

Chapter 2

Preliminaries

In this chapter, we will give a short overview on some basic concepts for the convenience of the reader. We start by introducing notation and summarize a few definitions as well as theorems from functional analysis, especially about linear operators, bilinear forms, and different kinds of function spaces. Moreover, several characterizations of the regularity of domains are given and a few theorems from vector analysis are recalled. The chapter concludes with some remarks on differential equations.

2.1 Basic Notation

As usual, we denote the set of positive integers by $\mathbb{N}$, the set of non-negative integers by $\mathbb{N}_0$, the set of all integers by $\mathbb{Z}$, the set of rational numbers by $\mathbb{Q}$, the set of real numbers by $\mathbb{R}$, positive real numbers by $\mathbb{R}^+$, non-negative real numbers by $\mathbb{R}_0^+$, and the set of complex numbers by $\mathbb{C}$. For any real number x , we define $[x]$ to be the largest integer n with $n \leq x$.

For any $n \in \mathbb{N}$, we define the n -dimensional real vector space as the cartesian product

$$\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n\text{-times}} = \mathbb{R}^n .$$

For elements $x = (x_1, x_2, \dots, x_n)^T, y = (y_1, y_2, \dots, y_n)^T \in \mathbb{R}^n$, we define the (Euclidean) inner or scalar product $x \cdot y$ and its induced (Euclidean) norm $\|x\|$ to be

$$x \cdot y = \sum_{i=1}^n x_i y_i , \tag{2.1a}$$

$$\|x\| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^n x_i^2} . \quad (2.1b)$$

Here, the upper index T means the transpose of a vector, and also denotes the transpose of a matrix later on. We frequently use Einstein's summation convention which means that using the same index variable twice within a single term implies summation of that term over all values of the index. For the Euclidean vector space $\mathbb{R}^3$, we also define the vector product $x \wedge y$ as

$$x \wedge y = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix} . \quad (2.1c)$$

For any subset $\Omega \subset \mathbb{R}^n$, $\partial\Omega$ denotes its boundary and its closure is denoted by $\overline{\Omega} = \Omega \cup \partial\Omega$. Special kinds of subsets are balls. A ball of radius r centered at x will be denoted by $B_r(x)$ and is defined by

$$B_r(x) = \{y \in \mathbb{R}^n : \|x - y\| < r\} . \quad (2.2)$$

The corresponding sphere is given by $\partial B_r(x)$ or more explicitly

$$\partial B_r(x) = \{y \in \mathbb{R}^n : \|x - y\| = r\} . \quad (2.3)$$

We introduce two special indexed symbols. For $k, l \in \mathbb{Z}$, the Kronecker symbol δ_{kl} is defined by

$$\delta_{kl} = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases} \quad (2.4)$$

The Levi-Civita tensor ϵ_{ijk} , $i, j, k \in \{1, 2, 3\}$, is the total anti-symmetric tensor of order 3, given by

$$\epsilon_{ijk} = \begin{cases} +1, & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3), \\ -1, & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3), \\ 0, & \text{otherwise} \end{cases} \quad (2.5a)$$

which yields

$$\epsilon_{ijk} = \frac{(i-j)(j-k)(k-i)}{2} . \quad (2.5b)$$

Here, we also introduced that we will use boldfaced letters for tensors of order 2 and higher.

2.2 Linear Operators and Bilinear Forms

Throughout this thesis, we will deal with a certain set of differential equations, i.e., differential operators and corresponding bilinear forms. Thus, in this section, we address linear operators and bilinear forms in a more abstract setting, starting with the definition of continuity and norms.

Definition 2.1 (Continuous Linear Operators and Operator Norm) Let $(V, \|\cdot\|_V)$ and $(Q, \|\cdot\|_Q)$ be two normed vector spaces. A linear operator $T : V \supset D(T) \rightarrow Q$ with domain $D(T) \subset V$ and range $R(T) \subset Q$ is continuous if and only if there exists a positive constant $C \in \mathbb{R}^+$ such that

$$\|Tx\|_Q \leq C \|x\|_V \text{ for all } x \in D(T) . \quad (2.6)$$

The space of all continuous linear operators from V to Q is denoted by $\mathcal{L}(V, Q)$. It is a normed space with norm given by

$$\|T\|_{\mathcal{L}(V, Q)} = \sup_{\|x\|_V=1} \|Tx\|_Q . \quad (2.7)$$

This norm is submultiplicative, i.e., for $T, S \in \mathcal{L}(V, Q)$, we have

$$\|TS\|_{\mathcal{L}(V, Q)} \leq \|T\|_{\mathcal{L}(V, Q)} \|S\|_{\mathcal{L}(V, Q)} . \quad (2.8)$$

If Q is a Banach space, then $\mathcal{L}(V, Q)$ is also a Banach space.

The kernel $\ker(T)$ and range $R(T)$ of a linear operator are defined as

$$\ker(T) = \{v \in V : Tv = 0\} , \quad (2.9)$$

$$R(T) = \{q \in Q : \exists v \in V : Tv = q\} . \quad (2.10)$$

Proof The submultiplicativity of the norm is proven, i.e., in [295, Chapter I.6].

Continuity of bilinear forms can be defined in a similar way.

Definition 2.2 (Continuous Bilinear Forms and their Norm) Let $(V, \|\cdot\|_V)$ and $(Q, \|\cdot\|_Q)$ be two normed vector spaces. A bilinear form $a(\cdot, \cdot) : V \times Q \rightarrow \mathbb{R}$ is continuous if and only if there exists a positive constant $C \in \mathbb{R}^+$ such that

$$|a(v, q)| \leq C \|v\|_V \|q\|_Q \text{ for all } v \in V, q \in Q . \quad (2.11)$$

The space of all continuous bilinear forms on $V \times Q$ is denoted by $\mathcal{L}(V \times Q, \mathbb{R})$ and is a normed space with a norm given by

$$\|a\|_{\mathcal{L}(V \times Q, \mathbb{R})} = \sup_{\|v\|_V=1} \sup_{\|q\|_Q=1} |a(v, q)| . \quad (2.12)$$

Proof The proof of submultiplicativity of the norm is done analogously to the previous one.

In order to establish a connection between bilinear forms and linear operators, we need to define the dual space V' .

Definition 2.3 (Dual Space and Dual Product) For a normed linear space $(V, \|\cdot\|_V)$, the set of all linear functionals on V is called its dual space and will be denoted by V' . The dual product of an element $v' \in V'$ and an element $v \in V$, denoted by

$$(v', v)_V = v'(v) , \quad (2.13)$$

is the value of the functional v' at the point v .

Lemma 2.4 (Linear Operator to a Bilinear Form) *For a continuous bilinear form $a(\cdot, \cdot) : V \times Q \rightarrow \mathbb{R}$ on the normed spaces $(V, \|\cdot\|_V)$ and $(Q, \|\cdot\|_Q)$ exists one and only one continuous linear operator $A : V \rightarrow Q'$ such that*

$$a(v, q) = (Av, q)_Q \quad (2.14)$$

for all $v \in V$ and $q \in Q$. The norms of A and $a(\cdot, \cdot)$ are equal.

Proof We can define $A : V \rightarrow Q'$ by imposing that for every $v \in V$ and $q \in Q$ the identity $(Av, q)_Q = a(v, q)$ is satisfied. Thus, A is well-defined. A is continuous and linear because $a(\cdot, \cdot)$ is continuous and (bi-)linear. Uniqueness of A results from (2.14).

In a similar way as the dual space allows us to define a linear operator corresponding to a bilinear form, it also allows us to define the adjoint operator of a linear operator.

Lemma 2.5 (Adjoint Operator) *Let $(V, \|\cdot\|_V)$ and $(Q, \|\cdot\|_Q)$ be two normed vector spaces. For any operator $T \in \mathcal{L}(V, Q)$, there exists a unique operator $T^* : Q' \rightarrow V'$ such that*

$$(T^*q', v)_V = (q', Tv)_Q \quad \text{for all } v \in V \text{ and } q' \in Q' . \quad (2.15)$$

The adjoint operator T^* is continuous, linear and its norm is the same as the norm of T .

If $(V, \|\cdot\|_V)$ and $(Q, \|\cdot\|_Q)$ are Banach spaces, the existence of an inverse operator $T^{-1} \in \mathcal{L}(Q, V)$ of T is equivalent to the existence of an inverse operator $(T^*)^{-1} \in \mathcal{L}(Q', V')$ of T^* . If the inverse operator exists, we have $(T^{-1})^* = (T^*)^{-1}$.

If T is only defined on a subset $D(T) \subsetneq V$, the above statements are still valid if T^* is defined on

$$D(T^*) = \{q' \in Q' : v \mapsto (q', Tv)_Q \text{ is continuous on } V\} . \quad (2.16)$$

Proof See, e.g., [295, Chapter VII].

It is sometimes necessary to not only consider linear operators, but families of linear operators, given by $(T(t))_{t \in \mathbb{R}_0^+}$. A useful restriction is to require such a family to be a strongly continuous semigroup [70, Definition 1.1]

Definition 2.6 (Strongly Continuous Semigroup) A family $(T(t))_{t \in \mathbb{R}_0^+}$ of bounded linear operators on a Banach space V is called a strongly continuous (one-parameter) semigroup if it satisfies

$$T(t+s) = T(t)T(s) \text{ for all } t, s \geq 0 , \quad (2.17)$$

$$T(0) = I , \quad (2.18)$$

with the identity operator I , and is strongly continuous, i.e., for every $v \in V$ the maps

$$t \mapsto T(t)v \quad (2.19)$$

are continuous from $\mathbb{R}^+$ to V for every $v \in V$.

In some cases, linear operators are not defined on a whole vector space V , but only on a subspace U . If the range of the operator is in $\mathbb{R}$, i.e., it is a linear functional, the Hahn-Banach Theorem gives a condition that an extension to the whole space exists.

Theorem 2.7 (Hahn-Banach Theorem) Let U be a subspace of a real vector space V , $p : V \rightarrow \mathbb{R}$ with

$$p(x+y) \leq p(x) + p(y) , \quad (2.20a)$$

$$p(tx) = tp(x) \quad (2.20b)$$

for all $x, y \in V$, $t \in \mathbb{R}_0^+$, and $f : U \rightarrow \mathbb{R}$ a linear functional with $f(x) \leq p(x)$ for all $x \in U$.

Then there exists a linear functional $F : V \rightarrow \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in U$ and $-p(-x) \leq F(x) \leq p(x)$ for all $x \in V$.

If V is locally convex and f is continuous on U , then F is continuous on V .

Proof [See, e.g., 238, Theorems 3.2 and 3.6].

2.3 Function Spaces

In order to deal with differential equations, we have to introduce some notation for differentiation and integration. As it turns out, the classical strong concept of differentiability is too restrictive. This leads to the definition of weak differentiability. Different kinds of requirements on the differentiability of functions yield different sets of functions which can be shown to be normed vector spaces. It turns out that there are some more remarkable properties of these spaces and the functions they contain as well as interesting relations between them.

Definition 2.8 ((Strong) Differentiation) Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, $u : \Omega \rightarrow \mathbb{R}$, $\gamma \in \mathbb{N}_0^n$, and $k \in \mathbb{N}_0$. Let x be a point in $\mathbb{R}^n$ with coordinates x_i , $i \in \mathbb{N}$, $i \leq n$. Throughout this thesis, we will use cartesian coordinates.

The partial derivative $D^\gamma u$ is defined by

$$(D^\gamma u)(x) = (\partial_{x_1}^{\gamma_1} \dots \partial_{x_n}^{\gamma_n} u)(x) = \left(\frac{\partial^{|\gamma|} u}{\partial x_1^{\gamma_1} \dots \partial x_n^{\gamma_n}} \right)(x), \quad x \in \Omega, \quad (2.21)$$

with $|\gamma| = \sum_{i=1}^n \gamma_i$ being the order of the derivative. The set of all derivatives of u of order k at point x is denoted by $(D^k u)(x) = \{(D^\gamma u)(x) : |\gamma| = k\}$.

Remark 2.9

- (i) Is u defined as a function on $\mathbb{R}^n$, $n \in \mathbb{N}$, such that $u : \mathbb{R}^n \rightarrow \mathbb{R}$, the restriction of u to $\Omega \subset \mathbb{R}^n$ is denoted by $u|_\Omega$.
- (ii) The gradient of a differentiable function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined as the vector of all first derivatives and denoted by

$$\nabla_x u(x) = (\partial_{x_1} u, \dots, \partial_{x_n} u)^T. \quad (2.22)$$

We will omit the index x if it is clear with respect to which variable the differentiation has to be carried out.

- (iii) The directional derivative of u with respect to a unit vector e is given by $(\nabla_x u) \cdot e$. The directional derivative with respect to the outer unit normal vector of a bounded domain Ω , i.e., a bounded connected open subset $\Omega \subset \mathbb{R}^n$ is denoted by $\partial_{n(x)} u$.

Please note the term “domain” as introduced here should not be confused with the domain of a linear operator as given in Definition 2.1.

As we now have introduced the notation for strong derivatives, we can define function spaces of continuously differentiable functions (see, e.g., [3]).

Definition 2.10 (Spaces of Continuously Differentiable Functions) Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a domain, i.e. a connected open set. For $k \in \mathbb{N}_0$, we denote by $C^k(\Omega)$ the vector space of all functions $u : \Omega \rightarrow \mathbb{R}$ which together with all their derivatives $D^\gamma u$ of order $|\gamma| \leq k$ are continuous on Ω . Analogously, we define the space $\mathcal{C}^k(\Omega)$ containing all vector-valued functions $v = (v_1, v_2, v_3)^T : \Omega \rightarrow \mathbb{R}^3$ which together with all their derivatives $D^\gamma v_i$ of order $|\gamma| \leq k$ for $i \in \{1, 2, 3\}$ are continuous on Ω . We write $C(\Omega)$ for $C^0(\Omega)$ and $\mathcal{C}(\Omega)$ for $\mathcal{C}^0(\Omega)$.

The spaces of infinitely continuously differentiable functions are given by $C^\infty(\Omega) = \bigcap_{k=0}^\infty C^k(\Omega)$ and $\mathcal{C}^\infty(\Omega) = \bigcap_{k=0}^\infty \mathcal{C}^k(\Omega)$, respectively.

The subspace of functions in $C^k(\Omega)$ that have compact support in Ω is denoted by $C_0^k(\Omega)$ and analogously for vector-valued functions $\mathcal{C}_0^k(\Omega)$. A function u has compact support in Ω if there is a compact set $K \subset \Omega$ such that

$$\text{supp}(u) = \overline{\{x \in \Omega : u(x) \neq 0\}} \subset K. \quad (2.23)$$

The spaces $C^k(\overline{\Omega})$ contain all functions $u \in C^k(\Omega)$ for which $D^\gamma u$ is bounded and uniformly continuous for all $\gamma \in \mathbb{N}_0^k$ with $0 \leq |\gamma| \leq k$. These spaces are Banach spaces when equipped with the norm

$$\|u\|_{C^k(\overline{\Omega})} = \max_{0 \leq |\gamma| \leq k} \sup_{x \in \Omega} |(D^\gamma u)(x)|. \quad (2.24)$$

The spaces $\mathcal{C}^k(\overline{\Omega})$ are defined accordingly.

Remark 2.11 Within this thesis, we reserve the term vector-valued for functions with values in $\mathbb{R}^2$ or $\mathbb{R}^3$.

In some cases, functions are required to be more regular than just being continuous, but requiring them to be continuously differentiable would be too much. Thus, we introduce the spaces of Hölder-continuous functions [3, 263].

Definition 2.12 (Hölder-Continuous Functions) Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a domain, $\gamma \in \mathbb{N}_0^k$, and $k \in \mathbb{N}_0$. The space $C^{k,s}(\overline{\Omega})$, $0 < s \leq 1$, is the subspace of functions $u \in C^k(\overline{\Omega})$ whose derivatives $D^\gamma u$ of order k satisfy

$$|(D^\gamma u)(x) - (D^\gamma u)(y)| \leq C \|x - y\|^s \quad \forall x, y \in \Omega \quad (2.25)$$

with a constant $C \in \mathbb{R}^+$. We say u has Hölder-continuous derivatives of order k with Hölder exponent s or, in the special case $s = 1$, u has Lipschitz-continuous derivatives of order k . $C^{k,s}(\overline{\Omega})$ is a Banach space if equipped with the norm

$$\|u\|_{C^{k,s}(\overline{\Omega})} = \|u\|_{C^k(\overline{\Omega})} + \max_{0 \leq |\gamma| \leq k} \sup_{\substack{x, y \in \Omega \\ x \neq y}} \frac{|(D^\gamma u)(x) - (D^\gamma u)(y)|}{\|x - y\|^s}. \quad (2.26)$$

For $r \geq s$, the inclusion $C^{k,r}(\overline{\Omega}) \subset C^{k,s}(\overline{\Omega})$ is valid.

If a function u satisfies

$$\lim_{\delta \rightarrow 0} \sup_{\substack{x, y \in \Omega \\ 0 < \|x - y\| < \delta}} \frac{|(D^\gamma u)(x) - (D^\gamma u)(y)|}{\|x - y\|^s} \quad (2.27)$$

for each $|\gamma| = k$, the derivatives of order k of u are called uniformly Hölder-continuous and u is an element of the space $C_u^{k,s}(\Omega)$.

Proof See, e.g., [3, Theorem 1.31] for the inclusion.

When looking for functions whose values on the boundary of a domain are prescribed, it is often useful, if not even necessary, to restrict the kind of domain under consideration to answer questions of existence and uniqueness. The domain is required to have some kind of regularity. In order to give different kinds of regularity properties, we need another definition [3, Section 3.34].

Definition 2.13 (m -smooth Transformation) Let Φ be a one-to-one transformation of a domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, onto a domain $G \subset \mathbb{R}^n$ with $\Psi = \Phi^{-1}$. We call Φ m -smooth if, writing $y = \Phi(x)$ and

$$\begin{aligned} y_1 &= \phi_1(x_1, \dots, x_n), & x_1 &= \psi_1(y_1, \dots, y_n), \\ y_2 &= \phi_2(x_1, \dots, x_n), & x_2 &= \psi_2(y_1, \dots, y_n), \\ &\vdots & &\vdots \\ y_n &= \phi_n(x_1, \dots, x_n), & x_n &= \psi_n(y_1, \dots, y_n), \end{aligned}$$

the functions $\phi_1, \dots, \phi_n$ belong to $C^m(\overline{\Omega})$ and the functions $\psi_1, \dots, \psi_n$ belong to $C^m(\overline{G})$.

The following summary of regularity conditions is taken from the book by Adams [3, pp. 66f] and is just slightly adapted. Here, a finite cone with vertex x is defined as the set

$$\mathcal{C}_x = B_{r_1}(x) \cap \{x + \lambda(y - x) : y \in B_{r_2}(z), \lambda > 0\} \quad (2.28)$$

with $B_{r_2}(z)$ being a ball such that $x \notin B_{r_2}(z)$ and an open cover $\{U_j\}$ of a set Ω is said to be locally finite if any compact set in $\mathbb{R}^n$ can intersect at most finitely many elements of $\{U_j\}$ [3, p. 65ff].

Definition 2.14 (Regularity of Domains) Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a domain. Ω has

- (i) The *segment property* if there exists a locally finite open cover $\{U_j\}$ of $\partial\Omega$ and a corresponding sequence $\{y_j\}$ of non-zero vectors such that if $x \in \overline{\Omega} \cap U_j$ for some j , then $x + \varepsilon y_j \in \Omega$ for $0 < \varepsilon < 1$,

- (ii) The *cone property* if there exists a finite cone $\mathcal{C}$ such that each point $x \in \Omega$ is the vertex of a finite cone $\mathcal{C}_x$ contained in Ω and congruent to $\mathcal{C}$.
- (iii) The *uniform cone property* if there exists a locally finite open cover $\{U_j\}$ of $\partial\Omega$ and a corresponding sequence $\{\mathcal{C}_j\}$ of finite cones, each congruent to some fixed finite cone $\mathcal{C}$, such that
 - (a) For some finite $M \in \mathbb{R}^+$, every U_j has a diameter less than M ,
 - (b) For some $\delta > 0$, $\bigcup_{j=1}^{\infty} U_j \supset \{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta\}$,
 - (c) For every j , $Q_j = \bigcup_{x \in \Omega \cap U_j} (x + \mathcal{C}_j) \subset \Omega$,
 - (d) For some finite $R \in \mathbb{N}$, every collection of $R + 1$ of the sets Q_j has an empty intersection.
- (iv) The *strong local Lipschitz property* if there exist positive numbers δ and M , a locally finite open cover $\{U_j\}$ of $\partial\Omega$, and for each U_j a real-valued function f_j of $n - 1$ real variables, such that
 - (a) For some finite $R \in \mathbb{N}$, every collection of $R + 1$ of the sets U_j has an empty intersection,
 - (b) For every pair of points $x, y \in \{z \in \Omega : \text{dist}(z, \partial\Omega) < \delta\}$ such that $\|x - y\| < \delta$, there exists j such that $x, y \in \{z \in U_j : \text{dist}(z, \partial U_j) > \delta\}$,
 - (c) Each function f_j satisfies a Lipschitz condition with constant M ,
 - (d) For some cartesian coordinate system $(\xi_{j,i})_{i=1}^n$ in U_j , the set $\Omega \cap U_j$ is represented by the inequality $\xi_{j,n} < f_j(\xi_{j,1}, \dots, \xi_{j,n-1})$.
- (v) The *uniform C^m -regularity property* if there exists a locally finite open cover $\{U_j\}$ of $\partial\Omega$ and a corresponding sequence $\{\Phi_j\}$ of m -smooth one-to-one transformations with Φ_j taking U_j onto $B_1(0) \subset \mathbb{R}^n$, such that
 - (a) For some $\delta > 0$, $\bigcup_{j=1}^{\infty} \Psi_j(B_{0.5}(0)) \supset \{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta\}$, where $\Psi = \Phi^{-1}$,
 - (b) For some finite $R \in \mathbb{N}$, every collection of $R + 1$ of the sets U_j has an empty intersection,
 - (c) For each j , $\Phi_j(U_j \cap \Omega) = \{y \in B_1(0) : y_n > 0\}$,
 - (d) If $(\phi_{j,1}, \dots, \phi_{j,n})$ and $(\psi_{j,1}, \dots, \psi_{j,n})$ denote the components of Φ_j and Ψ_j , respectively, then there exists a finite M such that for all $\gamma \in \mathbb{N}_0^n$, $|\gamma| \leq m$, for every $1 \leq i \leq n$, and for every j , we have $|D^\gamma \phi_{j,i}(x)| \leq M$, $x \in U_j$, and $|D^\gamma \psi_{j,i}(y)| \leq M$, $y \in B_1(0)$.

For the different kinds of regularity, we have $(v) \xrightarrow{m \geq 1} (iv) \implies (iii) \implies (i)$. These regularity properties require Ω to lie on only one side of its boundary, whereas the cone property does not impose this condition.

Remark 2.15

- (i) If Ω is bounded, the requirements for Ω being strong local Lipschitz reduce to the condition that for each point $x \in \partial\Omega$, there exists a neighborhood U of x such that $U \cap \partial\Omega$ is the graph of a Lipschitz-continuous function.

- (ii) In some cases it is necessary to require that the parts of the one-to-one transformation mentioned in the definition of the C^m -regularity property have not only bounded derivatives, but Hölder-continuous ones. This yields the $C^{m,s}$ -regularity property.

As already mentioned, the above introduced definition of strong differentiability with continuous or even Hölder-continuous derivatives is often too restrictive. Therefore, we need some other, weaker definition of derivatives. To define these weak derivatives, we need a definition of convergence in $C_0^\infty(\Omega)$ first (see, e.g., [3, Section 1.51]).

Definition 2.16 (Convergence in $C_0^\infty(\Omega)$) Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded domain, $\{\phi_l\}_{l \in \mathbb{N}_0} \subset C_0^\infty(\Omega)$, and $\phi \in C_0^\infty(\Omega)$. The sequence $\{\phi_l\}_{l \in \mathbb{N}_0}$ is said to converge towards ϕ in $C_0^\infty(\Omega)$ for $l \rightarrow \infty$ if there is a compact subset $K \subset \Omega$ such that

$$\text{supp}(\phi_l) \subset K \text{ for all } l \in \mathbb{N}_0, \quad (2.29a)$$

$$\text{supp}(\phi) \subset K, \quad (2.29b)$$

as well as all partial derivatives of ϕ_l of arbitrary order converge uniformly to those of ϕ , i.e.,

$$\sup_{x \in \Omega} |(D^\gamma \phi_l)(x) - D^\gamma \phi(x)| \xrightarrow{l \rightarrow \infty} 0 \text{ for all } \gamma \in \mathbb{N}_0^n. \quad (2.29c)$$

The equivalent definition holds in $\mathcal{C}_0^\infty(\Omega)$.

Remark 2.17 $C_0^\infty(\Omega)$ is often denoted by $\mathcal{D}(\Omega)$ and called the space of test functions, although the latter identification is not unique. It is a topological vector space, or to be more precise, a Fréchet space, but not normable [3, 238].

The above definition allows us to define distributions (see [234, Definition 5.8]).

Definition 2.18 (Distribution) Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded domain. A distribution (or generalized function) is a linear functional $f : C_0^\infty(\Omega) \rightarrow \mathbb{R}$ which is continuous in the following sense: If a sequence $\{\phi_l\}_{l \in \mathbb{N}_0} \subset C_0^\infty(\Omega)$ converges for $l \rightarrow \infty$ towards $\phi \in C_0^\infty(\Omega)$, then $f(\phi_l) = (f, \phi_l)_{C_0^\infty(\Omega)}$ converges for $l \rightarrow \infty$ towards $f(\phi)$.

The set of all distributions is denoted by $(C_0^\infty(\Omega))'$.

The space of vector-valued distributions is defined accordingly.

Remark 2.19 The space of distributions is often denoted by $\mathcal{D}'(\Omega)$. If we consider $C_0^\infty(\Omega)$ as a topological vector space, $(C_0^\infty(\Omega))'$ is its topological dual, i.e., it consists of all linear functionals on $C_0^\infty(\Omega)$ which are continuous with respect to the weak topology [3, 238].

There is another way to characterize functions that is useful to present here in anticipation of a more general concept that we will introduce later on. For this, we have to explain our interpretation of the integral of a function.

Within this thesis, all integrals are understood in the sense of Lebesgue integrals. Note that the choice of the appropriate Lebesgue measure depends on the domain of integration. We mention this at this point because we will later on encounter integrals over the boundary of three-dimensional domains which would, of course, vanish if integration would be performed with respect to the Lebesgue measure in $\mathbb{R}^3$.

Now, we can define [3, Section 1.53].

Definition 2.20 (Locally Integrable Functions) Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded domain. A function u is called locally integrable on Ω if for every compact subset $K \subset \Omega$ we have

$$\int_K |f(x)| \, dV_n(x) < \infty, \quad (2.30)$$

where dV_n denotes the volume element in $\mathbb{R}^n$.

Remark 2.21 If there is no confusion, we will omit the index n in dV_n .

For every locally integrable function u , we can define a corresponding distribution $T_u \in (C_0^\infty(\Omega))'$ simply by

$$T_u(\phi) = \int_\Omega u(x)\phi(x) \, dV(x), \quad \phi \in C_0^\infty(\Omega). \quad (2.31)$$

Usually, notation is a little bit abused by also using u instead of T_u to denote the corresponding distribution.

The reverse of the last statement is not true. There are many distributions for which no corresponding locally integrable function can be found. The most prominent example is the evaluation of a function ϕ at a certain point x , known as Dirac's delta distribution. If $0 \in \Omega$, the evaluation of a function $\phi \in C_0^\infty(\Omega)$ is given by

$$\delta(\phi) = \phi(0). \quad (2.32)$$

It is easy to prove that there is no locally integrable function for which

$$\int_\Omega \delta(x)\phi(x) \, dV(x) = \phi(0), \quad \phi \in C_0^\infty(\Omega). \quad (2.33)$$

However, δ obviously satisfies Definition 2.18.

In the course of this thesis, especially in Chap. 5, we need some operations on distributions. It is obvious how addition of two distributions and multiplication with a constant should be defined on distributions. Distributions may even be multiplied

by smooth functions [3, Section 1.58]. For $T \in (C_0^\infty(\Omega))'$ and $u \in C_0^\infty(\Omega)$, the product $uT \in (C_0^\infty(\Omega))'$ is defined by

$$(uT)(\phi) = T(u\phi), \quad \phi \in C_0^\infty(\Omega). \quad (2.34)$$

The support of a distribution is defined as follows [238, Definition 6.22].

Definition 2.22 (Support of a Distribution) Suppose $T \in (C_0^\infty(\Omega))'$ for an open bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$. If ω is a subset of Ω and if $T\phi = 0$ for all $\phi \in C_0^\infty(\omega)$, we say that T vanishes on ω . Let W be the union of all sets $\omega \subset \Omega$ on which T vanishes. The complement of W relative to Ω is the support of T .

Obviously, the definition of compact support now also holds for distributions.

Another operation which we need not only on distributions is convolution [238, Definition 6.25].

Definition 2.23 (Convolution) Let u be a function defined on $\mathbb{R}^n$, $n \in \mathbb{N}$, and $x \in \mathbb{R}^n$. We define

$$(\tau_x u)(y) = u(y - x), \quad y \in \mathbb{R}^n, \quad (2.35)$$

$$\check{u}(y) = u(-y), \quad (2.36)$$

$$(\tau_x \check{u})(y) = u(x - y). \quad (2.37)$$

Let v be another function on $\mathbb{R}^n$. The convolution $u * v$ is defined as

$$(u * v)(x) = \int_{\mathbb{R}^n} u(y)v(x - y) \, dV(y) = \int_{\mathbb{R}^n} u(y)(\tau_x \check{v})(y) \, dV(y) \quad (2.38)$$

if the integral exists for almost all $x \in \mathbb{R}^n$.

For a distribution $u \in (C_0^\infty(\mathbb{R}^n))'$ and $\phi \in C_0^\infty(\mathbb{R}^n)$, the function $u * \phi$ is given by

$$(u * \phi)(x) = u(\tau_x \check{\phi}). \quad (2.39)$$

Remark 2.24 If we have $u \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}_0^+)$, we define $\check{u}(x, t) = u(-x, -t)$.

Theorem 2.25 (Properties of Convolutions) Let $u \in (C_0^\infty(\mathbb{R}^n))'$, $\phi \in C_0^\infty(\mathbb{R}^n)$, $\psi \in C_0^\infty(\mathbb{R}^n)$. Then

- (i) $\tau_x(u * \phi) = (\tau_x u) * \phi = u * (\tau_x \phi)$ for all $x \in \mathbb{R}^n$,
- (ii) $u * \phi \in C^\infty(\mathbb{R}^n)$ and $u * (\phi * \psi) = (u * \phi) * \psi$.
- (iii) The operator L , defined by

$$L\phi = u * \phi, \quad \phi \in C_0^\infty(\mathbb{R}^n), \quad (2.40)$$

is a continuous linear mapping of $C_0^\infty(\mathbb{R}^n)$ into $C^\infty(\mathbb{R}^n)$ which satisfies $\tau_x L = L\tau_x$, $x \in \mathbb{R}^n$.

(iv) If L is a continuous linear mapping of $C_0^\infty(\mathbb{R}^n)$ into $C(\mathbb{R}^n)$ which satisfies $\tau_x L = L \tau_x$, $x \in \mathbb{R}^n$, then there exists a unique $u \in (C_0^\infty(\mathbb{R}^n))'$ such that L has a representation like (2.40).

Proof See [238, Theorems 6.30, 6.33].

Until now, convolutions for distributions are only declared if a distribution is convolved with an element of $C_0^\infty(\mathbb{R}^n)$. The next lemma extends convolution to elements of $C^\infty(\mathbb{R}^n)$.

Lemma 2.26 Let $u \in (C_0^\infty(\mathbb{R}^n))'$ have compact support, $\phi \in C^\infty(\mathbb{R}^n)$, $\psi \in C_0^\infty(\mathbb{R}^n)$. The convolution $u * \phi \in C^\infty(\mathbb{R}^n)$ is well-defined. Moreover,

- (i) $\tau_x(u * \phi) = (\tau_x u) * \phi = u * (\tau_x \phi)$ for all $x \in \mathbb{R}^n$,
- (ii) $u * (\phi * \psi) = (u * \phi) * \psi$,
- (iii) $u * \psi \in C_0^\infty(\mathbb{R}^n)$,
- (iv) $u * (\phi * \psi) = (u * \phi) * \psi = (u * \psi) * \phi$.

Proof See [238, Theorem 6.35].

Convolutions may also be defined between distributions.

Lemma 2.27 (Convolutions between Distributions) Let $u, v, w \in (C_0^\infty(\mathbb{R}^n))'$, $n \in \mathbb{N}$.

- (i) If at least one of u, v has compact support, the convolution $u * v$ is defined by $(u * v) * \phi = u * (v * \phi)$ for all $\phi \in C_0^\infty(\mathbb{R}^n)$ and $u * v = v * u$.
- (ii) If at least one of the supports $\text{supp}(u)$, $\text{supp}(v)$ is compact, we have $\text{supp}(u * v) \subset \text{supp}(u) + \text{supp}(v)$.
- (iii) If at least two of the supports $\text{supp}(u)$, $\text{supp}(v)$, $\text{supp}(w)$ are compact, we have $(u * v) * w = u * (v * w)$.

Proof See [238, Definition 6.36, Theorem 6.37].

We can now define weak derivatives by defining derivatives of distributions [3, Section 1.57], [238, Section 6.12].

Definition 2.28 (Weak Derivative) Let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, and $u \in (C_0^\infty(\Omega))'$. The weak derivative of u with respect to x_i , $i \in \{1, \dots, n\}$, is defined by

$$(\partial_{x_i} u, \phi)_{C_0^\infty(\Omega)} = -(u, \partial_{x_i} \phi)_{C_0^\infty(\Omega)}, \quad \phi \in C_0^\infty(\Omega). \quad (2.41)$$

For a multi-index $\gamma \in \mathbb{N}_0^n$, we have the generalization

$$(D^\gamma u, \phi)_{C_0^\infty(\Omega)} = (-1)^{|\gamma|} (u, D^\gamma \phi)_{C_0^\infty(\Omega)}, \quad \phi \in C_0^\infty(\Omega). \quad (2.42)$$

Remark 2.29 We use the same notation for weak derivatives and classical (strong) partial derivatives (based on the limit of difference quotients). If a continuous strong

derivative exists, it coincides with the weak derivative as can be seen by integration by parts.

Theorem 2.30 (Weak Derivatives and Convolution)

- (i) Suppose $u \in (C_0^\infty(\mathbb{R}^n))'$ and $\phi \in C_0^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$, or $u \in (C_0^\infty(\mathbb{R}^n))'$ with compact support and $\phi \in C^\infty(\mathbb{R}^n)$, then $D^\gamma(u * \phi) = (D^\gamma u) * \phi = u * (D^\gamma \phi)$ for all $\gamma \in \mathbb{N}_0^n$.
- (ii) Suppose $u \in (C_0^\infty(\mathbb{R}^n))'$ and δ is the delta distribution, then $D^\gamma u = (D^\gamma \delta)u$ for all $\gamma \in \mathbb{N}_0^n$. In particular, $u = \delta * u$.
- (iii) Suppose $u, v \in (C_0^\infty(\mathbb{R}^n))'$ and at least one of them has compact support, then $D^\gamma(u * v) = (D^\gamma u) * v = u * (D^\gamma v)$ for all $\gamma \in \mathbb{N}_0^n$.

Proof See [238, Theorems 6.30, 6.35, and 6.37].

A consequence of Theorem 2.30(ii) is that it allows to give an integral expression for Dirac's delta distribution and its derivatives, with a slight abuse of notation, by

$$\int_{\mathbb{R}^n} D^\gamma \delta(x - y) \phi(y) \, dV(y) = (-1)^{|\gamma|} (D^\gamma \phi)(x) . \quad (2.43)$$

It is easy to prove that the weak derivative of a distribution is also a distribution. Thus, for every distribution, there exist weak derivatives of arbitrary order. Nevertheless, classes of distributions and their derivatives can be distinguished if we introduce a new concept of regularity based on integrability. We begin by defining Lebesgue spaces.

Definition 2.31 (Lebesgue Spaces) Let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, and $p \in \mathbb{R}^+$. The Lebesgue space $L^p(\Omega)$ consists of all equivalence classes with respect to the Lebesgue measure of almost everywhere identical functions on Ω , whose representatives u satisfy

$$\int_{\Omega} |u(x)|^p \, dV(x) < \infty . \quad (2.44)$$

Moreover, the space $L^\infty(\Omega)$ contains all such equivalence classes whose representatives are measurable, essentially bounded functions $u : \Omega \rightarrow \mathbb{R}$ with

$$\operatorname{ess\,sup}_{x \in \Omega} |u(x)| < \infty . \quad (2.45)$$

Equivalent definitions hold for the spaces $\mathcal{L}^p(\Omega)$ and $\mathcal{L}^\infty(\Omega)$ of vector-valued functions.

Remark 2.32 It is convenient to identify a function with its respective equivalence class.

We summarize a few properties of the Lebesgue spaces.

Lemma 2.33 (Properties of Lebesgue Spaces) *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, and $1 \leq p < \infty$.*

(i) *The Lebesgue space $L^p(\Omega)$ is a Banach space with respect to the norm*

$$\|u\|_{L^p(\Omega)} = \left(\int_{\Omega} |u(x)|^p \, dV(x) \right)^{\frac{1}{p}}. \quad (2.46)$$

$L^\infty(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x \in \Omega} |u(x)|. \quad (2.47)$$

(ii) *The space $L^2(\Omega)$ is a Hilbert space if equipped with the scalar product*

$$(u, v)_{L^2(\Omega)} = \int_{\Omega} u(x)v(x) \, dV(x). \quad (2.48)$$

(iii) *For arbitrary $1 \leq p_1, p_2 < \infty$ with $p_1 \geq p_2$, we have $L^{p_1}(\Omega) \subset L^{p_2}(\Omega)$ and $L^{p_1}(\Omega) \subset L^1_{\text{loc}}(\Omega)$, with $L^1_{\text{loc}}(\Omega)$ being the space of locally integrable functions.*

(iv) *Let $1 < p_1 < \infty$ and p_2 such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. For $u \in L^{p_1}(\Omega)$, $v \in L^{p_2}(\Omega)$, we have $uv \in L^1(\Omega)$ and*

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^{p_1}(\Omega)} \|v\|_{L^{p_2}(\Omega)}. \quad (2.49)$$

This is known as Hölder's inequality. It also holds for $u \in L^1(\Omega)$ and $v \in L^\infty(\Omega)$; we have $uv \in L^1(\Omega)$.

(v) *Let $1 < p_1 < \infty$ and p_2 such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. The dual spaces of Lebesgue spaces are given by*

$$(L^{p_1}(\Omega))' = L^{p_2}(\Omega). \quad (2.50)$$

Moreover, $(L^1(\Omega))' = L^\infty(\Omega)$, but $(L^\infty(\Omega))' \neq L^1(\Omega)$.

(vi) *$C_0(\Omega)$ and $C_0^\infty(\Omega)$ are dense subspaces of $L^p(\Omega)$ for all $1 \leq p \leq \infty$.*

All of the above statements are also valid for spaces $\mathcal{L}^p(\Omega)$ with the obvious modifications.

Proof See, e.g., [3, Chapter 2].

The definition of Lebesgue spaces allows us to evaluate the regularity of a distribution by asking if it is also an element of some Lebesgue space. It comes naturally to extend this to a distribution's derivative. This gives rise to the definition of Sobolev spaces [194, 292].

Definition 2.34 (Sobolev Spaces) Let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, and $1 \leq p \leq \infty$. The Sobolev space $W^{k,p}(\Omega)$, $k \in \mathbb{N}_0$, is defined as the subspace of $L^p(\Omega)$ with

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\gamma u \in L^p(\Omega) \text{ for all } \gamma \in \mathbb{N}_0^n, |\gamma| \leq k\} . \quad (2.51)$$

$W^{k,p}(\Omega)$ is a separable Banach space with respect to the norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\gamma| \leq k} \int_{\Omega} |D^\gamma u(x)|^p \, dV(x) \right)^{\frac{1}{p}} . \quad (2.52)$$

For $p = 2$, we denote $H^k(\Omega) = W^{k,2}(\Omega)$. These spaces are separable Hilbert spaces with inner product

$$(u, v)_{H^k(\Omega)} = \sum_{|\gamma| \leq k} (D^\gamma u, D^\gamma v)_{L^2(\Omega)} . \quad (2.53)$$

Sobolev spaces of vector-valued functions are denoted by $\mathcal{W}^{k,p}(\Omega)$ as well as $\mathcal{H}^p(\Omega)$ and are defined analogously.

Proof Proofs of Sobolev spaces being Banach spaces or even Hilbert spaces can be found, e.g., in [3, Chapter III] or [292, Chapter 3].

Moreover, the concept of Hölder-continuity can also be transferred to weak derivatives in the following sense.

Definition 2.35 (Sobolev-Slobodeckij Spaces) Let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, and $1 \leq p \leq \infty$. The Sobolev-Slobodeckij space $W^{k,p}(\Omega)$ of fractional order $k = r + s$ with $r \in \mathbb{N}_0$ and $0 < s < 1$ is defined as the subspace of $W^{r,p}(\Omega)$ with

$$W^{k,p}(\Omega) = \{u \in W^{r,p}(\Omega) : |D^\gamma u|_{s,p,\Omega} < \infty \text{ for all } \gamma \in \mathbb{N}_0^n, |\gamma| = k\} , \quad (2.54)$$

where the semi-norm $|u|_{s,p,\Omega}$ is given by

$$|u|_{s,p,\Omega} = \left(\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{\|x - y\|^{n+ps}} \, dV(x) \, dV(y) \right)^{\frac{1}{p}} . \quad (2.55)$$

$W^{k,p}(\Omega)$ is a Banach space if equipped with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\|u\|_{W^{r,p}(\Omega)}^p + \sum_{|\gamma|=k} |D^\gamma u|_{s,p,\Omega}^p \right)^{\frac{1}{p}} . \quad (2.56)$$

$\mathcal{W}^{k,p}(\Omega)$ for vector-valued functions is defined accordingly.

Proof For the proof of completeness, see, e.g., [292, Chapter 3].

In what follows, we formulate results only for scalar-valued functions, although similar results are valid for vector-valued functions.

The following relations of Sobolev spaces and spaces of continuous differentiable functions are particularly useful when considering numerical solution schemes.

Lemma 2.36 *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, and $1 \leq p < \infty$. The Sobolev space $W^{k,p}(\Omega)$, $k \in \mathbb{N}_0$, is the completion of $C^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}(\Omega)}$.*

If Ω has the segment property, then the set of restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\Omega)$.

Proof See [3, Theorems 3.16, 3.18].

Lemma 2.36 suggests the definition of another class of Sobolev spaces.

Definition 2.37 Let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, and $1 \leq p < \infty$. The Sobolev space $W_0^{k,p}(\Omega)$, $k \in \mathbb{N}_0$, is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}(\Omega)}$.

Remark 2.38 In general, $W_0^{k,p}(\Omega) \neq W^{k,p}(\Omega)$. For conditions on Ω under which those spaces are equal, the reader is referred to, e.g., [3, p. 56ff].

With this definition, we can characterize Sobolev spaces with negative index.

Definition 2.39 (Sobolev Spaces with Negative Index) Let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, and $1 < p_1 < \infty$, p_2 such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. The Sobolev space $W^{-k,p_2}(\Omega)$, $k \in \mathbb{R}^+$, is defined as

$$W^{-k,p_2}(\Omega) = \left\{ f \in (C_0^\infty(\Omega))' : \|f\|_{W^{-k,p_2}(\Omega)} < \infty \right\}, \quad (2.57)$$

with

$$\|f\|_{W^{-k,p_2}(\Omega)} = \sup_{0 \neq u \in C_0^\infty(\Omega)} \frac{|f(u)|}{\|u\|_{W^{k,p_1}(\Omega)}}. \quad (2.58)$$

$W^{-k,p_2}(\Omega)$ is the dual space of $W^{k,p_1}(\Omega)$.

Proof For norm properties and the duality relation, see, e.g., [3, 3.10–3.13].

An essential property of Sobolev spaces is the existence of the following imbeddings:

Theorem 2.40 (Sobolev Imbedding Theorem) *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, $j, k \in \mathbb{N}_0$, $1 \leq p_1, p_2 < \infty$.*

(i) *If Ω has the cone property, the following imbeddings exist:*

(a) *Suppose $kp_1 < n$ and $p_1 \leq p_2 \leq \frac{np_1}{n-kp_1}$. Then*

$$W^{j+k,p_1}(\Omega) \hookrightarrow W^{j,p_2}(\Omega). \quad (2.59)$$

(b) Suppose $kp_1 = n$, $p_1 \leq p_2 < \infty$. Then

$$W^{j+k,p_1}(\Omega) \hookrightarrow W^{j,p_2}(\Omega). \quad (2.60)$$

Moreover, if $p_1 = 1$ and, thus, $k = n$, this also holds for $p_2 = \infty$.

(ii) If Ω has the strong local Lipschitz property, additional imbeddings hold:

(a) Suppose $kp_1 > n > (k-1)p_1$. Then

$$W^{j+k,p_1}(\Omega) \hookrightarrow C^{j,s}(\overline{\Omega}), \quad 0 < s < k - \frac{n}{p_1}. \quad (2.61)$$

(b) Suppose $n = (k-1)p_1$. Then

$$W^{j+k,p_1}(\Omega) \hookrightarrow C^{j,s}(\overline{\Omega}), \quad 0 < s < 1. \quad (2.62)$$

The last imbedding holds for $s = 1$ if $n = k-1$ and $p_1 = 1$.

Proof See, e.g., [3, Chapter V].

We will later on encounter the necessity to specify in some sense the values an element of a Sobolev space takes on the boundary of a domain Ω . This is not a trivial problem as elements of Sobolev spaces are equivalence classes like the elements of Lebesgue spaces on which the definition of Sobolev spaces is based. Lemma 2.36 allows us to find a solution to this dilemma by introducing the trace operator.

Theorem 2.41 (Trace Operator) Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded domain with the uniform $C^{m,s}$ -regularity property.

(i) Let $\frac{1}{2} < k \leq m + s$, whereas for $k \in \mathbb{N}$, $k = m - 1$, $s = 1$ is allowed. There is a continuous linear operator $T_0 : H^k(\Omega) \rightarrow H^{k-\frac{1}{2}}(\partial\Omega)$, called trace operator, such that

$$T_0 u = u|_{\partial\Omega} \text{ for all } u \in C^{[k]+1}(\overline{\Omega}). \quad (2.63)$$

If $k \in \mathbb{N}$, we have $u \in C^k(\overline{\Omega})$.

(ii) Let $k + 1 \leq m + s$, whereas for $k \in \mathbb{N}$, $m = k$ and $s = 1$ is allowed and $l \in \mathbb{N}$ such that $k - l > \frac{1}{2}$. There is another continuous linear trace operator $T_l : H^k(\Omega) \rightarrow \bigtimes_{i=0}^l H^{k-i-\frac{1}{2}}(\partial\Omega)$ such that

$$T_l u = \left(u|_{\partial\Omega}, \partial_{-n(x)} u|_{\partial\Omega}, \dots, \partial_{-n(x)}^l u|_{\partial\Omega} \right) \text{ for all } u \in C^{[k]+l+1}(\overline{\Omega}). \quad (2.64)$$

If $k \in \mathbb{N}$, we have $u \in C^{k+l}(\overline{\Omega})$. Here, $\partial_{-n(x)} u$ is the directional derivative with respect to the inner normal on $\partial\Omega$.

Proof See [292, Theorem 8.7], [175, Chapter I, Theorem 3.2] and the references therein.

Remark 2.42 Definition 2.37 and Theorem 2.41 are compatible, as we have

$$T_0 u = 0 \quad \text{on } \partial\Omega \quad \forall u \in W_0^{k,p}(\Omega) . \quad (2.65)$$

For $k \geq 2$, it is possible to show that

$$T_0 D^\gamma u = 0 \quad \text{on } \partial\Omega \quad \forall u \in W_0^{k,p}(\Omega) \quad \text{with } |\gamma| \leq k-1 . \quad (2.66)$$

The definition of all the above spaces can be generalized to functions which take values in a Banach space X [234, Chapter 10]. Let $I \subset \mathbb{R}$ be a bounded open interval and X be a Banach space with (topological) dual X' . We start by defining $C(I; X)$ to be the space of all bounded continuous functions $u : I \rightarrow X$, $t \mapsto u(t)$ and equip it with the norm

$$\|u\|_{C(I;X)} = \sup_{t \in I} \|u(t)\|_X . \quad (2.67)$$

Analogously, $C^k(I; X)$, $k \in \mathbb{N}$, is defined as the space of all functions $u : I \rightarrow X$ whose derivatives in I , i.e., with respect to t , up to order k are of class $C(I; X)$.

Moreover, the Lebesgue spaces $L^p(\Omega)$, $1 \leq p < \infty$, can be generalized to $L^p(I; X)$ by substituting the absolute value in their definition and the definition of their norms by the norm on X , thus, yielding the norm

$$\|u\|_{L^p(I;X)} = \left(\int_{t \in I} \|u(t)\|_X^p dt \right)^{\frac{1}{p}} . \quad (2.68)$$

Again, $L^2(I; X)$ is a Hilbert space. The space $L^\infty(I; X)$ consists of all measurable, essentially bounded functions $u : I \rightarrow X$. It is a Banach space with respect to the norm

$$\|u\|_{L^\infty(I;X)} = \operatorname{ess\,sup}_{t \in I} \|u(t)\|_X . \quad (2.69)$$

The spaces $C(\bar{I}; X)$ and $L^p(\bar{I}; X)$, $1 \leq p \leq \infty$, are defined accordingly.

The generalization of Sobolev spaces to X -valued functions is straight-forward. We show how this is done for $H^1(I; L^2(\Omega))$, where $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is an open bounded domain. The corresponding norm is given by

$$\|u\|_{H^1(I;L^2(\Omega))} = \left(\int_I \left(\int_\Omega (|u(x,t)|^2 + |\partial_t u(x,t)|^2) dV(x) \right) dt \right)^{\frac{1}{2}} . \quad (2.70)$$

For Hilbert spaces X , the following imbedding theorem can be established:

Lemma 2.43 (Sobolev Lemma for Hilbert Space Valued Functions) *Let $I \subset \mathbb{R}$ be a bounded open interval and X be a Hilbert space. Then any function $u \in H^1(I; X)$ corresponds to a function in $C(\bar{I}; X)$.*

Proof Analogous to [292, Theorem 6.2].

Remark 2.44 See, e.g., [253, Chapter III, Proposition 1.2] for a proof in a more general setting with Banach spaces instead of Hilbert spaces.

There is one more result which we use throughout this thesis regarding weak derivatives and difference quotients.

Lemma 2.45 (Weak Derivatives and Difference Quotients) *Let $\Omega \subset \mathbb{R}^n$ be open, $(0, t_{\text{end}}) \subset \mathbb{R}$, $1 < p \leq \infty$ and*

- (I) $w \in \mathcal{L}^p(\Omega)$, $k \in \{1, \dots, n\}$ or
- (II) $w \in L^p((0, t_{\text{end}}), X)$ with X being a Sobolev space on the set Ω .

Then the following assertions are true:

- (a) *If the weak partial derivatives satisfy $\partial_{x_k} w \in \mathcal{L}^p(\Omega)$ for some k in case (I) or $\partial_t w \in L^p((0, t_{\text{end}}), X)$ in case (II), we have for the corresponding difference quotients*

- (i) $\delta_{k,h} w \in \mathcal{L}^p(\Omega')$ for any compact subset Ω' of Ω , $h < \text{dist}(\Omega', \partial\Omega)$, and $\|\delta_{k,h} w\|_{\mathcal{L}^p(\Omega')} \leq \|\partial_{x_k} w\|_{\mathcal{L}^p(\Omega)}$,
- (ii) $\delta_{t,h} w \in L^p((a, b), X)$ for any compact subset $[a, b]$ of $(0, t_{\text{end}})$, $h < \min(a, t_{\text{end}} - b)$, and $\|\delta_{t,h} w\|_{L^p((a,b),X)} \leq \|\partial_t w\|_{L^p((0,t_{\text{end}}),X)}$.

This also holds for $p = 1$.

- (b) (i) *If there exists a constant $C \in \mathbb{R}^+$ such that for any compact subset $\Omega' \subset \Omega$ and $h < \text{dist}(\Omega', \partial\Omega)$ we have $\delta_{k,h} w \in \mathcal{L}^p(\Omega')$ and $\|\delta_{k,h} w\|_{\mathcal{L}^p(\Omega')} \leq C$, then $\partial_{x_k} w$ (which is a priori well defined as a distribution) is in fact in $\mathcal{L}^p(\Omega)$ and satisfies $\|\partial_{x_k} w\|_{\mathcal{L}^p(\Omega)} \leq C$. Moreover, when $h \rightarrow 0$, $\delta_{k,h} w \rightarrow \partial_{x_k} w$ in $\mathcal{L}^p_{\text{loc}}(\Omega)$.*
- (ii) *If there exists a constant C such that for any $[a, b] \Subset (0, t_{\text{end}})$ and $h < \min(a, t_{\text{end}} - b)$ we have $\delta_{t,h} w \in L^p((a, b), X)$ and $\|\delta_{t,h} w\|_{L^p((a,b),X)} \leq C$, then $\partial_t w$ (which is a priori well defined as a distribution) is in fact in $L^p((a, b), X)$ and satisfies $\|\partial_t w\|_{L^p((a,b),X)} \leq C$. Moreover, when $h \rightarrow 0$, $\delta_{t,h} w \rightarrow \partial_t w$ in $L^p_{\text{loc}}((a, b), X)$.*

Proof See [234, Lemmata 8.48 and 8.49], as well as [113, Lemmata 8.1 and 8.2] for case (I). Case (II) can be proven analogously.

2.4 Integral Calculus

In this section, we summarize some theorems from vector calculus about integrals on vector fields. In order to do this, we have to explain some more differential operators.

We already introduced the gradient in Remark 2.9 and the corresponding notation that uses the so-called nabla operator ∇ . The definition of the gradient can be generalized to differentiable vector-valued functions $u = (u_1, u_2, u_3)^T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by defining

$$(\nabla_x u)_{ij}(x) = \partial_{x_j} u_i(x), \quad i, j \in \{1, 2, 3\}, \quad (2.71)$$

for cartesian coordinates x_i . Thus, the gradient of a vector-valued function u is identical to the transpose of the Jacobian of u .

Other often used differential operators are the divergence of u , given by

$$\nabla_x \cdot u(x) = \sum_{i=1}^3 \partial_{x_i} u_i(x), \quad (2.72)$$

and the curl of u , given by

$$\nabla_x \wedge u(x) = \begin{pmatrix} \partial_{x_2} u_3(x) - \partial_{x_3} u_2(x) \\ \partial_{x_3} u_1(x) - \partial_{x_1} u_3(x) \\ \partial_{x_1} u_2(x) - \partial_{x_2} u_1(x) \end{pmatrix}. \quad (2.73)$$

If a function $u : \mathbb{R}^3 \rightarrow \mathbb{R}$ is differentiable in x and its gradient $\nabla_x u$ is also differentiable in x , we can calculate the divergence of the gradient. This is known as the Laplacian, given by

$$\nabla_x^2 u(x) = \nabla_x \cdot ((\nabla_x u)(x)) = \sum_{i=1}^3 \partial_{x_i}^2 u_i(x). \quad (2.74)$$

The Laplacian can be defined analogously for sufficiently differentiable vector-valued functions. Note that for a vector-valued function u the expression $\nabla_x \cdot ((\nabla_x u)(x))$ differs from $\nabla_x \cdot ((\nabla_x \cdot u)(x))$.

Remark 2.46

- (i) We will omit the index x in ∇_x whenever it is clear with respect to which variable the derivative is taken.
- (ii) It is not necessary for the application of any of the above differential operators that the function under consideration is defined on the whole space $\mathbb{R}^3$. It may instead be defined only on an open subset $\Omega \subset \mathbb{R}^3$.
- (iii) Except for the curl operator, all of the above operators can be defined not only for functions on $\mathbb{R}^3$ but $\mathbb{R}^n$, $n \in \mathbb{N}$, in a similar fashion.

With the above notation, we can formulate the following integral relations.

Theorem 2.47 (Gauß (Divergence) Theorem) *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. If the vector field $u : \overline{\Omega} \rightarrow \mathbb{R}^n$ is an element of $(C^1(\overline{\Omega}))^n$, we have*

$$\int_{\Omega} \nabla_x \cdot u(x) \, dV(x) = \int_{\partial\Omega} u(x) \cdot n(x) \, dS(x) \quad (2.75)$$

where $dV (= dV_n)$ denotes the volume element in $\mathbb{R}^n$, $n(x)$ denotes the outer (unit) normal field on $\partial\Omega$, and $dS (= dS_{n-1})$ denotes the surface element of $\mathbb{R}^n$.

Proof See, e.g., [68, §3.3] in a more general context.

As a consequence of the Gauß Theorem, we get by choosing $u = v \nabla w$ with $v \in C^1(\overline{\Omega})$, $w \in C^2(\overline{\Omega})$ the following theorem.

Theorem 2.48 (First Green Theorem) *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain on which the Gauß Theorem 2.47 is valid for all $u \in \mathcal{C}^1(\Omega)$. For $v \in C^1(\overline{\Omega})$ and $w \in C^2(\overline{\Omega})$, the relation*

$$\int_{\Omega} (v(x) \nabla_x^2 w(x) + (\nabla_x v(x)) \cdot (\nabla_x w(x))) \, dV(x) = \int_{\partial\Omega} v(x) \partial_{n(x)} w(x) \, dS(x) \quad (2.76)$$

holds.

Proof See, e.g., [97, Theorem 2.1].

If we insert $u = v \nabla w - w \nabla v$ with $v, w \in C^2(\overline{\Omega})$ into (2.75), we obtain

Theorem 2.49 (Second Green Theorem) *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain on which the Gauß Theorem 2.47 is valid for all $u \in \mathcal{C}^1(\Omega)$. For $v, w \in C^2(\overline{\Omega})$, the relation*

$$\begin{aligned} & \int_{\Omega} (v(x) \nabla_x^2 w(x) - w(x) \nabla_x^2 v(x)) \, dV(x) \\ &= \int_{\partial\Omega} (v(x) \partial_{n(x)} w(x) - w(x) \partial_{n(x)} v(x)) \, dS(x) \end{aligned} \quad (2.77)$$

holds.

Proof See, e.g., [97, Theorem 2.2].

As we are interested in time-dependent problems, there is one more integral identity which we need. For this, we first have to define a motion [188, Chapter 1, Definitions 1.2, 1.4, and 1.5].

Definition 2.50 (Motion, Configuration, Velocity) *Let $\Omega \subset \mathbb{R}^3$ be an open domain. A configuration of Ω is a mapping $\phi : \Omega \rightarrow \mathbb{R}^3$. A reference configuration*

is a certain fixed configuration of Ω . Points in Ω are called material points (more often material coordinates) and denoted by $X = (X_1, X_2, X_3)^T$. Points in $\mathbb{R}^3$ are called spatial points (more often spatial coordinates) and denoted by $x = (x_1, x_2, x_3)^T$.

A motion of Ω is a mapping $\phi : \Omega \times (a, b) \rightarrow \mathbb{R}^3$, $(X, t) \mapsto \phi(X, t) = x$ with an open interval $(a, b) \subset \mathbb{R}$. For $t \in (a, b)$ fixed, we write $\phi_t(X)$. For a subset $\mathcal{B} \subset \Omega$, we write $\mathcal{B}_t = \phi_t(\mathcal{B})$.

A motion ϕ_t is called regular or invertible if each $\phi_t(\Omega)$ is open and the inverse $\phi_t^{-1} : \Omega_t \rightarrow \Omega$ exists. ϕ_t is called $\mathcal{C}^k$ -regular, $k \in \mathbb{N}_0$, if $\phi \in \mathcal{C}^k(\Omega \times (a, b))$ and $\phi^{-1} \in \mathcal{C}^k(\Omega_t \times (a, b))$.

The material velocity $\mathcal{V}_t(X) = \mathcal{V}(X, t)$ is defined by

$$\mathcal{V}_t(X) = \partial_t \phi(X, t) \quad (2.78)$$

if the derivative exists. For $\mathcal{C}^1$ -regular motions, the spatial velocity $v_t(x)$ is defined by

$$v_t : \Omega_t \rightarrow \mathbb{R}^3, \quad v_t(x) = \mathcal{V}_t(\phi_t^{-1}(x)) . \quad (2.79)$$

We can now define how to calculate the absolute time derivative $\frac{d}{dt}$ of the spatial integral of some function u .

Theorem 2.51 (Reynolds Transport Theorem) *Let $\Omega \subset \mathbb{R}^3$, $(a, b) \subset \mathbb{R}$ be an open interval and ϕ_t be a $\mathcal{C}^1$ -regular motion. Let $u \in C^1(\Omega_t \times (a, b))$ and $\mathcal{B} \subset \Omega$ be open with a piecewise C^1 -regular boundary $\partial\mathcal{B}$. Then*

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{B}_t} u(x, t) \, dV(x) &= \int_{\mathcal{B}_t} \partial_t u(x, t) + \nabla_x \cdot (u(x, t)v(x, t)) \, dV(x) \\ &= \int_{\mathcal{B}_t} \frac{Du}{Dt}(x, t) + u(x, t) (\nabla_x \cdot v(x, t)) \, dV(x) , \end{aligned} \quad (2.80)$$

where $\frac{d}{dt}$ denotes the total derivative with respect to time, ∂_t denotes the partial derivative with respect to time and

$$\frac{Du}{Dt}(x, t) = \partial_t u(x, t) + (\nabla_x u(x, t)) \cdot v(x, t) \quad (2.81)$$

is the so-called material time derivative.

Proof See, e.g., [188, Chapter 2, Theorem 1.1] and [11, page 3].

Remark 2.52

- (i) Using the correct velocity v , the one as defined in Definition 2.50, and a reference configuration $\mathcal{B}$ is crucial here. If, for example, in the settings of fluid mechanics, the fluid is allowed to leave the volume $\mathcal{B}_t$, the velocity that has to be used in the Transport Theorem may differ from the velocity of the fluid.

(ii) Applying the Gauß Theorem 2.47 to (2.80) yields

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{B}_t} u(x, t) \, dV(x) &= \int_{\mathcal{B}_t} \partial_t u(x, t) \, dV(x) \\ &\quad + \int_{\partial \mathcal{B}_t} (u(x, t) v(x, t)) \cdot n(x) \, dS(x) . \end{aligned} \quad (2.82)$$

2.5 Differential Equations

Assume we have an open bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n > 1$, and a map

$$\mathcal{F} : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \dots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}, \quad k \in \mathbb{N}. \quad (2.83)$$

Then

$$\mathcal{F}((D^k u)(x), \dots, u(x), x) = 0 \text{ for all } x \in \Omega \quad (2.84)$$

is a partial differential equation (PDE) of order k if at least one derivative of order k is actually a part of the equation and no derivative of higher order than k is present. This can be done analogously for systems of differential equations. We only deal with linear PDEs here that can be written as

$$\sum_{|\gamma| \leq k} a_\gamma(x) (D^\gamma u)(x) = f(x), \quad \gamma \in \mathbb{N}_0^n, \quad (2.85)$$

with given coefficient functions $a_\gamma(x)$ and right-hand side $f(x)$. If $f \equiv 0$, the PDE is called homogeneous.

Within this thesis, we will deal with systems of linear PDEs of second order. There are three main classes of these PDEs. We start by defining an elliptic differential operator (cf., e.g., [72, Section 6.1.1.]).

Definition 2.53 (Elliptic PDEs) Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $u : \overline{\Omega} \rightarrow \mathbb{R}$. Let L be a linear differential operator of second order such that

$$u(x) \mapsto Lu(x) = \sum_{i,j=1}^n a_{ij}(x) \partial_{x_i} \partial_{x_j} u(x) + \sum_{i=1}^n b_i(x) \partial_{x_i} u(x) + c(x) u(x). \quad (2.86)$$

L is called uniformly elliptic if there exists a constant $C > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq C \|\xi\|^2 \quad (2.87)$$

for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^n$.

With this definition, we can now define the two other main classes (cf., e.g., [72, Sections 7.1.1. and 7.1.2.] or [101, Section 1.1]). For both, there is one distinguished variable, denoted by t rather than as a component of a vector x , which is usually the time being distinct from spatial variables summarized in x .

Definition 2.54 (Parabolic PDE) Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $(0, t_{\text{end}}) \subset \mathbb{R}$, $t_{\text{end}} > 0$, $u : \bar{\Omega} \times [0, t_{\text{end}}] \rightarrow \mathbb{R}$. Let L be a linear differential operator of second order such that

$$\begin{aligned} u(x, t) \mapsto Lu(x, t) = & \sum_{i,j=1}^n a_{ij}(x, t) \partial_{x_i} \partial_{x_j} u(x, t) + \sum_{i=1}^n b_i(x, t) \partial_{x_i} u(x, t) \\ & + c(x, t)u(x, t) - \partial_t u(x, t) . \end{aligned} \quad (2.88)$$

L is called uniformly parabolic if there exist positive constants $\bar{\lambda}_0, \bar{\lambda}_1$ such that

$$\bar{\lambda}_0 \|\xi\|^2 \leq \sum_{i,j=1}^n a_{ij}(x, t) \xi_i \xi_j \leq \bar{\lambda}_1 \|\xi\|^2 \quad (2.89)$$

for all $(x, t) \in \Omega \times (0, t_{\text{end}})$ and all $\xi \in \mathbb{R}^n$.

Definition 2.55 (Hyperbolic PDE) Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $(0, t_{\text{end}}) \subset \mathbb{R}$, $t_{\text{end}} > 0$, $u : \bar{\Omega} \times [0, t_{\text{end}}] \rightarrow \mathbb{R}$. Let L be a linear differential operator of second order such that

$$\begin{aligned} u(x, t) \mapsto Lu(x, t) = & \sum_{i,j=1}^n a_{ij}(x, t) \partial_{x_i} \partial_{x_j} u(x, t) + \sum_{i=1}^n b_i(x, t) \partial_{x_i} u(x, t) \\ & + c(x, t)u(x, t) - \partial_t^2 u(x, t) . \end{aligned} \quad (2.90)$$

L is called uniformly hyperbolic if there exists a constant $C > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x, t) \xi_i \xi_j \geq C \|\xi\|^2 \quad (2.91)$$

for all $(x, t) \in \Omega \times (0, t_{\text{end}})$ and all $\xi \in \mathbb{R}^n$.

Remark 2.56

- (i) It is possible that the character of a differential operator changes with x , e.g., when there is a function of x as coefficient of the time derivative term. Such equations can be locally elliptic, parabolic, or hyperbolic instead of uniformly, i.e., they are of one of these types on certain subdomains of Ω .

- (ii) Not all linear second order PDEs are of one of the above classes for $n > 2$. Indeed, the system of equations which we discuss in this thesis is of neither of the above types, but has similarities with some elliptic and parabolic equations.

We are interested in finding solutions of differential equations. Usually, we have some more information than just the differential equation itself in the form of initial and boundary conditions. From a theoretical point of view, these often help to guarantee uniqueness of solutions, or at least specify in which way solutions to the same problem may differ. Instead of discussing this in general, we will treat this problem in the next chapters for the set of equations in which we are interested in this thesis.

As already pointed out, the formulations of differential equations as given above with strong partial derivatives is often not suited to find answers to the questions of solvability, uniqueness of solutions, or their regularity. Instead, we would like to have a formulation based on weak derivatives.

Let us assume that a linear second order differential equation is given in its strong form by

$$Lu(x) = f(x) \text{ for all } x \in \Omega . \quad (2.92)$$

For simplicity, we equip this equation with the homogeneous Dirichlet boundary condition

$$u(x) = 0 \text{ for all } x \in \partial\Omega . \quad (2.93)$$

In the context of differential equations, we will often use the abbreviation $\Gamma = \partial\Omega$.

A classical strong solution of this PDE has to be in $C^2(\Omega)$ which is a rather restrictive requirement. We can relax this requirement in two points. First, we can get to a weak differentiable solution. For this purpose, suppose v is an arbitrary function belonging to $C_0^\infty(\Omega)$, multiply the differential equation by v and integrate over Ω . We obtain

$$(Lu, v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \text{ for all } v \in C_0^\infty(\Omega) . \quad (2.94)$$

Moreover, we can relax the requirements on differentiability of u by performing an integration by parts on the left-hand side, which gives us a bilinear form $a(u, v)$. Thus, a useful assumption on u is $u \in H_0^1(\Omega)$. Additionally, as $C_0^\infty(\Omega)$ is a dense subspace of $H_0^1(\Omega)$, we can extend the space of functions with which we multiply to $H_0^1(\Omega)$. This yields

$$a(u, v) = f(v) \text{ for all } v \in H_0^1(\Omega) . \quad (2.95)$$

Here, we interpreted the right-hand side as a linear functional on $H_0^1(\Omega)$. It is easy to see that every solution of the strong formulation is also a solution of the weak formulation. However, the opposite may not be true.

For other kinds of boundary conditions, the above procedure is changed in two points. On the one hand, if the values of u on the boundary are given and different from zero, u has to belong to another subspace of $H^1(\Omega)$. On the other hand, if normal derivatives of u are specified in a Neumann boundary condition, integration by parts yields some integrals over (parts of) the boundary Γ of Ω . These are usually incorporated into the linear form f on the right-hand side. Other modifications may be necessary for other boundary conditions.

The answer to the question, whether a solution of (2.95) exists, depends on the properties of $a(\cdot, \cdot)$.

Definition 2.57 (Coercitivity) Let V be a real Hilbert space with inner product $(\cdot, \cdot)_V$ and corresponding norm $\|\cdot\|_V$.

A bilinear form $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ is strict coercive if a constant $C > 0$ exists such that

$$a(u, u) \geq C \|u\|_V^2 \quad (2.96)$$

for all $u \in V$.

If $a(\cdot, \cdot)$ is a continuous coercive bilinear form and f a continuous linear functional on V , the task to determine $u \in V$ such that we have

$$a(u, v) = f(v) \text{ for every } v \in V \quad (2.97)$$

is called a variational problem. It is equivalent to determining $u \in V$ such that

$$\frac{1}{2}a(u, u) - f(u) = \min_{0 \neq v \in V} \left(\frac{1}{2}a(v, v) - f(v) \right). \quad (2.98)$$

For a proof of this equivalence, see, e.g., [55, Theorem 1.1.2].

Under the additional assumptions that $a(\cdot, \cdot)$ is symmetric and positive definite, i.e., it is an inner product, solvability of the variational problem is given by the Riesz Representation Theorem.

Theorem 2.58 (Riesz Representation Theorem) *Let V be a Hilbert space and $a(\cdot, \cdot)$ be an inner product on V . For every continuous linear functional $f \in V'$, there is a unique $u \in V$ such that (2.97) is valid.*

Proof See, e.g., [295, Section III.6].

For non-symmetric bilinear forms, we have

Theorem 2.59 (Lax-Milgram Theorem) *Let V be a Hilbert space and $a(\cdot, \cdot)$ be a continuous and strict coercive bilinear form on $V \times V$. For every continuous linear functional $f \in V'$, there is a unique $u \in V$ such that (2.97) is valid.*

Proof See, e.g., [295, Section III.7].

In problems in which a special role is assigned to time, a different setting, in which time is regarded as a parameter, is advantageous [175, Chapter IV]. Moreover, we consider two Hilbert spaces V and $\mathcal{H}$ such that V is densely imbedded in $\mathcal{H}$, $\mathcal{H}' = \mathcal{H}$, and, thus, $V \subseteq \mathcal{H} \subseteq V'$. The pair $(V, \mathcal{H})$ is called a rigged Hilbert space or Gelfand triple. For $t \in (0, t_{\text{end}})$, $t_{\text{end}} > 0$, we consider a family $a(t; \cdot, \cdot)$ of continuous bilinear forms on V and assume that for $u, v \in V$ the function $t \mapsto a(t; u, v)$ is measurable and that there is a constant $C \in \mathbb{R}^+$, independent of t, u , and v , such that (cf. [175, Chapter IV, (1.1)])

$$|a(t; u, v)| \leq C \|u\|_V \|v\|_V. \quad (2.99)$$

Using such a family of bilinear forms, the weak formulation of a parabolic PDE may be given as the following problem [175, Chapter IV, Problem 1.1]: Find $u \in L^2((-\infty, t_{\text{end}}), V)$ such that $u(t) = 0$ for almost every $t < 0$ and

$$\frac{d}{dt} (u(t), v)_V + a(t; u(t), v) = (f(t), v)_V + (u^0, v)_V \delta(t) \quad (2.100)$$

with $f \in L^2((-\infty, t_{\text{end}}), \mathcal{H})$ vanishing for $t < 0$ and u^0 is given in $\mathcal{H}$. For this problem, we have the following theorem.

Theorem 2.60 (Lions) *Let $V, \mathcal{H}$ be as defined at the beginning of this paragraph and let $a(t; \cdot, \cdot)$ be a family of continuous bilinear forms on V for $t \in [0, t_{\text{end}}]$ with (2.99). If $a(t; \cdot, \cdot)$ satisfies*

$$a(t; v, v) \geq c \|v\|_V - \beta \|v\|_{\mathcal{H}} \text{ for all } v \in V \quad (2.101)$$

with some constant $c \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$, there exists a unique solution u of the above given problem and the map $\{f, u^0\} : L^2((-\infty, t_{\text{end}}), \mathcal{H}) \times \mathcal{H} \rightarrow V$ is continuous.

The condition (2.101) is called modified coercitivity or Gårding inequality.

Proof See, e.g., [175, Chapter IV, Theorem 1.1].

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