Chapter 2
Chiral Tensor Power Spectrum from Quantum Gravity

Quantum fluctuations that are produced during inflation freeze out after leaving the horizon and can survive until today, as was described in Sect. 1.2.5. These fluctuations, having been produced in the very early universe, might carry some information about the quantum nature of gravity. The theory of loop quantum gravity does not use the metric as its fundamental gravitational variable, but a (generally) complex connection. Therefore, deriving the power spectrum of tensor perturbations in this framework, which was done in Sect. 1.2.5 in the standard second order formalism, could lead to a different result. Considering new variables to describe spacetime is always interesting from a quantum mechanical point of view, as different quantum theories can give rise to equivalent classical theories [1]. We cannot know from first principles which description is the correct one, and experiments that involve quantum mechanical observables like power spectra might be the only way of finding out.

I will first outline general principles of the canonical quantization of gravity in Sect. 2.1, starting with the usual approach taken in quantum field theory, and then describing the framework of loop quantum gravity. I will finish by comparing the canonical and covariant approaches to quantization.

In the Sect. 2.2, I will describe different formalisms used in general relativity. In particular, the tetrad formalism, the first order formalism which results in the Palatini action, and the Ashtekar formalism which forms the basis of loop quantum gravity will be discussed.

Section 2.3 is based on work that has been published in [2, 3]. I will describe how using the Ashtekar variables instead of the standard metric variables to find a perturbed gravitational action during inflation leads to a chirality in the tensor power spectrum, which could leave an observable signature in the CMB. Even though the Ashtekar variables are motivated by loop quantum gravity, they are interesting to study regardless of the success of the theory. If we were to observe a chiral tensor spectrum, it might not necessarily mean that LQG is the correct description of quantum gravity, but it would definitely give us insight into the quantum nature of spacetime.

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2.1 Canonical Quantization of Gravity

In this section, I will briefly discuss the main aspects of quantum field theory (QFT), especially regarding the quantization of gravity, before giving some background on loop quantum gravity, highlighting its successes and shortcomings. I will finish by stressing why it might be interesting for Cosmology to consider the Ashtekar variables, which are motivated by the canonical theory of LQG, as the fundamental variables describing spacetime.

2.1.1 Quantum Field Theory

Quantum field theory is the union of quantum mechanics and special relativity, where instead of considering single particle states, we consider fields which are quantized over a (typically) flat, Minkowski background [4].

When one first studies QFT as an undergraduate, one probably learns how to quantize a scalar field canonically, i.e. using the formalism of Sect. 1.2.4. The canonical quantization procedure [5] has been very successful in the context of QFT, and is used in particular to build the theory of quantum electrodynamics (QED), which has made experimentally verified predictions with astonishing accuracy [6]. One conceptual problem with the approach is the lack of manifest Lorentz invariance due to the splitting of space and time, although the Feynman rules one derives to describe interactions obey the Lorentz symmetry [4].

An alternative approach to quantization is the path integral formalism [7], which uses the Lorentz invariant Lagrangian as its central dynamical variable. It also preserves all other symmetries of the theory and is therefore more suited to treating non-Abelian gauge theories like quantum chromodynamics (QCD) [6].

Although the two approaches lead to equivalent results, depending on the situation, one might be more suitable than the other [4], although the path integral formalism is usually the method of choice for the most developed theory of quantum gravity to date, string theory [1, 8].

In all realistic field theories, ultraviolet divergences arise; which means that at very high energies certain quantities of interest become infinite. Using the procedure of renormalization, we can deal with these divergences and arrive at a physically meaningful theory [4, 6]. It is a well known fact that this procedure fails in the case of gravity: When one tries to quantize the graviton field by treating it as a perturbation around flat spacetime [4], divergences arise that cannot be renormalized. This is probably not surprising; most field theories are effective in the sense that their regime of validity does not extend to the highest energy scales [4]. For gravity itself, it seems we cannot simply cut off the highest energy modes and ignore the nature of spacetime at the Planck scale.

2.1.2 Loop Quantum Gravity

Loop Quantum gravity is an attempt to find a quantum theory of gravity in the most "conservative" [9] way: Its aim is to quantize gravity in a background independent (as the background itself is quantized), non-perturbative manner, without resorting to new physics like higher dimensions, supersymmetry or trying to arrive at a unified description of all fundamental forces. This is in contrast to string theory, which incorporates all these aspects and is also based on the standard QFT approach of quantizing over a fixed, flat background spacetime. LQG, on the other hand, uses a canonical quantization method.

In LQG, we do not want to consider gravitons propagating on a fixed background as one would do in standard QFT, but rather define operators corresponding to spacetime itself. Therefore, the canonical variables should describe spacetime, and indeed the metric was chosen as the central gravitational variable (with its conjugate being related to the extrinsic curvature) in the first attempt of defining a canonical quantum theory of gravity, the ADM formalism [10].

In all canonical theories of GR we need to satisfy a number of constraints, which correspond to the quantum Einstein equations [11] and incorporate diffeomorphism invariance. Appendix A.2 gives some background on Hamiltonian constrained systems, and the specific constraints arising in LQG are given in Sect. 2.3.2. In particular, the Hamiltonian constraint, which corresponds to invariance under time translations, on a quantum level leads to the Wheeler-DeWitt equation $\mathcal{H}|\Psi\rangle = 0$ [12], where the quantum Hamiltonian $\mathcal{H}$ acts on the "wave function of the universe" $|\Psi\rangle$. It is constrained to vanish to reflect the fact that there is no global time variable in GR (this is simply the analogue of the Schrödinger equation in canonical quantum gravity).

Within the ADM formalism, it was very difficult to solve this constraint with the chosen quantum operators. In 1986, Ashtekhar introduced a set of new variables [13, 14], discussed in Sect. 2.2.3, where the central canonical variable is a connection, and its conjugate a (densitised) metric field. Further work by [15, 16] led to the definition of the loop representation (hence the name LQG): The actual variables promoted to field operators were the holonomy (parallel transport around a closed loop) of the connection, and a flux of the densitised metric [11]. Like the creation and annihilation operators of particle states in Sect. 1.2.4, these operators create and destroy "loop states", quantum excitations of spacetime along a single loop [9] (the idea of a loop basis was also used in the context of Yang Mills theory in terms of the Wilson loop [17]).

This approach greatly simplified solving the constraint equations [18], especially after work by Thiemann [19]. The Hilbert space these loop states live in has a basis in terms of spin network states [20, 21]. It is possible to define area and volume operators acting on these spin networks (which can be regarded as building blocks of spacetime [9]) with discrete spectra [22, 23], showing that spacetime is fundamentally discrete in LQG.

Kinematically, the theory is well developed: There exists a well defined scalar product [24, 25] and matter can be coupled to the theory [26, 27]. Progress has also recently been made on identifying n-point functions [28], and therefore an expression for the graviton propagator can be obtained [29]. However, the dynamics of the theory are still not well understood and the low-energy limit that should yield GR has not been established [30].

LQG also has some applications to other areas of physics. It provides a way to calculate the Bekenstein-Hawking entropy [31] and has also spawned the field of loop quantum Cosmology [32, 33]. Loop quantum Cosmology contains some interesting results, including a possible mechanism for driving inflation [34], the absence of singularities [35] and the replacement of the Big Bang by a Big Bounce [36]. However, the approach I will take below is not comparable; I will only be using the Ashtekhar variables, not the loop representation which is the foundation of the LQG formalism.

2.1.3 Different Approaches in Quantum Gravity

In canonical quantum gravity spacetime has to be foliated into spacelike slices evolving in time to be able to define the canonical variables [10]. This introduces an explicit time dependence which manifestly breaks Lorentz invariance. The initial lack of covariance (invariance under general coordinate transformations) and the related problem of defining dynamics are the main criticisms faced by this approach.

Although a path integral formulation of LQG now exists using spinfoams [37, 38], it is still in its infancy and work remains to be done trying to link the different formalisms [30]. Arguably the most successful attempt at trying to find a fundamental theory of quantum gravity to date is string theory [1, 8], which is a covariant approach and therefore does not suffer from the same problems as LQG (although proponents of the latter theory will claim that on the other hand, string theory does not address the principle of background independence in GR, needing to define a fixed background [9]).

Of course, there are many other approaches to tackling the problem of quantum gravity, for example causal dynamical triangulation [39] (which is similar in nature to the spinfoam formalism) or causal set theory [40, 41], where the causal structure of spacetime is taken as the most important physical ingredient.

While a mathematically consistent theory of quantum gravity would obviously be a major breakthrough in theoretical physics, any consistent theory will suffer from the problem that it seems impossible with current technology to make testable predictions: The energy scales at which quantum gravity effects play a role are far too high to be probed directly by experiment. Indirect evidence seems to be the best we can hope for at the moment, and Cosmology is a great candidate to provide just that. Clearly, the conditions right after the Big Bang were such that quantum gravity effects must have played a central role, and they might have left an imprint in the CMB through inflation, which explicitly describes how quantum fluctuations become classical observables. Deriving the spectrum of tensor perturbations using the Ashtekar formalism would provide a test for the predictive power of the theory.

2.2 Different Formalisms for General Relativity

Usually, the protagonist of GR is the metric $g_{\mu\nu}$, and the dynamics are defined by the Einstein-Hilbert action (1.8). However, we can also describe gravitational degrees of freedom using a formulation in terms of tetrads (which requires introducing the language of differential forms), as described in Sect. 2.2.1. The content of this section is based on section 2.9 and Appendix J of [42]. I will continue by introducing the first order formalism in 2.2.2, where the metric and connection are taken to be independent initially, giving the Palatini action. Combining both of these ingredients makes it possible to define the Ashtekar formalism in Sect. 2.2.3.

2.2.1 The Tetrad Formalism

It is sometimes useful, especially when trying to treat GR as a gauge theory, to use a non-coordinate basis as opposed to the standard basis vectors dx^μ, ∂_μ. Motivated by the fact that you can always define a local inertial frame in GR which looks flat, consider the tetrad basis e^I, $I = 1\ldots4$, that satisfies $ds^2 = \eta_{IJ}e^Ie^J$, where η_{IJ} is the Minkowski metric. I is an "internal" index and transforms under the vector representation of the Lorentz group SO(3,1) [43]. We can write the basis vectors e^I in terms of the old coordinate basis as

$$e^I = e^I{}_\mu dx^\mu\,, \tag{2.1}$$

so the defining condition for the tetrad basis can be written in components as

$$g_{\mu\nu} = \eta_{IJ}e^I{}_\mu e^J{}_\nu\,. \tag{2.2}$$

The spacetime indices, denoted by Greek letters, can be raised and lowered using the metric $g_{\mu\nu}$ and transform by general coordinate transformations, while the internal indices, denoted by capital Latin letters, can be raised and lowered using the Minkowski metric η_{IJ} and transform by local Lorentz transformations. The components satisfy orthogonality conditions,

$$e^I{}_\mu e^\mu{}_J = \delta^I_J\,, \quad e^\mu{}_I e^I{}_\nu = \delta^\mu_\nu\,. \tag{2.3}$$

We can also use the components $e^I{}_\mu$ of the tetrad basis to relate the components of a vector V in each basis:

$$V^I = e^I{}_\mu V^\mu\,. \tag{2.4}$$

To be able to use covariant derivatives in this formalism, we need to define the spin connection $\omega_\mu{}^I{}_J$. The covariant derivative of some tensor $A^I{}_J$ is then given by

$$\nabla_\mu A^I{}_J = \partial_\mu A^I{}_J + \omega_\mu{}^I{}_K A^K{}_J - \omega_\mu{}^K{}_J A^I{}_K\,. \tag{2.5}$$

To obtain the defining relations for the spin connection and the curvature in the tetrad basis, it helps to simplify expressions if we use the language of differential forms. Let me define them and list some of their properties.

A differential p-form is a $(0, p)$ antisymmetric tensor (i.e. a 0-form is a scalar, and a one-form is a dual vector $\omega = \omega_\mu dx^\mu$). The (components of the) wedge product between a p-form A and q-form B is an antisymmetrised tensor product,

$$(A \wedge B)_{\mu_1 \ldots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \ldots \mu_p} B_{\mu_{p+1} \ldots \mu_{p+q}]} \,. \tag{2.6}$$

A basis for p-forms can be written using the wedge product as $\frac{1}{p!} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$. A p-form A is then given by

$$A = \frac{1}{p!} A_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \,, \tag{2.7}$$

where the components $A_{\mu_1 \ldots \mu_p}$ are totally antisymmetric.

We will also need the exterior derivative which is an antisymmetrised partial derivative that maps a p-form into a $p + 1$-form [42]:

$$(dA)_{\mu_1 \ldots \mu_{p+1}} = (p+1) \partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+q}]} . \tag{2.8}$$

Specifically, for zero and one-forms, i.e. a scalar ϕ and vector $\omega = \omega_\mu dx^\mu$, the exterior derivative is

$$(d\phi) = \partial_\mu \phi \, dx^\mu \,, \quad (d\omega) = \partial_{[\mu} \omega_{\nu]} dx^\mu \wedge dx^\nu \,. \tag{2.9}$$

Since partial derivatives commute, and the exterior derivative is antisymmetric, we have $d(dA) = d^2 A = 0$ for any p-form A.

An important property of the exterior derivative is its action on the wedge product of two forms. If A is a p-form,

$$d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB \,. \tag{2.10}$$

Finally, one can use n-forms ω in n dimensions to define integration on the manifold, specifically $\int \omega = \int \omega_{0123} d^n x$. As differential forms are completely antisymmetrised, there is only one independent component for an n-form in n dimensions.

We can write the tetrad basis and the spin-connection as one-forms e^I and $\omega^I{}_J$ by suppressing their spacetime indices. The Cartan equations provide defining relations for the torsion and the Riemann tensor in the tetrad basis:

$$T^I \equiv de^I + \omega^I{}_J \wedge e^J \,, \tag{2.11}$$

$$R^I{}_J \equiv d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J \,. \tag{2.12}$$

Note that $R^I{}_J$ is a two-form; it specifies the entire Riemann tensor (not the Ricci tensor). It can be regarded as the field strength of the spin-connection [43]. The Christoffel connection, Eq. (1.5), that is commonly used in GR is torsion-free and ensures $\nabla_\alpha g_{\mu\nu} = 0$. The first property leads to Eq. (2.11) being zero, which gives a condition for the spin connection in terms of the tetrad, and the second implies that the spin connection must be antisymmetric, $\omega^\mu_{IJ} = -\omega^\mu_{JI}$.

The tetrad formalism actually makes calculating metric components, spin connection and Riemann tensor a lot simpler than the usual coordinate approach. As we will make use of them in Sect. 2.3, I will derive a tetrad basis and the associated spin connection for a flat Friedmann background (see also Appendix J of [42]).

For a flat FRW metric (1.12) using conformal time, we have $g_{\mu\mu} = a^2$ (no sum), with all off-diagonal components zero. We need to satisfy Eq. (2.2), and clearly the choice $e^0{}_0 = e^1{}_1 = e^2{}_2 = e^3{}_3 = a$ does the job (any other choice will be related to this by a local Lorentz transformation [42]). The four tetrad forms $e^I = e^I_\mu dx^\mu$, $I = 0, i$, can then be written as

$$e^0 = a d\eta\,, \tag{2.13}$$

$$e^i = a dx^i\,. \tag{2.14}$$

We can derive the components of the spin connection $\omega^I{}_J$ using the torsion free condition (2.11). First though, due to the antisymmetry $\omega_{IJ} = -\omega_{JI}$, we see that

$$\omega^0{}_0 = 0\,, \tag{2.15}$$

$$\omega^0{}_i = \omega^i{}_0\,, \tag{2.16}$$

$$\omega^i{}_j = -\omega^j{}_i\,, \tag{2.17}$$

where we had to raise and lower indices with the Minkowski metric.

Let us solve Eq. (2.11) separately for the $I = 0$ and $I = i$ components (which all have the same form) using the solutions for the tetrad. To take the exterior derivatives, regard the forms in Eqs. (2.13) and (2.14) as a product of a scalar and a one-form and then use the product rule in Eq. (2.10) to obtain (remembering that $d^2 = 0$) $de^0 = a' d\eta \wedge d\eta = 0$ and $de^i = a' d\eta \wedge dx^i$. For $I = 0$, the torsion free condition then gives

$$a\omega^0{}_i \wedge dx^i = 0\,, \tag{2.18}$$

where I used $\omega^0{}_0 = 0$. For $I = i$, we obtain

$$a' d\eta \wedge dx^i + a\omega^i{}_0 \wedge d\eta + a\omega^i{}_j \wedge dx^j = 0\,. \tag{2.19}$$

The only solution compatible with the antisymmetry of the spin connection is to set $\omega^i{}_j = 0$ as well, with the only non-zero component being $\omega^i{}_0 = (a'/a)dx^i = He^i$ [42]. This clearly solves the torsion free conditions, Eqs. (2.18) and (2.19).

2.2.2 The Palatini Formalism

We can rewrite the Einstein-Hilbert action (1.8) using tetrads (remember integration over 4-forms is well defined in four dimensions). The result is [43]

$$S_{\mathrm{EH}}\left(g_{\mu\nu}(e)\right) = \frac{1}{2}\int \epsilon_{IJKL} e^I \wedge e^J \wedge R^{KL}\left(\omega(e)\right) . \tag{2.20}$$

This makes the internal gauge symmetry under local Lorentz transformation more apparent [43]. Now, consider the following change in viewpoint: Instead of thinking of the action as a function of the tetrad e^I only, we can initially regard it as a function of both e^I and $\omega^I{}_J$, and keep metric and connection independent. The resulting action,

$$S_{\mathrm{PK}}\left(e^I_\mu, \omega^{IJ}_\mu\right) = \frac{1}{2}\int \epsilon_{IJKL} e^I \wedge e^J \wedge R^{KL}(\omega) , \tag{2.21}$$

is called the Palatini-Kibble action [19]. Varying it with respect to the metric gives the usual Einstein equations, and varying with respect to $\omega^I{}_J$ shows that it is indeed the spin connection $\omega(e)$ we defined, i.e. it satisfies the torsion-free Cartan equation (2.11) and it is manifestly antisymmetric. This is also known as the first order formalism [44] as the equations of motion only contain first derivatives of metric and connection, while the second order formalism of the Einstein-Hilbert action contains second derivatives of $g_{\mu\nu}$.

2.2.3 The Ashtekar Formalism

We can make a further generalization of the Palatini action and add a term $\delta_{IJKL} e^I \wedge e^J \wedge R^{KL}(\omega)$, where $\delta_{IJKL} = \delta_{I[K}\delta_{L]J}$. This term is compatible with the symmetries and vanishes on-shell, when we use the equation of motion for the spin connection $\omega(e)$ [43]. This gives the Holst action [45]

$$S_{\mathrm{H}}\left(e^I_\mu, \omega^{IJ}_\mu\right) = \left(\frac{1}{2}\epsilon_{IJKL} + \frac{1}{\gamma}\delta_{IJKL}\right)\int e^I \wedge e^J \wedge R^{KL}(\omega) , \tag{2.22}$$

where the coupling constant introduces the Immirzi parameter γ. This parameter will not appear in the classical theory; however, it does play a role in the quantum theory as we will show later in Sect. 2.3, and also appears in the black hole entropy formula derived for LQG [31].

The Holst action is the fundamental action of loop quantum gravity and can be used to derive the new set of canonical variables in terms of a connection A and its conjugate E, which is related to the metric. This choice greatly simplified the constraint algebra [30] compared to the old ADM formalism [10]. We can write Eq. (2.22) as [43]

$$S(A, E, N, N^a) = \frac{m_{\rm Pl}^2}{\gamma} \int d^4x \left[\dot{A}_a^i E_i^a - A_0^i G_i - NH - N^a H_a \right] , \qquad (2.23)$$

where (A, E) are the canonically conjugated variables, and A_0^i, N and N^a are Lagrange multipliers for the first class constraints (see appendix A.2 for details on constrained systems). The Hamiltonian H and the space diffeomorphism constraint H^a encode the invariance of the action under time translations and spatial diffeomorphisms, and the Gauss constraint G_i generates SU(2) gauge transformations. As we have made a specific choice for the time coordinate, the local Lorentz symmetry is broken to a local SO(3)$\sim$SU(2) symmetry transforming the objects E_i^a and A_a^i.

The canonical variables satisfy commutation relations [43]

$$\{A_a^i(\mathbf{x}), E_j^b(\mathbf{y})\} = \frac{\gamma}{m_{\rm Pl}^2} \delta_a^b \delta_j^i \delta(\mathbf{x} - \mathbf{y}) \, . \qquad (2.24)$$

Specifically, E corresponds to the densitized inverse triad

$$E_i^a = \det \left(e_b^j \right) e_i^a \, , \qquad (2.25)$$

where $i = 1, 2, 3$ is an internal index, and $a = 1, 2, 3$ a spatial index; and A to the SU(2) connection (as opposed to a Lorentz connection) [43]

$$A_a^i = -\frac{1}{2} \epsilon^{ijk} \omega^{jk}{}_a + \gamma \omega^{0i}{}_a \, , \qquad (2.26)$$

where $\omega^I{}_J$ is the spin connection satisfying the torsion-free condition. Defining a mapping (see [19], p.127)

$$\omega^i = -\frac{1}{2} \epsilon^{ijk} \omega^{jk} \, , \qquad (2.27)$$

the connection can also be written as

$$A^i = \omega^i + \gamma \omega^{0i} \, . \qquad (2.28)$$

The original variables chosen by Ashtekar [13] were defined for an Immirzi parameter $\gamma = \pm i$. They are special in the sense that the symmetry group of the connection can be identified with the self dual (SD) SU(2) subgroup of the Lorentz symmetry for $\gamma = i$, and the anti-self dual (ASD) SU(2) for $\gamma = -i$ [43]. These subgroups correspond to the isomorphism between the complexified Lorentz group and SU(2)$\times$SU(2). Hence, I will refer to A^i as the SD connection if $\gamma = i$, and as the ASD connection if $\gamma = -i$.

2.3 Spectrum of Tensor Perturbations Using Ashtekar Variables

In this section I will study the tensor perturbations and calculate their power spectrum within the Ashtekar formalism. First, I will identify the canonical variables, perturbed to first order to describe metric perturbations, in Sect. 2.3.1.

The Hamiltonian description is discussed in 2.3.2: The constraints arising in the Ashtekar formalism will be discussed and Hamilton's equations will be derived for the full and the perturbed variables. Finally, I will derive the second order Hamiltonian describing the dynamics of gravitons (and therefore encoding tensor perturbations). Although classically it reduces to the well-known result presented in Sect. 1.2.5, it is still very instructive to carry out the calculation explicitly as a number of subtleties need to be taken into account which had not been previously identified in the literature.

In Sect. 2.3.3, I will expand the perturbation variables in Fourier space. As the connection is complex, there will be separate positive and negative frequency modes corresponding to gravitons and anti-gravitons, which are related by reality conditions. I will end the section by deriving the commutation relations for the modes.

The quantum theory can then be discussed in Sect. 2.3.4. The Fourier space Hamiltonian can be written in terms of graviton creation and annihilation operators which are linear combinations of the metric and connection. Having identified these operators, we can set up a Hilbert space of graviton states. The states with negative energy are not normalisable under the chosen inner product, which is fixed by the reality conditions. Therefore, half of the graviton operators are unphysical and should be removed, after which we are left with the usual two graviton polarizations. I will show that after normal ordering, we obtain a chiral vacuum energy, the first real novelty compared to standard perturbation theory.

The chirality will be explored in more detail in Sect. 2.3.5 where I will derive the main result: The power spectrum of tensor perturbations in the Ashtekar formalism is chiral, if the Immirzi parameter γ has an imaginary part. This would lead to a non-zero TB correlator in the CMB and therefore potentially be observable.

I will finish by discussing the case of a purely real γ in Sect. 2.3.6 before concluding. Note that in the following, in general a complex value of γ will be considered, which can be split into a real and imaginary part,

$$\gamma = \gamma_R + i\gamma_I \; . \tag{2.29}$$

It will sometimes be instructive to focus on the SD/ASD connection for which $\gamma = \pm i$, or a purely imaginary γ, as these cases exhibit special behaviour. The case of a purely real γ, which renders the connection real, will not be considered until Sect. 2.3.6.

2.3.1 The Canonical Variables

To study the tensor perturbations during inflation within the Ashtekar formalism, we will consider the metric

$$ds^2 = a^2[-d\eta^2 + (\delta_{ab} + h_{ab})dx^a dx^b]\,, \tag{2.30}$$

where $a = -\frac{1}{H\eta}$ for a de Sitter background and we have omitted the TT superscript in the perturbation h_{ab}. Note that we will use the following index convention: I and μ refer to 4D internal and space-time indices, respectively, while i, j, ... and a, b, ... denote the corresponding 3D indices.

We need to express the perturbations in the tetrad basis to relate it to the Ashtekar variables. To zeroth order, the metric is given by (see Sect. 2.2.1) $e^{I\,(0)}_{\mu} = a\delta^I_\mu$ and the non-zero spin connection forms are ${\omega^i}_0^{(0)} = He^i$.

Now consider a tetrad basis of the spacetime (2.30) including perturbations, $e^I = e^{I\,(0)} + \delta e^I$. Clearly, the time component is not perturbed, so we only care about the triads e^i. A solution for the triad components that satisfies the defining relation (2.2) to first order (i.e. ignoring second order perturbations) is

$$e^i_a = a\left(\delta^i_a + \frac{1}{2}h^i_a\right). \tag{2.31}$$

Instead of referring to the metric perturbation h_{ab}, we will simply write the perturbation in the triad as

$$e^i_a = a\delta^i_a + \delta e^i_a\;. \tag{2.32}$$

The inverse triad, which needs to satisfy Eq. (2.3) to first order is then given by

$$e^a_i = \frac{1}{a}\delta^a_i - \frac{1}{a^2}\delta e^a_i\;. \tag{2.33}$$

If we remember that δe^i_a is defined as the perturbation in the triad (2.32), we do not need to distinguish between i and a indices and can simply raise and lower them with the Kronecker delta. Although this mixes internal group and spatial indices, we can always unambiguously recover the initial perturbation δe^i_a. We will therefore refer to the perturbed triad as δe_{ij} (and simply call it the metric), and the perturbed Ashtekar connection as a_{ij}. Note that with this convention δe_{ij} will turn out to be proportional to the variable $\tilde{h}^r$ used in Sect. 1.2.5, whose mode functions v obeyed Eq. (1.62).

Like the unperturbed spin connection, its perturbation $\delta{\omega^I}_J$ must satisfy the conditions (2.15), (2.16) and (2.17) due to antisymmetry. We need to expand the torsion free equation (2.11) to first order in terms of $\delta{\omega^I}_J$, $\delta{e^I}_J$ and the unperturbed quantities ${\omega^I}_J^{(0)}$ and $e^{I\,(0)}_J$. For $I = 0$, we have to solve

$${\omega^0}_i^{(0)} \wedge \delta e^i + \delta{\omega^0}_i \wedge e^{i\,(0)} = 0\,, \tag{2.34}$$

where we only kept non-zero spin connection terms and used $\delta e^0 = 0$. Similarly, for $I = i$, we obtain

$$d\delta e^i + \delta\omega^i{}_0 \wedge e^{0\,(0)} + \delta\omega^i{}_j \wedge e^{j\,(0)} = 0\,. \tag{2.35}$$

Using the rules in Sect. 2.2.1, after some algebra we find

$$\delta\omega^0_i = \frac{1}{a}\delta e'_{ij}\, dx^j\,, \tag{2.36}$$

$$\delta\omega_{ij} = -\frac{2}{a}\partial_{[i}\delta e_{j]k}\, dx^k\,, \tag{2.37}$$

where we lowered spatial indices with the Kronecker delta.

We can now define the Ashteker variables perturbed to first order, Eqs. (2.25) and (2.28). Using the background solutions for the triad and spin connection, the definition of the perturbed triad in Eq. (2.33) and noting that $\det\left(e^j_b\right) = a^3$, we obtain

$$E^a_i = a^2\delta^a_i - a\delta e^a_i\,, \tag{2.38}$$

$$A^i_a = \gamma H a\delta^i_a + \frac{a^i_a}{a}\,. \tag{2.39}$$

The classical solution for the perturbed connection a^i_a is given by the perturbed spin connections, (2.36) and (2.37):

$$a_{ij} = \epsilon_{ikl}\partial_k\delta e_{lj} + \gamma\delta e'_{ij}\,. \tag{2.40}$$

Note that this condition is only supposed to be satisfied on-shell, as initially we treat metric and connection as separate variables according to the first order formalism.

To obtain the Poisson brackets for the perturbation variables (which will be promoted to commutators when quantizing), we simply need to plug in expressions (2.38) and (2.39) into the full Poisson brackets (2.24). This results in four Poisson bracket terms of which only the last one is non-zero, which determines the Poisson bracket for fluctuations as

$$\{a^i_a(\mathbf{x}), \delta e^b_j(\mathbf{y})\} = -\frac{\gamma}{m^2_{\rm Pl}}\delta^b_a\delta^i_j\delta(\mathbf{x}-\mathbf{y})\,. \tag{2.41}$$

2.3.2 Hamiltonian Formalism

As we know that the Holst action (2.22) is classically equivalent to the ordinary Einstein-Hilbert action (1.8), the perturbed Ashtekar variables must lead to an equation of motion for the tensor perturbations that is identical to the one you would obtain in the second order formalism. The triad satisfies $\delta e_{ij} = ah_{ij}/2$, which has the same form as the field redefinition of the tensor modes in Sect. 1.2.5, $\tilde{h}^r_{\mathbf{k}} \equiv \frac{a}{2} m_{\mathrm{Pl}} h^r_{\mathbf{k}}$, up to a factor of m_{Pl}. It should therefore also obey the mode equation (1.62). We can obtain the equation of motion for the perturbation δe_{ij} from Hamilton's equations (derived for the full Ashtekar variables) by keeping only the first order part. Later in this section I will derive the same equations from a perturbed Hamiltonian instead.

The Hamiltonian constraint in the Ashtekar formalism for a general γ is given by [19]:

$$\mathcal{H} = \frac{m_{\mathrm{Pl}}^2}{2} \int d^3x N E^a_i E^b_j \Big[\epsilon_{ijk} (F^k_{ab} + H^2 \epsilon_{abc} E^c_k) - 2(1+\gamma^2) K^i_{[a} K^j_{b]} \Big] . \quad (2.42)$$

Let me define the new quantities appearing in (2.42): The field strength F^i of the Ashtekar connection A^i is given by

$$F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^{ijk} A^j_a A^k_b \,, \quad (2.43)$$

K^i is the extrinsic curvature of the spatial surfaces,

$$K^i_a = \frac{A^i_a - \omega^i_a(E)}{\gamma} \quad (2.44)$$

(on shell this becomes $K^i_a \approx \omega^{0i}_a$) and $N = 1/a^2$ is the lapse density. For a SD/ASD connection, $\gamma = \pm i$, the term involving the extrinsic curvature vanishes, greatly simplifying the constraint.

We also need to take into account a Hamiltonian boundary term [46–48],

$$\mathcal{H}_{BT} = -m_{\mathrm{Pl}}^2 \int d\Sigma_a N \epsilon_{ijk} E^a_i E^b_j A_{bk} \,. \quad (2.45)$$

Although the boundary term is often ignored by imposing fall-off condition at infinity [48, 49], this cannot be done in general, e.g. when using a plane wave expansion. Therefore, it will turn out to be essential to include the boundary term in order to recover the correct classical solution.

The full Hamiltonian has two other constraints [19] [as was shown in the Holst action (2.22)], the Gauss constraint

$$G_i = D_a E^a_i = \partial_a E^a_i + \epsilon_{ijk} A^j_a E^a_k \approx 0 \,, \quad (2.46)$$

and the vector constraint

$$V_b = E^a_i F^i_{ab} \approx 0 \,, \quad (2.47)$$

which is a linear combination of the Gauss and diffeomorphism constraint. Both constraints are satisfied by the background solution. It can be checked that they are also satisfied to first order using the perturbed variables (2.38) and (2.39). We will usually not be concerned with these constraints, as they do not encode the dynamics of the theory, but I will comment on their significance when perturbing the Hamiltonian to second order later.

2.3.2.1 Hamilton's Equations

To derive Hamilton's equations for the full Ashtekar variables, we need to make use the Poisson brackets in Eq. (2.24) and remember the rule $\{A, BC\} = \{A, B\}C + B\{A, C\}$. Hamilton's equations for $\gamma = \pm i$ (where the terms proportional to $(1+\gamma^2)$ in Eq. (2.42) can be ignored) take a fairly concise form:

$$A_a^{i\,\prime} = \{A_a^i, \mathcal{H}\} = \gamma N \epsilon_{ijk} E_j^b \left(F_{ab}^k + \frac{3}{2} H^2 \epsilon_{abc} E_k^c \right), \tag{2.48}$$

$$E_i^{a\prime} = \{E_i^a, \mathcal{H}\} = -\gamma \epsilon_{ijk} D_b (N E_j^a E_k^b) , \tag{2.49}$$

where D_a is the covariant derivative taken with the connection A^i. We can obtain evolution equations for the perturbations by plugging Eqs. (2.38) and (2.39) into (2.48) and (2.49) and expanding to first order. This gives the Hamilton equations for the perturbations,

$$a_{ij}' = 2\gamma H^2 a^2 \delta e_{ij} - \gamma \epsilon_{inm} \partial_n a_{mj} , \tag{2.50}$$

$$\delta e_{ij}' = \frac{1}{\gamma} (a_{ij} - \epsilon_{inm} \partial_n \delta e_{mj}) . \tag{2.51}$$

Hamilton's equation for δe_{ij} is the same as (2.40), i.e. it simply encodes the torsion free condition which must be satisfied on shell. Taking the derivative of (2.51), and eliminating the time derivative of the perturbed connection through (2.50), makes it possible to obtain a second order equation for δe_{ij}, independent of the connection:

$$\delta e_{ij}'' - \left(\nabla^2 + \frac{2}{\eta^2} \right) \delta e_{ij} = 0 . \tag{2.52}$$

This is the same as Eq. (1.62) in real space, proving that classically, the standard formalism of cosmological perturbation theory and the Ashtekar framework are equivalent, at least for the case $\gamma = \pm i$. Note that γ has dropped out of the equation, as it should not affect any classical results.

The Hamiltonian (2.42) of the Ashtekar formalism has been chosen such that it can be related to the ordinary Einstein-Hilbert action by a change of variables, for any choice of γ. Therefore we know that Eq. (2.52) needs to hold in the general case as well. This will help us in deriving Hamilton's equations for the perturbations. For a general γ, Hamilton's equations, derived for the full Ashtekar variables, are a lot more complicated than in the SD/ASD case. Taking the Poisson brackets with the

Hamiltonian (2.42), we obtain the same expression as in Eqs. (2.48) and (2.49), plus additional terms proportional to $(1 + \gamma^2)$:

$$A_a^{i\,\prime} = \gamma N \epsilon_{ijk} E_j^b \left(F_{ab}^k + \frac{3}{2} H^2 \epsilon_{abc} E_k^c \right) - \gamma(1+\gamma^2) N E_j^b (K_b^j K_a^i - K_a^j K_b^i)$$
$$-m_{\rm Pl}^2 (1+\gamma^2) \int d^3y N E_j^b E_k^c \{A_a^i(x), \omega_{[b}^j \omega_{c]}^k\} \tag{2.53}$$

$$E_i^{a\prime} = -\gamma \epsilon_{ijk} D_b (N E_j^a E_k^b) + (1+\gamma^2) N (E_i^a E_j^b - E_j^a E_i^b) K_b^j \ . \tag{2.54}$$

The Poisson bracket $\{A, \omega(E)\}$ is a very long and messy expression, so the last term of Eq. (2.53) is left unexpanded. As for the case $\gamma = \pm i$, we can obtain the evolution equations for the perturbations by substituting the definition of the Ashtekar variables into (2.53) and (2.54) and expanding to first order.

In the case of the triad, this yields the same expression as before, Eq. (2.51). We would like to avoid having to work out Hamilton's equation for a_{ij} explicitly as it would involve having to compute the unexpanded Poisson bracket in (2.53). As we know that e_{ij} needs to satisfy the equation of motion (2.52), we can actually avoid doing the explicit calculation and simply use Eqs. (2.51) and (2.52) to deduce Hamilton's equation for the connection. It should contain the terms on the RHS of Eq. (2.50) plus additional terms proportional to $(1 + \gamma^2)$, such that it reduces to the old expression for $\gamma = \pm i$. Carrying out these manipulations, we finally obtain Hamilton's equations for the perturbations for a general value of γ:

$$a'_{ij} = 2\gamma H^2 a^2 \delta e_{ij} - \gamma \epsilon_{inm} \partial_n a_{mj} + \frac{1+\gamma^2}{\gamma} \epsilon_{inm} \partial_n (a_{mj} - \epsilon_{mkl} \partial_k \delta e_{lj}) \ , \tag{2.55}$$

$$\delta e'_{ij} = \frac{1}{\gamma} (a_{ij} - \epsilon_{inm} \partial_n \delta e_{mj}) \ . \tag{2.56}$$

At first glance, it might seem odd that these expressions yield the same equation of motion for the triad [Eq. (2.52)] as in the case $\gamma = \pm i$, considering the connection equation has acquired an additional term in $1 + \gamma^2$ compared to Eq. (2.50). However, this is necessary as terms proportional to $1 + \gamma^2$ do actually appear in the derivation of the result for the $\gamma = \pm i$ case, where they can be set to zero. These terms must be present in the case of general γ.

2.3.2.2 Second Order Hamiltonian

We have found the equations of motion for the perturbations by perturbing the full Hamilton equations. However, to be able to quantize the theory, we need to identify the perturbed Hamiltonian. This exercise is not trivial; as we will see in the following, a fair number of subtleties need to be taken into account before arriving at the correct result.

The perturbed Hamiltonian should contain tensor perturbations and encode the dynamics of gravitons. Therefore, we know that the constraint $\mathcal{H} \approx 0$, which demonstrates the lack of dynamics, cannot apply to the perturbative Hamiltonian which we would like to quantize. Let us think about the Hamiltonian to different orders in the perturbative expansion.

The first order Hamiltonian is trivially zero (once the other constraints are used). The second order Hamiltonian, on the other hand, includes two terms,

$$ {}^2\mathcal{H} = {}^2_1\mathcal{H} + {}^2_2\mathcal{H}\,, \tag{2.57} $$

where ${}^2_1\mathcal{H}$ contains products of first order perturbations, and ${}^2_2\mathcal{H}$ is linear in second order perturbations in the triad and connection. Only the sum of these terms vanishes on shell, ${}^2\mathcal{H} \approx 0$. We can therefore identify the first term, ${}^2_1\mathcal{H}$, with the dynamical Hamiltonian to second order, while the second term ${}^2_2\mathcal{H}$ simply encodes the backreaction or compensation due to the non-linearity of the gravitational field, which ensures that the Hamiltonian constraint is satisfied. Therefore, we will need to calculate ${}^2_1\mathcal{H}$ to understand graviton dynamics.

Let me also stress that in the Ashtekar formulation, off-shell, *the Hamiltonian is not real*, due to the presence of the complex Immirzi parameter γ. Of course, imposing the constraints, the Hamiltonian becomes weakly zero and is therefore manifestly real. However, as the constraint does not apply to the dynamical second order Hamiltonian, ${}^2_1\mathcal{H}$ is indeed complex. The complexity of ${}^2_1\mathcal{H}$ will have an effect on perturbation theory, and the novelties I will describe can be traced back to this fact. Even though a complex Hamiltonian might seem strange, the quantum theory we set up later (Sect. 2.3.4) will still be well defined. All classical results can be recovered and the quantum Hamiltonian will turn out to be hermitian after fixing the inner product.

Before proceeding, note that the other constraints are also not zero when considering only the second order part that is quadratic in first order perturbations. Specifically, for the Gauss constraint we get

$$ {}^2_1G_i = -\epsilon_{ijk} a^j_a \delta e^a_k \neq 0\,. \tag{2.58} $$

When deriving (2.42) from the usual ADM action, the Gauss constraint and the torsion free condition are used [19]. Therefore, non-zero terms proportional to ${}^2_1G_i$ and ${}^2_1T^a$ will appear in the expression for ${}^2_1\mathcal{H}$. However, it can be checked that these additional terms result in a full divergence and can therefore be ignored.

By expanding the Hamiltonian (2.42) to second order we obtain:

$$ \begin{aligned} {}^2_1\mathcal{H} = \frac{m_{\rm Pl}^2}{2}\int d^3x \Bigg\{ & \frac{1}{\gamma^2} a_{ij}a_{ij} + 2\epsilon_{ijk}\delta e_{li}\partial_j a_{kl} - 2H^2a^2\delta e_{ij}\delta e_{ij} \\ & + \frac{2}{\gamma} Ha\delta e_{ij}a_{ij} - 2\frac{1+\gamma^2}{\gamma} Ha\delta e_{ij}\epsilon_{ikl}(\partial_k\delta e_{lj}) \\ & - \frac{1+\gamma^2}{\gamma^2}\Big[\epsilon_{ikl}(\partial_k\delta e_{lj})a_{ij} + \epsilon_{ikl}a_{ij}(\partial_k\delta e_{lj}) - \epsilon_{ikl}\epsilon_{jmn}(\partial_k\delta e_{lj})(\partial_m\delta e_{ni})\Big]\Bigg\}\,, \end{aligned} \tag{2.59} $$

where we kept the ordering as it appeared in the calculation, as it will affect the quantization. Only the first four terms survive for $\gamma = \pm i$. This expression is not the correct perturbative Hamiltonian, however: it does not reduce to the Hamiltonian (1.59) obtained for tensor perturbations in the second order formalism on shell, i.e. when using the torsion free condition (2.40). This is due to two reasons.

First, we have not yet included the boundary term (2.45) at the same order and level in perturbation theory (second order terms quadratic in first order variables). It is given by

$$ {}^{2}_{1}\mathcal{H}_{BT} = m_{\rm Pl}^2 \int d\Sigma_i \epsilon_{ijk} \delta e_{lj} a_{lk} \,. \tag{2.60}$$

To make this into a volume instead of a surface integral, we use Stokes' theorem [50] which introduces a divergence,

$$ {}^{2}_{1}\mathcal{H}_{BT} = \frac{m_{\rm Pl}^2}{2} \int d^3x \; 2\epsilon_{ijk} \partial_i (\delta e_{lj} a_{lk}) \,, \tag{2.61}$$

where we introduced factors of two to obtain the same pre-factor as in (2.59). The derivative term can be split into two contributions, one of which cancels the second term of Eq. (2.59) and the other is $-2\epsilon_{ijk}(\partial_j \delta e_{li}) a_{kl}$.

The second issue is more subtle and related to the terms proportional to H in (2.59). There should not be any terms linear in the Hubble rate, as we want to rederive the Hamiltonian for ordinary tensor perturbations, Eq. (1.59), where the only explicitly time dependent term is a''/a, which in de Sitter is given by $2/\eta^2 = 2a^2H^2$ and is therefore quadratic in H.

To understand what has gone wrong, recall the perturbed expression for the triad and connection:

$$ A_a^i = \gamma H a \delta_a^i + \frac{a_a^i}{a} \,, \tag{2.62}$$

$$ E_i^a = a^2 \delta_i^a - a \delta e_i^a \,. \tag{2.63}$$

Instead of thinking of this as a zero order part plus a perturbation, you can also regard it as a canonical transformation [51]: we have replaced variables (A_a^i, E_j^b) with variables $(a_a^i, \delta e_j^b)$, which have the same symplectic structure as the original variables (the fact that the Poisson brackets (2.41) have a minus sign compared to (2.24) is related to the fact that we defined the perturbation δe_j^b in the densitized triad, not its inverse, initially. We could also absorb the minus sign into a field redefinition of the triad perturbation). Such a transformation can always be performed for canonical systems, regardless of whether the new variables are small perturbations. In this viewpoint, instead of "freezing" the background and considering spacetime perturbations around it, we regard the perturbed variables as equivalent to the full Ashtekar variables.

If the canonical transformation is explicitly time dependent (which it is as a is a function of time), the Hamiltonian in terms of the new variables, denoted by $\mathcal{K}$, is related to the old Hamiltonian by a generating function F [51]:

$$\mathcal{K} = \mathcal{H} + \frac{\partial F}{\partial \eta} \,. \tag{2.64}$$

To obtain the correct Hamiltonian, in principal we therefore need to compute the generating function. However, again it is possible to "cheat" slightly by using consistency arguments instead of performing explicit calculations. We know that it should be possible to derive Hamilton's equations for the perturbations by taking the Poisson brackets (2.41) with the (correct) perturbed Hamiltonian to second order. By demanding consistency with Eqs. (2.55) and (2.56), which were obtained from perturbing the full Hamilton's equations, we find that the appropriate generating function must be

$$\frac{\partial F}{\partial \eta} = -\frac{m_{\text{Pl}}^2}{\gamma} \int d^3x H a \delta e_{ij} \left[a_{ij} - (1+\gamma^2)\epsilon_{ikl}\partial_k \delta e_{lj} \right] \,.$$

Adding this term to the Hamiltonian in (2.59) eliminates the second line, i.e. the terms proportional to H. The final expression, taking the boundary term (2.61) into account, is therefore:

$$\begin{aligned} \mathcal{H}_{eff} = \frac{m_{\text{Pl}}^2}{2} \int d^3x \Bigg[& \frac{1}{\gamma^2} a_{ij}a_{ij} - 2H^2 a^2 \delta e_{ij}\delta e_{ij} + \left(1 - \frac{1}{\gamma^2}\right)\epsilon_{ikl}(\partial_k \delta e_{lj}) a_{ij} \\ & - \left(1 + \frac{1}{\gamma^2}\right)\epsilon_{ikl} a_{ij}(\partial_k \delta e_{lj}) + \left(1 + \frac{1}{\gamma^2}\right)\epsilon_{ikl}\epsilon_{jmn}(\partial_k \delta e_{lj})(\partial_m \delta e_{ni}) \Bigg] . \end{aligned} \tag{2.65}$$

This corresponds to the effective perturbative Hamiltonian, which can be used to quantize the theory in terms of graviton states.

By using the on-shell condition (2.40), we can derive the Hamiltonian in terms of the triad only, remembering $2a^2H^2 = a''/a$:

$$\mathcal{H}_{eff}|_{\text{on-shell}} = \frac{m_{\text{Pl}}^2}{2} \int d^3x \left[\delta e'_{ij}\delta e'_{ij} + (\partial_k \delta e_{ij})^2 - \frac{a''}{a}\delta e_{ij}\delta e_{ij} \right] . \tag{2.66}$$

After identifying the two physical polarizations of the triad by using appropriate mode expansions in the next section, it will be clear that this is exactly the same as expression as (1.59), the second order Hamiltonian for tensor modes derived in the second order formalism.

2.3.3 Fourier Space Expansion

To be able to quantize the theory, we need to expand the perturbed variables in terms of Fourier modes. However, we need to be careful that we perform this expansion correctly, by taking into account two separate, but related points.

Firstly, note that in the Ashtekar formalism, the connection is initially complex and we are not enforcing any reality conditions before quantizing. Therefore, we must have graviton and anti-graviton modes in the expansion. This means that the negative and positive frequencies in the field expansion are initially independent (so compared to Eq. (1.45), we should have a different operator $b_{\mathbf{k}}$ associated with the second term). Secondly, we will make the common field theory choice stipulating that the spatial vector $\mathbf{k}$ points in the direction of propagation for both positive and negative frequency terms. The reality conditions will then identify gravitons and anti-gravitons moving in the same direction, not in opposite directions.

This choice not always been made in previous literature on the subject, where non-physical couplings between $\mathbf{k}$ and $-\mathbf{k}$ modes appeared in the physical Hamiltonian inside the horizon [49, 52]. These should only be present outside the horizon, where they represent the production of particle pairs by the gravitational field (with the particles in each pair moving in opposite directions) [53].

We therefore make the following Fourier expansion:

$$\begin{aligned}\delta e_{ij} &= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \sum_r \epsilon^r_{ij}(\mathbf{k})\tilde{e}_{r+}(\mathbf{k},\eta)e^{i\mathbf{k}\cdot\mathbf{x}} + \epsilon^{r\star}_{ij}(\mathbf{k})\tilde{e}^\dagger_{r-}(\mathbf{k},\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}\,,\\ a_{ij} &= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \sum_r \epsilon^r_{ij}(\mathbf{k})\tilde{a}_{r+}(\mathbf{k},\eta)e^{i\mathbf{k}\cdot\mathbf{x}} + \epsilon^{r\star}_{ij}(\mathbf{k})\tilde{a}^\dagger_{r-}(\mathbf{k},\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \end{aligned} \tag{2.67}$$

where $\tilde{e}_{rp}(\mathbf{k},\eta) = e_{rp}(\mathbf{k})\Psi_e(k,\eta)$ and $\tilde{a}_{rp}(\mathbf{k},\eta) = a_{rp}(\mathbf{k})\Psi^{rp}_a(k,\eta)$, and ϵ^r_{ij} are polarization tensors. In a frame where the direction $i=1$ is aligned with $\mathbf{k}$, they are given by [54]:

$$\epsilon^{(r)}_{ij} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \pm i \\ 0 & \pm i & -1 \end{pmatrix}\,. \tag{2.68}$$

Equation (2.67) has the same form as the mode expansion for tensor perturbations (1.57) performed in Sect. 1.2.5, but now with an additional negative frequency term which is independent of the first, as required. The amplitudes $a_{rp}(\mathbf{k})$ and $e_{rp}(\mathbf{k})$ have two indices (in contrast with some of the previous literature [49, 52]): $r=\pm 1$ for right (R) and left (L) helicities, and p for graviton ($p=1$) and anti-graviton ($p=-1$) modes (which were not present in Eq. (1.57), where the tensor perturbations were manifestly real).

We can assume that the amplitudes a_{rp} and e_{rp}, which will correspond to annihilation operators upon quantization, are equal, and the differences can be absorbed into the mode functions Ψ_e and Ψ_a. Imposing the on-shell condition we will find that while Ψ_e is independent of helicity and graviton states, the mode functions for the connection, $\Psi_a(k, \eta)$, must carry an r, p dependence.

2.3.3.1 Mode Functions

As we have seen that the Ashtekar formalism is equivalent to the second order formalism of Sect. 1.2.5, we know that the mode functions of the triad will satisfy the equation of motion (1.62)

$$\Psi_e'' + \left(k^2 - \frac{2}{\eta^2}\right)\Psi_e = 0, \tag{2.69}$$

where $'$ denotes differentiation with respect to conformal time. This has the Bunch-Davies solution given in Eq. (1.62),

$$\Psi_e = \frac{e^{-ik\eta}}{2\sqrt{k}}\left(1 - \frac{i}{k\eta}\right) . \tag{2.70}$$

The boundary condition in the far past, $|k\eta| \gg 1$, is

$$\Psi(k, \eta) \sim e^{-ik\eta} . \tag{2.71}$$

This shows that $\mathbf{k}$ can be regarded as the direction of propagation of the wave as the exponentials in which $\mathbf{k}$ appears can be written in four-vector form as $e^{-ik\eta}e^{i\mathbf{k}\cdot\mathbf{x}} = e^{ik_\mu x^\mu}$, $k_\mu k^\mu = 0$.

Let us find an expression for the mode functions of the connection on-shell. We need to plug the Fourier space expansion (2.67) into the classical solution of the connection derived from the torsion free condition, Eq. (2.40). Making use of the identity

$$\epsilon_{inl} k_n \epsilon_{lj}^{(r)} = -irk\epsilon_{ij}^{(r)} , \tag{2.72}$$

we find

$$\Psi_a^{rp} = (\gamma_R + p\gamma_I)\Psi_e' + rk\Psi_e . \tag{2.73}$$

This expression can be simplified inside the horizon ($k|\eta| \gg 1$), when the boundary condition (2.71) holds:

$$\Psi_a^{rp} = \Psi_e k \left(r - i\gamma_R + p\gamma_I\right) , \tag{2.74}$$

There is only a dependence on p if γ has an imaginary part and for a purely real γ, Ψ_a^r would be the same for gravitons and anti-gravitons. This is to be expected, as

Table 2.1 Relationship between helicity and duality states

	$r = +$ [R]	$r = -$ [L]
$p = +$ $[G]$	SD	ASD
$p = -$ $[\overline{G}]$	ASD	SD

for a manifestly real theory we would not have needed to expand in terms of two different operators a_{r+} and a_{r-}, but just a single a_r.

Before carrying on with the quantization of the perturbations, let us briefly investigate the relationship between the helicity states, labelled by r, and the duality states, defined by $\gamma = \pm i$. In this case, Eq. (2.74) becomes

$$\Psi_a^{rp} = (r - ip\gamma)k\Psi_e \ . \tag{2.75}$$

For an SD connection, $i\gamma = -1$, and the quantity in brackets is simply $(r + p)$. This is clearly zero if r and p have different signs. Therefore, the only components of the connection that survive in the self dual case are the right handed ($r = 1$) positive frequency of the graviton ($p = 1$) and the left handed ($r = -1$) negative frequency of the anti-graviton ($p = -1$). The ASD connection has $i\gamma = +1$ and therefore contains the remaining degrees of freedom, right handed anti-graviton and left-handed graviton. The split of the states into SD and ASD parts is summarized in Table 2.1.

This analysis shows that helicity modes and duality modes do not align, i.e. the SD connection carries both right and left-handed helicity states and similarly for the ASD connection. This point has been highlighted in [55], but it requires performing the correct Fourier space expansion including graviton and anti-gravitons states and was therefore missed in [49, 52].

2.3.3.2 Reality Conditions

When we set up the Hilbert space of quantum states in Sect. 2.3.4, we will need to impose reality conditions to relate graviton and anti-graviton states (and their Hermitian conjugates), which will enable us to obtain the physical degrees of freedom. The reality conditions will eventually be used to fix the inner product, but it is instructive to obtain the corresponding conditions on the operators.

The metric is real, $\delta e_{ij} = \overline{\delta e}_{ij}$. Imposing this on the Fourier expansion, we find

$$e_{r+}(\mathbf{k}) = e_{r-}(\mathbf{k}) \ . \tag{2.76}$$

Therefore, graviton and anti-graviton are identified for each polarization and each mode $\mathbf{k}$. This is a good check that the expansion in Eq. (2.67) is physically sensible, as we do not get relations between different polarizations or wavevectors $\mathbf{k}$ and $-\mathbf{k}$.

On-shell, the triad therefore only needs one set of creation and annihilation operators in its Fourier expansion.

For the connection, the torsion free condition and the reality condition are linked: Although the connection can be complex, it must satisfy the torsion-free condition, which will ensure that the metric is real. From the defining expression for the Ashtekar connection, Eq. (2.28), we know that

$$\Re A^i = \omega^i + \gamma_R \omega^{0i} \,, \tag{2.77}$$

$$\Im A^i = \gamma_I \omega^{0i} \,. \tag{2.78}$$

There are two reality conditions for the connection, but we only need to impose one as a constraint, as the dynamical evolution (described by Hamilton's equations) will make sure that the second condition is satisfied. Let us see what this implies for the perturbations a_{ij}. Using the solutions for the perturbed spin connection components, Eqs. (2.36) and (2.37), we obtain

$$a_{ij} + \bar{a}_{ij} = 2a\left(\delta\omega_{ij} + \gamma_R \delta\omega^0_{ij}\right) = 2\epsilon_{ikl}\partial_k \delta e_{lj} + 2\gamma_R \delta e'_{ij} \,, \tag{2.79}$$

$$a_{ij} - \bar{a}_{ij} = 2ai\gamma_I \delta\omega^0_{ij} = 2i\gamma_I \delta e'_{ij} \,. \tag{2.80}$$

Using the expansion (2.67), in Fourier space this becomes

$$\tilde{a}_{r+}(\mathbf{k},\eta) + \tilde{a}_{r-}(\mathbf{k},\eta) = 2rk\tilde{e}_{r+}(\mathbf{k},\eta) + 2\gamma_R \tilde{e}'_{r+}(\mathbf{k},\eta) \,, \tag{2.81}$$

$$\tilde{a}_{r+}(\mathbf{k},\eta) - \tilde{a}_{r-}(\mathbf{k},\eta) = 2i\gamma_I \tilde{e}'_{r+}(\mathbf{k},\eta) \,, \tag{2.82}$$

where $\tilde{a}_{rp} = a_{rp}\Psi_a^{rp}$ and $\tilde{e}_{rp} = e_{rp}\Psi_e$. The reality condition for the connection we want to impose as a constraint should be non-dynamical, so let us eliminate the time derivative of the metric by combining Eqs. (2.81) and (2.82):

$$i\gamma^* \tilde{a}_{r+}(\mathbf{k},\eta) - i\gamma \tilde{a}_{r-}(\mathbf{k},\eta) = 2rk\gamma_I \tilde{e}_{r+}(\mathbf{k},\eta) \,. \tag{2.83}$$

Its Hermitian conjugate is:

$$-i\gamma \tilde{a}^\dagger_{r+}(\mathbf{k},\eta) + i\gamma^* \tilde{a}^\dagger_{r-}(\mathbf{k},\eta) = 2rk\gamma_I \tilde{e}^\dagger_{r-}(\mathbf{k},\eta) \,, \tag{2.84}$$

where we have used Eq. (2.76) to turn $p = 1$ into $p = -1$ on the RHS. This shows that for each r and $\mathbf{k}$ there are two independent conditions upon the four operators $a_{rp}(\mathbf{k})$ and $e_{rp}(\mathbf{k})$. We will use them later when we define the inner product.

On shell, we can use the full torsion-free conditions, Eq. (2.73), which can be written as a weak identity on the operators:

$$\tilde{a}_{r-}(\mathbf{k},\eta) \approx rk\tilde{e}_r + \gamma^* \tilde{e}'_r \rightarrow \tilde{e}_r(r - i\gamma^*)k \,, \tag{2.85}$$

$$\tilde{a}_{r+}(\mathbf{k},\eta) \approx rk\tilde{e}_r + \gamma \tilde{e}'_r \rightarrow \tilde{e}_r(r - i\gamma)k \,, \tag{2.86}$$

where the latter expression is valid in the limit $k|\eta| \gg 1$, c.f. Eq. (2.74). These identities will be useful later when deriving the graviton operators for this theory, as they will show that one of the graviton modes is unphysical.

2.3.3.3 Commutation Relations

Before we can set up a quantum theory in terms of graviton operators we need to define the commutation relations for the modes. To do this, we first promote the Poisson brackets (2.24) and (2.41) of the connection and metric in position space to commutators:

$$\left[A_a^i(\mathbf{x}), E_j^b(\mathbf{y})\right] = i\frac{\gamma}{m_{\rm Pl}^2}\delta_a^b\delta_j^i\delta(\mathbf{x}-\mathbf{y})\,, \tag{2.87}$$

$$\left[a_a^i(\mathbf{x}), \delta e_j^b(\mathbf{y})\right] = -i\frac{\gamma}{m_{\rm Pl}^2}\delta_a^b\delta_j^i\delta(\mathbf{x}-\mathbf{y})\,. \tag{2.88}$$

Note that these commutators have been derived from the fundamental Poisson brackets of the Ashtekar variables and hence have not been gauge fixed yet, i.e. the TT projection has not been carried out and we therefore have not identified the two physical polarizations of tensor perturbations. The Fourier expansion (2.67), on the other hand, assumed by construction that there are only two helicity states $r = \pm 1$. It was shown in [56] that the appropriate form of the commutator (2.88), taking care of the gauge fixing, is

$$[a_{ij}(\mathbf{x}), \delta e_{kl}(\mathbf{y})] = -i\frac{\gamma}{m_{\rm Pl}^2}P_{ijkl}(\mathbf{x}-\mathbf{y})\,, \tag{2.89}$$

where the delta function is replaced by a function $P_{ijkl}(\mathbf{x})$ which takes care of the TT projection and is given by

$$P_{ijkl}(\mathbf{x}) = \int\frac{d^3k}{(2\pi)^3}\sum_r \epsilon_{ij}^r(\mathbf{k})\epsilon_{kl}^{r\star}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\,. \tag{2.90}$$

To obtain the equivalent of Eq. (2.89) for modes, let us first consider the unprojected commutator (2.88) again. Dropping the indices, we can split the metric and connection into separate positive and negative frequency parts, $\delta e = \delta e^+ + \delta e^-$, $a = a^+ + a^-$, which are given by

$$\delta e^+(\mathbf{x},\eta) = \int\frac{d^3k}{(2\pi)^{\frac{3}{2}}}e^+(\mathbf{k},\eta)e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{2.91}$$

$$\delta e^-(\mathbf{x},\eta) = \int\frac{d^3k}{(2\pi)^{\frac{3}{2}}}e^{-\dagger}(\mathbf{k},\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \tag{2.92}$$

and similarly for a.

Therefore there are four terms in the commutator and, as is standard in QFT [6], the only non-vanishing equal-time commutators must be given by positive and negative frequency parts,

$$[a^+(\mathbf{x}), \delta e^-(\mathbf{y})] = [a^-(\mathbf{x}), \delta e^+(\mathbf{y})] = -i\frac{\gamma}{2m_{\text{Pl}}^2}\delta(\mathbf{x}-\mathbf{y}) \; . \tag{2.93}$$

For the modes, this implies

$$[a^+(\mathbf{k}), e^{-\dagger}(\mathbf{k}')] = [a^-(\mathbf{k}), e^{+\dagger}(\mathbf{k}')] = -i\frac{(\gamma_R + pi\gamma_I)}{2m_{\text{Pl}}^2}\delta(\mathbf{k}-\mathbf{k}') \; , \tag{2.94}$$

Taking expression (2.89) for the TT projected position space commutators into account, we see that the operators we have defined in the Fourier expansion (2.67) have commutation relation

$$[\tilde{a}_{rp}(\mathbf{k}), \tilde{e}^{\dagger}_{sq}(\mathbf{k}')] = -i\frac{(\gamma_R + pi\gamma_I)}{2m_{\text{Pl}}^2}\delta_{rs}\delta_{p\bar{q}}\delta(\mathbf{k}-\mathbf{k}') \; , \tag{2.95}$$

where $\overline{q} = -q$.

The dependence on $\delta_{p\bar{q}}$ shows that we only get non-vanishing commutators when considering the positive frequency of one variable and the negative frequency of the other. As before, when we considered the mode functions of the connection, there is no p dependence if $\gamma_I = 0$, as for a real field there is no distinction between gravitons and anti-gravitons.

2.3.4 *Quantum Hamiltonian*

We now have all the ingredients to set up the Hamiltonian in Fourier space which will be the starting point for the quantum theory. We want to express it in the standard form where it just reduces to a creation times an annihilation operator, counting the number of states, c.f. Eq. (1.44). In our case, these states will be graviton states and the operators will create and annihilate gravitons. As we have not imposed the torsion-free condition yet, the graviton operators will be linear combinations of the metric and connection, and only reduce to metric variables on-shell. Due to the complexity of the Hamiltonian, this exercise is non-trivial. We will find twice as many particle states as expected as well as unphysical particle production terms. However, once the correct inner product has been identified, we will reproduce the expected form of the Hamiltonian.

Note that from now on we will consider the inside horizon limit $k\eta \gg 1$ for which terms in H can be neglected, as we are not interested in the behaviour of tensor perturbations outside the horizon where they freeze out.

Inserting the expansion (2.67) into (2.65) and making use of the relations

$$\epsilon^r_{ij}(\mathbf{k})\epsilon^{s\star}_{ij}(\mathbf{k}) = 2\delta^{rs}\,, \quad \epsilon^r_{ij}(-\mathbf{k}) = \epsilon^{r\star}_{ij}(\mathbf{k})\,, \tag{2.96}$$

we obtain a lengthy expression for the Fourier space Hamiltonian:

$$\begin{aligned}
\mathcal{H}_{eff} = m^2_{\mathrm{Pl}} \int d^3k \sum_r \frac{1}{\gamma^2} \Big\{ \\
& \Big[k^2\left(\gamma^2+1\right)\tilde{e}_{r+}(\mathbf{k}) - kr\left(\gamma^2+1\right)\tilde{a}_{r+}(\mathbf{k})\Big]\tilde{e}_{r+}(-\mathbf{k}) \\
& +\Big[k^2\left(\gamma^2+1\right)\tilde{e}_{r+}(\mathbf{k}) - kr\left(\gamma^2+1\right)\tilde{a}_{r+}(\mathbf{k})\Big]\tilde{e}^\dagger_{r-}(\mathbf{k}) \\
& +\Big[k^2\left(\gamma^2+1\right)\tilde{e}^\dagger_{r-}(\mathbf{k}) - kr\left(\gamma^2+1\right)\tilde{a}^\dagger_{r-}(\mathbf{k})\Big]\tilde{e}_{r+}(\mathbf{k}) \\
& +\Big[k^2\left(\gamma^2+1\right)\tilde{e}^\dagger_{r-}(\mathbf{k}) - kr\left(\gamma^2+1\right)\tilde{a}^\dagger_{r-}(\mathbf{k})\Big]\tilde{e}^\dagger_{r-}(-\mathbf{k}) \\
& +\Big[kr\left(\gamma^2-1\right)\tilde{e}_{r+}(\mathbf{k}) + \tilde{a}_{r+}(\mathbf{k})\Big]\tilde{a}_{r+}(-\mathbf{k}) \\
& +\Big[kr\left(\gamma^2-1\right)\tilde{e}_{r+}(\mathbf{k}) + \tilde{a}_{r+}(\mathbf{k})\Big]\tilde{a}^\dagger_{r-}(\mathbf{k}) \\
& +\Big[kr\left(\gamma^2-1\right)\tilde{e}^\dagger_{r-}(\mathbf{k}) + \tilde{a}^\dagger_{r-}(\mathbf{k})\Big]\tilde{a}_{r+}(\mathbf{k}) \\
& +\Big[kr\left(\gamma^2-1\right)\tilde{e}^\dagger_{r-}(\mathbf{k}) + \tilde{a}^\dagger_{r-}(\mathbf{k})\Big]\tilde{a}^\dagger_{r-}(-\mathbf{k})\Big\}\,.
\end{aligned} \tag{2.97}$$

2.3.4.1 Hamiltonian for $\gamma = \pm i$

Before trying to make sense of this monstrosity, it is instructive to study the case of a SD/ASD connection for which $\gamma^2 = -1$. In this case, Eq. (2.97) reduces to a much more tractable form:

$$\begin{aligned}
\mathcal{H}_{eff} = {} & m^2_{\mathrm{Pl}} \int d^3k \sum_r g_{r-}(\mathbf{k})g_{r+}(-\mathbf{k}) + g_{r-}(\mathbf{k})g^\dagger_{r-}(\mathbf{k}) \\
& + g^\dagger_{r+}(\mathbf{k})g_{r+}(\mathbf{k}) + g^\dagger_{r+}(\mathbf{k})g^\dagger_{r-}(-\mathbf{k})\,,
\end{aligned} \tag{2.98}$$

where

$$g_{r+}(\mathbf{k}) = \tilde{a}_{r+}(\mathbf{k})\,, \tag{2.99}$$
$$g^\dagger_{r+}(\mathbf{k}) = -\tilde{a}^\dagger_{r-}(\mathbf{k}) + 2kr\tilde{e}^\dagger_{r-}(\mathbf{k})\,, \tag{2.100}$$
$$g_{r-}(\mathbf{k}) = -\tilde{a}_{r+}(\mathbf{k}) + 2kr\tilde{e}_{r+}(\mathbf{k})\,, \tag{2.101}$$
$$g^\dagger_{r-}(\mathbf{k}) = \tilde{a}^\dagger_{r-}(\mathbf{k})\,, \tag{2.102}$$

which can be identified as the the graviton ($p = 1$) and anti-graviton ($p = -1$) creation and annihilation operators $g^{\dagger}_{rp}$, g_{rp}. Note that the creation and annihilation operators for each index r, p are only hermitian conjugates of each after the reality conditions (2.83) and (2.84) have been imposed.

Their commutation relations can be derived from Eq. (2.95):

$$[g_{rp}(\mathbf{k}), g^{\dagger}_{sq}(\mathbf{k}')] = -\frac{i\gamma}{m_{\mathrm{Pl}}^2}(pr)k\delta_{rs}\delta_{pq}\delta(\mathbf{k}-\mathbf{k}') \; . \tag{2.103}$$

The Hamiltonian (2.98) has some unusual features. Firstly, for each $\mathbf{k}$ we find four independent modes ($r = \pm 1$ and $p = \pm\, 1$), instead of two as would be expected for tensor perturbations. Half of these states have negative energy (those with $i\gamma = pr$, which leads to a minus sign in the commutator instead of the usual plus sign). For example, for the SD connection $\gamma = i$ the left "graviton" ($r = -1$ and $p = 1$) and the right "anti-graviton" ($r = 1$ and $p = -1$) carry negative energy. Secondly, there are unphysical production terms in the Hamiltonian (2.98) which couple $\mathbf{k}$ and $-\mathbf{k}$ modes. These pump terms represent pair production [53], and should not be present in the subhorizon limit $k|\eta| \gg 1$ where spacetime is approximately flat.

Both of these pathological features are not present for classical solutions, as they vanish on-shell when imposing the conditions (2.85) and (2.86). For example, for $\gamma = i$, the on-shell conditions imply $a_{R-} \approx 0$ and $a_{L+} \approx 0$. When also imposing the reality conditions such that we can consider the creation and annihilation operators as hermitian conjugates of one another, $g^{\dagger}_{rp} = \left(g_{rp}\right)^{\dagger}$, we find that two of the operators are eliminated. Only $g^{\dagger}_{R+}$, $g^{\dagger}_{L-}$, which create positive energy states, are non-zero. Thus, the negative energy modes do not exist classically and you can check that the pump terms also vanish.

As mentioned previously, quantum mechanically we do not want to treat the reality conditions as operator conditions but impose them on the inner product, which should also remove the unphysical states from the Hilbert space. We will use a holomorphic representation where we consider the states as analytic functions over the complex domain as introduced by Bargmann [57].

As mentioned above, the reality conditions simply ensure that $g^{\dagger}_{rp}$ is indeed the hermitian conjugate of g_{rp}. This condition is sufficient to fix the inner product [49, 58, 59]. A holomorphic representation for wavefunctions $\Phi = \langle z|\Phi\rangle$ is defined as one which diagonalises $g^{\dagger}_{rp}$ [57]:

$$\langle z|g^{\dagger}_{rp}|\Phi\rangle = z_{rp}\langle z|\Phi\rangle \; , \tag{2.104}$$

where $z_{rp}(\mathbf{k})$ are complex eigenvalues. Similarly to the case of deriving the action of the momentum operator on states when working in the usual position space representation, we can derive the action of g_{rp} from the commutator (2.103):

$$\langle z|g_{rp}|\Phi\rangle = -i\frac{\gamma}{m_{\mathrm{Pl}}^2}(pr)k\frac{\partial}{\partial z_{rp}}\langle z|\Phi\rangle \; . \tag{2.105}$$

We want to define an inner product in this representation. The decomposition of the unity operator for the complex eigenvectors $|z\rangle$ is given by [59]

$$\mathbb{1} = \int dz d\bar{z} e^{\mu(z,\bar{z})} , \tag{2.106}$$

where $e^{\mu(z,\bar{z})}$ is a positive integration measure (for the normal position representation with eigenstates $|x\rangle$, it is just equal to 1). The inner product can then be written as

$$\langle \Phi_1 | \Phi_2 \rangle = \int dz d\bar{z} e^{\mu(z,\bar{z})} \bar{\Phi}_1(\bar{z}) \Phi_2(z) . \tag{2.107}$$

The defining condition of the hermitian conjugate of an operator is $\langle \Phi_1 | g^\dagger_{rp} | \Phi_2 \rangle = \overline{\langle \Phi_2 | g_{rp} | \Phi_1 \rangle}$, which can be used to derive an expression for the measure. Using the defining relations for the creation and annihilation operators, Eqs. (2.104) and (2.105), and the definition of the inner product (2.107), we obtain a differential equation for $\mu(z, \bar{z})$:

$$\frac{i\gamma}{m^2_{\text{Pl}}}(pr)k \frac{\partial \mu}{\partial \bar{z}_{rp}} = z_{rp} . \tag{2.108}$$

This can be integrated to give

$$\mu(z, \bar{z}) = \int d\mathbf{k} \frac{m^2_{\text{Pl}}}{k} \sum_{rp} \frac{pr}{i\gamma} z_{rp}(\mathbf{k}) \bar{z}_{rp}(\mathbf{k}) , \tag{2.109}$$

which fixes $\langle \Phi_1 | \Phi_2 \rangle$. The vacuum of this representation is defined by $g_{rp}\Phi_0 = 0$ which gives

$$\Phi_0 = \langle z | 0 \rangle = 1 , \tag{2.110}$$

and particle states are monomials in the respective variables,

$$\Phi_n = \langle z | n \rangle \propto (g^\dagger_{rp})^n \Psi_0 = z^n_{rp} . \tag{2.111}$$

These states are not normalisable for $i\gamma = pr$, as in this case the measure is positive and the exponential in (2.107) blows up. Hence, these states should be removed from the physical Hilbert space and therefore their associated operators g_{rp} should not appear in the Hamiltonian. For $\gamma = i$, this only leaves two physical modes $g^{ph}_R = g_{R+}$ and $g^{ph}_L = g_{L-}$.

For the SD connection we therefore obtain the physical Hamiltonian

$$\mathcal{H}^{ph}_{eff} \approx m^2_{\text{Pl}} \int d\mathbf{k} \, (g^{ph}_L g^{ph\,\dagger}_L + g^{ph\,\dagger}_R g^{ph}_R) . \tag{2.112}$$

This looks like the standard Hamiltonian for a harmonic oscillator, with the difference that only the left handed graviton needs to be normal ordered and produces a vacuum energy. For the ASD connection only the right handed graviton produces vacuum energy. Left and right handed gravitons are not on the same footing, and the theory is chiral. We will explore this chirality in more detail after finding the graviton operators for general γ.

2.3.4.2 Hamiltonian for Complex Values of γ

Let us focus on the general Hamiltonian (inside the horizon) in terms of modes again, Eq. (2.97). We need to identify linear combinations of metric and connection that can act as graviton operators, equivalent to g_{rp} and $g^{\dagger}_{rp}$ for $\gamma = \pm i$. We want to end up with two physical operators corresponding to the two independent polarizations, however initially there should be four different operators. Two of them will be zero on-shell, representing the unphysical modes, while the other two should commute with them [c.f. Eq. (2.103)] and reduce to metric variables on-shell.

To find the general expression, consider the graviton operators for $\gamma = \pm i$ and find linear combinations of them that satisfy these conditions. After some algebraic manipulations that make use of the on-shell conditions (2.85) and (2.86), we can identify suitable operators:

$$G_{r\mathcal{P}_+} = \frac{(r - i\gamma)g_{r+} - (r + i\gamma)g_{r-}}{-2\gamma i}, \tag{2.113}$$

$$G_{r\mathcal{P}_-} = \frac{(r + i\gamma)g_{r+} - (r - i\gamma)g_{r-}}{-2\gamma i}, \tag{2.114}$$

where the new index $\mathcal{P} = \mathcal{P}_+, \mathcal{P}_-$ labels physical and non-physical modes. This notation is used to avoid confusions with $p = \pm 1$ used for positive and negative frequencies, and except for the cases of $\gamma = \pm i$, the two indices do not align. Using the on-shell conditions, we find that $G_{r\mathcal{P}_-} \approx 0$ and $G_{r\mathcal{P}_+} \approx 2rke_r$ as required, and you can check that their commutator is zero. The index $\mathcal{P} = \mathcal{P}_+ = 1$ therefore denotes physical modes, which reduce to the metric classically (and quantum mechanically will have positive energy and norm), and $\mathcal{P} = \mathcal{P}_- = -1$ denotes modes that vanish on-shell (and quantum mechanically will have negative energy and norm).

We can use the expressions in Eqs. (2.99) to (2.102) to write the new operators $G_{r\mathcal{P}}$ in terms of metric and connection variables. We can find expressions for the creation operators by demanding that they are hermitian conjugates of the annihilation operators once the reality conditions (2.81) and (2.82) are imposed. The operators and their commutators are listed in Table 2.2.

The Hamiltonian (2.97) can be written in terms of the new graviton operators as

$$\mathcal{H}_{eff} = \frac{m_{\rm Pl}^2}{2}\int d^3k \sum_r -(1+i\gamma r)G_{r\mathcal{P}_+}(\mathbf{k})G_{r\mathcal{P}_-}(-\mathbf{k}) - (1-i\gamma r)G_{r\mathcal{P}_-}(\mathbf{k})G_{r\mathcal{P}_+}(-\mathbf{k})$$
$$+(1+i\gamma r)G_{r\mathcal{P}_+}(\mathbf{k})G^{\dagger}_{r\mathcal{P}_+}(\mathbf{k}) + (1-i\gamma r)G^{\dagger}_{r\mathcal{P}_+}(\mathbf{k})G_{r\mathcal{P}_+}(\mathbf{k})$$

$$+(1-i\gamma r)G_{r\mathcal{P}_-}(\mathbf{k})G^\dagger_{r\mathcal{P}_-}(\mathbf{k})+(1+i\gamma r)G^\dagger_{r\mathcal{P}_-}(\mathbf{k})G_{r\mathcal{P}_-}(\mathbf{k})$$
$$-(1-i\gamma r)G^\dagger_{r\mathcal{P}_+}(\mathbf{k})G^\dagger_{r\mathcal{P}_-}(-\mathbf{k})-(1+i\gamma r)G^\dagger_{r\mathcal{P}_-}(\mathbf{k})G^\dagger_{r\mathcal{P}_+}(-\mathbf{k})\,. \tag{2.115}$$

This is the generalization of Eq. (2.98). As before, there are too many graviton states as well as unphysical pair production terms. They all vanish on shell where the operator corresponding to $\mathcal{P}_-$ is zero. We can now set up the Hilbert space, fixing the inner product by requiring that the operators in Table 2.2 are indeed hermitian conjugates of one another.

Again, we use a holomorphic representation which diagonalises $G^\dagger_{r\mathcal{P}}$, i.e.:

$$\langle z|G^\dagger_{r\mathcal{P}}|\Phi\rangle = z_{r\mathcal{P}}\langle z|\Phi\rangle\,. \tag{2.116}$$

The commutation relations in Table 2.2 determine the action of the annihilation operators,

$$\langle z|G_{r\mathcal{P}}|\Phi\rangle = \mathcal{P}\frac{k}{m^2_{\rm Pl}}\frac{\partial}{\partial z_{r\mathcal{P}}}\langle z|\Phi\rangle\,. \tag{2.117}$$

This is formally very similar to the case $\gamma=\pm i$, but note that the variables $z_{r\mathcal{P}}$ are not the same as before. Using the definition of the inner product Eq. (2.107), and the same formal condition $\langle\Phi_1|G^\dagger_{r\mathcal{P}}|\Phi_2\rangle=\overline{\langle\Phi_2|G_{r\mathcal{P}}|\Phi_1\rangle}$, we arrive at an expression for the measure:

$$\mu(z,\bar z)=\int d\mathbf{k}\frac{m^2_{\rm Pl}}{k}\sum_{r\mathcal{P}}\mathcal{P}z_{r\mathcal{P}}(\mathbf{k})\bar z_{r\mathcal{P}}(\mathbf{k})\,. \tag{2.118}$$

The vacuum state

$$\Phi_0=\langle z|0\rangle=1\,, \tag{2.119}$$

and the particle states

$$\Phi_n=\langle z|n\rangle\propto(G^\dagger_{r\mathcal{P}})^n\Psi_0=z^n_{r\mathcal{P}}\,, \tag{2.120}$$

have the same form as before (but are defined in terms of new variables $z_{r\mathcal{P}}$). The measure implies that states with $\mathcal{P}=\mathcal{P}_-=-1$ are not normalisable and the

Table 2.2 Physical and unphysical graviton modes

Physical $\mathcal{P}=\mathcal{P}_+=1$	Unphysical $\mathcal{P}=\mathcal{P}_-=-1$
$G_{r\mathcal{P}_+}=\frac{-r}{i\gamma}(\tilde a_{r+}-k(r+i\gamma)\tilde e_{r+})$	$G_{r\mathcal{P}_-}=\frac{-r}{i\gamma}(\tilde a_{r+}-k(r-i\gamma)\tilde e_{r+})$
$G^\dagger_{r\mathcal{P}_+}=\frac{r}{i\gamma}(\tilde a^\dagger_{r-}-k(r-i\gamma)\tilde e^\dagger_{r-})$	$G^\dagger_{r\mathcal{P}_-}=\frac{r}{i\gamma}(\tilde a^\dagger_{r-}-k(r+i\gamma)\tilde e^\dagger_{r-})$
$\left[G_{r\mathcal{P}_+}(\mathbf{k}),G^\dagger_{s\mathcal{P}_+}(\mathbf{k}')\right]=\frac{k}{m^2_{\rm Pl}}\delta_{rs}\delta(\mathbf{k}-\mathbf{k}')$	$\left[G_{r\mathcal{P}_-}(\mathbf{k}),G^\dagger_{s\mathcal{P}_-}(\mathbf{k}')\right]=-\frac{k}{m^2_{\rm Pl}}\delta_{rs}\delta(\mathbf{k}-\mathbf{k}')$

operators corresponding to $\mathcal{P}_-$ should be removed from the Hamiltonian. The physical Hamiltonian for a general value of γ is therefore:

$$\mathcal{H}_{eff}^{ph} \approx \frac{m_{\rm Pl}^2}{2} \int d\mathbf{k} \sum_r [G_r^{ph} G_r^{ph\dagger}(1+ir\gamma) + G_r^{ph\dagger} G_r^{ph}(1-ir\gamma)] \,, \tag{2.121}$$

where $G_r^{ph} = G_r \mathcal{P}_+$.

2.3.4.3 Vacuum Energy

Only the first term in the physical Hamiltonian (2.121) needs to be normal ordered, using the commutation relation in Table 2.2. This leads to a chiral (r-dependent) term corresponding to the vacuum energy, $V_r \propto 1 + ir\gamma$. The asymmetry in the vacuum energy between the right- and left-handed gravitons is given by

$$\frac{V_R - V_L}{V_R + V_L} = i\gamma \; . \tag{2.122}$$

This equation is valid for any complex γ. There are a few points of interest to note. If γ is purely imaginary and $|\gamma| > 1$, the vacuum energy $V_r \propto 1 + ir\gamma$ of one of the modes becomes negative. Negative vacuum energy is often associated with fermionic degrees of freedom [60], but this will not be investigated further here.

More importantly, if γ has a real part the VE for each r is complex. When right and left helicities are added together, however, we simply obtain $V_R + V_L \propto 1 + i\gamma + 1 - i\gamma = 2$, so the total vacuum energy is indeed real.

The reason we obtain a chiral, complex vacuum energy is because the Hamiltonian is not hermitian before normal ordering: Although it is real on-shell for any value of γ (which does not appear in any on-shell expressions) and the graviton operators themselves are hermitian, unless γ is imaginary, taking the hermitian conjugate of the perturbative physical Hamiltonian (2.121) does not yield $\mathcal{H}^\dagger = \mathcal{H}$.

Hermiticity is restored after normal ordering, when γ drops out of the Hamiltonian and is only present in the vacuum energy term. As the latter is not physically measurable (and when coupled to the Einstein equations, we need to consider the total which is indeed real), this result might not be too concerning. However it might also imply that it is more physical to consider only a purely imaginary γ or that we should use a symmetric ordering for the Hamiltonian: When we first defined the Hamiltonian in Eq. (2.42), we picked an ordering of the form EEF (the field strength contains connection terms, which do not commute with metric terms). Knowing which ordering in quantum mechanics is “correct” is an issue which can ultimately only be resolved by experiment. It can be checked that using an EFE or $\frac{1}{2}(EEF + FEE)$ ordering would satisfy $\mathcal{H} = \mathcal{H}^\dagger$ on and off-shell, for any value of γ. In this case there would be no chirality in the vacuum energy. However, note that we would obtain the same graviton operators regardless of ordering, and as will see now, chirality will still be present in the vacuum fluctuations.

2.3.5 Chiral Vacuum Fluctuations

The central gravitational variable in the Ashtekar formalism is the connection, not the metric, which can be seen from the Holst action (2.22). Therefore, the power spectrum of tensor perturbations should be derived from the (TT-projected) perturbations of the connection as opposed to the metric. As in the second order formalism, the Ashtekar tensor perturbations will have an effect on the CMB fluctuations, especially on the polarization. We will not need to worry about the exact normalization of the tensor fluctuations, as we are mainly interested to see whether the complex nature of the connection will play a role.

The analogous expression to the tensor power spectrum (1.65) is given by

$$\langle 0|A_r^\dagger(\mathbf{k})A_r(\mathbf{k}')|0\rangle = P_r(k)\delta(\mathbf{k}-\mathbf{k}') \ , \tag{2.123}$$

where $A_r(\mathbf{k})$ represents Fourier space connection variables with handedness r, i.e.

$$A_r(\mathbf{k}) = \tilde{a}_{r+}(\mathbf{k})e^{-ik\cdot x} + \tilde{a}_{r-}^\dagger(\mathbf{k})e^{ik\cdot x} \ . \tag{2.124}$$

Note that we could have picked a different ordering in the 2-point function (2.123), so in general we have to consider

$$A^\dagger A \to \alpha A^\dagger A + \beta A A^\dagger \ , \tag{2.125}$$

with $\alpha+\beta = 1$ and $\alpha, \beta > 0$. As opposed to the vacuum energy, we will see that the power spectrum (2.123), being a measurable variance, is always real and positive.

To compute the physical power spectrum, we need to relate the connection variables to the physical graviton modes labelled by $\mathcal{P}_+$ in Table 2.2. As we need to go on-shell to define physical states, we can use conditions (2.85) and (2.86) to express the metric variables in terms of the connection:

$$\tilde{e}_{r+} = \frac{\tilde{a}_{r+}}{k(r-i\gamma)} \ , \quad \tilde{e}_{r-}^\dagger = \frac{\tilde{a}_{r-}^\dagger}{k(r+i\gamma)} \ . \tag{2.126}$$

These relations can be subsituted into the equations for $G^\dagger_{r\mathcal{P}_+}$, $G_{r\mathcal{P}_+}$ in Table 2.2, which gives expressions for the physical connection modes a_{r+}^{ph} and $a_{r-}^{ph\dagger}$. The remaining modes can be obtained by taking hermitian conjugates (as we are on-shell, the reality conditions have been imposed). We find

$$a_{r+}^{ph} = \frac{r-i\gamma}{2r}G_{r\mathcal{P}_+} \ , \tag{2.127}$$

$$a_{r+}^{ph\dagger} = \frac{r+i\gamma^*}{2r}G^\dagger_{r\mathcal{P}_+} \ , \tag{2.128}$$

$$a_{r-}^{ph} = \frac{r-i\gamma^*}{2r}G_{r\mathcal{P}_+} \ , \tag{2.129}$$

$$a_{r-}^{ph\dagger} = \frac{r+i\gamma}{2r} G^{\dagger}_{r\mathcal{P}_+} \,. \tag{2.130}$$

We can see that the physical connection modes depend solely on the graviton operators, so they will be the same for any ordering of the Hamiltonian. Plugging these expressions into (2.124) we obtain for the two connection helicity states:

$$A_r^{ph}(\mathbf{k}) = \frac{r-i\gamma}{2r} G_{r\mathcal{P}_+}(\mathbf{k}) e^{-ik\cdot x} + \frac{r+i\gamma}{2r} G^{\dagger}_{r\mathcal{P}_+}(\mathbf{k}) e^{ik\cdot x} \,,$$
$$A_r^{ph\dagger}(\mathbf{k}) = \frac{r-i\gamma^*}{2r} G_{r\mathcal{P}_+}(\mathbf{k}) e^{-ik\cdot x} + \frac{r+i\gamma^*}{2r} G^{\dagger}_{r\mathcal{P}_+}(\mathbf{k}) e^{ik\cdot x} \,.$$

This means that the power spectrum (2.123) is given by (using $G_{r\mathcal{P}_+}(\mathbf{k}')|0\rangle = 0$)

$$\langle 0| A_r^{ph\dagger}(\mathbf{k}) A_r^{ph}(\mathbf{k}')|0\rangle = P_r(\gamma) \langle 0| G_{r\mathcal{P}_+}(\mathbf{k}) G^{\dagger}_{r\mathcal{P}_+}(\mathbf{k}')|0\rangle \,. \tag{2.131}$$

We could eliminate the expectation value of graviton operators by using their commutator to give us an expression in terms of delta functions. However, we are only interested in the chiral dependence of the power spectrum P_r, which is given by

$$P_r(\gamma) = \frac{(r+i\gamma)(r-i\gamma^*)}{4} = \frac{1-2\gamma_I r + |\gamma|^2}{4} \,. \tag{2.132}$$

If $\gamma_I r < 0$, $P_r(\gamma)$ is obviously positive. Otherwise,

$$P_r(\gamma) \propto 1 - 2|\gamma_I| + \gamma_I^2 + \gamma_R^2 = (1-|\gamma_I|)^2 + \gamma_R^2 \,, \tag{2.133}$$

which is also positive for any complex γ. Therefore, the 2-point function is indeed always real and positive, as required. The chiral asymmetry in the power spectrum can be written as

$$\frac{P_R - P_L}{P_R + P_L} = -\frac{2\gamma_I}{1+|\gamma|^2} \,, \tag{2.134}$$

or, for a general ordering of the 2-point function as in (2.125),

$$\frac{P_R - P_L}{P_R + P_L} = \frac{2(\beta-\alpha)\gamma_I}{1+|\gamma|^2} \,. \tag{2.135}$$

The chirality in the power spectrum of tensor fluctuations is the main new result of this work, and a big difference to the standard second order formalism described in Sect. 1.2.5 (which corresponds to the limit $|\gamma| \to \infty$, for which the Holst action reduces to the Palatini action).

We can see that if γ was purely real there would be no asymmetry in the vacuum fluctuations for right and left gravitons. The chirality is related to the fact that for a γ with an imaginary part the connection is a complex field and therefore we must expand it in terms of graviton and anti-graviton modes. Note, however, that a real

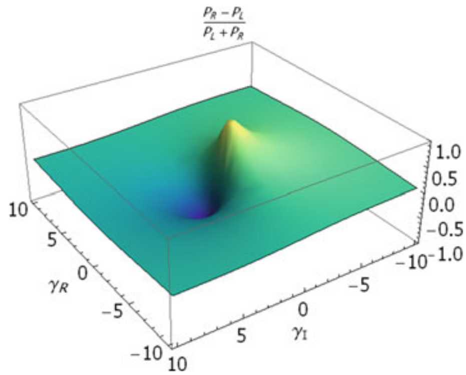

Fig. 2.1 Power spectrum asymmetry as a function of a generally complex Immirzi parameter γ

part in the Immirzi parameter does affect the absolute value of the asymmetry due to the factor $|\gamma|$ in the denominator of (2.134). We can also see that for a completely symmetric ordering of the 2-point function, $\alpha = \beta$, the RHS of Eq. (2.135) is zero. Hence, even if the Ashtekar formalism was the correct description of gravity, we would not obtain a chiral power spectrum if γ was real or the ordering symmetric. Not measuring chirality would therefore not be able to rule out the theory.

We can plot the power spectrum asymmetry (2.134) against the real and imaginary parts of γ, see Fig. 2.1. It is obviously antisymmetric in γ_I, and the minimum and maximum are at $\gamma = \pm i$ respectively which are the values that correspond to a SD/ASD connection. They display the maximum chirality because the Palatini action can naturally be split into a SD and ASD part [19]. The axis $\gamma_I = 0$ corresponds to a real γ and therefore displays no asymmetry.

2.3.5.1 Measuring a Chiral Tensor Spectrum

As was mentioned in Sect. 1.3.2, in the absence of parity violation, the TB power spectrum of the CMB would be zero. In the situation we have just considered, the chirality of the power spectrum (2.134) breaks parity. The effect of parity violation on the CMB power spectra was investigated in [61]. It was found that the ratio between the quadrupole of the TB correlator (zero in standard cosmological approaches) and the BB correlator is given by

$$\frac{C_2^{\mathrm{TB}}}{C_2^{\mathrm{BB}}} \approx f_{\mathrm{PB}} \alpha_2 \,, \tag{2.136}$$

where $\alpha_2 \approx 200$ parametrises the relative strength between the TB and BB spectra and $f_{\mathrm{PB}} = 2\frac{P_R - P_L}{P_R + P_L}$ is the parity breaking parameter, which is zero if no chirality is present. In our case, we therefore find

$$\frac{C_2^{\mathrm{TB}}}{C_2^{\mathrm{BB}}} \approx 800 \frac{(\beta - \alpha)\gamma_I}{1 + |\gamma|^2} \,, \tag{2.137}$$

for the ratio of tensor induced TB and BB quadrupole modes. Not only would chirality render the TB correlator non-zero, it would also be easier to detect TB rather than BB correlation ($C_2^{\mathrm{TB}} > C_2^{\mathrm{BB}}$) for a wide range of values of γ, given approximately by

$$\frac{1}{800} < |\gamma| < 800\,. \tag{2.138}$$

BICEP2 has recently detected B-modes [62] that might have arisen due to tensor perturbations from inflation, however we do not yet have tensor power spectra over a large number of multipoles as the experiment only took data from a small patch of sky. Although there was no hint of parity violation in their analysis so far, this might change once the full power spectrum becomes available. It will therefore be possible in the near future to constrain the model I have described. If the TB correlator is consistent with zero, we know that for Ashtekar gravity to be correct, γ must be either quite far from the range in Eq. (2.138) or real. If a chirality was detected, on the other hand, it could indeed have originated from this mechanism.

2.3.6 A Purely Real γ

Before I conclude, let us quickly consider the case of a purely real theory for which $\Im(\gamma) = 0$. Although it will turn out that we can take the limit $\Im(\gamma) \to 0$ in all of our main results to obtain the answer in the real theory, it is not initially obvious why this would work, as a real theory is very different from a complex one. I will describe the main differences and show why our results are still well defined in the real case.

A purely real theory would require Fourier mode expansions using operators a_r and e_r without a p index, as there is no need to consider separate sets of creation and annihilation operators. We therefore would only get two modes for each $\mathbf{k}$ and r as usual in the second order theory. As we ignore the p index, what used to be reality conditions in the complex theory, where we related modes with different p, are now just operator conditions, $\tilde{e}_{r+} = \tilde{e}_{r-}$ and $\tilde{a}_{r+} = \tilde{a}_{r-}$. Similarly, the commutation relations (2.95) have one less index and must be replaced by

$$[\tilde{a}_r(\mathbf{k}), \tilde{e}_s^\dagger(\mathbf{k}')] = -i\frac{\gamma}{2m_{\mathrm{Pl}}^2}\delta_{rs}\delta(\mathbf{k}-\mathbf{k}')\ . \tag{2.139}$$

The Hamiltonian, on the other hand, will still have the same form, as $p = -1$ modes always appear with a dagger and $p = +1$ modes without, see Eq. (2.97). This enables us to define the same physical and unphysical graviton operators as before, however without a p index on the RHS, e.g.

$$G_{r\mathcal{P}_+} = \frac{-r}{i\gamma}(\tilde{a}_r - k(r + i\gamma)\tilde{e}_r)\ . \tag{2.140}$$

Note that as opposed to the complex case, were the graviton operators were only hermitian conjugates of each other after the reality conditions had been imposed, for the real theory the reality conditions are satisfied by the metric and connection operators. Therefore, $G_{r\mathcal{P}}$ and $G^{\dagger}_{r\mathcal{P}}$ are automatically conjugates of one another, which can be trivially seen from their definitions.

We still have a non-physical mode, however, which can be eliminated by imposing the torsion free condition which relates a_r to e_r. As before, we can define a holomorphic representations and an inner product, which will show that the non-physical modes have negative energy and should therefore be excluded. Our Hamiltonian and Hilbert space will therefore have the same structure as for a general complex γ. Hence, the real theory can be viewed as the limit $\Im(\gamma) \to 0$ in the sections above.

2.4 Conclusions

I have shown that using the Ashtekar formalism in cosmological perturbation theory leads to a number of interesting results.

Classically, rederiving the second order Hamiltonian corresponding to tensor perturbations is far from trivial. We saw that we need to take boundary terms into account, as well as regard the change from the full Ashtekar variables to the perturbations as a canonical transformation in order to arrive at the correct form of the Hamiltonian. I was then able to reproduce the standard result for the equation of motion of tensor modes, as obtained in the second order formalism.

On the quantum mechanical front there were several novelties. First of all, the fact that the connection is complex makes the exercise a lot more involved than in the usual case. We need to expand the fields in terms of positive and negative frequency operators, which are related by reality conditions. These are not supposed to be imposed on the operators, but only at the very end when choosing the inner product of the Hilbert space. We can write the Hamiltonian in terms of graviton creation and annhilation operators, which are linear combinations of metric and connection. When fixing the inner product, we find that half of the operators are unphysical, demonstrated by them being zero-on shell, when the torsion free condition relating metric and connection is imposed. This also gets rid of unphysical coupling terms between $\mathbf{k}$ and $-\mathbf{k}$ in the Hamiltonian.

As the connection is complex, so is the dynamical, perturbed Hamiltonian. This is not a problem as we ensure actual observables are real by requiring the Hamiltonian to be hermitian through the choice of inner product, at least after normal ordering. The complexity of the Hamiltonian is, however, the origin of the chiral effects we observe.

Before normal ordering, if $\gamma_R \neq 0$, the Hamiltonian is not hermitian, which results in an imaginary vacuum energy for each helicity. Non-hermitian Hamiltonians have been studied before [63] and are not necessarily regarded as problematic. In our case, the total vacuum energy for both helicities is real, and therefore the non-hermitian nature might not be physically significant.

The main result of this chapter is the chiral power spectrum of tensor perturbations, which is described in terms of perturbed connection variables. This chirality is present as long as γ is not purely real, and the strongest effect occurs for the SD/ASD connection for which $\gamma = \pm i$. The chirality in the power spectrum is a novelty compared to the standard second order formalism, and demonstrates that using different variables to describe spacetime does not necessarily lead to equivalent results.

A chiral graviton would break parity and therefore lead to a non-zero TB correlator, which can be probed by CMB measurements. As the Planck collaboration will release their polarization results later this year, it is only a matter of time until the full power spectrum can be obtained, which will enable us to constrain the value of the Immirzi parameter.

Although gravitational chirality can be produced in other ways [60, 64, 65], the mechanism presented here is by far the simplest. If a chiral tensor power spectrum was to be observed, it would hint at the Ashtekar formalism being the correct fundamental description of gravity.

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