\limits_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\omega(t-t')}\mathrm{d}\omega \\ &\qquad \text{interchange integration} \\ &= \int\limits_{-\infty}^{+\infty} f(t')\delta(t-t')\mathrm{d}t' = f(t)\,. \end{aligned} \tag{2.12}$$

Q.e.d.[2] Here we have used (2.8) and (2.9). For $f(t) = 1$ we get:

$$\mathrm{FT}(\delta(t)) = 1. \tag{2.13}$$

[2]In Latin: "quod erat demonstrandum", "what we've set out to prove".

The impulse therefore requires all frequencies with unity amplitude for its Fourier representation ("white" spectrum). Conversely:

$$\mathrm{FT}(1) = 2\pi\delta(\omega). \tag{2.14}$$

The constant 1 can be represented by a single spectral component, viz. $\omega = 0$. No others occur. As we have integrated from $-\infty$ to $+\infty$, naturally an $\omega = 0$ will also result in infinity for intensity.

We realise the dual character of the forward and inverse transformations: a very slowly varying function $f(t)$ will have a very high spectral density for very small frequencies; the spectral density will go down quickly and rapidly approaches 0. Conversely, a quickly varying function $f(t)$ will show spectral density over a very wide frequency range: Fig. 2.1 explains this once again.

Let's discuss a few examples now.

Example 2.1 ("Rectangle, even")

$$\begin{aligned} f(t) &= \begin{cases} 1 \text{ for} & -T/2 \le t \le T/2 \\ 0 \text{ else} & \end{cases}. \\ F(\omega) &= 2\int_0^{T/2} \cos\omega t \,\mathrm{d}t = T\frac{\sin(\omega T/2)}{\omega T/2}. \end{aligned} \tag{2.15}$$

The imaginary part is 0, as $f(t)$ is even. The Fourier transformation of a rectangular function therefore is of the type $\frac{\sin x}{x}$. Some authors use the expression $\mathrm{sinc}(x)$

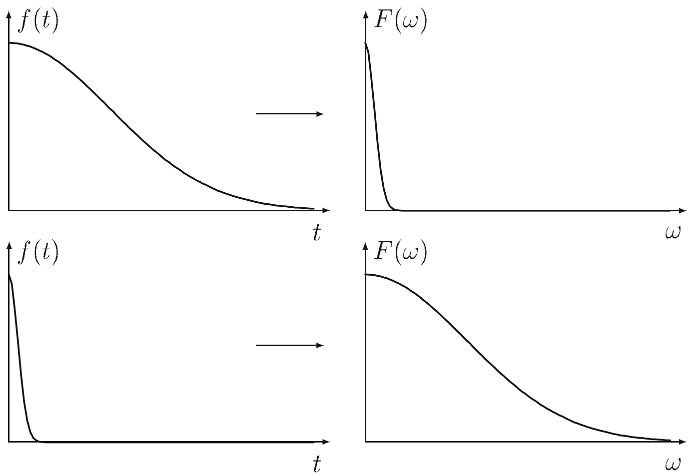

Fig. 2.1 A slowly-varying function has only low-frequency spectral components (*top*); a rapidly-falling function has spectral components spanning a wide range of frequencies (*bottom*)

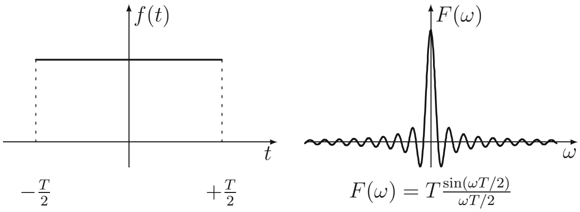

Fig. 2.2 "Rectangular function" and Fourier transformation of type $\frac{\sin x}{x}$

for this case. What the "c" stands for, I don't know.[3] The "c" already has been "used up" when defining the complementary error-function $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$. That's why we'd rather stick to $\frac{\sin x}{x}$. These functions $f(t)$ and $F(\omega)$ are shown in Fig. 2.2. They'll keep us busy for quite a while.

Keen readers would have spotted the following immediately: if we made the interval smaller and smaller, and did not fix $f(t)$ at 1 in return, but let it grow at the same rate as T decreases ("so the area under the curve stays constant"), then in $\lim_{T\to\infty}$ we would have a new representation of the δ-function. Again, we get the case where over- and undershoots on the one hand get closer to each other when T gets smaller, but on the other hand, their amplitude doesn't decrease. The shape $\frac{\sin x}{x}$ will stay the same. As we're already familiar with Gibbs' phenomenon in the case of steps, this naturally won't surprise us any more. Contrary to the discussion in Sect. 1.4.3 we don't have a periodic continuation of $f(t)$ beyond the integration interval, i.e. there are two steps (one up, one down). It's irrelevant that $f(t)$ on average isn't 0. It's important that for:

$$\omega \to 0 \qquad \sin(\omega T/2)/(\omega T/2) \to 1$$

(use l'Hospital's rule or $\sin x \approx x$ for small x).

Now, we calculate the Fourier transform of important functions. Let's start with the Gaussian.

Example 2.2 (The normalised Gaussian) The prefactor is chosen in such a way that the area is 1.

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\frac{t^2}{\sigma^2}}.$$

$$F(\omega) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{2}\frac{t^2}{\sigma^2}} \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}t \tag{2.16}$$

[3]It stands for "sinus cardinalis", but what is "cardinalis"? Has nothing to do with the catholic church, I guess.

$$= \frac{2}{\sigma\sqrt{2\pi}} \int_0^{+\infty} e^{-\frac{1}{2}\frac{t^2}{\sigma^2}} \cos \omega t \, dt$$
$$= e^{-\frac{1}{2}\sigma^2\omega^2}.$$

Again, the imaginary part is 0, as $f(t)$ is even. The Fourier transform of a Gaussian results in another Gaussian. Note that the Fourier transform is not normalised to area 1. The $1/2$ occurring in the exponent is handy (could also have been absorbed into σ), as the following is true for this representation:

$$\begin{aligned} \sigma &= \sqrt{2 \ln 2} \times \text{HWHM (half width at half maximum = HWHM)} \\ &= 1.177 \times \text{HWHM}. \end{aligned} \tag{2.17}$$

$f(t)$ has σ in the exponent's denominator, $F(\omega)$ in the numerator: the slimmer $f(t)$, the wider $F(\omega)$ and vice versa (cf. Fig. 2.3).

Example 2.3 (Bilateral exponential function)

$$\begin{aligned} f(t) &= e^{-|t|/\tau}. \\ F(\omega) &= \int_{-\infty}^{+\infty} e^{-|t|/\tau} e^{-i\omega t} dt = 2 \int_0^{+\infty} e^{-t/\tau} \cos \omega t \, dt = \frac{2\tau}{1+\omega^2\tau^2}. \end{aligned} \tag{2.18}$$

As $f(t)$ is even, the imaginary part is 0. The Fourier transform of the exponential function is a Lorentzian (cf. Fig. 2.4).

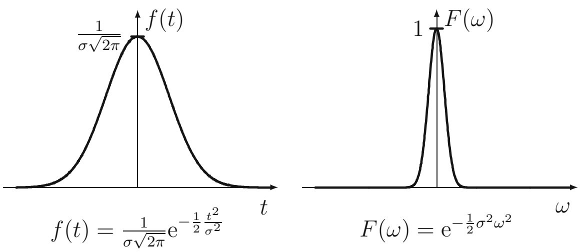

Fig. 2.3 Gaussian and Fourier transform (=equally a Gaussian)

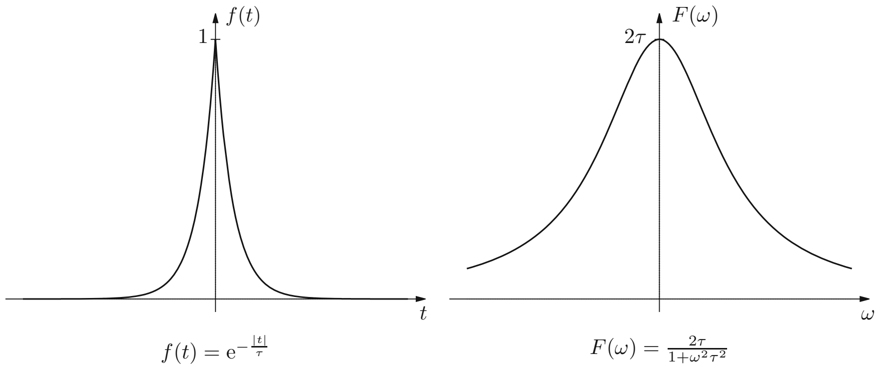

Fig. 2.4 Bilateral exponential function and Fourier transformation (=Lorentzian)

Example 2.4 (Unilateral exponential function)

$$f(t) = \begin{cases} \mathrm{e}^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{else} \end{cases}. \tag{2.19}$$

$$F(\omega) = \int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t = \left.\frac{\mathrm{e}^{-(\lambda+\mathrm{i}\omega)t}}{-(\lambda+\mathrm{i}\omega)}\right|_0^{+\infty} \tag{2.20}$$

$$= \frac{1}{\lambda+\mathrm{i}\omega} = \frac{\lambda}{\lambda^2+\omega^2} + \frac{-\mathrm{i}\omega}{\lambda^2+\omega^2}. \tag{2.21}$$

(*Sorry*: When integrating in the complex plane, we really should have used the Residue Theorem[4] instead of integrating in a rather cavalier fashion. The result, however, is correct all the same.)

$F(\omega)$ is complex, as $f(t)$ is neither even nor odd. We now can write the real and the imaginary parts separately. The real part has a Lorentzian shape we're familiar with by now, and the imaginary part has a dispersion shape. Often the so-called polar representation is used, too, so we'll deal with that one in the next section.

Examples in physics: the damped oscillation that is used to describe the emission of a particle (for example a photon, a γ-quantum) from an excited nuclear state with a lifetime of τ (meaning, that the excited state depopulates according to $\mathrm{e}^{-t/\tau}$), results in a Lorentzian-shaped emission-line. Exponential relaxation processes will result in Lorentzian-shaped spectral-lines, for example in the case of nuclear magnetic resonance.

[4]The Residue Theorem is part of the theory of functions of complex variables.

2.1.4 Polar Representation of the Fourier Transform

Every complex number $z = a + \mathrm{i}b$ can be represented in the complex plane by its magnitude and phase φ (Fig. 2.5):

$$z = a + \mathrm{i}b = \sqrt{a^2 + b^2}\,\mathrm{e}^{\mathrm{i}\varphi} \text{ with } \tan\varphi = b/a.$$

This allows us to represent the Fourier transform of the "unilateral" exponential function as in Fig. 2.6.

Alternatively to the polar representation we can also represent the real and imaginary parts separately (cf. Fig. 2.7).

Please note that $|F(\omega)|$ is no Lorentzian! If you want to "stick" to this property, you better represent the square of the magnitude: $|F(\omega)|^2 = 1/(\lambda^2 + \omega^2)$ is a Lorentzian again. This representation is often also called the power representation: $|F(\omega)|^2 = (\text{real part})^2 + (\text{imaginary part})^2$. The phase goes to 0 at the maximum of $|F(\omega)|$, i.e. when "in resonance".

Warning: The representation of the magnitude as well as of the squared magnitude does away with the *linearity* of the Fourier transformation!

Finally, let's try out the inverse transformation and find out how we return to the "unilateral" exponential function (the Fourier transform didn't look all that "unilateral"!).

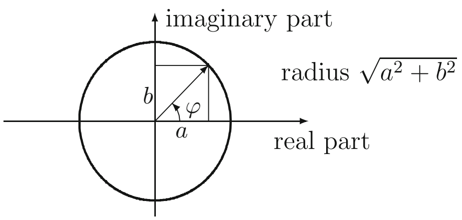

Fig. 2.5 Polar representation of a complex number $z = a + \mathrm{i}b$

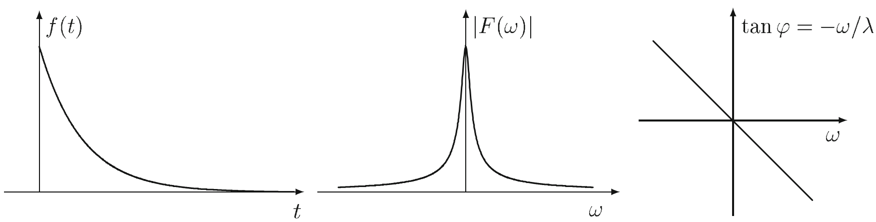

Fig. 2.6 Unilateral exponential function, magnitude of the Fourier transform and phase (imaginary part/real part)

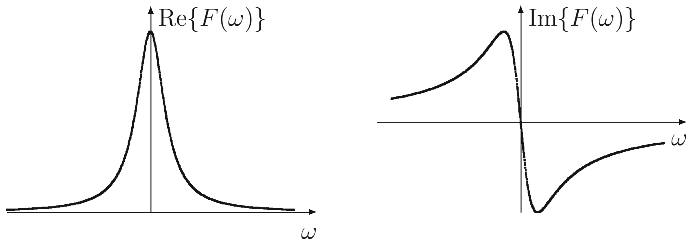

Fig. 2.7 Real part and imaginary part of the Fourier transform of a unilateral exponential function

$$
\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\lambda - \mathrm{i}\omega}{\lambda^2+\omega^2}\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\left\{2\lambda\int_{0}^{+\infty}\frac{\cos\omega t}{\lambda^2+\omega^2}\mathrm{d}\omega + 2\int_{0}^{+\infty}\frac{\omega\sin\omega t}{\lambda^2+\omega^2}\mathrm{d}\omega\right\} \\
&= \frac{1}{\pi}\left\{\frac{\pi}{2}\mathrm{e}^{-|\lambda t|} \pm \frac{\pi}{2}\mathrm{e}^{-|\lambda t|}\right\},\ \text{where}\ \begin{matrix}\text{“}+\text{” for } t \geq 0 \\ \text{“}-\text{” for } t < 0\end{matrix}\ \text{is valid} \\
&= \begin{cases} \mathrm{e}^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{else} \end{cases}.
\end{aligned}
\tag{2.22}
$$

2.2 Theorems and Rules

2.2.1 Linearity Theorem

For completeness’ sake, once again:

$$
\begin{aligned}
f(t) &\leftrightarrow F(\omega), \\
g(t) &\leftrightarrow G(\omega), \\
a\cdot f(t) + b\cdot g(t) &\leftrightarrow a\cdot F(\omega) + b\cdot G(\omega).
\end{aligned}
\tag{2.23}
$$

2.2.2 The First Shifting Rule

We already know: shifting in the time domain means modulation in the frequency domain:

$$
\begin{aligned}
f(t) &\leftrightarrow F(\omega), \\
f(t-a) &\leftrightarrow F(\omega)\mathrm{e}^{-\mathrm{i}\omega a}.
\end{aligned}
\tag{2.24}
$$

The proof is quite simple.

Example 2.5 ("Rectangular function")

$$f(t) = \begin{cases} 1 \text{ for } T/2 \le t \le T/2 \\ 0 \text{ else} \end{cases} .$$

$$F(\omega) = T\,\frac{\sin(\omega T/2)}{\omega T/2}. \tag{2.25}$$

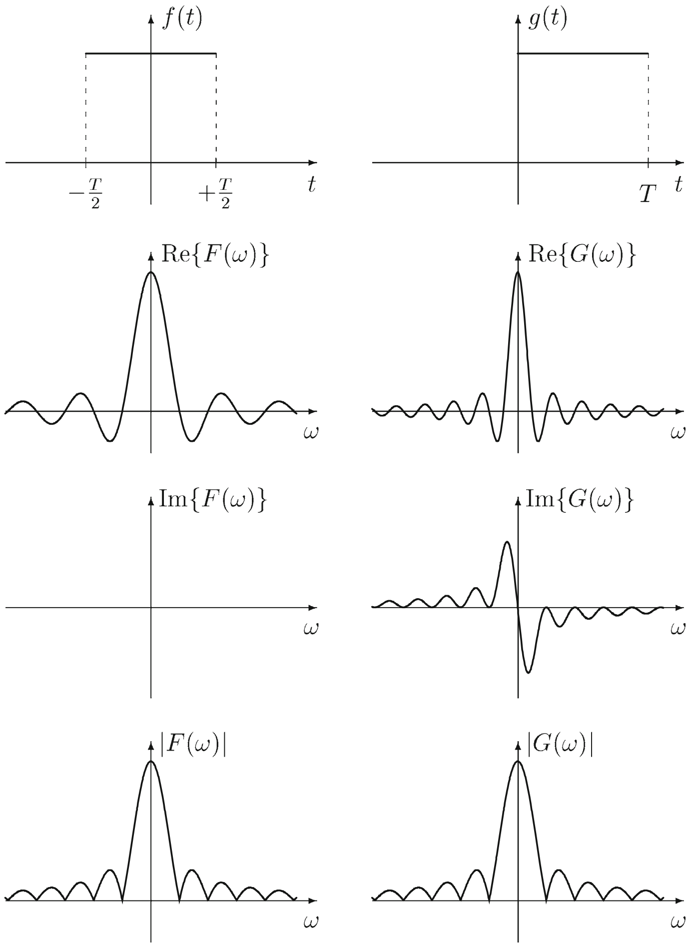

Fig. 2.8 "Rectangular function", real part, imaginary part, magnitude of Fourier transform (*left from top to bottom*); for the "rectangular function", shifted to the *right* by $T/2$ (*right from top to bottom*)

Now we shift the rectangle $f(t)$ by $a = T/2 \to g(t)$, and then get (see Fig. 2.8):

$$\begin{aligned} G(\omega) &= T\frac{\sin(\omega T/2)}{\omega T/2}\mathrm{e}^{-\mathrm{i}\omega T/2} \\ &= T\frac{\sin(\omega T/2)}{\omega T/2}(\cos(\omega T/2) - \mathrm{i}\sin(\omega T/2)). \end{aligned} \tag{2.26}$$

The real part gets modulated with $\cos(\omega T/2)$. The imaginary part which before was 0, now is unequal to 0 and "complements" the real part exactly, so $|F(\omega)|$ stays the same. Equation (2.24) contains "only" a phase factor $\mathrm{e}^{-\mathrm{i}\omega a}$, which is irrelevant as far as the magnitude is concerned. As long as you only look at the power spectrum, you may shift the function $f(t)$ along the time-axis as much as you want: you won't notice any effect. In the phase of the polar representation, however, you'll see the shift again:

$$\begin{aligned} &\tan\varphi = \frac{\text{imaginary part}}{\text{real part}} = -\frac{\sin(\omega T/2)}{\cos(\omega T/2)} = -\tan(\omega T/2) \\ &\text{or} \ \ \varphi = -\omega T/2. \end{aligned} \tag{2.27}$$

Don't worry about the phase φ overshooting $\pm\pi/2$.

2.2.3 *The Second Shifting Rule*

We already know: a modulation in the time domain results in a shift in the frequency domain:

$$\begin{aligned} f(t) &\leftrightarrow F(\omega), \\ f(t)\mathrm{e}^{\mathrm{i}\omega_0 t} &\leftrightarrow F(\omega - \omega_0). \end{aligned} \tag{2.28}$$

If you prefer real modulations, you may write:

$$\begin{aligned} \mathrm{FT}(f(t)\cos\omega_0 t) &= \frac{F(\omega + \omega_0) + F(\omega - \omega_0)}{2}, \\ \mathrm{FT}(f(t)\sin\omega_0 t) &= \mathrm{i}\frac{F(\omega + \omega_0) - F(\omega - \omega_0)}{2}. \end{aligned} \tag{2.29}$$

This follows from Euler's identity (1.22) straight away.

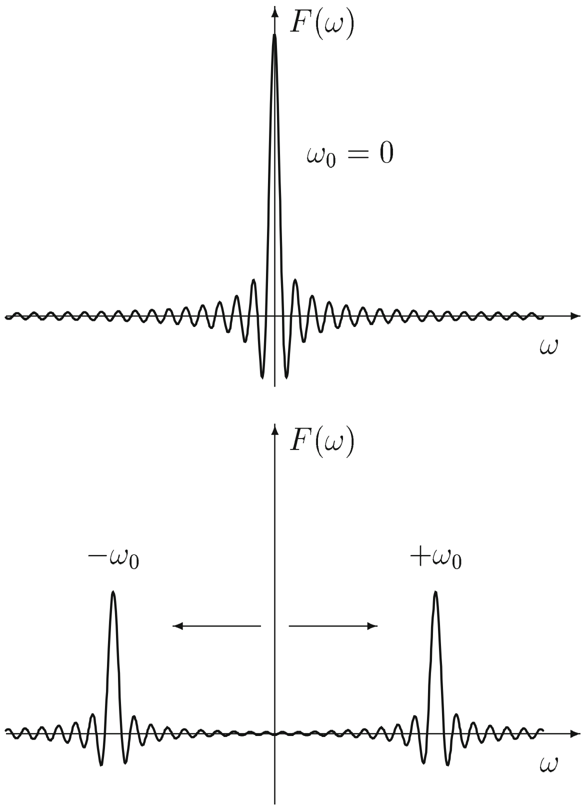

Fig. 2.9 Fourier transform of $g(t) = \cos \omega t$ in the interval $-T/2 \leq t \leq T/2$

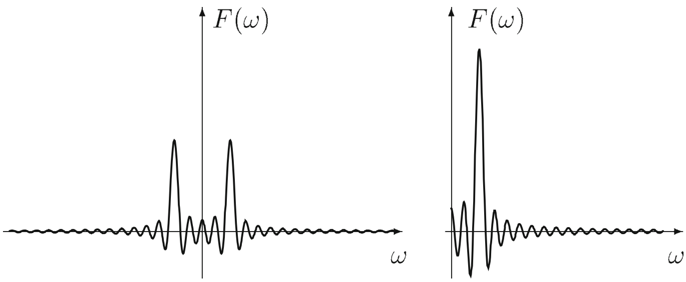

Fig. 2.10 Superposition of $\frac{\sin x}{x}$ sidelobes at small frequencies for negative and positive (*left*) and positive frequencies only (*right*)

Example 2.6 ("Rectangular function")

$$f(t) = \begin{cases} 1 \text{ for } -T/2 \leq t \leq +T/2 \\ 0 \text{ else} \end{cases}.$$

$$F(\omega) = T\frac{\sin(\omega T/2)}{\omega T/2} \qquad \text{(cf. (2.15))}$$

and

$$g(t) = \cos\omega_0 t. \tag{2.30}$$

Using $h(t) = f(t) \cdot g(t)$ and the Second Shifting Rule we get:

$$H(\omega) = \frac{T}{2}\left\{\frac{\sin[(\omega+\omega_0)T/2]}{(\omega+\omega_0)T/2} + \frac{\sin[(\omega-\omega_0)T/2]}{(\omega-\omega_0)T/2}\right\}. \tag{2.31}$$

This means: the Fourier transform of the function $\cos\omega_0 t$ within the interval $-T/2 \leq t \leq T/2$ (and outside equal to 0) consists of two frequency peaks, one at $\omega = -\omega_0$ and another one at $\omega = +\omega_0$. The amplitude naturally gets split evenly ("between brothers"). If we had $\omega_0 = 0$, then we'd get the central peak $\omega = 0$ once again; increasing ω_0 splits this peak into two peaks, moving to the left and the right (cf. Fig. 2.9).

If you don't like negative frequencies, you may flip the negative half-plane, so you'll only get *one* peak at $\omega = \omega_0$ with twice (that's the original) intensity.

Caution: For small frequencies ω_0 the sidelobes of the function $\frac{\sin x}{x}$ tend to "rub shoulders", meaning that they interfere with each other. Even flipping the negative half-plane won't help that. Figure 2.10 explains the problem.

2.2.4 Scaling Theorem

Similar to (1.41) the following is true:

$$\begin{aligned} f(t) &\leftrightarrow F(\omega), \\ f(at) &\leftrightarrow \frac{1}{|a|}F\left(\frac{\omega}{a}\right). \end{aligned} \tag{2.32}$$

Proof (Scaling) Analogously to (1.41) with the difference that here we cannot stretch or compress the interval limits $\pm\infty$:

$$\begin{aligned}
F(\omega)^{\text{new}} &= \frac{1}{T}\int_{-\infty}^{+\infty} f(at)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \\
&= \frac{1}{T}\int_{-\infty}^{+\infty} f(t')\mathrm{e}^{-\mathrm{i}\omega t'/a}\frac{1}{a}\mathrm{d}t' \qquad \text{with } t' = at \\
&= \frac{1}{|a|}F(\omega)^{\text{old}} \qquad \text{with } \omega = \frac{\omega^{\text{old}}}{a}. \quad \square
\end{aligned}$$

Here, we tacitly assumed $a > 0$. For $a < 0$ we would get a minus sign in the prefactor; however, we would also have to interchange the integration limits and thus get together the factor $\frac{1}{|a|}$. This means: stretching (compressing) the time-axis results in the compression (stretching) of the frequency-axis. For the special case $a = -1$ we get:

$$\begin{aligned}
f(t) &\to F(\omega), \\
f(-t) &\to F(-\omega).
\end{aligned} \tag{2.33}$$

Therefore turning around the time axis ("looking into the past") results in turning around the frequency axis. This profound secret will stay hidden to all those unable to think in anything but positive frequencies.

2.3 Convolution, Cross Correlation, Autocorrelation, Parseval's Theorem

2.3.1 Convolution

The convolution of a function $f(t)$ with another function $g(t)$ means:

Definition 2.3 (*Convolution*)

$$f(t) \otimes g(t) \equiv \int_{-\infty}^{+\infty} f(\xi)g(t-\xi)\mathrm{d}\xi. \tag{2.34}$$

Please note there is a minus sign in the argument of $g(t)$. The convolution is commutative, distributive, and associative. This means:

$$\text{commutative:} \qquad f(t) \otimes g(t) = g(t) \otimes f(t).$$

Here we have to take into account the sign!

Proof (Convolution, commutative) Substituting the integration variables:

$$f(t) \otimes g(t) = \int_{-\infty}^{+\infty} f(\xi)g(t-\xi)\mathrm{d}\xi = \int_{-\infty}^{+\infty} g(\xi')f(t-\xi')\mathrm{d}\xi'$$

$$\text{with } \xi' = t - \xi. \quad \square$$

$$\text{Distributive:} \qquad f(t) \otimes (g(t) + h(t)) = f(t) \otimes g(t) + f(t) \otimes h(t)$$

(Proof: *Linear operation*!).

$$\text{Associative:} \qquad f(t) \otimes (g(t) \otimes h(t)) = (f(t) \otimes g(t)) \otimes h(t)$$

(the convolution sequence doesn't matter; proof: double integral with interchange of integration sequence).

Example 2.7 (Convolution of a "rectangular function" with another "rectangular function") We want to convolve the "rectangular function" $f(t)$ with another "rectangular function" $g(t)$:

$$\begin{aligned} f(t) &= \begin{cases} 1 \text{ for } -T/2 \le t \le T/2 \\ 0 \text{ else} \end{cases}, \\ g(t) &= \begin{cases} 1 \text{ for } 0 \le t \le T \\ 0 \text{ else} \end{cases}. \\ h(t) &= f(t) \otimes g(t). \end{aligned} \tag{2.35}$$

According to the definition in (2.34) we have to mirror $g(t)$ (minus sign in front of ξ). Then we shift $g(t)$ and calculate the overlap (cf. Fig. 2.11).

We get the first overlap for $t = -T/2$ and the last one for $t = +3T/2$ (cf. Fig. 2.12).

At the limits, where $t = -T/2$ and $t = +3T/2$, we start and finish with an overlap of 0, the maximum overlap occurs at $t = +T/2$: there the two rectangles are exactly on top of each other (or below each other?). The integral then is exactly T; in between the integral rises/falls at a linear rate (cf. Fig. 2.13).

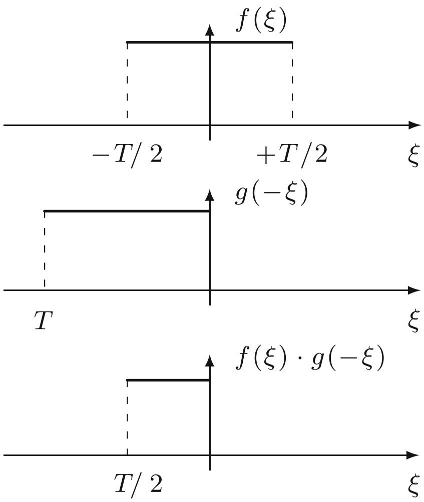

Fig. 2.11 "Rectangular function" $f(\xi)$, mirrored rectangular function $g(-\xi)$, overlap (*from top to bottom*). The area of the overlap gives the convolution integral

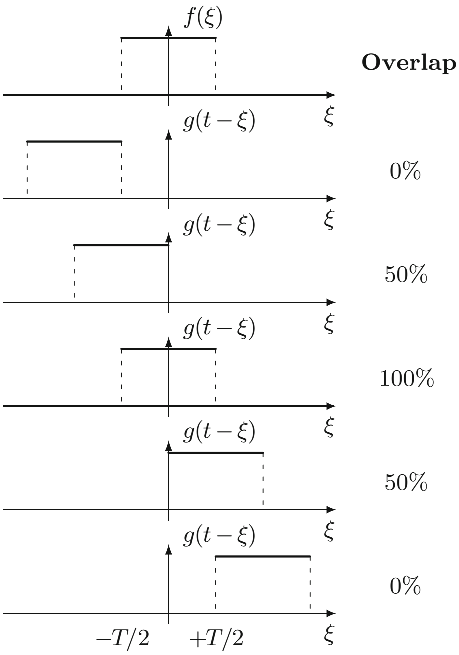

Fig. 2.12 Illustration of the convolution process of $f(t)$ and $g(t)$ with $t = -T/2$, 0, $+T/2$, $+T$, $+3T/2$ (*from top to bottom*)

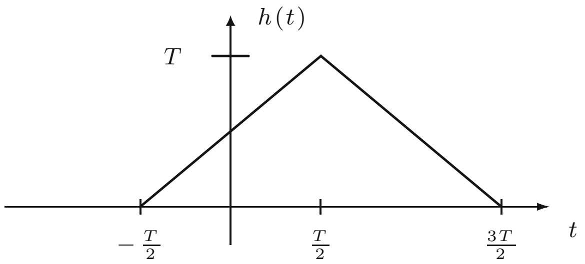

Fig. 2.13 Convolution $h(t) = f(t) \otimes g(t)$

Please note the following: the interval, where $f(t) \otimes g(t)$ is unequal to 0, now is twice as big: $2T$! If we had defined $g(t)$ symmetrically around 0 in the first place (I didn't want to do that, so we can't forget the mirroring!), then also $f(t) \otimes g(t)$ would be symmetrical around 0. In this case we would have convolved $f(t)$ with itself.

Now to a more useful example: let's take a pulse that looks like a "unilateral" exponential function:

$$f(t) = \begin{cases} \mathrm{e}^{-t/\tau} & \text{for } t \geq 0 \\ 0 & \text{else} \end{cases}. \tag{2.36}$$

Any device that delivers pulses as a function of time, has a finite rise-/decay-time, which for simplicity's sake we'll assume to be a Gaussian (see Fig. 2.14):

$$g(t) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\frac{t^2}{\sigma^2}}. \tag{2.37}$$

That is how our device would represent a δ-function—we can't get sharper than that. The function $g(t)$ therefore is the device's resolution function, which we'll have to use for the convolution of *all* signals we want to record. An example would be the bandwidth of an oscilloscope. We then need:

$$\mathrm{S}(t) = f(t) \otimes g(t), \tag{2.38}$$

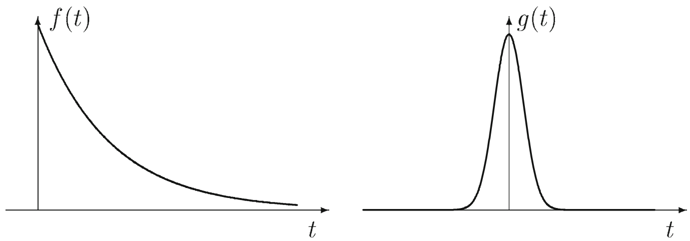

Fig. 2.14 Illustration of convolution: the Gaussian will be shifted over the unilateral exponential-function

where S(t) is the experimental, "smeared" signal. It's obvious that the rise at $t = 0$ will not be as steep, and the peak of the exponential function will get "ironed out". We'll have to take a closer look:

$$\begin{aligned}
S(t) &= \frac{1}{\sigma\sqrt{2\pi}} \int_0^{+\infty} \mathrm{e}^{-\xi/\tau} \mathrm{e}^{-\frac{1}{2}\frac{(t-\xi)^2}{\sigma^2}} \mathrm{d}\xi \\
&= \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\frac{t^2}{\sigma^2}} \int_0^{+\infty} \exp \underbrace{\left[-\frac{\xi}{\tau} + \frac{t\xi}{\sigma^2} - \frac{1}{2}\xi^2/\sigma^2\right]}_{\text{form quadratic complement}} \mathrm{d}\xi \\
&= \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\frac{t^2}{\sigma^2}} \mathrm{e}^{\frac{t^2}{2\sigma^2}} \mathrm{e}^{-\frac{t}{\tau}} \mathrm{e}^{\frac{\sigma^2}{2\tau^2}} \int_0^{+\infty} \mathrm{e}^{-\frac{1}{2\sigma^2}\left(\xi-\left(t-\frac{\sigma^2}{\tau}\right)\right)^2} \mathrm{d}\xi \qquad (2.39)\\
&= \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{t}{\tau}} \mathrm{e}^{+\frac{\sigma^2}{2\tau^2}} \int_{-(t-\sigma^2/\tau)}^{+\infty} \mathrm{e}^{-\frac{1}{2\sigma^2}\xi'^2} \mathrm{d}\xi' \quad \text{with } \xi' = \xi - \left(t - \frac{\sigma^2}{\tau}\right) \\
&= \frac{1}{2}\mathrm{e}^{-\frac{t}{\tau}} \mathrm{e}^{+\frac{\sigma^2}{2\tau^2}} \operatorname{erfc}\left(\frac{\sigma}{\sqrt{2}\tau} - \frac{t}{\sigma\sqrt{2}}\right).
\end{aligned}$$

Here, $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$ is the complementary error-function with the defining equation:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2} \mathrm{d}t. \qquad (2.40)$$

The functions $\operatorname{erf}(x)$ and $\operatorname{erfc}(x)$ are shown in Fig. 2.15.

The function $\operatorname{erfc}(x)$ represents a "smeared" step. Together with the factor $1/2$ the height of the step is just 1. As the time in the argument of $\operatorname{erfc}(x)$ in (2.39) has a negative sign, the step of Fig. 2.15 is mirrored and also shifted by $\sigma/\sqrt{2}\tau$. Figure 2.16 shows the result of the convolution of the exponential function with the Gaussian.

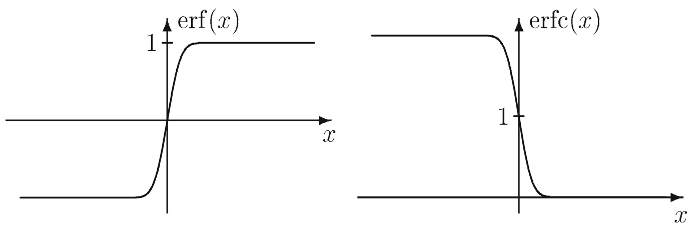

Fig. 2.15 The functions $\operatorname{erf}(x)$ and $\operatorname{erfc}(x)$

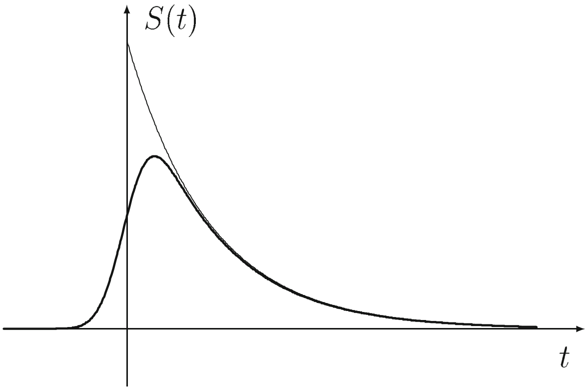

Fig. 2.16 Result of the convolution of a unilateral exponential function with a Gaussian. Exponential function without convolution (*thin line*)

The following properties immediately stand out:

i. the finite time resolution ensures that there also is a signal at negative times, whereas it was 0 before convolution,
ii. the maximum is not at $t = 0$ any more,
iii. what can't be seen straight away, yet is easy to grasp, is the following: the center of gravity of the exponential function, which was at $t = \tau$, doesn't get shifted at all upon convolution. An *even* function won't shift the center of gravity! Have a go and check it out!

It's easy to remember the shape of the curve in Fig. 2.16. Start out with the exponential function with a "90°-vertical cliff", and then dump "gravel" to the left and to the right of it (equal quantities! it's an even function!): that's how you get the gravel-heap for $t < 0$, demolish the peak and make sure there's also a gravel-heap for $t > 0$, that slowly gets thinner and thinner. Indeed, the influence of the step will become less and less important if times get larger and larger, i.e.:

$$\frac{1}{2}\mathrm{erfc}\left(\frac{\sigma}{\sqrt{2}\tau} - \frac{t}{\sigma\sqrt{2}}\right) \to 1 \qquad \text{for } t \gg \frac{\sigma^2}{\tau}, \tag{2.41}$$

and only the unchanged $\mathrm{e}^{-t/\tau}$ will remain, however with the constant factor $\mathrm{e}^{+\frac{\sigma^2}{2\tau^2}}$. This factor is always >1 because we always have more "gravel" poured downwards than upwards.

Now we prove the extremely important Convolution Theorem:

$$\begin{aligned} f(t) &\leftrightarrow F(\omega), \\ g(t) &\leftrightarrow G(\omega), \\ h(t) = f(t) \otimes g(t) &\leftrightarrow H(\omega) = F(\omega) \cdot G(\omega), \end{aligned} \tag{2.42}$$

i.e., the *convolution integral* becomes, through Fourier transformation, a *product* of the Fourier-transformed ones.

Proof (Convolution Theorem)

$$
\begin{aligned}
H(\omega) &= \iint f(\xi)g(t-\xi)\mathrm{d}\xi \times \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t \\
&= \int f(\xi)\mathrm{e}^{-\mathrm{i}\omega\xi}\left[\int g(t-\xi)\mathrm{e}^{-\mathrm{i}\omega(t-\xi)}\mathrm{d}t\right]\mathrm{d}\xi \\
&\qquad\quad \uparrow \qquad \text{expanded} \qquad \uparrow \\
&= \int f(\xi)\mathrm{e}^{-\mathrm{i}\omega\xi}\mathrm{d}\xi \times G(\omega) \\
&= F(\omega)\times G(\omega). \quad \square
\end{aligned}
\tag{2.43}
$$

In the step before the last one, we substituted $t' = t - \xi$. The integration boundaries $\pm\infty$ did not change by doing that, and $G(\omega)$ does not depend on ξ.

The inverse Convolution Theorem then is:

$$
\begin{aligned}
f(t) &\leftrightarrow F(\omega), \\
g(t) &\leftrightarrow G(\omega), \\
h(t) = f(t)\cdot g(t) &\leftrightarrow H(\omega) = \tfrac{1}{2\pi}F(\omega)\otimes G(\omega).
\end{aligned}
\tag{2.44}
$$

Proof (Inverse Convolution Theorem)

$$
\begin{aligned}
H(\omega) &= \int f(t)g(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t \\
&= \int\left(\frac{1}{2\pi}\int F(\omega')\mathrm{e}^{+\mathrm{i}\omega' t}\mathrm{d}\omega' \times \frac{1}{2\pi}\int G(\omega'')\mathrm{e}^{+\mathrm{i}\omega'' t}\mathrm{d}\omega''\right)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t \\
&= \frac{1}{(2\pi)^2}\int F(\omega')\int G(\omega'')\underbrace{\int \mathrm{e}^{\mathrm{i}(\omega'+\omega''-\omega)t}\mathrm{d}t}_{=2\pi\delta(\omega'+\omega''-\omega)}\mathrm{d}\omega'\mathrm{d}\omega'' \\
&= \frac{1}{2\pi}\int F(\omega')G(\omega-\omega')\mathrm{d}\omega' \\
&= \frac{1}{2\pi}F(\omega)\otimes G(\omega). \quad \square
\end{aligned}
$$

Caution: Contrary to the Convolution Theorem (2.42), in (2.44) there is a factor of $1/2\pi$ in front of the convolution of the Fourier transforms.

A widely popular exercise is the de-convolution of data: the instruments' resolution function "smears out" the quickly varying functions, but we naturally want to reconstruct the data to what they would look like if the resolution function was infinitely good—provided we precisely knew the resolution function. In principle, that's a good idea—and thanks to the Convolution Theorem, not a problem: you

Fourier-transform the data, divide by the Fourier-transformed resolution function and transform it back. For practical applications it doesn't quite work that way. As in real life, we can't transform from $-\infty$ to $+\infty$, we need low-pass filters, in order not to get "swamped" with oscillations resulting from cut-off errors. Therefore the advantages of de-convolution are just as quickly lost as gained. Actually, the following is obvious: whatever got "smeared" by finite resolution, can't be reconstructed unambiguously. Imagine that a very pointed peak got eroded over millions of years, so there's only gravel left at its bottom. Try reconstructing the original peak from the debris around it! The result might be impressive from an artist's point of view, an artefact, but it hasn't got much to do with the original reality (unfortunately the word artefact has negative connotations among scientists).

Two useful examples for the Convolution Theorem:

Example 2.8 (Gaussian frequency distribution) Let's assume we have $f(t) = \cos\omega_0 t$, and the frequency ω_0 is not precisely defined, but is Gaussian distributed:

$$P(\omega) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\frac{\omega^2}{\sigma^2}}.$$

What we're measuring then is:

$$\tilde{f}(t) = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\frac{\omega^2}{\sigma^2}} \cos(\omega - \omega_0) t \, \mathrm{d}\omega, \tag{2.45}$$

i.e. a convolution integral in ω_0. Instead of calculating this integral directly, we use the inverse of the Convolution Theorem (2.44), thus saving work and gaining higher enlightenment. But watch it! We have to handle the variables carefully. The time t in (2.45) has nothing to do with the Fourier transformation we need in (2.44). And the same is true for the integration variable ω. Therefore we rather use t_0 and ω_0 for the variable pairs in (2.44). We identify:

$$\begin{aligned} F(\omega_0) &= \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\frac{\omega_0^2}{\sigma^2}} \\ \frac{1}{2\pi} G(\omega_0) &= \cos\omega_0 t \qquad \text{or } G(\omega_0) = 2\pi\cos\omega_0 t. \end{aligned}$$

The inverse transformation of these functions using (2.11) gives us:

$$f(t_0) = \frac{1}{2\pi} \mathrm{e}^{-\frac{1}{2}\sigma^2 t_0^2}$$

(cf. (2.16) for the inverse problem; don't forget the factor $1/2\pi$ when doing the inverse transformation!),

$$g(t_0) = 2\pi \left[\frac{\delta(t_0 - t)}{2} + \frac{\delta(t_0 + t)}{2} \right]$$

(cf. (2.9) for the inverse problem; use the First Shifting Rule (2.24); don't forget the factor $1/2\pi$ when doing the inverse transformation!).

Finally we get:

$$h(t_0) = \mathrm{e}^{-\frac{1}{2}\sigma^2 t_0^2} \left[\frac{\delta(t_0 - t)}{2} + \frac{\delta(t_0 + t)}{2} \right].$$

Now the only thing left is to Fourier-transform $h(t_0)$. The integration over the δ-function actually is fun:

$$\begin{aligned} \tilde{f}(t) \equiv H(\omega_0) &= \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{2}\sigma^2 t_0^2} \left[\frac{\delta(t_0 - t)}{2} + \frac{\delta(t_0 + t)}{2} \right] \mathrm{e}^{-\mathrm{i}\omega_0 t_0} \mathrm{d}t_0 \\ &= \mathrm{e}^{-\frac{1}{2}\sigma^2 t^2} \cos \omega_0 t. \end{aligned}$$

Now, this was more work than we'd originally thought it would be. But look at what we've gained in insight!

This means: the convolution of a Gaussian distribution in the frequency domain results in exponential "damping" of the cosine term, where the damping happens to be the Fourier transform of the frequency distribution. This, of course, is due to the fact that we have chosen to use a cosine function (i.e. a basis function) for $f(t)$. $P(\omega)$ makes sure that oscillations for $\omega \neq \omega_0$ are slightly shifted with respect to each other, and will more and more superimpose each other destructively in the long run, averaging out to 0.

Example 2.9 (Lorentzian frequency distribution) Now naturally we'll know immediately what a convolution with a Lorentzian distribution:

$$P(\omega) = \frac{\sigma}{\pi} \frac{1}{\omega^2 + \sigma^2} \tag{2.46}$$

would do:

$$\begin{aligned} \tilde{f}(t) &= \int_{-\infty}^{+\infty} \frac{\sigma}{\pi} \frac{1}{\omega^2 + \sigma^2} \cos(\omega - \omega_0) t \mathrm{d}\omega, \\ h(t_0) = \mathrm{FT}^{-1}(\tilde{f}(t)) &= \mathrm{e}^{-\sigma t_0} \left[\frac{\delta(t_0 - t)}{2} + \frac{\delta(t_0 + t)}{2} \right]; \\ \tilde{f}(t) &= \mathrm{e}^{-\sigma t} \cos \omega_0 t. \end{aligned} \tag{2.47}$$

This is a damped wave. That's how we would describe the electric field of a Lorentz-shaped spectral line, sent out by an "emitter" with a life time of $1/\sigma$.

These examples are of fundamental importance to physics. Whenever we probe with plane waves, i.e. $e^{i\boldsymbol{qx}}$, the answer we get is the Fourier transform of the respective distribution function of the object. A classical example is the elastic scattering of electrons at nuclei. Here, the form factor $F(\boldsymbol{q})$ is the Fourier transform of the distribution function of the nuclear charge density $\rho(\boldsymbol{x})$. The wave vector $\boldsymbol{q}$ is, apart from a prefactor, identical with the momentum.

Example 2.10 (Gaussian convolved with Gaussian) We perform a convolution of a Gaussian with σ_1 with another Gaussian with σ_2. As the Fourier transforms are Gaussians again—yet with σ_1^2 and σ_2^2 in the *numerator* of the exponent—it's immediately obvious that $\sigma_{\text{total}}^2 = \sigma_1^2 + \sigma_2^2$. Therefore, we get another Gaussian with geometric addition of the widths σ_1 and σ_2.

2.3.2 Cross Correlation

Sometimes we want to know if a measured function $f(t)$ has anything in common with another measured function $g(t)$. Cross correlation is ideally suited to that.

Definition 2.4 (*Cross correlation*)

$$h(t) = \int_{-\infty}^{+\infty} f(\xi) g^*(t+\xi) \mathrm{d}\xi \equiv f(t) \star g(t). \tag{2.48}$$

Watch it: Here, there is a plus sign in the argument of g, therefore we don't mirror $g(t)$. For even functions $g(t)$, this, however, doesn't matter.

The asterisk * means: complex conjugated. We may disregard it for real functions. The symbol $\star$ means: cross correlation, and is not to be confounded with $\otimes$ for convolution. Cross correlation is associative and distributive, yet *not* commutative. That's not only because of the complex-conjugated symbol, but mainly because of the plus sign in the argument of $g(t)$. Of course we want to convert the integral in the cross correlation to a product by using Fourier transformation.

$$\begin{aligned} f(t) &\leftrightarrow F(\omega), \\ g(t) &\leftrightarrow G(\omega), \\ h(t) = f(t) \star g(t) &\leftrightarrow H(\omega) = F(\omega) G^*(\omega). \end{aligned} \tag{2.49}$$

Proof (Fourier transform of cross correlation)

$$\begin{aligned}
H(\omega) &= \iint f(\xi)g^*(t+\xi)\mathrm{d}\xi \cdot \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t \\
&= \int f(\xi)\left[\int g^*(t+\xi)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t\right]\mathrm{d}\xi \\
&\qquad \text{First Shifting Rule complex conjugated with } \xi = -a \\
&= \int f(\xi)G^*(+\omega)\mathrm{e}^{-\mathrm{i}\omega\xi}\mathrm{d}\xi \\
&= F(\omega)G^*(\omega). \quad \square
\end{aligned} \tag{2.50}$$

Here we used the following identity:

$$\begin{aligned}
G(\omega) &= \int g(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t \\
&\qquad \text{(take both sides complex conjugated)} \\
G^*(\omega) &= \int g^*(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t \\
G^*(-\omega) &= \int g^*(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t \\
&\qquad (\omega \text{ to be replaced by} - \omega).
\end{aligned} \tag{2.51}$$

The interpretation of (2.49) is simple: if the spectral densities of $f(t)$ and $g(t)$ are a good match, i.e. have much in common, then $H(\omega)$ will become large on average, and the cross correlation $h(t)$ will also be large, on average. Otherwise if $F(\omega)$ would be small e.g., where $G^*(\omega)$ is large and vice versa, so that there is never much left for the product $H(\omega)$. Then also $h(t)$ would be small, i.e. there is not much in common between $f(t)$ and $g(t)$.

A maybe somewhat extreme example is the technique of "Lock-in amplification", used to "dig up" small signals buried deeply in the noise. In this case we modulate the measured signal with a so-called carrier frequency, detect an extremely narrow spectral range—provided the desired signal does have spectral components in exactly this spectral width—and often additionally make use of phase information, too. Anything that doesn't correlate with the carrier frequency, gets discarded, so we're only left with the noise close to the working frequency.

2.3.3 Autocorrelation

The autocorrelation function is the cross correlation of a function $f(t)$ with itself. You may ask, for what purpose we'd want to check for what $f(t)$ has in common with $f(t)$. Autocorrelation, however, seems to attract many people in a magic manner. We often hear the view, that a signal full of noise can be turned into something really good by using the autocorrelation function, i.e. the signal-to-noise ratio would improve a lot. Don't you believe a word of it! We'll see why shortly.

Definition 2.5 (*Autocorrelation*)

$$h(t) = \int f(\xi) f^*(\xi + t) \mathrm{d}\xi. \tag{2.52}$$

We get:

$$\begin{aligned} f(t) &\leftrightarrow F(\omega), \\ h(t) = f(t) \star f(t) &\leftrightarrow H(\omega) = F(\omega)F^*(\omega) = |F(\omega)|^2. \end{aligned} \tag{2.53}$$

We may either use the Fourier transform $F(\omega)$ of a noisy function $f(t)$ and get angry about the noise in $F(\omega)$. Or we first form the autocorrelation function $h(t)$ from the function $f(t)$ and are then happy about the Fourier transform $H(\omega)$ of function $h(t)$. Normally, $H(\omega)$ does look a lot less noisy, indeed. Instead of doing it the roundabout way by using the autocorrelation function, we could have used the square of the magnitude of $F(\omega)$ in the first place. We all know, that a squared representation in the ordinate always pleases the eye, if we want to do cosmetics to a noisy spectrum. Big spectral components will grow when squared, small ones will get even smaller (cf. New Testament, Matthew 13:12: "For to him who has will more be given but from him who has not, even the little he has will be taken away."). But isn't it rather obvious that squaring doesn't change anything to the signal-to-noise ratio? In order to make it "look good", we pay the price of losing linearity.

Then, what is autocorrelation good for? A classical example comes from femtosecond measuring devices. A femtosecond is one part in a thousand trillion (US)—or a thousand billion (British)—of a second, not a particularly long time, indeed. Today, it is possible to produce such short laser pulses. How can we measure such short times? Using electronic stop-watches we can reach the range of 100 ps; hence, these "watches" are too slow by 5 orders of magnitude. Precision engineering does the job! Light travels in a femtosecond a distance of about 300 nm, i.e. about $1/100$ of a hair diameter. Today you can buy positioning devices with nanometer precision. The trick: Split the laser pulse into two pulses, let them travel a slightly different optical length using mirrors, and combine them afterwards. The detector is an "optical coincidence" which yields an output only if both pulses overlap. By tuning the optical path (using the nanometer screw!) you can "shift" one pulse over the other, i.e. you perform a cross correlation of the pulse with itself (for purists: with its exact copy). The entire system is called autocorrelator.

2.3.4 Parseval's Theorem

The autocorrelation function also comes in handy for something else, namely for deriving Parseval's theorem. We start out with (2.52), insert especially $t = 0$, and get Parseval's theorem:

$$h(0) = \int |f(\xi)|^2 \mathrm{d}\xi = \frac{1}{2\pi} \int |F(\omega)|^2 \mathrm{d}\omega. \tag{2.54}$$

We get the second equal sign by inverse transformation of $|F(\omega)|^2$, in which for $t = 0$ the factor $\mathrm{e}^{\mathrm{i}\omega t}$ becomes unity.

Equation (2.54) states, that the "information content" of the function $f(x)$—defined as integral over the square of the magnitude—is just as large as the "information content" of its Fourier transform $F(\omega)$—defined as integral over the square of the magnitude of $F(\omega)$ divided by 2π. Let's check this out straight away using an example, namely our much-used "rectangular function"!

Example 2.11 ("Rectangular function")

$$f(t) = \begin{cases} 1 \text{ for } -T/2 \le t \le T/2 \\ 0 \text{ else} \end{cases}.$$

We get on the one hand:

$$\int_{-\infty}^{+\infty} |f(t)|^2 \mathrm{d}t = \int_{-T/2}^{+T/2} \mathrm{d}t = T$$

and on the other hand:

$$\begin{aligned} F(\omega) &= T\frac{\sin(\omega T/2)}{\omega T/2}, \text{ thus} \\ \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|^2 \mathrm{d}\omega &= 2\frac{T^2}{2\pi}\int_{0}^{+\infty} \left[\frac{\sin(\omega T/2)}{\omega T/2}\right]^2 \mathrm{d}\omega \\ &= 2\frac{T^2}{2\pi}\frac{2}{T}\int_{0}^{+\infty} \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x = T \\ &\text{with } x = \omega T/2. \end{aligned} \tag{2.55}$$

It's easily understood that Parseval's theorem contains the squared magnitudes of both $f(t)$ and $F(\omega)$: anything unequal to 0 has information, regardless if it's positive or negative. The power spectrum is important, the phase doesn't matter. Of course we can use Parseval's theorem to calculate integrals. Let's simply take the last example for integration over $\left(\frac{\sin x}{x}\right)^2$. We need an integration table for that one, whereas integrating over 1, that's determining the area of a square, is elementary.

2.4 Fourier Transformation of Derivatives

When solving differential equations, we can make life easier using Fourier transformation. The derivative simply becomes a product:

$$\begin{aligned} f(t) &\leftrightarrow F(\omega), \\ f'(t) &\leftrightarrow \mathrm{i}\omega F(\omega). \end{aligned} \tag{2.56}$$

Proof (Fourier transformation of derivatives with respect to t) The abbreviation FT denotes the Fourier transformation:

$$\begin{aligned} \mathrm{FT}(f'(t)) &= \int_{-\infty}^{+\infty} f'(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t = \underbrace{f(t)\mathrm{e}^{-\mathrm{i}\omega t}\Big|_{-\infty}^{+\infty} - (-\mathrm{i}\omega)\int_{-\infty}^{+\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t}_{\text{partial integration}} \\ &= \mathrm{i}\omega F(\omega). \quad \square \end{aligned}$$

The first term in the partial integration is discarded, as $f(t) \to 0$ for $t \to \pm\infty$. Otherwise we would run into trouble with the integration, like we did at the end of Sect. 2.1.2. This game can go on:

$$\mathrm{FT}\left(\frac{\mathrm{d}f^n(t)}{\mathrm{d}^n t}\right) = (\mathrm{i}\omega)^n F(\omega). \tag{2.57}$$

For negative n we may also use the formula for integration. We can also formulate in a simple way the derivative of a Fourier transform $F(\omega)$ with respect to the frequency ω:

$$\frac{\mathrm{d}F(\omega)}{\mathrm{d}\omega} = -\mathrm{i}\mathrm{FT}(tf(t)). \tag{2.58}$$

Proof (Fourier transformation of derivatives with respect to ω)

$$\frac{\mathrm{d}F(\omega)}{\mathrm{d}\omega} = \int_{-\infty}^{+\infty} f(t)\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t = -\mathrm{i}\int_{-\infty}^{+\infty} f(t)t\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t = -\mathrm{i}\mathrm{FT}(tf(t)). \quad \square$$

Weaver [2] gives a neat example for the application of Fourier transformation:

Example 2.12 (Wave equation) The wave equation:

$$\frac{\mathrm{d}^2u(x,t)}{\mathrm{d}t^2} = c^2\frac{\mathrm{d}^2u(x,t)}{\mathrm{d}x^2} \tag{2.59}$$

can be made into an oscillation equation using Fourier transformation of the local variable, which is much easier to solve. We assume:

$$U(\xi, t) = \int_{-\infty}^{+\infty} u(x, t)\mathrm{e}^{-\mathrm{i}\xi x}\mathrm{d}x.$$

Then we get:

$$\begin{aligned} \mathrm{FT}\left(\frac{\mathrm{d}^2 u(x,t)}{\mathrm{d}x^2}\right) &= (\mathrm{i}\xi)^2 U(\xi, t), \\ \mathrm{FT}\left(\frac{\mathrm{d}^2 u(x,t)}{\mathrm{d}t^2}\right) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} U(\xi, t), \end{aligned} \tag{2.60}$$

and all together:

$$\frac{\mathrm{d}^2 U(\xi, t)}{\mathrm{d}t^2} = -c^2\xi^2 U(\xi, t).$$

The solution of this equations is:

$$U(\xi, t) = P(\xi)\cos(c\xi t),$$

where $P(\xi)$ is the Fourier transform of the starting profile $p(x)$:

$$P(\xi) = \mathrm{FT}(p(x)) = U(\xi, 0).$$

The inverse transformation gives us two profiles propagating to the left and to the right:

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} P(\xi)\cos(c\xi t)\mathrm{e}^{\mathrm{i}\xi x}\mathrm{d}\xi \\ &= \frac{1}{2\pi}\frac{1}{2}\int_{-\infty}^{+\infty} P(\xi)\left[\mathrm{e}^{\mathrm{i}\xi(x+ct)} + \mathrm{e}^{\mathrm{i}\xi(x-ct)}\right]\mathrm{d}\xi \\ &= \frac{1}{2}p(x+ct) + \frac{1}{2}p(x-ct). \end{aligned} \tag{2.61}$$

As we had no dispersion term in the wave equation, the profiles are conserved (cf. Fig. 2.17).

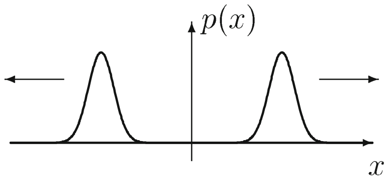

Fig. 2.17 Two starting profiles $p(x)$ propagating to the *left* and the *right* as solutions of the wave equation

2.5 Pitfalls

2.5.1 "Turn 1 into 3"

Just for fun, we'll get into magic now: let's take a unilateral exponential function:

$$\begin{aligned} f(t) &= \begin{cases} \mathrm{e}^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{else} \end{cases} \\ \text{with } F(\omega) &= \frac{1}{\lambda + \mathrm{i}\omega} \\ \text{and } |F(\omega)|^2 &= \frac{1}{\lambda^2 + \omega^2}. \end{aligned} \tag{2.62}$$

We put this function (temporarily) on a unilateral "pedestal":

$$\begin{aligned} g(t) &= \begin{cases} 1 \text{ for } t \geq 0 \\ 0 \text{ else} \end{cases} \\ \text{with } G(\omega) &= \frac{1}{\mathrm{i}\omega}. \end{aligned} \tag{2.63}$$

We arrive at the Fourier transform of Heaviside's step function $g(t)$ from the Fourier transform for the exponential function for $\lambda \to 0$. We therefore have: $h(t) = f(t) + g(t)$. Because of the linearity of the Fourier transformation:

$$H(\omega) = \frac{1}{\lambda + \mathrm{i}\omega} + \frac{1}{\mathrm{i}\omega} = \frac{\lambda}{\lambda^2 + \omega^2} - \frac{\mathrm{i}\omega}{\lambda^2 + \omega^2} - \frac{\mathrm{i}}{\omega}. \tag{2.64}$$

This results in:

$$\begin{aligned}
|H(\omega)|^2 &= \left(\frac{\lambda}{\lambda^2+\omega^2} - \frac{\mathrm{i}\omega}{\lambda^2+\omega^2} - \frac{\mathrm{i}}{\omega}\right) \times \left(\frac{\lambda}{\lambda^2+\omega^2} + \frac{\mathrm{i}\omega}{\lambda^2+\omega^2} + \frac{\mathrm{i}}{\omega}\right) \\
&= \frac{\lambda^2}{(\lambda^2+\omega^2)^2} + \frac{1}{\omega^2} + \frac{\omega^2}{(\lambda^2+\omega^2)^2} + \frac{2\omega}{(\lambda^2+\omega^2)\omega} \\
&= \frac{1}{\lambda^2+\omega^2} + \frac{1}{\omega^2} + \frac{2}{\lambda^2+\omega^2} \\
&= \frac{3}{\lambda^2+\omega^2} + \frac{1}{\omega^2}.
\end{aligned}$$

Now we return $|G(\omega)|^2 = 1/\omega^2$, i.e. the square of the Fourier transform of the pedestal, and have gained, compared to $|F(\omega)|^2$, a factor of 3. And we only had to temporarily "borrow" the pedestal to achieve that?! Of course (2.64) is correct. Returning $|G(\omega)|^2$ wasn't. We borrowed the interference term we got when squaring the magnitude, as well, and have to return it, too. This inference term amounts to just $2/(\lambda^2+\omega^2)$.

Now let's approach the problem somewhat more academically. Assuming we have $h(t) = f(t) + g(t)$ with the Fourier transforms $F(\omega)$ and $G(\omega)$. We now use the polar representation:

$$\begin{aligned}
F(\omega) &= |F(\omega)|\mathrm{e}^{\mathrm{i}\varphi_f} \\
&\text{and} \\
G(\omega) &= |G(\omega)|\mathrm{e}^{\mathrm{i}\varphi_g}.
\end{aligned} \tag{2.65}$$

This gives us:

$$H(\omega) = |F(\omega)|\mathrm{e}^{\mathrm{i}\varphi_f} + |G(\omega)|\mathrm{e}^{\mathrm{i}\varphi_g}, \tag{2.66}$$

which is, due to the linearity of the Fourier transformation, entirely correct. However, if we want to calculate $|H(\omega)|^2$ (or the square root of it), we get:

$$\begin{aligned}
|H(\omega)|^2 &= \left(|F(\omega)|\mathrm{e}^{\mathrm{i}\varphi_f} + |G(\omega)|\mathrm{e}^{\mathrm{i}\varphi_g}\right)\left(|F(\omega)|\mathrm{e}^{-\mathrm{i}\varphi_f} + |G(\omega)|\mathrm{e}^{-\mathrm{i}\varphi_g}\right) \\
&= |F(\omega)|^2 + |G(\omega)|^2 + 2|F(\omega)| \times |G(\omega)| \times \cos(\varphi_f - \varphi_g)\,.
\end{aligned} \tag{2.67}$$

If the phase difference $(\varphi_f - \varphi_g)$ doesn't happen to be 90° (modulo 2π), the interference term does not cancel. Don't think you're on the safe side with real Fourier transforms. The phases are then 0, and the interference term reaches a maximum. The following example will illustrate this:

Example 2.13 (Overlapping lines) Let's take two spectral lines—say of shape $\frac{\sin x}{x}$—that approach each other. $H(\omega)$ simply is a linear superposition[5] of the two lines, yet not $|H(\omega)|^2$. As soon as the two lines start to overlap, there also will be an interference term. To use a concrete example, let's take the function of (2.31) and,

[5] i.e. addition.

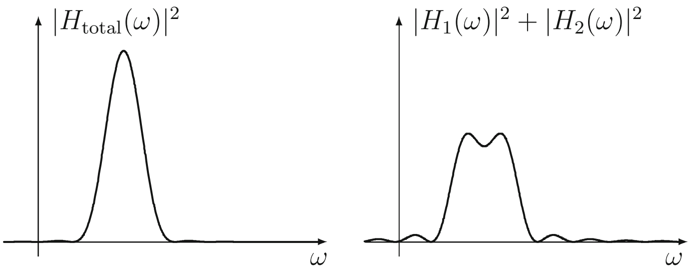

Fig. 2.18 Superposition of two $\left(\frac{\sin x}{x}\right)$-functions. Power representation with interference term (*left*); power representation without interference term (*right*)

for simplicity's sake, flip the negative frequency axis to the positive axis. Then we get:

$$\begin{aligned} H_{\text{total}}(\omega) &= H_1 + H_2 \\ &= T\left(\frac{\sin[(\omega-\omega_1)T/2]}{(\omega-\omega_1)T/2} + \frac{\sin[(\omega-\omega_2)T/2]}{(\omega-\omega_2)T/2}\right). \end{aligned} \tag{2.68}$$

The phases are 0, as we have used two cosine functions $\cos\omega_1 t$ and $\cos\omega_2 t$ for input. So $|H(\omega)|^2$ becomes:

$$\begin{aligned} |H_{\text{total}}(\omega)|^2 &= T^2\left\{\left(\frac{\sin[(\omega-\omega_1)T/2]}{(\omega-\omega_1)T/2}\right)^2 + \left(\frac{\sin[(\omega-\omega_2)T/2]}{(\omega-\omega_2)T/2}\right)^2\right. \\ &\qquad \left. + 2\frac{\sin[(\omega-\omega_1)T/2]}{(\omega-\omega_1)T/2} \times \frac{\sin[(\omega-\omega_2)T/2]}{(\omega-\omega_2)T/2}\right\} \\ &= T^2\left\{|H_1(\omega)|^2 + H_1^*(\omega)H_2(\omega)\right. \\ &\qquad \left. + H_1(\omega)H_2^*(\omega) + |H_2(\omega)|^2\right\}. \end{aligned} \tag{2.69}$$

Figure 2.18 backs up the facts: for overlapping lines, the interference term makes sure that in the power representation the lineshape is *not* the sum of the power representation of the lines. *Fix*: Show real and imaginary parts separately. If you want to keep the linear superposition (it's so useful), then you have to stay clear of the squaring!

2.5.2 Truncation Error

We now want to look at what will happen, if we truncate the function $f(t)$ somewhere—preferably where it isn't large any more—and then Fourier-transform it. Let's take a simple example:

Example 2.14 (Truncation error)

$$f(t)=\begin{cases}\mathrm{e}^{-\lambda t} & \text{for } 0\le t\le T\\ 0 & \text{else}\end{cases}. \tag{2.70}$$

The Fourier transform then is:

$$F(\omega)=\int_0^T \mathrm{e}^{-\lambda t}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t=\frac{1}{-\lambda-\mathrm{i}\omega}\mathrm{e}^{-\lambda t-\mathrm{i}\omega t}\Big|_0^T=\frac{1-\mathrm{e}^{-\lambda T-\mathrm{i}\omega T}}{\lambda+\mathrm{i}\omega}. \tag{2.71}$$

Compared to the untruncated exponential function, we're now saddled with the additional term $-\mathrm{e}^{-\lambda T}\mathrm{e}^{-\mathrm{i}\omega T}/(\lambda+\mathrm{i}\omega)$. For large values of T it isn't all that large, but to our grief, it oscillates. Truncating the smooth Lorentzian gave us small oscillations in return. Figure 2.19 explains that (cf. Fig. 2.7 without truncation).

The morale of the story: don't truncate if you don't have to, and most certainly neither brusquely nor brutally. How it should be done—if you've got to do it—will be explained in the next chapter.

Finally, an example how not to do it:

Example 2.15 (Exponential on pedestal) We'll once again use our truncated exponential function and put it on a pedestal, that's only nonzero between $0\le t\le T$. Assume a height of a:

$$\begin{aligned} f(t)&=\begin{cases}\mathrm{e}^{-\lambda t} & \text{for } 0\le t\le T\\ 0 & \text{else}\end{cases} && \text{with } F(\omega)=\frac{1-\mathrm{e}^{-\lambda T}\mathrm{e}^{-\mathrm{i}\omega T}}{\lambda+\mathrm{i}\omega},\\ g(t)&=\begin{cases}a \text{ for } 0\le t\le T\\ 0 \text{ else}\end{cases} && \text{with } G(\omega)=a\frac{1-\mathrm{e}^{-\mathrm{i}\omega T}}{\mathrm{i}\omega}.\end{aligned} \tag{2.72}$$

Here, to calculate $G(\omega)$, we've again used $F(\omega)$, with $\lambda=0$. $|F(\omega)|^2$ we've already met in Fig. 2.19. $\mathrm{Re}\{G(\omega)\}$ as well as $\mathrm{Im}\{G(\omega)\}$ are shown in Fig. 2.20.

Finally, in Fig. 2.21 $|H(\omega)|^2$ is shown, decomposed into $|F(\omega)|^2$, $|G(\omega)|^2$ and the interference term.

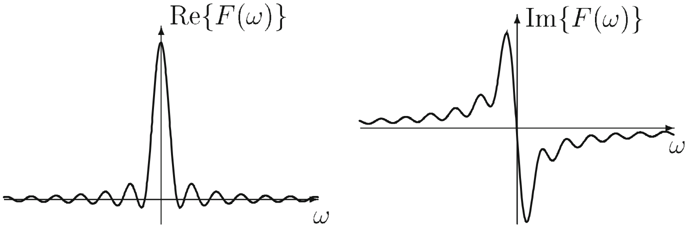

Fig. 2.19 Fourier transform of the truncated unilateral exponential function

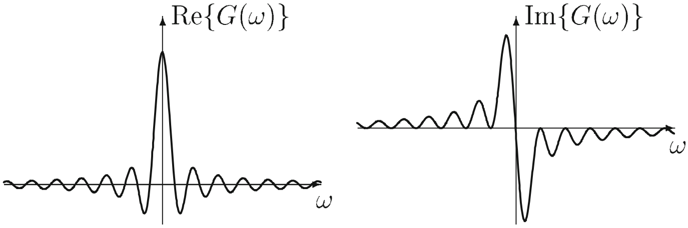

Fig. 2.20 Fourier transform of the pedestal

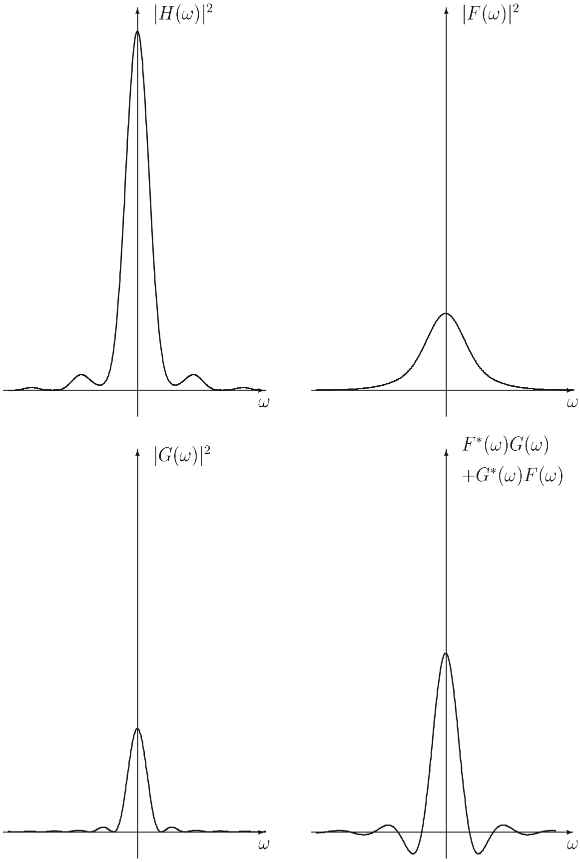

Fig. 2.21 Power representation of Fourier transform of a unilateral exponential function on a pedestal (*top left*), the unilateral exponential function (*top right*); Power representation of the Fourier transform of the pedestal (*bottom left*) and representation of the interference term (*bottom right*)

For this figure we picked the function $5e^{-5t/T} + 2$ in the interval $0 \leq t \leq T$. The exponential function therefore already dropped to e^{-5} at truncation, the step with $a = 2$ isn't all that high either. Therefore neither $|F(\omega)|^2$ nor $|G(\omega)|^2$ look all that terrible either, but $|H(\omega)|^2$ does. It's the interference term's fault. The truncated exponential function on the pedestal is a prototypic example for "bother" when doing Fourier transformations. As we'll see in Chap. 3, even using window functions would be of limited help. That's only the—overly popular—power representation's and interference term's fault.

Fix: Subtract the pedestal before transforming. Usually we're not interested in it anyway. For example a logarithmic representation helps, giving a straight line for the e-function, which then becomes "bent" and runs into the background. Use extrapolation to determine a. It would be best to divide by the exponential, too. You are presumably interested in (possible) small oscillations only. In case you have no data for long times, you will run into trouble. You will also get problems if you have a superposition of several exponentials such that you won't get a straight line anyhow. In such cases, I guess, you will be stumped with Fourier transformation. Here, Laplace transformation helps which we shall not treat here.

Playground

2.1 Black Magic

The Italian mathematician Maria Gaetana Agnesi—appointed in 1750 to the faculty of the University of Bologna by the Pope—constructed the following geometric locus, called "versiera":

i. draw a circle with radius $a/2$ at $(0; a/2)$
ii. draw a straight line parallel to the x-axis through $(0; a)$
iii. draw a straight line through the origin with a slope $\tan\theta$
iv. the geometric locus of the "versiera" is obtained by taking the x-value from the intersection of both straight lines while the y-value is taken from the intersection of the inclined straight line with the circle.

a. Derive the x- and y-coordinates as a function of θ, i.e. in parameterised form.
b. Eliminate θ using the trigonometric identity $\sin^2\theta = 1/(1+\cot^2\theta)$ to arrive at $y = f(x)$, i.e. the "versiera".
c. Calculate the Fourier transform of the "versiera".

2.2 The Phase Shift Knob

On the screen of a spectrometer you see a single spectral component with non-zero patterns for the real and imaginary parts. What shift on the time axis, expressed as a fraction of the oscillation period T, must be applied to make the imaginary part vanish? Calculate the real part which then builds up.

2.3 Pulses

Calculate the Fourier transform of:

$$f(t) = \begin{cases} \sin\omega_0 t & \text{for } -T/2 \le t \le T/2 \\ 0 & \text{else} \end{cases} \qquad \text{with } \omega_0 = n\frac{2\pi}{T/2}.$$

What is $|F(\omega_0)|$, i.e. at "resonance"? Now, calculate the Fourier transform of two of such "pulses", centered at $\pm\Delta$ around $t = 0$.

2.4 Phase-Locked Pulses

Calculate the Fourier transform of:

$$f(t) = \begin{cases} \sin\omega_0 t \ \text{ for } & \begin{array}{c} -\Delta - T/2 \le t \le -\Delta + T/2 \\ \text{and } +\Delta - T/2 \le t \le +\Delta + T/2 \end{array} \\ 0 & \text{else} \end{cases} \qquad \text{with } \omega_0 = n\frac{2\pi}{T/2}.$$

Choose Δ such that $|F(\omega)|$ is as large as possible for all frequencies ω! What is the full width at half maximum (FWHM) in this case?

Hint: Note that now the rectangular pulses "cut out" an integer number of oscillations, not necessarily starting/ending at 0, but being "phase-locked" between left and right "pulses" (Fig. 2.22).

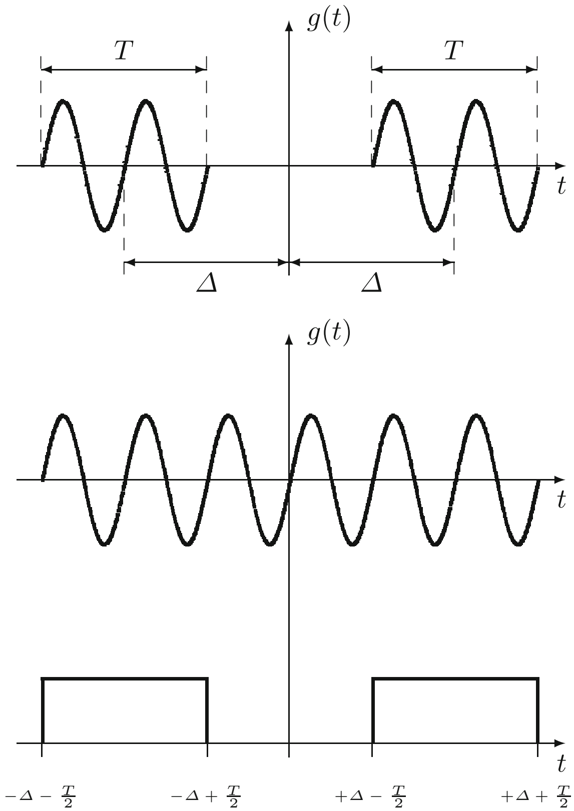

Fig. 2.22 Two pulses 2Δ apart from each other (*top*). Two "phase-locked" pulses 2Δ apart from each other (*bottom*)

2.5 Tricky Convolution

Convolve a normalised Lorentzian with another normalised Lorentzian and calculate its Fourier transform.

2.6 Even Trickier

Convolve a normalised Gaussian with another normalised Gaussian and calculate its Fourier transform.

2.7 Voigt Profile (for Gourmets only)

Calculate the Fourier transform of a normalised Lorentzian convolved with a normalised Gaussian. For the inverse transformation you need a good integration table, e.g. [8, No 3.953.2].

2.8 Derivable

What is the Fourier transform of:

$$g(t) = \begin{cases} t\mathrm{e}^{-\lambda t} & \text{for } 0 \leq t \\ 0 & \text{else} \end{cases}.$$

Is this function even, odd, or mixed?

2.9 Nothing Gets Lost

Use Parseval's theorem to derive the following integral:

$$\int_0^\infty \frac{\sin^2 a\omega}{\omega^2} \mathrm{d}\omega = \frac{\pi}{2} a \qquad \text{with } a > 0.$$

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