Chapter 2
From Lagrange to Frege: Functions and Expressions

Marco Panza

2.1 Introduction

Part I of Frege's *Grundgesetze* is devoted to the "exposition [*Darlegung*]" of his formal system. It opens with the following claim ([97], Sect. I.1, p. 5; [110], p. 5_1):

> If the task is to give the original reference [*Bedeutung*] of the word 'function' in its mathematical usage, then it is easy to slip into calling a function of x any expression [*Ausdruck*] that is formed from 'x' and certain determinate numbers by means of the notations [*Bezeichnungen*] for sum, product, power, difference, etc. This is inappropriate [*unzutreffend*], since in this way a function is depicted [*hingestellt*] as an *expression*, as a combination of signs [*Verbindung von Zeichen*], and not as what is designated [*Bezeichnete*] thereby. One will therefore be tempted to say 'reference of an expression' instead of 'expression'.

Frege does not explicitly ascribe this inappropriateness to anyone, though he could have ascribed it to many.[1] One is Lagrange, who, a little less than one century earlier, defined functions as follows, both in the *Théorie des fonctions analytiques* and in the *Leçons sur le calcul des fonctions* ([134], Sect. 1, p. 1; [140], *Introduction*, p. 1; [137], p. 6; [138], p. 6; I quote from the *Théorie*; the analogous passage of the *Leçons* presents some inessential changes):

> One calls a 'function' of one or several quantities any calculational expression [*expression de calcul*] into which these quantities enter in any way whatsoever combined or not with other quantities which are regarded as having given and invariable values, whereas the quantities of the function may receive any possible value.

[1]Baker [5] has considered, but finally rejected, the idea that Frege could also have ascribed this inappropriateness to himself, by referring to Sects. 9–10 of his [92]. My purpose here is not to describe the evolution of Frege's views. I shall rather confine myself to considering his mature views, as they emerge from *Grundgesetze* or other contemporary works.

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According to our modern view, this definition is based on an inadmissible conflation of syntactical items and their *designata*. Frege's warning against a definition like this appears, then, to be not merely motivated by a different approach, but rather mandated by conceptual clarity and rigour. Historically speaking, things are not so simple, however: if considered in context, Lagrange's definition reflects a quite precise conception of mathematics, which is, in some crucial respects, close to Frege's.

Like Frege's *Grundgesetze*, Lagrange's treatises also pursue a foundational program. Still, the former's program is not only crucially different from the latter's, it also depends on a different idea of what a foundation of mathematics should be like.[2] Despite both this contrast and that between warning and Lagrange's definition, the notion of a function plays similar roles in their respective programs. My purpose is to emphasise this similarity. In doing so, I hope to contribute to a better understanding of Frege's logicism, especially in relation to its crucial differences from a set-theoretic foundational perspective. This should also shed some light on a question raised by Hintikka and Sandu in a widely discussed paper [129], namely whether Frege should or should not be credited with the notion of an arbitrary function that underlies our standard interpretation of second-order logic.[3]

In Sect. 2.2, I shall recount Lagrange's notion of a function.[4] In Sect. 2.3, I shall advance some remarks on connected historical matters. This will provide an appropriate framework for discussing the role played by the notion of a function in Frege's *Grundgesetze*, to which Sect. 2.4 is devoted. Some concluding remarks will close the chapter.

2.2 Lagrange's Notion of a Function

Lagrange's treatises aim at offering a non-infinitesimalist interpretation of the differential formalism. For this purpose, the calculus is embedded in a general theory of functions, often termed 'algebraic analysis'.[5] Though this theory originated with Euler's *Introductio* [77], Lagrange suggests a new way of integrating the calculus within it, quite different from that suggested by Euler himself in his *Institutiones* [80]. This rests on a more general conception of mathematics, that, though close to Euler's, significantly differs from it.[6]

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[2]Broadly speaking, a foundation of mathematics aims at a reorganisation of mathematics according to a suitable order. For Frege, such an order ought both to reflect an objective order of truths ([93], Sect. 2) and to provide mathematics, especially arithmetic and real analysis, with an epistemically sound basis (*ibid*., Sect. 3). For Lagrange, it should rather obey an ideal of purity (as I and my coauthor G. Ferraro have largely argued for in [89]).

[3]Cf. Sect. 3.1 of Chap. 3, below.

[4]A more comprehensive discussion is offered in [89].

[5]This is suggested by the complete title of the *Théorie*: 'Théorie des fonctions analytiques contenant les principes du calcul différentiel [...] réduits à l'analyse algébrique des quantités finies'.

[6]For a discussion of Euler's conception, I refer the reader to [152].

The basic idea is that of recasting all mathematics within algebraic analysis, understood as a general theory of functions. Hence, functions are not merely conceived as objects to be studied by a branch of mathematics; they are taken to be what all mathematics is about.

This idea is structurally similar to that which underlies the set-theoretic foundational program. But, while this program is often understood ontologically—i.e. it is taken to entail the requirement that all mathematical objects be ultimately identified with appropriate sets—Lagrange does not argue that all that mathematics deals with has to be ultimately identified with functions. For him, mathematics is the science of quantities, as classically maintained, and not every quantity is ultimately a function: this is not so for numbers or geometrical and mechanical magnitudes, which are endowed with a specific, irreducible nature. Concerning them, Lagrange's point is rather that mathematics should study their mutual relations, and, for this purpose, it should look at them as functions of each other. Still, according to him, this is possible only if a general theory of functions is provided in which functions are considered as such, and identified with abstract quantities, i.e. quantities lacking any specific nature. Algebraic analysis is just this theory. And Lagrange's definition of functions is aimed at fixing what it is about.

Following this definition, quantities enter into expressions. Hence, these expressions are said to include that which, according to our familiar distinction between *designans* and *designatum*, the terms composing them designate.

One could argue that this is simply because of a slip of the pen. But this is implausible, not only because Lagrange would then have been the victim of the same slip in both editions of the *Théorie* and of the *Leçons* as well, but also because this slip would have then been also very common among other contemporary mathematicians. It would have also affected, for example, the definition of function offered in Euler's *Introductio* ([77], vol. I, Sect. 4, p. 4; [83], vol. I, p. 3):

> A function of a variable quantity is an analytical expression composed in any way whatever of this variable quantity and numbers or constant quantities.

Once the possibility of a slip of the pen has been discarded, it only remains to accept that, for both Lagrange and Euler, quantities are not what the terms entering into a "calculational" or "analytical expression" designates, but are these very terms.

A hint for understanding how this is possible can be found in a criticism Lagrange addresses to Newton's conception of the calculus ([134], Sect. 5, p. 4; [140], *Introduction*, p. 3):

> [...] Newton considered mathematical quantities as generated by motion [...]. But [...] introducing motion in a calculation whose object is nothing but algebraic quantities is the same as introducing an extraneous idea [...].

The calculation Lagrange is referring to is the calculus. His point is thus that the calculus should not concern quantities generated by motion, but algebraic quantities. Criticising Newton for introducing motion within pure mathematics was usual. But arguing that the calculus should concern algebraic quantities was new. For Lagrange, the calculus is to be immersed within algebraic analysis. Hence, for him, also algebraic analysis should be about algebraic quantities. But what are algebraic quantities?

In Lagrange's setting, there is no room for identifying them as that which algebra is about, supposing that this is independent of (and prior to) algebraic analysis. In other words, there is no room for taking the functions entering into algebraic analysis to involve quantities that algebra supplies. This would result in a structural duplication (algebra on one side, with its own formulas, and algebraic analysis on the other side, with its functions), of which there is no trace either in the *Théorie* or in the *Leçons*. Rather, Lagrange repeatedly claims or implies that algebra and algebraic analysis do not differ essentially. The following quotation, from his treatise on numerical equations, provides an example ([135], p. vii; [139], p. 15):

> Taken in the most comprehensive sense, algebra is the art of determining unknowns through functions of known quantities or quantities regarded as known; and general solution of equations consists in finding, for all the equations of any degree, the functions of the coefficients of these equations that are able to represent all their roots.

Furthermore, for Lagrange, functions not only include (algebraic) quantities, but are also (algebraic) quantities. Here is what he writes in the second edition of the *Théorie*:

> Through the character 'f' or 'F' placed before a variable, we shall designate in general any function of this variable, that is, any quantity depending on this variable and which varies with it according to a given law.

Another passage where he is quite clear on this is the following ([134], Sect. 2; [140], *Introduction*, pp. 1–2; cf. also [137], p. 4; [138], p. 4):

> The word 'function' has been employed by the first analysts in order to designate in general the powers of a same quantity. Then its meaning has been extended to any quantity however formed by another quantity. Leibniz and the Bernoullis employed it firstly in this general sense, and it is today generally adopted.

Doubtless, Lagrange takes his definition to be consistent with this "generally adopted" sense, which Johann Bernoulli had fixed already in 1718, by stating that a "function of a variable quantity" is a "quantity however composed by this variable quantity and constants" ([15], p. 106).

There is thus no doubt that, for Lagrange, functions are both expressions that contain quantities and quantities. Insofar as this view is openly incompatible with the *designans/designatum* distinction, it can be grasped only if this distinction is thrown away. Two new quotations (respectively from [137], p. 4 and [138], p. 4, and, from [136], 235) suggest a way of doing this:

> [...] one should regard algebra as the science of functions, and it is easy to see that, in general, the solution of equations does not consist but in finding the values of unknown quantities as determined functions of known quantities. These functions represent, then, the different operations that have to be performed on the known quantities in order to obtain the values of those which are sought, and they are properly only the last result of the calculation.
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> Strictly speaking, algebra in general is nothing but the theory of functions. In Arithmetic, one looks for numbers according to given conditions between these numbers and other

> numbers; and the numbers that are found meet these conditions without preserving any trace of the operations that were needed in order to form them. In algebra, instead, the sought after quantities have to be functions of given quantities, that is, expressions representing the different operations that have to be performed on these quantities in order to get the values of the sought after quantities. In algebra *stricto sensu*, one only considers primitive functions that result from ordinary algebraic operations; this is the first branch of the theory of functions. In the second branch, one considers derivative functions, and it is this branch that we simply designate with the name 'theory of analytical functions' [...].

Lagrange's terminology is fluctuating and imprecise. But it is clear that he considers algebraic analysis to be a general subject including at least two interrelated branches that are not distinguished because of their objects, which are always functions, but rather because of by the way these objects are considered. The former is the theory of algebraic equations; the latter the theory of analytical functions, i.e. Lagrange's own version of the calculus. Arithmetic results when functions are instantiated on numbers. In this case, they can be computed and this produces new numbers whose operational relations with those from which they result is lost. In algebraic analysis, instead, functions are not instantiated on independently given quantities, but are the relevant quantities themselves; they can only be transformed, and, whatever their form might be, they maintain a trace of the operational relations that link them to the quantities of which they are functions. This is just what makes algebraic analysis pure and general. Its subject matter is the system of relations induced by the (indefinite) composition of some (elementary) operations applied to previously indeterminate arguments. Precisely because these arguments are taken as being previously indeterminate, they are subsequently characterised by nothing other than the network of relations they enter into. These relations are immediately displayed, or, as Lagrange improperly says, "represented",[7] by appropriate expressions, which are taken to constitute a *sui generis* sort of quantity: algebraic quantities, or functions. These are endowed with a purely relational identity and lack any intrinsic nature, though being capable of being studied as such, and of being instantiated on numbers and geometric or mechanical magnitudes.

It follows that, according to Lagrange, neither operations nor their arguments precede symbols: at the beginning there are only symbols submitted to appropriate rules; operations and quantities appear next, whenever these symbols are supposed to acquire a mathematical meaning. For example, the symbol '$+$' is not taken to designate the independently given operation of addition. This operation is rather fixed by the rules of composition and transformations relative to this symbol: it is not because addition is commutative that '$a + b$' can be transformed into '$b + a$',

[7] As Lagrange uses it, the verb 'to represent' is not intended to indicate a relation between two distinct entities, one of which is taken to stand for the other under an appropriate respect. Expressions do not "represent" operations because they stand for them or present them afresh. They display at once these operations, their results, and the corresponding relations: 'x^2' displays, for example, the operation of taking the square, the quantity related to x according to this operation, and the relation between it and x.

but the other way around. The universe of Lagrange's general theory of functions is thus a universe of symbols governed by rules of composition and transformation, not a universe of objects, operations, and relations to which these symbols refer.

All this makes clear that, for Lagrange, the notion of a function is mathematically primitive: his definition is intended as a clarification of this notion that is based on no previous mathematical development. All that is required for understanding this definition is taking for granted an appropriate extension of the algebraic formalism originating with Viète and Descartes. But, as such, this formalism is not yet supposed to be a mathematical system; mathematics only begins when the formulas of this formalism are understood as quantities, i.e. just when functions are introduced.

A last point has to be clarified: if things are this way, how can Lagrange maintain that algebraic quantities or functions are quantities in a genuine sense of this term? Partly, this is because they are arguments of operations with the same formal properties as the usual operations on numbers and geometric magnitudes.[8] Furthermore, this is because Lagrange tacitly assigns to them some properties that do not depend on their being constituted by appropriate expressions: he attributes to them a linear order and some metric relations, and also supposes they comply with continuity conditions. This is essential for his reductionist program to succeed. But it also produces a discrepancy between the understanding of functions as expressions and their understanding as quantities. This is one of the reasons why this program ultimately failed.[9]

2.3 Arbitrary Functions and the Arithmetisation of Analysis

A notorious shortcoming of Lagrange's theory is relevant to my purpose. To see it, consider an example ([134], Sect. 96; [140] Sect. I.84).

Let

$$z = ax + by + c \tag{2.1}$$

be a function of two variables x and y, involving the constants a, b and c. Insofar as

$$z'_x = a \qquad \text{and} \qquad z'_y = b \tag{2.2}$$

this function provides the complete primitive of the following partial differential equation:

$$z - xz'_x - yz'_y - c = 0 \tag{2.3}$$

[8]Though partial, this answer is not simple. It reveals a crucial feature that Lagrange's program shares with any foundational reductionist program in mathematics: this program stipulates a new start for mathematics, without being free to forget what mathematics was before its advent. Hence, this start has both to be taken as primitive and to be so shaped as to allow a reformulation of what was there independently of it.

[9]Arguing for this is one of the main purpose of [89].

This is not the only primitive of this equations, however. To get another primitive, suppose that a be a function of x and y, and b a function of a. From (2.1), by taking the derivatives with respect to x and y, one gets, respectively:

$$z'_x = a + a'_x \left[x + yb'_a\right] \qquad \text{and} \qquad z'_y = b + a'_y \left[x + yb'_a\right]$$

It is, then, enough to also suppose that

$$x + yb'_a = 0 \tag{2.4}$$

to get the equalities (2.2), again. Insofar as we have supposed that b is a function of a, from (2.4) it follows that a is a function of $\frac{x}{y}$, and so is $a\frac{x}{y} + b$. Taking this last function to be $\varphi\left(\frac{x}{y}\right)$, from (2.1) one gets

$$z = y\varphi\left(\frac{x}{y}\right) + c \tag{2.5}$$

This is the other primitive of (2.3) we were looking for.[10] In Lagrange terminology ([137], continuation of lect. XX; [138], lect. XX), this is the "general primitive" of (2.3), and it is, indeed, in a quite clear sense, a primitive much more general than (2.1).

The relevant point here is that the function designated by 'φ', as well as the functions that a is supposed to be of x and y, and b of a, are, as Lagrange himself admits ([134], Sect. 95; [140], Sect. I.83), "absolutely arbitrary", in the sense that they are not only susceptible of being displayed by whatsoever expression, but they are even not required to be displayed by any expression at all. All that is required for the argument to proceed is that b be a function of a, a be a function of x and y, and some operational conditions about derivatives functions be met, so as to ensure, for example, that the derivative of by with respect to x is $yb'_a a'_x$. Hence, all that is required for (2.5) to be a solution of (2.3) is that $\varphi\left(\frac{x}{y}\right)$ be a function of $\frac{x}{y}$ and that a function satisfy these operational conditions, which is perfectly independent of its being displayed by any expression.

In order to account for the existence of general primitives of partial differential equations, Lagrange is then forced to deal with arbitrary functions, which are not so

[10] Verification is easy. From (2.5) it follows:

$$z'_x = y\varphi'\left(\frac{x}{y}\right)\frac{1}{y} = \varphi'\left(\frac{x}{y}\right) \qquad \text{and} \qquad z'_y = \varphi\left(\frac{x}{y}\right) - y\varphi'\left(\frac{x}{y}\right)\frac{x}{y^2} = \varphi\left(\frac{x}{y}\right) - \varphi'\left(\frac{x}{y}\right)\frac{x}{y}$$

Replacing in (2.3), one gets, then

$$y\varphi\left(\frac{x}{y}\right) + c + x\varphi'\left(\frac{x}{y}\right) = x\varphi'\left(\frac{x}{y}\right) + y\varphi\left(\frac{x}{y}\right) + c$$

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because they are waiting for a further possible determination through an appropriate expression, but are rather intrinsically indeterminate insofar as they are not expressions, but just quantities that are supposed to depend on other quantities and respect the appropriate operational conditions.

Lagrange cautiously avoids remarking on this. But the question was not ignored at the time. Euler openly tackled it many years earlier [78, 79, 81]. The details of Euler's arguments are not relevant for the present purpose.[11] It is enough to say that in these works, he takes functions to be "quantities somehow determined by some variable" ([81], p. 3). This fits with the definition he provides in the *Institutiones* ([80], p. VI; [84], p. VI):

> Those quantities that depend on others in [...][such a] way that if these are changed, they also undergo a change, are usually said to be functions of these latter [quantities]. This is a quite broad denomination and encompasses in itself all ways in which one quantity can be determined by others. Hence, if 'x' denotes a variable quantity, all quantities that in any way depend on x or are determined by it, are said to be functions of it.

This definition has often been opposed to those offered by Lagrange and by Euler himself in the *Introductio*.[12] It is also mentioned by Hintikka and Sandu in their paper on Frege's notion of function ([129], pp. 296–297) as an early manifestation of the "concept of arbitrary function". Hintikka and Sandu are interested in the question "whether Frege assumed the standard interpretation of higher-order quantifiers or a non-standard one" (*ibid.*, p. 298), i.e. whether, for him, the range of second-order quantifiers is "the entire power set $P(do(M))$ of the relevant domain $do(M)$ of individuals", or "only some designed subset of $P(do(M))$" (*ibid.*, p. 290). For Hintikka and Sandu, "the conception of the standard interpretation [...] is, to all purposes, equivalent with the notion of an arbitrary function or the notion of an arbitrary set" (*ibid.*, p. 298). They argue that "Frege lacked both the idea of arbitrary function and the idea of arbitrary set, and hence in effect opted for a non-standard interpretation" (*ibid.*). The definition of the *Institutiones* is mentioned as evidence that the "idea of an arbitrary function" dates back to long before Frege (*ibid.*, p. 296).

This suggests that in the evolution of the notion of function, two camps opposed each other: on the one side, those that admitted the notion of an arbitrary function, like the Euler of the *Institutiones*, and many others, among whom Hintikka and Sandu mention Dirichlet, Lobachevsky, and Cantor; on the other side, those who rejected or lacked this notion, like the Euler of the *Introductio*, Lagrange—at least for the definition he explicitly provides—and Frege, to whom they also add Weierstrass and Kronecker (*ibid.*, pp. 296–298). At first glance, my claim that the notion of function plays similar roles in Lagrange's and Frege's foundational programs seems to support this account. This is only partially true, however. What follows will explain why.[13]

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[11] On these arguments and the mathematical discussion they were part of, cf.: [195], pp. 237–300; [117], pp. 1–21; [64]; [31], pp. 21–33; and [151], 256–264.

[12] For example in [207].

[13] Hintikka and Sandu's theses have generated a sharp controversy: cf. [34, 62, 124], for example. This largely depended on their arguing that it is "unfortunate that philosophers habitually

My first remark is that the apparent generality of the definition of the *Institutiones* is limited by the notion of quantity it is based on. In the same treatise, Euler argues that "every quantity, by its nature, can increase and decrease up to infinity" ([80], p. IV; [84], p. V). This echoes the classical, Aristotelian conception of quantity (*Metaphysics*, Δ, 13, 1020*a*, 7–14, and *Categories*, part 6), on which d'Alembert focuses by claiming that a quantity is "that which can be increased or decreased" ([2], p. 653). This is a quite vague conception, however. When the definition of the *Institutiones* is related to it, all that one understands from it is that a function is anything that can increase or decrease insofar as this depends on the increasing or decreasing of something else. Now, this idea is not only quite different from the modern one involved in the standard interpretation of higher-order quantifiers. More importantly, it is also a poor basis for any mathematical argument. To explain his notion of arbitrary function—which, in fact, reduces to arguing that a solution of a partial differential equation can involve something such as a discontinuous function—Euler is forced to rely on the representation of functions through curves. Hence, though perhaps more general than that of the *Introductio*, the definition of the *Institutiones* is more imprecise and less effective: it is inappropriate as a starting point for a general theory of functions, as algebraic analysis was intended to be.

This is why Lagrange preferred grounding his theory on another definition. His attempt failed, largely because of shortcomings like that mentioned above. But this failure did not result in the general admission of the definition of the *Institutiones*, but rather fostered the shaping of a new notion. Cauchy's *Cours d'analyse* [42] was the manifesto of the new course.[14]

As well as Lagrange's *Théorie*, Cauchy's treatise presents itself, according to its title, as a treatise of algebraic analysis. But this last term here takes a quite different meaning than in Lagrange's treatise. For Cauchy, algebraic analysis is a preliminary part of analysis (to be followed by the calculus), and analysis is a particular branch of mathematics. It is then essentially distinct both from algebra and geometry, but it is expected to be as rigorous as the latter, which is possible only insofar as it never relies on "arguments drawn from the generality" of the former (*ibid*., p. ii; [32], p. 1).[15]

This is already quite far from Lagrange's conceptions. But a more radical difference depends on the fact that Cauchy does not open his treatise by fixing the notion

(Footnote 13 continued)
go to Frege", since "Frege was far too myopic to be a fruitful source for concepts, idea and problems" ([129], p. 315). I shall not deal with this allegation, and confine myself to giving an account of Frege's views in their historical context.

[14] An essentially different reaction was promoted by a group of British mathematicians including Woodhause, Babbage, and Peacock. Though their conceptions were highly influential in the history of logic, considering them is not relevant to my present purpose.

[15] A similar view had been endorsed by Ampère, almost twenty years earlier, in a memoir presented to the *Institut des Sciences* in 1803 and appearing in 1806 ([3], p. 496): "That which is termed a fact of analysis has always to be reduced to the metaphysical principles of this science if one wants to have a right idea of it. It is evident, indeed, that one has always to find the reasons for all the results obtained through calculation in the attentive examination of the conditions of any question, since the use of algebraic characters can add nothing to the ideas that they represent.".

of a function, but rather by independently explaining the notion of a quantity. Though his explanation[16] is far from perspicuous, his general strategy is clear enough.

Analysis starts by inheriting the notion of a magnitude from an independent source. This notion is taken as primitive in analysis, since analysis neither requires nor is capable of providing any further clarification of it. Analysis no longer deals with magnitudes, directly. It is rather concerned with their measures (which is also an unanalysed notion). These result from two sources. Each magnitude can be compared to another of the same species which is taken as a unit; but also its increase or decrease can be taken into account. In the former case, its measure is a number; in the latter, it is a quantity. Taken as such, numbers are neither positive nor negative, but only greater or smaller than each other. Quantities, instead, are either positive or negative: they are so insofar as they are respectively measures of the increase or the decrease of a magnitude. But, insofar as the increase and decrease of any magnitude can only be estimated by comparison with an appropriate unit, quantities are associated with numbers, namely they are signed numbers, which are not positive or negative numbers, but numbers preceded either by the sign '$+$' or by the sign '$-$'. Hence, any quantity has a numerical value, which is nothing but the number that is got when its sign is omitted.

It is only after having fixed these notions that Cauchy comes to functions. He begins by distinguishing variable from constant quantities: a quantity is variable if it is supposed "to take on successively several values different from each other", while it is constant if it "takes on a fixed and determined value" ([42], p. 4; [32], p. 6). He then introduces functions as follows ([42], p. 19; [32], p. 17):

> When variable quantities are so related to each other that the value of one of them being given, one can infer the values of all the others, one usually conceives these various quantities to be expressed by means of one of them, which therefore is called the 'independent variable'. The other quantities, expressed by means of the independent variable, are those which one terms functions of that variable.

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[16]Cf. [42], pp. 1–2, and [32], pp. 5–6:

> First of all, we shall indicate what idea it seems appropriate to us to attach to the two words 'number' and 'quantity'. We shall always take the denomination of numbers in the sense in which it is used in arithmetic, by making the numbers to arise from the absolute measure of magnitudes [*grandeurs*], and we shall only apply the denomination of quantities to real positive or negative quantities, i.e. to numbers preceded by the signs '$+$' or '$-$'. Furthermore, we shall regard quantities as intended to express an increase or decrease, so that a given magnitude will simply be represented by a number, if one only means to compare it with another magnitude of the same species taken as a unity, and by the same number preceded by the sign '$+$' or the sign '$-$', if one considers it as to be used for increasing or decreasing a fixed magnitude of the same species [*comme devant servir à l'accroissemment ou la diminuition d'une grandeur fixe de la même espèce*]. [...] We shall call: the 'numerical value' of a quantity that number which forms its basis; 'equal quantities' those that have the same sign and the same numerical value; and 'opposite quantities' two quantities with the same numerical value affected by opposite signs.

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Immediately after this, an analogous definition is offered for functions of several variables. Much later ([42], pp. 246–247; [32], p. 163), Cauchy makes clear that these explanations only concern "real functions", to which "imaginary" ones are opposed: these latter are defined as "expressions" of the form '$\phi(x, y, z, \ldots) + \chi(x, y, z, \ldots)\sqrt{-1}$', where '$\phi(x, y, z, \ldots)$' and '$\chi(x, y, z, \ldots)$' designate real functions of $x, y, z, \ldots$.

At first glance, Cauchy's idea of a real function seems close to that of the Euler of the *Introductio*: the adverb 'usually', occurring in his definition, suggests that, for him, real functions are quantities depending on other quantities though they are not necessarily expressed in terms of these latter quantities. But a crucial difference appears when one focuses on the notion of quantity: for Cauchy, a quantity is what we would today term a real number (though real numbers are only informally defined by him, as measures of magnitudes). Hence, in modern parlance, his real functions are functions of real variables. Imaginary functions, instead, are a symbolic generalisation of real ones: they are just "symbolic expressions": combinations of algebraic signs that do not mean anything by themselves ([42], p. 173; [32], p. 117).

This provides the starting point of the so called arithmetisation of analysis. Put briefly, and using modern terminology, this is a development of mathematical analysis based on the idea that functions have to be defined on real and complex numbers. This program differs from Lagrange's in many respects. Two of them are relevant for my purpose. On the one side, the notion of function is no longer mathematically primitive: before introducing it, a (more or less) appropriate notion of real and complex numbers has to be fixed. On the other side, this notion is now confined within a quite narrow disciplinary context, i.e. a particular branch of mathematics. The failure of Lagrange's program resulted, then, in the removal of the notion of function from the basic foundational role that his program had conferred on it.

But something else is also relevant. According to Cauchy's definition, a real function is identified with a real number, namely a variable one, whose variation depends (in any way whatsoever) on the variation of another real number. Though this conception was later refined, several manuals of real analysis continued to base on it. A late example is Czuber's *Vorlesungen über Differential- und Integralrechnung* [45]. Here is how he (*ibid.*, Sect. 3, p. 15) defines real functions:

> If to every value of the real variable x that belongs to its domain [*Bereich*] a definite number y is correlated, then in general y also is defined as a variable, and is said to be a function of the real variable x.

The basic idea is the same as Cauchy's: a real function is a variable real number. This conception is flawed, at least if it is not offered a clear explanation of what it is for a real number to be variable (an explanation which neither Cauchy nor Czuber were able to offer). But it also contains the crucial idea of conceiving functions extensionally, that is, not for the way they realise a connection between appropriate items, but for their connecting certain items to certain other items. In other words, the idea is that of making the identity of a function rest on what it connects rather than on the way it realises the connection. This idea depends on the dissociation of *relata* from relations—which in Lagrange's idea of algebraic quantity are instead kept together. Furthermore, it depends on the admission that the *relata* come before the relation. This is the idea that, through a gradual and difficult evolution, has finally resulted in the modern extensional set-theoretic notion of function: the notion on which that of an arbitrary function considered by Hintikka and Sandu is based.

Mentioning Czuber in this respect is relevant, since Frege takes his definition into account and openly rejects it, in his [100], to which I shall return at the end of Sect. 2.4.4. As we shall see, Frege's objection does not head him to suggest some refinement of the conception of real functions as real numbers, but rather results in his rejection of the very extensional conception of functions. In this way, Frege radically contrasts the more fundamental ground the program of the arithmetisation of analysis was based on, and certainly does not do it by taking a set-theoretic perspective. He rather comes back, in a sense, to Lagrange's attitude. Emphasising this double contrast of Frege's ideas on functions with the arithmetisation of analysis on one side, and with a set-theoretic perspective on another side, is the aim of the next section, where I shall try to show that the views Frege expounds in [100] are perfectly in agreement with the way he deals with functions in the *Grundgesetze*.

2.4 Functions in Frege's *Grundgesetze*

In a recent paper, Tappenden [192] has called it a "myth" that, so far as it is relevant to Frege, nineteenth century mathematics could be reduced to the arithmetisation of analysis, this being conceived as a process "exemplified by Weierstrass", and essentially consisting in "a series of reductions", such as those of derivatives to limits of reals, reals and limits of reals to sets of rationals, rationals to sets of pairs of integers, and integers to sets (*ibid.*, pp. 99–101). But, for Tappenden, denouncing this myth should not result in endorsing the "countermyth" that Frege was "crucially different from Weierstrass and, by extension, from nineteenth-century mathematics generally", in that he was moved by "philosophical desiderata" rather than "mathematical considerations" (*ibid.*, p. 102). According to Tappenden, Frege's views did differ from Weierstrass's, but "this does not reflect a divide between Frege and mathematicians", since "Weierstrass differed from many mathematicians", especially from Riemann, and "Frege was in the Riemannian tradition" (*ibid.*, pp. 106–107).

Doubtless, Frege cannot be enrolled in the process of successive reductions just mentioned (though he was certainly concerned with the rigorisation of analysis: [58]). There are various reasons for this. Among many others, one is relevant for my purpose: Frege's foundational program neither involves the reduction of natural numbers to sets, nor indulges in the conviction that a prior definition of natural, real and complex numbers is required for the notion of function to be clarified. The contrary is true: for Frege, natural and real numbers have to be defined within a formal system conceived as a system of logic, to be set up before any sort of mathematics, and to be expounded by the appeal to a few (non-mathematical) fundamental notions, including that of function.

As far as the notion of function is concerned, Frege's association with the Riemannian tradition is doubtful, instead. In recounting the differences between Riemann's and Weierstrass's "styles", Tappenden argues that, whereas Weierstrass's mathematics is concerned with "explicit given representations of functions", Riemann's requires proving the existence of functions having certain properties "without producing an explicit expression", and is then "committed" to a "wider conception", according to which functions are not "connected to available expressions" ([192], pp. 107 and 121). For Tappenden, Frege's "treatment of function quantification presupposes the most general notion of function, irrespective of available expressions and definitions" (*ibid.*, p. 114). I disagree. Tappenden provides several pieces of evidence showing that both Frege's scientific milieu and his intellectual sympathy were with Riemann's (*ibid.*, pp. 123–130), but he recognises that there is no evidence supporting the claim that the notion of a function "which Frege takes as basic and unreduced" is just the Riemannian one. Tappenden seems to suggest that the best clue for this is merely given by Frege's exposure to the "mathematics around him" (*ibid.*, p. 132). Still, there is a good reason for doubting that Frege's notion coincides with Riemann's: the latter is a mathematical notion; the former cannot be so intended.

Undoubtedly, Frege was aware of most of the mathematical discussions taking place around him, and it is highly plausible that the crucial role he assigned to functions resulted from his "reflection on the function concept in mathematical analysis" ([58], p. 238; [106], vol. 1, p. 129). But it does not follow from this that Frege just imported his own notion of a function from the contemporary mathematical discussion. He could not have been able to appeal to the notion of a function in the exposition of his logical system, if this notion had not been both perfectly independent of any sort of number, and not in need of any possible mathematical proof of existence, more generally, if it had not been a non-mathematical notion. Hence, this notion could have been neither Weierstrass's, nor Riemann's one.

2.4.1 Elucidating the Notion of a Function

But no more could it have been Lagrange's. The main reason for this is not that Lagrange's notion is based on a conflation of syntactical items and their *designata*. It rather pertains to Frege's very conception of a formal system. His own formal system, the *Begriffsschrift*, is usually presented as a system of second-order logic. But it is, in fact, quite different from a formal system in the modern sense. A crucial difference is that the syntax/semantic distinction, as we conceive it today, is lacking: there is nothing like a purely syntactical level of symbols, formulas and rules, and a subsequent level in which an interpretation is provided. The *Begriffsschrift* is, *ipso facto*, a meaningful system. Hence, to introduce it, more than a simple presentation of its language (merely fixing the syntactical behaviour of its elements) is required.

Fixing the meaning of the relevant symbols and formulas and justifying the relevant rules is also needed.

The exposition of the *Begriffsschrift* that occupies part I of the *Grundgesetze* ([97], pp. 5–69), and opens with the passage I quoted at the beginning of this chapter, is just devoted to this latter task. This is what Frege sets forth in a short "*Einleitung*" (*ibid*., pp. 1–4) that precedes it.

Calling on the notion of function is part of this task. Hence, Frege could have not required that understanding this notion depended on taking the *Begriffsschrift* for granted. But, insofar as the *Begriffsschrift* is for him a model, or better a source, for any scientific formalism, neither could he have admitted that understanding this notion depended on taking any previous formalism for granted. The more fundamental difference between Frege's and Lagrange's notions rests on this.

Frege explains that, for his enterprise to succeed, some relevant "notions [*Begriffe*]" have to be "made clear [*scharf gefasst*]" ([97], p. 1). This is especially the case for the notion underlying the use that mathematicians make of the words 'set [*Menge*]', or 'system [*System*]', the latter case being that of Dedekind (*ibid*.). Frege takes some explanations offered by Dedekind and Schröder [49, 178] into critical account, and argues that what is actually meant with this use is the "subordination of a concept under a concept or the falling of an object under a concept" ([97], p. 2; [110], p. 2_1). Similar considerations, he adds, hold for the word 'correlation [*Zuordnung*]', which, in the context of a reduction of arithmetic to logic, would better be replaced with 'relation [*Beziehung*]' ([97], p. 3; [110], p. 3_1). It follows, he says, that at the grounds of his own "construction [*Bau*]" there have to be the logical notions of a concept and a relation (*ibid*.). In other words: founding arithmetic on logic means reducing the mathematical notions of a set and a correspondence to the logical ones of a concept and a relation.[17] This is just the aim of the *Begriffsschrift*. But for Frege, a necessary

[17] In Chap. 1 of this book, Benis Sinaceur argues that Dedekind's logicism, if any, should not be assimilated to Frege's. The previous remarks should be enough to confirm that this was also Frege's conviction. These remarks fit, moreover, with another that Frege already makes in the Preface of the same *Grundgesetze*, also quoted by Benis Sinaceur, in Sect. 1.5.3 of Chap. 1, above ([97], *Vorwort*, p. VIII; [110]; p. VIII_1): "Mr Dedekind too is of the opinion that the theory of numbers is a part of logic; but his essay barely contributes to the confirmation of this opinion since his use of the expressions 'system' 'a thing belongs to a thing' are neither customary in logic nor reducible to something acknowledged as logical". Frege's point is then that the notions of set and set membership are not logical as such, but should rather be reduced to logical ones, which is just what Dedekind does not do. It follows that, for Frege, Dedekind's view that "the unique and therefore absolutely indispensable foundation […][for] the whole science of numbers" is "the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing", and that without this ability "no thinking is possible" ([49], p. VIII; [53], p. 14), do not coincide with the idea that "arithmetic belongs to logic", as Stein maintains, by taking this last claim to be the same as the claim that "the principles of arithmetic are essentially involved in all thought" ([185], p. 246). The ability to which Dedekind refers is, indeed, a basic cognitive capacity, which, for Frege, does not pertains to logic at all.

condition for articulating this reduction is making these basic logical notions clear, which should result in conveying a logical content before the reduction begins. This is Frege's main point: insofar as logic is to come first, it cannot result from a further reduction to something which is prior to it; still, for it to begin, a content is to be conveyed. What is needed is not a reduction; still it is something suitable for conveying a content. This is an "exposition" ([97], *Einleitung*, pp. 3–4; [110], pp. pp. 3_1–4_1)[18]:

> Yet even after the concepts are sharply circumscribed, it would be hard, almost impossible, to satisfy the demands necessarily imposed here on the conduct of proof without special auxiliary means. Such an auxiliary means is my *Begriffsschrift*, whose exposition [*Darlegung*] will be my first task. It will not always be possible to give a proper definition of everything, simply because our ambition has to be to go back to what is logically simple, and this as such allows of no proper definition. In such a case, I have to make do with gesturing at what I mean.

What Frege means here by 'exposition [*Darlegung*]' is close to what elsewhere (for example, in: the same *Grundgesetze*, Sect. I.1, footnote, and I.34–35; [95], p. 193; [101], pp. 301–302 and 305–306; [106], vol. 1, p. 232, and vol. 2, p. 63) he means by 'elucidation [*Erläuterung*]'. The crucial role of elucidation in "Frege's project" has been recently emphasised by Weiner ([201], Chap. 6; [202], especially pp. 58–61). This is neither a logical nor a scientific procedure. Still, it is a necessary "propaedeutic" ([101], p. 301; [109], p. 300) for logic, and, then, for any science, including mathematics. Its task is communicating basic contents that, insofar as they are purported to be part of logic, and even provide grounds for it, cannot be communicated by logical means, that is, through indefectible definitions (that for Frege could only be explicit ones). In some cases, these contents are reducible, and elucidation can be plain and unequivocal (provided of course that other contents, also communicated through elucidation, are grasped), and can even result in some sort of explicit (though informal) definitions. That's the case with the notions of a concept and a relation, since Frege takes both concepts and relations to be functions, respectively of one and several arguments, whose values are truth-values ([97], Sect. I.3–4). In some other cases, these contents are irreducible, or ineffable, with the effect that their elucidation is successful only if one can count "on a little goodwill and cooperative

[18] The same point is also made in "Über Begriff unf Gegenstand", concerning concepts: [95], p. 193; [104], pp. 42–43.

understanding, even guessing" ([101], p. 301; [109], p. 301).[19] That's the case with the notion of a function.[20]

Functions are opposed to objects, for Frege. Thus, they cannot be expressions. And, for the very same reason, they cannot be quantities, numbers, or sets. Concerning quantities and numbers, this is also a consequence of the requirement that the notion of a function come before mathematics. Concerning sets, things are more entangled, since it is far from certain that Frege considered the notion of a set to be mathematical (though his considerations about the use of the words 'set' and 'system' suggest he did).[21] In any case, the requirements that the notion of a function be logically primitive, and that its elucidation belong then to a propaedeutic for logic are enough for excluding the possibility of understanding functions as sets of pairs.

But this is not all. For Frege, all that which is not an object is a function, to the effect that there is no room for specifying which sorts of entities functions are. Indeed,

[19] Cf. also [100], p. 665; [104], p. 115: "The peculiarity of functional signs, which we here called 'unsaturatedness', naturally has something answering to it in the functions themselves. They too may be called 'unsaturated', and in this way we mark them out as fundamentally different from numbers. Of course this is no definition; but likewise none is here possible. I must confine myself to hinting at what I have in mind by means of a metaphorical expression [*bildlichen Ausdruck*], and here I rely on the charitable discernment of the reader." According to several scholars (cf., for instance: [44, 65]), the view that elucidation can convey ineffable content, and that this is an essential task for philosophy is Frege's, and it manifests an important aspect of Frege's influence on Wittgenstein (this view is often said to go back to [110], though Geach does not explicitly mention elucidation and limits himself to arguing that "Frege already held, and his philosophy of logic would oblige him to hold, that there are logical category-distinctions which will clearly show themselves in a well-constrcuted language, but which cannot properly be asserted in language": *ibid.*, p. 55). Usually, these scholars admit that Frege calls on different species of elucidation, and take the elucidation of "what is logically primitive" ([44], p. 182) to be the species in which ineffable content is conveyed, the prototypical example being the elucidation of the concept/object distinction. Despite this, it seems to me that if the notions of function and truth-value are taken for granted, the claim that concepts are first-level functions of one argument whose values are truth-values is fully unproblematic. The prototypical example of elucidation's conveying ineffable content is rather that of the function/object distinction. The case of the elucidation of the notions of a concept and a relation also shows that, if the exposition of the *Begriffsschrift* is assimilated to elucidation, then elucidation is opposed to definition only if this last term is taken in a quite strict technical sense (which is proper to Frege), according to which it only refers to the explicit formal definitions admitted within the *Begriffsschrift*. In a broader sense, definitions, even explicit ones, can enter in an elucidation.

[20] The question whether the exposition that occupies part I of the *Grundgsetze* has or not a semantic extent—namely whether one can take it or not to provide "semantic justifications of axioms and rules" ([122], p. 365, where Heck is arguing for the affirmative, in contrast with what is argued by Ricketts [168])—is not fully relevant here. What seems to me relevant is that this semantic extent, if any, is quite different from that which would be involved in any discussion about the interpretation of a formal system, and, overall, that this exposition not only aims at showing that "the rules of the system are truth-preserving and that the axioms are true" ([122], p. 365), but also includes the elucidation of fundamental notions like those of an object, a function, a truth-value, a concept, and a relation (on this claim, cf. also [169], Sect. 6, esp. pp. 191–193). This elucidation is "required if one is to master the notation of [...][Frege's] symbolism and properly understand its significance" ([44], p. 181), namely it is "necessary for explaining how Frege's notation [i.e. his *Begriffsschrift*] is to be used in the expression of thoughts" ([201], pp. 251).

[21] Cf. footnote (17).

provided that they are not objects, to wonder which sorts of entities they are, would be the same as wondering which sort of functions they are… Hence, elucidating the notion of a function cannot consist in telling us what functions are. All that Frege can do towards elucidating this notion is to try to account for the way functions work in already given languages (the natural one, and its codified versions used in mathematics), and expounding how they are intended to work in the *Begriffsschrift*.

In my view, this is connected with a point that the mere assertion that functions are not objects only partially accounts for: according to Frege, appealing to functions is indispensable in order to fix the way his formal language is to run, but functions are not as such actual components of this language. More generally, functions manifest themselves in our referring to objects—either concrete or abstract—and making statements about them, but they are not as such actual inhabitants of some world of *concreta* and *abstracta*. Briefly: Frege's formal language, as well as ordinary ones, display functions, but there are no functions as such. As he writes to A. Marty on August 29th 1882 ([106], vol. 2, p. 164; [108], p. 101): "A concept is unsaturated, in that it requires something that falls under it; hence it cannot subsist [*bestehen*] by itself ".[22] *Mutatis mutandis*, the same holds for functions in general.

This is not at all to deny Frege's antipsychologism and objectivism about functions (and concepts or relations). It is merely to argue that it is not part of these theses that functions actually exist as such in some realm of *abstracta*. What is part of these theses is that functions pertain to an objective account of the way language actually works, rather than the way we subjectively think, with the effect that one must appeal to them in order to account for the logical structure of language and thought. As pointed out by Picardi ([159], p. 53): functions are to be conceived as "objective pattern[s] that we discern in the world", rather than as "separate ingredient[s] of it".

2.4.2 How (First-Level) Functions Work in the **Begriffsschrift**

To clarify all this, let me briefly sketch the role that functions play in the *Begriffsschrift*.

In Sect. I.5 of *Grundgesetze*, Frege establishes that statements (*Satze*) are formed in this system by letting the special sign '$\vdash$' precede appropriate terms. These are either names of a truth-value— i.e. either of the True or of the False—or appropriate formulas. These latter formulas involve Latin letters and are suitable for being transformed into a name of a truth-value through appropriate replacements of these letters. I shall better specify this condition pretty soon. For the time being it is enough to say that, though he does not say it explicitly, Frege implies that a statement in which the sign '$\vdash$' precedes a name of a truth-value asserts that what makes up this name is

[22]Cf. also [99], p. 34; [109], p. 282: "It is clear that we cannot put down [*hinstellen*] a concept as independent, like an object; rather it can occur only in a connection. One may say that it can be distinguished within, but it cannot be separated from the context in which it occurs".

a name of the True obtains,[23] whereas a statement in which the sign '$\vdash$' precedes a formula involving Latin letters asserts that it obtains what makes up this formula transforms into a name of the True under any licensed replacement of these letters. The former case is fundamental; the latter reduces to it through the appropriate stipulations on the replacement of Latin letters. Let us then begin with the former.

For Frege, the True and the False are two peculiar objects whose existence is taken for granted. Hence, a name of a truth-value is a name of an object, or a "proper name [*Eigenname*]" as he says ([97], Sect. I.3; [110], p. 7_1). But not any proper name is suitable for yielding a statement of the *Begriffsschrift*, if preceded by the sign '$\vdash$', and it is no more so for any name of a truth-value. For a proper name to be suitable for this, it has to belong to the language of the *Begriffsschrift* (or to an appropriate extension of it), and to be appropriately formed within this language. The former requirement is obvious and already sufficient for excluding names like 'the True' or 'the False', which do not belong to this language. The latter is what matters here. It could be so rephrased: a proper name of the language of the *Begriffsschrift* is suitable for yielding a statement of this system (if preceded by the sign '$\vdash$') if it is formed so as to be the name of the value of a concept or relation, i.e. of a function whose values are truth-values. Hence, such a name not only refers to a truth-value, it also refers to it in a certain way, which depends on the nature of the relevant function (and it is just because of this nature that this name is possibly warranted to be a name of the True, as it happens if the corresponding statement is a theorem).

But what does it mean, in the context of the *Begriffsschrift*, that a function has a certain nature? Though Frege is never explicit on this matter, his exposition leaves no doubt: it means that this function is either one of the few primitive ones admitted in this system, or is generated in a certain way by reiteratively composing these primitive functions[24], and, possibly, by relying on some auxiliary explicit definitions.

This is still not clear enough, since, provided that functions are not actual components of the language of the *Begriffsschrift*, the problem of understanding how something which is not such an actual component can be either a primitive item of this language or be composed by reiteratively composing primitive items of it is still open. Part of the answer is that the foregoing condition has to be understood as follows: in the context of the *Begriffsschrift*, a function has a certain nature if the names of its values are either names of values of a certain primitive function, or are generated in a certain way by reiteratively composing these latter names and, possibly, by relying on some other proper names introduced by explicit definition.

But this is not the end of the story, yet. It is still necessary to explain, what does it mean that a proper name is a name of a value of a certain function. For my present purpose, I can restrict the answer to primitive functions (to pass to composed ones, it would be enough to specify which rules of composition are licensed, which is a question that we can leave aside, here).

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[23] For example, in the same Sect. I.5, Frege argues that '$2^2 = 4$' is a name of the True, and that the statement '$\vdash 2^2 = 4$' asserts that the square of 2 is 4.

[24] Cf. footnote (4) of the Introduction.

These functions are introduced through appropriate informal but explicit definitions.[25] Four of them (two concepts and two relations) are introduced in Sects. I.5–7, and I.12. They are the horizontal, the negation, the identity, and the implication. These are first-level functions: functions whose arguments are objects. For the time being, I primarily restrict the discussion to these functions. I shall explicitly consider higher-level functions in Sect. 2.4.4 (especially pp. 87–89). Up to that point, I shall use the terms 'function', 'concept' and 'relation' to primarily speak of first-level functions. In order to generalise some of the things I shall say about them to functions of any level, some changes would be necessary. But what I shall later say of higher-level functions is intended to show that these changes would not effect what is essential for my purposes.

Consider then, as examples, the four aforementioned functions. They are defined through the following stipulation schemas:

$$\begin{array}{llll} \text{—}\,\Delta \text{ is } \begin{cases} \mathsf{T} \text{ if } \Delta \text{ is } \mathsf{T} \\ \mathsf{F} \text{ if } \Delta \text{ is not } \mathsf{T} \end{cases} & & \text{—}\!\!\!\top\!\!\text{—}\,\Delta \text{ is } \begin{cases} \mathsf{F} \text{ if } \Delta \text{ is } \mathsf{T} \\ \mathsf{T} \text{ if } \Delta \text{ is not } \mathsf{T} \end{cases} & \\ \Gamma = \Delta \text{ is } \begin{cases} \mathsf{T} \text{ if } \Gamma \text{ is } \Delta \\ \mathsf{F} \text{ if } \Gamma \text{ is not } \Delta \end{cases} & \begin{array}{l} \text{—}\!\!\!\top\!\!\text{—}\,\Delta \\ \quad\llcorner\,\Gamma \end{array} \text{ is } & \begin{cases} \mathsf{F} \text{ if } \Gamma \text{ is } \mathsf{T} \text{ and } \Delta \text{ is not } \mathsf{T} \\ \mathsf{T} \text{ if } \Gamma \text{ is not } \mathsf{T} \text{ or } \Delta \text{ is } \mathsf{T} \end{cases}, & \end{array} \tag{2.6}$$

where 'T' and 'F' refer to the True and the False, respectively, and 'Γ' and 'Δ' are schematic letters for objects.

These definitions involve nothing but schematic names of objects, among which '$\text{—}\Delta$', '$\text{—}\!\!\!\top\!\!\text{—}\,\Delta$', '$\Gamma = \Delta$', and '$\begin{array}{l} \text{—}\!\!\!\top\!\!\text{—}\,\Delta \\ \quad\llcorner\,\Gamma \end{array}$' are schematic names of values of the relevant functions. It is then clear that these functions are defined by fixing the reference of the names of all their possible values.

Of course, this definition belongs to the language of the exposition of the *Begriffsschrift*. Indeed, though Frege largely uses Greek capital letters, like 'Γ' and 'Δ' in such an exposition, they are not part of the language of *Begriffsschrift* itself, and this is then neither the case of the schematic names involving them. Within the *Begriffsschrift*, Greek capital letters are replaced either by names of particular objects or by Latin letters. These last letters are used to "express generality" ([97], Sect. I.17; [110], p. 31_1). Some Frege scholars (for example [115], p. 67) take them to be free variables and suggest understanding the formulas involving them as abbreviations of universally quantified statements. It seems to me more faithful to Frege's views to understand them as special schematic letters, differing from the Greek capital ones for being used within the *Begriffsschrift*. Insofar as, in this system, any formula

[25] These definitions are informal insofar as they belong to the exposition of the *Begriffsschrift*, rather than to the *Begriffsschrift*, itself. Hence, they reduce to stipulations stated in the natural language, as clearly as possible (under the supposition that what is involved in them has been previously elucidated). This is, thus, another example of the fact that, if the exposition of the *Begriffsschrift* is assimilated to elucidation, the latter is not necessarily opposed to definition in the broad sense: cf. see the footnote (19).

occurs within a statement or an explicit definition—which can be taken to be a sort of statement—within the *Begriffsschrift,* Latin letters only enter into statements. This makes it possible to fix their use by stipulating that a statement of the *Begriffsschrift* in which they occur asserts that things are such that the schematic proper names resulting from this same statement by omitting the sign '$\vdash$', and replacing each Latin letter with a Greek capital one, is a schematic name of the True. For example, the statement

$$\text{`}\vdash\!\!\!\begin{array}{l} a \\ \llcorner\, a \end{array}\text{'}, \tag{2.7}$$

asserts that things are such that '$—\!\!\begin{array}{l} \Gamma \\ \llcorner\, \Gamma \end{array}$' is a schematic name of the True.[26] This does not entail that the formula '$—\!\!\begin{array}{l} a \\ \llcorner\, a \end{array}$' is, in turn, a name of the True. This is simply because, taken alone, it is not a well-formed formula of the *Begriffsschrift*, where it can only occur within a statement. This is the same for any formula involving Latin letters.[27]

This should be enough to make clear how functions are supposed to enter into statements in the *Begriffsschrift*. But this is not by far the end of the story, since the exposition of this system —although not this very system—also involves "names of functions [*Functionsnamen*]" ([97], Sect. I.2; [110] p. 6_1), or *f*-names, as I shall say from now on. On the one hand, this is natural, since it is easy to imagine a situation in which, by speaking about the *Begriffsschrift*, one has to mention some particular functions, as I have just done myself using the terms 'horizontal', 'negation', 'identity', and 'implication'. On the other hand, this is puzzling, since functions are not objects, and it is then difficult to understand how they can have names. The puzzle has two aspects, at least: a notational and a substantial one.

As far as only the former is taken into account, Frege's solution merely depends on the introduction of a special sort of letter, whose purpose is just that of entering into *f*-names. These are Greek small letters, like 'ξ' and 'ζ'. By replacing 'Γ' and 'Δ' with them in the left hand sides of stipulations (2.6), one gets the following names of the relevant functions:

$$—\,\xi \quad ; \quad —\!\!\top\!\!—\,\xi \quad ; \quad \xi = \zeta \quad ; \quad —\!\!\begin{array}{l} \xi. \\ \llcorner\, \zeta \end{array} \tag{2.8}$$

Like the Greek capital letters 'Γ' and 'Δ', neither these names nor the Greek small letters 'ξ' and 'ζ' belong to the language of the *Begriffsschrift*, but only to

[26] In fact, appropriate conventions relative to the "scope of the generality" have to be also made ([97], Sect. I.8 and I.17; [110], pp. 11_1–12_1, 31_1). Here, I cannot enter into this matter, and merely observe that Frege's use of Latin letters is such that generality cannot be expressed in the *Begriffsschrift* only through them: universal quantifiers are also necessary.

[27] Frege emphasises this fact by stipulating that a Latin letter for objects "indicates [*andeute*]" an object rather than refers to it ([97], Sect. I.17; [110], p. 31_1).

that of its exposition. But, unlike Greek capital letters, Greek small ones are not schematic, and are neither variables nor constants. They are merely used to hold places open for being occupied, both in the language of the *Begriffsschrift* and in that of its exposition, by other appropriate letters, so as to get either names of values of the relevant functions, or formulas suitable for entering into statements. Consider implication: the former case obtains if 'ξ' and 'ζ' are replaced by 'Δ ' and 'Γ' or by names of determined objects like '2' and '3', so as to get the schematic names '$\begin{array}{l}\overline{\;\;\top\;\;}\;\Delta\\ \quad\llcorner\;\Gamma\end{array}$' or '$\begin{array}{l}\overline{\;\;\top\;\;}\;2\\ \quad\llcorner\;3\end{array}$' (which is a name of the Truth, since 3 is not $\top$); the latter case obtains if 'ξ' and 'ζ' are replaced by 'b' and 'a', so as to get the formula '$\begin{array}{l}\overline{\;\;\top\;\;}\;b\\ \quad\llcorner\;a\end{array}$' suitable for entering into the statement '$\begin{array}{l}\vdash\!\overline{\;\top\;}\;b\\ \quad\llcorner\;a\end{array}$'. One could then say that f-names are tools to be used in the *Begriffsschrift* for forming proper names and statements, or for analysing them.

This account of the role of functions in the *Begriffsschrift* and in its exposition could be completed in many respects. But from the little I have said, it should be clear that for functions to manifest themselves in the *Begriffsschrift*, there is no need for them to be actual components of its language. Though things are less clear for the language of the exposition of this system, because of the presence of f-names, there is no doubt that, for Frege, these names are tools for forming proper names and statements, or for analysing them. In my understanding, this is just what he means with his well known metaphor about the unsaturated nature of functions.

The point is made, for example, at the very beginning of part I of the *Grundgesetze*, with respect to the example of the numerical function $\left(2+3x^2\right)x$ ([97], Sect. I.1; [110], pp. 5_1–6_1). Frege claims that the "essence [*Wesen*]" of this function both "reveals itself [...] in the connection [*Zusamengehörigkeit*] it bestows between the numbers whose signs we put for 'x' and the numbers that then result as the reference of the expression" resulting from this replacement, and "lies [...] in the part of the expression that is there besides the 'x'". Then he adds that "the expression of a function is in need of completion, unsaturated" and that 'x' (which, according to him, should be used in mathematics like 'ξ' is used in the *Begriffsschrift*) is there "to hold open places for a numeral", and then to "to make know the particular mode of need for completion that constitutes the peculiar essence" of the function. Despite his using the term 'essence', Frege says nothing here about what he considers functions to be. He only says something about the way the corresponding expressions are intended to work.

2.4.3 (First-level) Functions and Names of Functions

This cannot be all, however, since the substantial aspect of the puzzle about f-names remains still unsettled. Do these names refer to something? And, what does it mean that two f-names are names of the same function or of distinct functions?

These questions concern particular aspects of a more general problem. For Frege, identity only applies to objects. Hence, strictly speaking, no identity condition for functions is conceivable. Does this mean that functions can meet some other sort of sameness conditions, or that there are no such conditions at all[28]?

The matter is connected to the paradox of the concept ⌜horse⌝: in "Über Begriff und Gegenstand" Frege famously holds that "the three words 'the concept ⌜horse⌝' do designate [*bezeichnen*] an object, and, on account of that, they do not designate a concept" ([95], p.195; [104], p. 45). One could think that this merely depends on the awkwardness of natural language, and that this is just what Frege implies by saying that "it is impossible to ignore that there is an unavoidable linguistic hardship [*unvermeidbare sprachliche Härte*] if we claim that the concept ⌜horse⌝ is not a concept" ([95], p. 196; [104], p. 46). Still, this hardship is unavoidable for him, which suggests that he takes the inconvenience of natural language to be a symptom of a deeper problem.

This is confirmed by his raising the problem also in the *Grundgesetze* (in a footnote to Sect. I.4, [97], p. 8; [110], p. 8_1):

> There is a difficulty [...] which can easily obscure the true state of affairs and thereby arouse suspicion concerning the correctness of my conception. If we compare the expression 'the truth-value of Δ's falling under the concept $\Phi(\xi)$' with '$\Phi(\Delta)$' we see that '$\Phi()$' really corresponds to 'the truth-value of ()'s falling under the concept $\Phi(\xi)$', and not to 'the concept $\Phi(\xi)$'. So the latter words do not really designate a concept (in our sense), even though the linguistic form makes it look as if they do. On the inescapable situation [*Zwangslage*] in which language here finds itself, cf. my essay "Über Begriff und Gegenstand".

The relevant language here is that of the exposition of the *Begriffsschrift*. The "inescapable situation" or "unavoidable hardship" in which it finds itself is then a symptom of a problem relative to the basic notions of this system. In this language, 'Φ' works as a schematic letter for functions. Hence '$\Phi(\Delta)$' and 'the truth-value of

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[28]That identity only applies to objects is a point that Frege makes on many occasions; he often argues as well that a "corresponding relation" applies to concepts or functions. But he does not use a fixed compact vocabulary for this purpose. In his review of Husserl's *Philosophie der Arithmetik* ([98], p. 320; [109], p. 200), he argues that "coincidence [*Zusammenfallen*] in extension is a necessary and sufficient condition for the occurrence between concepts of the relation that corresponds to equality [*Gleichheit*] between objects" (I shall come back later to this claim, at p. 83), then remarks: "it should be noted in this connection that I'm using the word 'equal [*gleich*]' without further addition in the sense of 'not different [*nich verschieden*]', 'coinciding [*zusamenfallend*]', 'identical [*identisch*]'". In "Ausführungen über Sinn und Bedeutung" ([106], vol. 1, pp. 132; [107], p. 122), he also argues that "the word 'the same [*derselbe*]' used to designate a relation between objects cannot properly be used to designate the corresponding relation between concepts". Hence, speaking of sameness conditions for functions is not faithful to Frege's parlance. Still, I use this expression for short, to speak of the conditions under which a certain function is this very function rather than some other one.

Δ's falling under the concept $\Phi(\xi)$' are proper names that refer to the same object (either the True or the False). Frege's point is then the following: insofar as the former of these proper names is formed by filling a blank in '$\Phi()$', the role of 'the concept $\Phi(\xi)$' in the latter cannot but be that of contributing to form this name, rather than that of designating a concept. But, then, what does 'the concept $\Phi(\xi)$' mean? Or, more generally, what is one speaking about by saying something of the concept $\Phi(\xi)$, rather than of some other concept?

To better appreciate the nature of the problem, consider another quandary, only apparently related to it: from the supposition that any function has a value-range, it follows that the concept ⌜horse⌝ has an extension; but, if 'the concept ⌜horse⌝' refers to an object, the statement 'the concept ⌜horse⌝ has an extension' cannot be true. To solve this quandary, it is enough to pass to the language of the *Begriffsschrift*. Let '$Hrs(\xi)$' be a name of the concept ⌜horse⌝ in an appropriate extension of this language. The statement

$$`\vdash\!\!\!\top\!\!\smile^{\mathfrak{a}}\!\!\top\; \mathfrak{a} = \grave{\varepsilon}Hrs(\varepsilon)'$$

is then a rendering of 'the concept ⌜horse⌝ has an extension', and it is an immediate consequence of

$$`\vdash\!\!\smile^{\mathfrak{f}}\!\!\top\!\!\smile^{\mathfrak{a}}\!\!\top\; \mathfrak{a} = \grave{\varepsilon}\mathfrak{f}(\varepsilon)'$$

which is a rendering of 'any function has a value-range'.

The problem that Frege tackles in "Über Begriff und Gegenstand" is essentially different, since it is not solvable by passing to the language of the *Begriffsschrift*. It is not about the way some statements have to be appropriately formulated: it is rather about the way functions and f-names have to be understood.

Possibly, a clearer way to state this is the following. Consider the phrase 'the function $\Phi(\xi)$', or also the mere f-name '$\Phi(\xi)$', by supposing that it is just used for naming a certain function, and replace in them 'ξ' with 'Δ', so as to get 'the function $\Phi(\Delta)$' and '$\Phi(\Delta)$'. The former expression is misguided. The latter is not, but, clearly, it is no more suitable for naming the relevant function. It follows that both in 'the function $\Phi(\xi)$' and in '$\Phi(\xi)$'—supposing that this last name is just used for naming a certain function—'ξ' is not used to hold a place open.[29] Hence, in spite of being used to name functions, these expressions are not unsaturated, and are then unsuitable for this purpose.

[29]In order to show that the paradox does not depend on the use of expressions like 'the concept _', Wright has stated it as follows ([206], pp. 74–77; for clarity, I adapt his argument to my setting; on this matter, cf. also [68], pp. 212 *seq.*): (i) the expression 'That which is named by '$\Phi(\xi)$' ' is a singular term; (ii) hence, its reference, if any, is an object; (iii) the reference of 'That which is named by '$\Phi(\xi)$' ' is that which is named by '$\Phi(\xi)$'; (iv) hence, that which is named by '$\Phi(\xi)$' is an object. It follows that the problem cannot be solved by merely jettisoning expressions like 'the concept _'.

The solution that Frege offers in "Über Begriff und Gegenstand" matches up with the nature of the problem, since it does not merely consist in suggesting some linguistic tricks. It goes as follows ([95], p. 197; [104], pp. 46–47):

> In logical enquiries one often needs to assert [*auszusagen*] something about a concept, and to shape it in the usual form for it, namely to put the content of the assertion into the grammatical predicate. Consequently, one would expect that the reference of the grammatical subject would be the concept; but, because of its predicative nature, this cannot play this part; it must first be converted into an object, or, speaking more precisely, represented [*vertreten*] by an object, which we designate by the prefix 'the concept', as in 'the concept ⌜man⌝ is not empty'.

The problem with this solution is that it is begging the question, at least partially. As the same point could and should also be made about functions in general, it requires that for each function whose name supplies the grammatical subject of an assertion about itself, there is an object "representing" this same function, to which this name refers, in the context of this assertion. But, for this to provide an effective solution to the problem, one should also require that the truth-conditions of this assertion depend on the relevant function, i.e. that the object representing this function reflects what makes it a certain particular function. And this requires, in turn, that appropriate conditions for singling out this function be provided.

Frege acknowledges that the objects representing functions should be of "a quite special kind" ([95], p. 201; [104]; p. 50). But he is silent not only on their very nature, but also on the way they might reflect the relevant features of the functions they represent, and on the sameness conditions of these functions.

It is quite tempting to take these objects to be the value-ranges of the corresponding functions, and even to argue that this is what Frege himself implies when he claims to have never "identified concept and extension of concept" and adds that he "merely expressed [...][the] view that in the expression 'the number that applies to the concept F is the extension of the concept ⌜like-numbered to the concept F⌝, the words 'extension of the concept' could be replaced by 'concept'" ([95], p. 199; [104]; p. 48). But there are many reasons for resisting this temptation.[30]

Let me advance two of them, both of which depend on taking the relevant problem to be not merely that of providing a reference for 'the function $\Phi(\xi)$' or '$\Phi(\xi)$' in the context of an assertion about a certain function, but rather that of explaining what makes this assertion hold of this very function rather than of some other one. On this understanding, admitting that 'the function $\Phi(\xi)$' or '$\Phi(\xi)$' refer, in the context of this assertion, to the value-range of $\Phi(\xi)$ results in admitting both that, with respect to this context, $\Phi(\xi)$ is to be taken to be the same function as $\Psi(\xi)$ if and only if the value-range of $\Phi(\xi)$ is the same of that of the function $\Psi(\xi)$, and that the truth conditions of this assertion just depend on the value-range of the function $\Phi(\xi)$.

The first reason is that, if this were so, many distinctions and assertions that one would plausibly like to make would collapse and have quite odd truth-conditions. For example, one should conclude that, with respect to the context of an assertion about the function $—\xi$, this last function is to be taken to be the same function as

[30] Some of these reasons have been offered in [174, 175]. For a critical discussion of them, cf. [170].

$\xi = (\xi = \xi)$, and that the assertions '$\xi = (\xi = \xi)$ is an elementary function of the *Begriffsschrift*' and 'the function $\xi = (\xi = \xi)$ is called 'horizontal' and enters into any statement of the *Begriffsschrift*' are true insofar as '$\text{—}\,\xi$ is an elementary function of the *Begriffsschrift*' and 'the function $\text{—}\,\xi$ is called 'horizontal' and enters into any statement of the *Begriffsschrift*' are true.

These conclusions are not only odd. They also seem to go against Frege's claims. For example, in Sect. I.10 of *Grundgesetze*, he undertakes to offer a "more precise determination of what the value-range of a function is supposed to be" ([97], *Inhaltsverzeichniss*, p. XXVII; [110], p. XVII_1). To this purpose, he considers the three functions introduced in the previous sections, namely $\text{—}\,\xi$, $\text{—}\!\!\!\top\!\!\!\text{—}\,\xi$ and $\xi = \zeta$, and remarks that "we can reduce [*zurückführen*] the function $\text{—}\,\xi$ to the function $\xi = \zeta$", since "the function $\xi = (\xi = \xi)$ has the same value as the function $\text{—}\,\xi$ for every argument" ([97], Sect. I.10; [110]: p. 16_1), which seems to imply that, with respect to the context of these assertions, he takes the functions $\text{—}\,\xi$ and $\xi = (\xi = \xi)$ to be two distinct functions with the same value-range.

The second reason is as follows.[31] Let '$\Phi(\xi)$' and '$\Psi(\xi)$' be two (distinct) f-names. To say that the value-range of $\Phi(\xi)$ is the same as the value-range of $\Psi(\xi)$ means, for Frege, that $\Phi(\Delta)$ is the same object as $\Psi(\Delta)$, whatever the object Δ might be, as Basic Law V prescribes.[32] Hence, admitting that $\Phi(\xi)$ is the same function as $\Psi(\xi)$ if and only if the value-range of $\Phi(\xi)$ is the same as that of $\Psi(\xi)$ results in admitting that $\Phi(\xi)$ is the same function as $\Psi(\xi)$ if and only if $\Phi(\Delta)$ is the same object as $\Psi(\Delta)$, whatever the object Δ might be. But what does it mean that $\Phi(\Delta)$ is the same object as $\Psi(\Delta)$, whatever the object Δ might be? Insofar as Frege has no way to understand the totality of values of a function otherwise than as the value-range of this function, and has no other identity condition for value-ranges of functions than that stated by Basic Law V, according to him this cannot but mean that the proper names '$\Phi(\Delta)$' and '$\Psi(\Delta)$' are identified as the names of the values of two functions $\Phi(\xi)$ and $\Psi(\xi)$ for Δ as argument, and that these functions are associated to appropriate rules, procedures or capabilities which, besides being apt to identify the names of their values, are also apt to warrant that, whatever the object Δ might be, the reference of the proper name '$\Phi(\Delta)$', identified as the name of a value of the function $\Phi(\xi)$, cannot but be the same as the reference of the proper name '$\Psi(\Delta)$' identified as the name of a value of the function $\Psi(\xi)$. It would follow that admitting that 'the function $\Phi(\xi)$' or '$\Phi(\xi)$' refer, in the context of an assertion about the function $\Phi(\xi)$, to the value-range of this function would result in admitting that, with respect to this context, $\Phi(\xi)$ is to be taken to be the same function as $\Psi(\xi)$ if and only if these functions are associated to rules, procedures or capabilities that

[31] I develop here a remark of Hintikka and Sandu ([129], p. 299: for Frege, "the extension of a concept can only be apprehended by our logical faculties starting out from the concept".

[32] For simplicity, I only consider here first-level functions with one argument. It is easy to generalise Basic Law V to first-level functions with several arguments. But, if functions of higher-levels are considered, it is not perfectly clear what it would mean, for Frege, that these functions have the same or different value-ranges (on this matter, cf. [174], p. 32), and it would then be hard to allege that, in order to provide sameness conditions for these functions, it would be enough to stipulate that these conditions reduce to the identity conditions of the value-ranges of these functions.

provide such a warrant. But, if it is admitted that functions are associated to such rules, procedures or capabilities, it seems much more natural to maintain that, with respect to the context of an assertion about the function $\Phi(\xi)$, what enforces that this function be taken to be the same as $\Psi(\xi)$ directly pertains to these very rules, procedures or capabilities, without appealing to the value-ranges of these functions.

This looks like a *reductio ad absurdum* of the identification of Frege's objects of a quite special kind with value-ranges of functions. But what about the view that, with respect to the context of an assertion about the function $\Phi(\xi)$, what arranges matters so that this function is to be taken to be the same as $\Psi(\xi)$ directly pertains to the appropriate rules, procedures or capabilities associated to these functions? Answering this question requires taking other elements into account.

The passage of "Über Begriff und Gegenstand" quoted above is not the only one where Frege implies, or even openly claims, that f-names—or, more specifically, concept-words [*Begriffsworten*]—have both sense and reference. He does it, for example, in a letter to Husserl of May, 24th 1891 ([106], vol. 2, pp. 94–98), in "Ausführungen über Sinn und Bedeutung", probably written between 1892 and 1895 ([106], vol. 1, pp. 128–136), and in "Einleitung in die Logik", of August 1906 ([106], vol. 1, p. 208–212; [107], pp. 191–196). In all these cases, he also argues that the reference of a concept-word is the concept itself, and, in the third of these texts, he goes as far as to imply that a function or concept is just the reference of an f-name or a concept-word, respectively.

In this last case, his argument depends on the principle of compositionality, and goes as follows ([106], vol. 1, pp. 209–212; [107], p. 193 and 195). If we say 'Jupiter is larger than Mars', we are saying that the references of 'Jupiter' and 'Mars' stand to one other in a certain relation, and we do this through the words 'is larger than'. Insofar as this relation holds between references of proper names, it "belongs to the realm of references". Hence, one has to admit that also the phrase 'is larger than Mars' is "endowed by reference [*bedeutungsvoll*]". So, if a statement is split up into a proper name and the remainder, then the latter "has for its sense an unsaturated part of a thought, and we call 'concept' its reference". In more generality, there are many proper names that can be analysed into a saturated part, namely, a proper name, and an unsaturated part. If the latter is such that by saturating it with a proper name having a reference, one gets another such proper name, then "we call 'function' the reference of this unsaturated part".[33]

This being said, Frege cannot but remark that claims like these bring us back to the paradox tackled in "Über Begriff unf Gegenstand". In "Einleitung in die Logik", he confines himself to arguing that "language forces upon us" the "mistake [*Fehler*]" or "inaccuracy [*Ungenauigkeit*]" these claims involve, with the result that we cannot avoid them but by bearing the difficulty in mind and insisting that concepts are unsaturated or "predicative in character" ([106], vol. 1, pp. 209–210; [107], p. 193). In "Ausführungen über Sinn und Bedeutung", he says, or at least implies, something

[33] We find a similar claim already in "Über Begriff und Gegenstand" ([95], p. 198; [104], pp. 47–48): "We must say in brief, taking […] 'predicate' in the linguistic sense: a concept is the reference of a predicate".

more. He specifies ([106], vol. 1, p. 128–132; [107], pp. 118–121) that "a concept-word refers to a concept, if the word is used as it is appropriate for logic". Then he adds, as a clarification, that "in any statement, we can substitute *salva veritate* one concept-word for another if they have the same extension, so that it is also the case that in relation to inference and to the laws of logic, concepts differ only insofar as their extensions are different". To reinforce these claims, Frege observes that the unsaturatedness of functions also comes out in the case of concepts entering into the subject of a statement [*Subjektsbegriffen*], such as in 'all equilateral triangles are equiangular', which he takes to be the same as 'if anything is an equilateral triangle, than it is an equiangular triangle'. To consider a simpler example, this means that a statement like 'the morning star is a planet' should be rephrased, in good logic, as 'the object that is the morning star is a planet', with the result that its subject involves the concept-word '...is the morning star', whose reference is the concept ⌜Morning Star⌝. Finally, Frege goes on to argue that the identity of the extensions of concepts results in a second-level relation holding between the concepts themselves and corresponding to the identity of objects.[34]

What seems to me important here is that Frege relativises his claims to the case where concept-words, and plausibly f-names in general, are "used as it is appropriate for logic" and inferences and laws of logic are concerned, which means, I suggest, that these names occur (as unsaturated components) within some proper names or statements used for affirming and inferring truths about objects, as always happens in the *Begriffsschrift*.[35] Hence, his point seems to be that, when language is used in order to affirm and infer truths about objects and f-names occur as unsaturated parts of proper names and sentences, the former names have references and refer to functions, and functions differ only if their value-ranges differ, so that a second-level relation analogous to the identity between objects applies to functions when they have the same value-range.

These claims should not be taken as evidence for identifying Frege's objects of a quite special kind with the value-ranges of functions, and even less as evidence for arguing that, for Frege, the identity of value-ranges provides the sameness of the corresponding functions. It seems quite clear, indeed, that these claims only apply insofar f-names are used as it is appropriate for logic, i.e only insofar as functions are involved in affirming and inferring truths about objects, namely about their values. This leaves open the problem of understanding what makes it that an assertion about a function is about this function rather than some other function, or, more generally,

[34]Frege even arrives at suggesting a special sign for this relation (to be used, of course, in the language of the exposition of the *Begriffsschrift*). Let $\Phi\,(\xi)$ and $\Psi\,(\xi)$ two concepts with the same extension. Frege suggests writing '$\Phi\,(\alpha) \overset{\alpha}{\underset{\frown}{\smile}} \Psi\,(\alpha)$' arguing that this expresses the same thing as '$\underset{\smile}{\overset{\alpha}{-}}\; \Phi\,(\alpha) = \Psi\,(\alpha)$'.

[35]That logic is concerned with truths about objects is, in my view, the distinctive mark of Frege's extensionalist conception of logic (which he emphasises in "Ausführungen über Sinn und Bedeutung" by repeatedly observing that his remarks favour the "logician of extension against that of intension" ([106], vol. 1, p. 128 and 133–134; [107], p. 118 and 122–123). But this conception does not entail at all an extensionalist conception of functions.

what makes it that a certain function is this very function rather than some other function.

Frege argues that a concept-word has a reference and this is just what he calls 'concept' also in the 1903 paper on "Über die Grundlagen der Geometrie". But in this case, he adds that "this is not a definition, since the decomposition [of a proper name or statement] into a saturated and an unsaturated part must be considered as a logically primitive phenomenon that must simply be recognised but not reduced to something simpler". This is a hint for a better understanding of Frege's view. In the language of the exposition of the *Begriffsschrift*—the only one in which Frege grants to himself the licence to speak about functions—one can describe what functions and f-names do in the language of the *Begriffsschrift*, or in any other language used for affirming truths about objects. But one cannot say what functions are, since, though being at work in these latter languages, functions are not, as such, actual components of them. If the account of what functions and f-names do is intended to be fine-grained enough for identifying the contribution of single functions, then unavoidably we fall into inaccuracy. However, this should not be so bad as to blur what is essential, namely that functions manifest themselves in the way we refer to objects through proper names and use statements to affirm truths about objects. Saying that the reference of f-names used in these languages (as unsaturated expressions) are functions is then nothing more than saying that f-names contribute to form (molecular) proper names and statements, or can be recognised through an analysis of (molecular) proper names and statements, and that functions "establish connections"[36] between the objects whose names are recognised as (saturated) parts of the relevant (molecular) proper names and statements and those that these latter proper names refer to and these statements are about. This suggests that the sense of an f-name depends on the way the references of the (molecular) proper names involving this f-name (as an unsaturated part of it) are to be determined on the basis of the references of the proper names which are recognised as (saturated) parts of the former proper names, i.e. on the way functions establish connections between objects. The value-ranges of functions merely depend, instead, on which objects are connected to which others. And, insofar as the same objects can be connected in different ways, two f-names can have different senses though referring to functions with the same value-range.

In this picture, the sense of an f-name essentially differs from the function this name refers to, since the former depends on the way the latter does what it does: in other words, functions act, and senses differ if the ways they act differ. And, both the sense and the reference of an f -name differ from the value-range of the corresponding function, since value-ranges neither act, nor differ if the ways the functions act differ.[37] Still, it seems obvious that the same function cannot connect the same objects in two different ways. Hence, though functions differ (i.e. produce different outcomes) only insofar as their value-ranges differ, when their names are used as it is appropriate for logic, when these same names are used in the context

[36] Cf. the quote from Sect. I.1 of *Grundgesetze* at the end of Sect. 2.4.2 below.

[37] I'm indebted to F. Schmitz for this account of the distinction between sense and reference of an f-name and the value-range of the corresponding function.

of an assertion about particular functions (necessarily made in the language of the exposition of the *Begriffsschrift*), these functions differ insofar as the senses of these names (when used as it is appropriate for logic) differ.

Now for Frege, in the language of the *Begriffsschrift*, and in any other language appropriate for expressing and inferring truths about objects, f-names cannot be used appropriately unless it is determinate which objects the relevant functions connect to which other objects. This is a requirement that Frege often advances. For example in the "Ausführungen über Sinn und Bedeutung": "it must be determinate [*bestimmet*] for every object whether it falls under a concept or not; a concept-word which does not meet this requirement on its reference is not endowed with a reference [*bedeutungslos*]" ([106], vol. 1, p. 133; [107], p. 122). I do not see any other way to understand this requirement than by taking it as demanding that an appropriate use of f-names requires the capability of deciding which object the corresponding function connects to any given object. But, at least in the context of a codified language suitable for being used in science (like the *Begriffsschrift* whatsoever extended), this capability cannot be conceived as a mere subjective ability, but rather depends on the availability of appropriate rules or procedures. And, if this is so, it is natural to admit that the sense of an f-name (when used as it is appropriate for logic) just depends on these rules or procedures, so that, in the context of an assertion about functions (which cannot but be an assertion about what functions and f-names do in the language *Begriffsschrift*), also the sameness of functions depend on these same rules or procedures.

This brings us back to the view I have above contrasted to the identification of Frege's objects of a quite special kind with value-ranges of functions. According to this view, these objects should somehow reflect the distinctive features of these rules or procedures (i.e. the differences among them), even when they connect the same objects to the same other objects and result then in the same value-ranges.[38]

[38]In a letter to Husserl of October 30th–November 1st, 1906 ([106], vol. 2, pp. 101–105), Frege argues both that the thought expressed by a statement is what it has in common with any other equipollent [*äequipollent*] statement (*ibid.*, p. 102) and that 'if A then B' and 'it is not the case that A without B' are equipollent (*ibid.*, pp. 103–104). Insofar as the thought expressed by a statement is its sense, this means that these last statements have the same sense, so that also the f-names 'if ξ then ζ' and 'it is not the case that ξ without ζ' should have the same sense. This might appear to conflict with the view that two statements have different senses if one can "understand" both of them at the same time "while coherently taking different [epistemic] attitudes towards them", that Evans has ascribed to Frege ([85], pp. 18–19). To solve this conflict, C. Penco has suggested distinguishing the semantic from the epistemic sense of a statement, arguing that the latter "could be represented by the different procedures through which each formula is given a truth condition" ([158], pp. 104–105). This suggests that 'if ξ then ζ' and 'it is not the case that ξ without ζ' have the same semantic sense but different epistemic senses, since these f-names are related to different procedures. Though Penco's notion of the epistemic sense of a statement fits with my understanding of Frege's notion of the sense of an f-name, it seems to me relevant to observe that, in the language of the *Begriffsschrift*, conjunction is expressed through implication and negation ([97], Sect. I.12), so that 'ξ and ζ' is, by convention, a shortcut for 'it is not the case that if ξ then non ζ'. One could then argue that the previous statements have the same

2.4.4 Compositionality of Functions, Higher-Level Functions, and the Notion of an Arbitrary Function

This picture seems to fit perfectly with the compositional approach to functions that is at work in the *Grundgesetze*. This approach is evident from the way the exposition of the *Begriffsschrift* proceeds. Here I cannot but limit myself to consider another example that manifests this approach quite clearly and should be enough to complete what I have said on this matter so far.

I have already mentioned the Sect. I.10 of *Grundgesetze*, where Frege tries to determine as precisely as he can what is the value-range of a function, since, he says ([97], Sect. I.10; [110], p. 16_1), "we have admittedly by no means yet completely fixed the reference of a name such as 'ἐ$\Phi(\varepsilon)$' ". This lack of determination, he argues, can be overcome if, "for each function, it is determined, when it is introduced, what values it takes on for value-ranges as arguments". This claim makes it already clear that, for Frege, any function that receives a name in his system is to be introduced in a way that makes it possible to determine its values. But this is not all. What is also relevant, to show Frege's attitude towards functions, is that he considers appropriate to tackle the problem by considering the three first-level functions considered up to that point, namely $\xi = \zeta$, $—\xi$ and $\mathrel{-\!\!\top\!\!-}\,\xi$. The argument Frege develops concerning these functions and the reasons for he takes this argument appropriate as a response to the problem are far from crystal-clear. R. Heck has submitted both Frege's argument and the response he draws from it to a very subtle analysis ([121]; [123], Chap. 4). I cannot enter this matter here. What is relevant is that, as Heck observes, the argument "works only because Frege's formal language has certain expressive resources, and does not have others—because, that is, for each of the functions introduced before Sect. 10, the question what values it takes on for value-ranges as arguments can be reduced, in one way or another, to the corresponding question about identity" ([121], p. 277; [123], pp. 98–99), where 'identity' refers, of course, to the function $\xi = \zeta$. It is, then, the specific nature of the logical formalism that has been chosen, and, in particular, the nature of its primitive first-level functions, that, in Frege's mind, allows him to begin to respond to a general question about functions and their value-ranges. And the initial response is, moreover, capable of generalisation, just because of the way other functions are formed out from the primitive ones or are explicitly introduced thanks to appropriate stipulations. Since, in concluding his argument, after having remarked that what he has established through it is enough for determining "the *value-ranges* as far as is possible here", he remarks ([97], Sect. I.10; [110]: p. 18_1):

> Only when the further issue arises of introducing a function that is not completely reducible to the functions already known will we be able to stipulate what values it should have for

(Footnote 38 continued)
sense (without specification), since they correspond to the same procedure, so that one could also say (in the language of the exposition of the *Begriffsschrift*) that the f-names 'it is not the case that ξ without ζ' and 'if ξ then ζ' refer, once appropriately rendered, to the same function.

value-ranges as arguments; and this can then be viewed as a determination of the value-ranges as well as of that function.

This claim and the way Frege reasons in Sect. I.10, before concluding this way clearly manifests a compositional approach to functions. But they still do not provide enough evidence for concluding that the universe of Frege's functions includes only functions that are introduced or purported to be introduced in such a way that they result in being *ipso facto* associated with a rule or procedure to be used for determining the references of the names of their values.

After all, all what has been said up to now only applies to functions that have a name in the language of the exposition of the *Begriffsschrift* and whose values have a name in the *Begriffsschrift* itself. The fact that these functions are associated to such a rule or procedure is an obvious consequence of the fact that these functions are elementary functions introduced through stipulations like (2.6), or functions formed out from elementary functions so introduced. Hence—one could argue—wondering about the sameness conditions of these functions is essentially different from wondering about the sameness conditions of all functions whatsoever.

But is this distinction appropriate for the case of Frege? For us, the notion of any function whatsoever is not to be reduced to that of a function having any name whatsoever in an appropriate language, or whose values have any whatsoever name in an(other) appropriate language. But it seems to me that this cannot be also the case with Frege. Insofar as functions are not actual components of some world of *concreta* and *abstracta*, but merely manifest themselves in the way we refer to objects, they can only be distinguished by looking at the way their names contribute to the formation of proper names and statements, or can be recognised as (unsaturated) components of proper names and statements.

To this, one can retort that in the language of the *Begriffsschrift*, functions are also supposed to provide arguments of other functions of a higher-level. To see the problem, take the way Frege introduces the first-order universal quantifier in Sect. I.8 of *Grundgesetze* ([97], Sect. I.8; [110]: p. 12_1):

'$\overset{\mathfrak{a}}{\smile}\,\Phi(\mathfrak{a})$' refers to the True if the value of the function $\Phi(\xi)$ is the True for every argument, and otherwise to the False.

This stipulation introduces a second-level concept: the concept $\overset{\mathfrak{a}}{\smile}\,\varphi(\mathfrak{a})$. The empty place in the name of this concept is marked by 'φ', which works in names of second-level functions as 'ξ' works in names of first-level ones. Now, it seems that, as a stipulation introducing first-level functions implicitly relies on the totality of objects (which provides the range of the relevant schematic letters for objects), a stipulation introducing a second-level function implicitly relies on the totality of first-level functions with the appropriate number of arguments (which would provide the range of the relevant schematic letters for functions, like 'Φ' in the foregoing stipulation). If this were so, the question would be obvious: does Frege hold that this totality includes only functions having names, or whose values have names—these names being either appropriately introduced in the relevant languages, or composed on the basis of names appropriately introduced—or does he hold that this totality is larger? This question is similar to that which Hintikka and Sandu answer in their paper mentioned in Sect. 2.3.[39] There is then no need to consider second-order quantifiers to advance such a question. The consideration of first-order quantifiers, or, more generally, of second-level functions, is enough.

But, is this question appropriate? The following quotation drawn again from "Über Begriff und Gegenstand" makes me doubt that it is ([95], p. 201; [104]: pp. 50–51):

> [...] the assertion that is made about a concept does not suit an object. Second-level concepts, under which concepts fall, are essentially different from first-level concepts, under which objects fall. The relation of an object to a first-level concept under which it falls is different from the relation, certainly analogous, of a first-level to a second-level concept. To do justice at once to that distinction and to the analogy, we might perhaps say that an object falls *under* a first-level concept, a concept falls *within* a second-level concept. The distinction of concept and object thus still holds, with all its sharpness.

At first glance, this is a quite vague distinction. But one could perhaps clarify it by suggesting that what Frege means here is that, in the language of the *Begriffsschrift*, names of second-level functions occur within proper names or statements, as unsaturated components of them, insofar as these latter names result, or are taken to result, from saturating the former names with appropriate names of first-level functions. If any possible saturation of a name of a second-level function with a name of a fist level one results in a name of a truth-value, the corresponding second-level function is a concept or a relation, and the relevant first-level functions fall *within* it if these names refer to the True. Consider the previous example. The f-name '$-\!\overset{\mathfrak{a}}{\smile}\!-\ \varphi(\mathfrak{a})$' of a second-level function occurs in the proper name '$-\!\overset{\mathfrak{a}}{\smile}\!-\ \mathfrak{a} = \mathfrak{a}$' insofar as the latter results from saturating the former with the name '$\xi = \xi$' of a first-level function.[40] Now, insofar as '$-\!\overset{\mathfrak{a}}{\smile}\!-\ \mathfrak{a} = \mathfrak{a}$' is a name of a truth-value, and this is also the case for any other proper name resulting from saturating '$-\!\overset{\mathfrak{a}}{\smile}\!-\ \varphi(\mathfrak{a})$' with a name of a first-level function, $-\!\overset{\mathfrak{a}}{\smile}\!-\ \varphi(\mathfrak{a})$ is a concept. Furthermore, insofar as '$-\!\overset{\mathfrak{a}}{\smile}\!-\ \mathfrak{a} = \mathfrak{a}$' refers to the True, the function '$\xi = \xi$' falls *within* this concept.

More generally, if a proper name results, or is taken to result, from saturating the name of a second-level function with appropriate names of first-level functions, then the first-level functions named by these latter f-names are said to be arguments of the second-level function named by the former f-name. The difference from the case of first-level functions is clear: for a proper name resulting from saturating a name of a first-level function with names of objects to belong to the language of the

[39] The question has also been considered by Dummett, who has argued ([74], pp. 219–220) that "there is meagre evidence" for attributing to Frege the conception that his function-variables "range over the entire classical totality of [appropriate] functions", and that "his formulation make it more likely that he thought of his function-variable as ranging over only those functions that could be referred to by functional expressions in his symbolism".

[40] To understand what I mean by speaking of proper names or statements which are taken to result (rather than merely resulting) from saturating names of second-level functions with appropriate names of first-level functions, consider the example of a proper name like '$\Phi(\Delta)$' or '$\Psi(\Delta, \Gamma)$'. These can be either taken to result from saturating the names '$\Phi(\xi)$' and '$\Psi(\xi, \zeta)$' of first-level functions with the proper names 'Δ' and 'Γ', or taken to result from saturating the names '$\varphi(\Delta)$' and '$\psi(\Delta, \Gamma)$' of second-level functions with these same names of first-level functions.

Begriffsschrift, these names of objects have in turn to belong, as such, to this same language; a proper name belonging to the language of the *Begriffsschrift* cannot result, instead, from saturating a name of a second-level function with f-names belonging, as such, to this same language, for the simple reason that this language does not include f-names (otherwise than as unsaturated components of proper names or statements).

This syntactical difference is structurally relevant but does not undermine what is essential for my present purpose: Frege's treatment of functions both of first and of higher-level, in the exposition of the *Begriffsschrift*, focuses on the way proper names (namely names of values of functions) and statements are formed by saturating f-names with other appropriate names. Hence, though one could and should say that a second-level function connects first-level functions to objects, rather than objects to objects, the work that first and second-level functions carry out within the language of the *Begriffsschrift* is essentially the same, and essentially depends on their having a name, or on the fact that their values have a name.

Mutatis mutandis, all that has been said for second-level functions also applies to higher-level ones. An example is given by the second-order universal quantifier, which Frege introduces in all its generality in the *Grundgesetze*, I.24 as the third-level function $\smile\!\!^{\mathfrak{f}}\!\!-\ \mu_\beta\,(\mathfrak{f}\,(\beta))$ defined by stipulating that '$\Omega_\beta\,(\Phi\,(\beta))$' refers to the True whatever the first-level function $\Phi\,(\xi)$ might be.[41] Hence, the possibility of functions which are not endowed with a name, or are not at least associated to appropriate rules to be used for forming the names of their values and determining their reference, merely lies outside the horizon of Frege's use of the notion of function in the exposition of the *Begriffsschrift*. Any argument to be used for arguing that he admits functions like these would be not only merely speculative, but also intrinsically vague, since Frege give us no hint for understanding what he could mean by claiming that certain functions exist, if this were not merely intended as a metaphoric way for saying that their names or the names of their values are at work in some appropriate language.

In Sect. I.2 of *Grundgesetze*, Frege describes a process through which the mathematical notion of a function was gradually extended ([97], Sect. I.2; [110]: p. 6_1): firstly, functions were taken to be formed only by the fundamental arithmetic operations; then operations involving a passage to a limit were admitted; finally, "the word 'function' was so generally understood that in some cases the connection between the argument and the value of a function could no longer be expressed through the signs of analysis, but only through words",[42] and complex numbers were admitted both as arguments and values of functions. Frege add then that, "in both direction [...][he has] gone still further", for having introduced new signs or used old ones for a new purpose, and for having admitted other objects than numbers as arguments

[41] The letter 'μ' is here used here to hold a place empty for a second-level function with one argument, the index 'β' is used to make it clear that the arguments whose places are indicated by 'β', both in '$\mathfrak{f}\,(\beta)$' and in '$\Phi\,(\beta)$ ', are bound, and 'Ω' and 'Φ' serve as schematic letters for functions of the second and first-level, respectively.

[42] This is enough evidence for concluding that, contrary to what Hintikka and Sandu seem to imply ([129], p. 311) Frege admitted the possibility of non-differentiable functions in real analysis (on this matter, cf. also [124], p. 41, and [36], pp. 90, 99–100).

and values of functions. But he neither says nor implies that functions and/or their values could lack names or not be associated to rules or procedures to be used for forming the names of their values and determining their reference.

In another passage, close to this one, drawn from "Function und Begriff" ([94], p. 12; [104] p. 28), Frege mentions Dirichlet's function as an example of a mathematical function merely described through ordinary language, by describing it as a function "whose value is 1 for rational and 0 for irrational arguments". According to Burgess, this is enough for inferring that it is not part of Frege's notion of function "that a function must be definable in a *symbolic* language". Hence, he continues, even if Frege had been convinced that "there is a finitary *symbolic* language [...] in which for every function there is an expression", he could have not based this conviction on purely conceptual grounds, being rather forced to appeal, at least partially, to inductive evidence relative to the possibility of defining, in some appropriate language, suitable functions to be used as witnesses for the existence theorems of contemporary mathematical analysis ([35], p. 106). This would have been a mistake, but, for Burgess, it is plausible to ascribe such a mistake to Frege, provided that it was only some years later that "it became more or less established orthodoxy in the mathematical community that functions are not restricted to be definable", and he "was largely unaware of the bearing of Cantor's cardinality theorems" entailing that "there are more Fregean concepts than Fregean objects": *ibid*., p. 107 and 101–102).

This mention of Cantor's cardinality theorems suggests that, for Burgess, Frege's notion of a function could have been undermined by considerations about sets' cardinality, which seems to have been possible only if this notion had been extensional in nature. But all that I have said up to now should provide evidence for concluding that this is not so.

Burgess's discussion is largely based on a passage—also quoted by Hintikka and Sandu as a major piece of evidence for their main thesis ([129], p. 312–313)—drawn from "Was ist eine Funktion?", the 1904 paper that I mentioned at the end of Sect. 2.3 ([100], pp. 662–663; [104], pp. 112–113), where Frege argues against Czuber's definition of function. In this paper, he remarks that the idea of function as a law of correlation expressed by an equation "has been found too narrow", but suggests that the difficulty "could be easily avoided by introducing new signs into the symbolic language of arithmetic". This passage is open to many interpretations not necessarily fitting with Hintikka and Sandu's thesis.[43] But it seems to me that another passage drawn from this same paper is much more explicit.[44] In my view, it makes manifest in a nutshell the main feature of Frege's notion of function, by making it clear that it is not extensional at all. This is just the passage where Frege critically discusses Czuber's definition of real functions. Here is what he says ([100], p. 661–662; [104], pp. 111–12):

[43] According to [62], pp. 142–145, the context of this passage suggests that Frege is here merely arguing for the possibility of extending the class of analytically representable functions so as "to include all functions of a particular class": a quite common view among mathematicians of his time. Heck and Stanley ([124], p. 419–421) have considered, instead, that Frege's only point, here, is that functions are unsaturated, which seems to me a quite implausible interpretation.

[44] This passage is also partially quoted in ([129], p. 312).

> It would be simpler and clearer to state the matter as follows. 'With every number of an x-domain is correlated a number. I call the totality of these numbers the y-domain'. Here we certainly have a y-domain, but we have no y of which we could say that it is a function of the real variable x. Now, the delimitation of the domain appears irrelevant to the question of the nature of the function [*Wesen der Funktion*]. Why could we not at once take the domain to be the totality of real numbers, or the totality of complex numbers, including real numbers? The heart of the matter really lies in a quite different place, viz. hidden in the word 'correlated'. Now, how do I acknowledge whether the number 5 is correlated with the number 4? The question is unanswerable unless it is somehow completed. […] Correlation […] takes place according to a law [*Gesetz*], and different laws of this sort can be thought of. Hence, the expression 'y is a function of x' has no sense, unless it is completed by the statement [*Angabe*] of the law according to which the correlation takes place. This is a mistake in the definition. And is not the law, which this definition treats as not being given, the main thing? […] Distinctions between laws of correlation will go along with distinctions between functions; and these cannot any longer be regarded as quantitative. If we just think of algebraic functions, the logarithmic function, elliptic functions, we conceive ourselves immediately that these are qualitative differences […].

2.5 Concluding Remarks

In a sense, my account of Frege's notion of function fits with Hintikka and Sandu's conclusions.[45] But it hinges on different concerns. In my view, the relevant question is not whether Frege endorsed the standard or a non-stardard interpretation of second-order logic. What is relevant is rather the way functions are supposed to work both in his formal system and in his exposition of it.[46] For Frege, functions are neither defined on sets, nor conceived as pairs of sets. So, it is out of order to wonder whether he takes the range of second-order quantifiers to coincide or not with the whole power set of the range of the first order-quantifiers. The notion of set (however conceived) is not a resource Frege considers himself to be licensed to appeal to in the exposition of his formal system, which is not, by the way, in need of any semantic interpretation, since it is *ipso facto* presented as an already interpreted system ([188], p. 4).

In replying to Hintikka and Sandu's paper, Heck and Stanley have argued that "Frege would not have accepted any of the familiar arguments in favour of a non-

[45]More recently, Sandu has reiterated his and Hintikka's major theses, and added that, for Frege, intensions have "logical primacy" over extensions ([173], pp. 241–243). To support this, Sandu argues that Ramsey's efforts for reforming logicism [165] were manly motivated by an understanding of Frege's and Russell's conceptions about functions, which is close to that outlined in his joint paper with Hintikka. In opposition to this, Ramsey would have aimed to conciliate logicism with "the extensional attitude of the mathematics of his days (Cantorian set theory)", and this resulted in his grasping of "the concept of arbitrary function in extension" ([173], pp. 238 and 250). This has convinced Demopoulos to reconsider the objections to Hintikka and Sandu's theses advanced in a joint paper with Bell [62], and conclude that "Frege's functions" should be distinguished from "arbitrary correspondences" i.e. arbitrary functions in set-theoretic sense ([61], especially p. 6).

[46]I agree then with Demopoulos, according to whom, "the interest of Hintikka and Sandu's paper has less to do with standard versus nonstandard interpretations of second-order logic than with Frege's concept of a function" ([61], p. 6).

standard interpretation", with the result that, if he "did interpret his higher-order quantifiers non-standardly, then a study of his reasons for doing so would presumably provide a entirely new set of motivations for rejecting the standard interpretation" ([124], p. 417–418). This argument depends on the admission that Frege could have had positive reasons for rejecting the standard interpretation, and could have then conceived it. But this is just what he could not have done. He was faced neither with the choice between the standard and any nonstandard interpretation of second-order logic, nor with the choice between accepting or rejecting our extensional set-theoretic notion of arbitrary function. His way of conceiving functions was simply such as to make this idea unavailable to him.[47]

Rather than projecting Frege's conception onto the modern (set-theoretic) setting, we should instead try to understand the intrinsic motivations of his own approach, by placing it in the appropriate historical and philosophical framework. This framework is provided by his reaction to the program of the arithmetisation of analysis, namely to the requirement that numbers and magnitudes should be defined within a formal logical system whose exposition is to depend on the appeal to a quite general, and then non-mathematical notion of function, the elucidation of which should moreover result in the elucidation of the notions of concept and a concept's extension, and, then, in the clarification of the very nature of logic.

In Sect. 2.3, we saw that the program of the arithmetisation of analysis came in turn from a reaction to Lagrange's foundational program, which also ascribed a basic role to the notion of function. A comparison between Lagrange's and Frege's views on functions is then quite natural.

Though both of them take the notion of function to be primitive, they take it to be so in two quite different ways. For Lagrange, functions are objects, namely expressions of an appropriate formalism which is taken for granted. The essential purpose for focusing on them is that of providing a purely relational construal of the notion of quantity, whose aim is to free this notion from any specific essence and make it perfectly formal, and then general. For Frege, functions are opposed to objects, so that they cannot be expressions. Furthermore, they are supposed to act in any language appropriate for expressing truths about objects, and any language is supposed to be meaningful, so that the elucidation of the notion of function is conceived as a prerequisite for the exposition of any appropriate formalism. The essential purpose for focusing on this notion is that of making clear the way in which we refer to objects and express truths about them, so as to provide a formal construal of logic, just conceived as the general framework in which truths about objects can be expressed.

Despite these crucial differences, Lagrange's and Frege's conceiving the notion of function as primitive and taking it as a basis for developing the respective foundational programs also results in important analogies and makes both these programs

[47]Despite their focusing on the question of whether Frege adopted the standard or a nonstandard interpretation of second-order logic, Hintikka and Sandu also suggest something like this when they claim ([129], p. 313) that "there is no niche in [...][Frege's] world for [...][our] notion of an arbitrary function", and that in Frege's logic "there is no room for the idea of a arbitrary function-in-extension".

essentially different from both the arithmetisation of analysis and from set-theoretic reduction. What opposes the two former programs to the latter ones is, so to say, an intensional approach: the idea that an appropriate foundation of mathematics necessarily depends on the clarification of the way the relevant items mathematics is about are related to each other. Furthermore, both for Lagrange and Frege, this clarification depends on the identification of a formal expression for the relevant relations within an appropriate formalism. In both cases, mathematics is then conceived as a system of appropriate expressions. For Lagrange, these expressions are functions; for Frege, they merely make the role and nature of functions manifest. Still, in both cases the idea of a function detached from any appropriate expression is merely inconceivable. If my account is correct, this is not the effect of Lagrange's and Frege's intellectual myopia.[48] It rather depends on the intrinsic nature of their respective programs.

[48]cf. see the footnote (13).

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