Chapter 2
Green's Functions for Laplace and Wave Equations

This chapter shows the solution method for Green's functions of 1, 2 and 3D Laplace and wave equations. Lengthy and detailed explanations are given in order to instruct the basic technique of the integral transform. Especially, Fourier inversion integral for the time-harmonic Green's function is discussed in detail, and three evaluation techniques are introduced in Sect. 2.5.

2.1 1D Impulsive Source

We start from the simplest wave equation that has two variables: a single space variable x and the time t,

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - P\delta(x)\delta(t) \tag{2.1.1}$$

The nonhomogeneous term represents a wave source with magnitude P. The two Dirac's delta functions, $\delta(x)$ and $\delta(t)$, show the location and the impulsive nature of the source. The Green's function is a particular solution of the differential equation corresponding to the impulsive source. The Green's function is sought under the quiescent condition at an initial time,

$$\phi|_{t=0} = \left.\frac{\partial \phi}{\partial t}\right|_{t=0} = 0 \tag{2.1.2}$$

and the convergence condition at infinity,

$$\phi|_{x\to\pm\infty} = \left.\frac{\partial \phi}{\partial x}\right|_{x\to\pm\infty} = 0 \tag{2.1.3}$$

To obtain the particular solution of the wave equation (2.1.1), we apply the integral transforms and reduce the differential equation to an algebraic equation in the transformed domain. Since the unknown function $\phi(x,t)$ has two variables, we

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apply the double transform: Laplace transform with respect to the time variable t: $0 \le t < +\infty$,

$$f^*(s) = L[f(t)] = \int_0^{+\infty} f(t)\exp(-st)dt \tag{2.1.4}$$

and Fourier transform with respect to the space variable x: $-\infty < x < +\infty$,

$$\bar{f}(\xi) = \int_{-\infty}^{+\infty} f(x)\exp(+i\xi x)dx, \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{f}(\xi)\exp(-i\xi x)d\xi \tag{2.1.5}$$

Firstly, we multiply the kernel of the Laplace transform $\exp(-st)$ to both sides of the differential equation (2.1.1),

$$\int_0^{\infty}\left(\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - P\delta(x)\delta(t)\right)\exp(-st)dt \tag{2.1.6}$$

and perform the Laplace transform integral term by term. The order of integration and differentiation are interchanged for the first term in the left hand side of the equation. The first term in the right hand side is left in its order and then integrated by parts. The last nonhomogeneous term, which has the delta function, is evaluated by using the formula (1.2.3). Then, Eq. (2.1.6) is rewritten as

$$\frac{d^2}{dx^2}\int_0^{\infty}\phi\,\exp(-st)dt = \frac{1}{c^2}\int_0^{\infty}\frac{\partial^2\phi}{\partial t^2}\exp(-st)dt - P\delta(x)\int_0^{\infty}\delta(t)\exp(-st)dt \tag{2.1.7}$$

We define the Laplace transform of the unknown function as

$$\phi^* = \int_0^{\infty}\phi\,\exp(-st)dt \tag{2.1.8}$$

The time-derivative term is integrated by parts,

$$\int_0^{\infty}\frac{\partial^2\phi}{\partial t^2}\exp(-st)dt = \left[\frac{\partial\phi}{\partial t}\exp(-st)\right]_{t=0}^{t\to\infty} + [s\phi\,\exp(-st)]_{t=0}^{t\to\infty} + s^2\int_0^{\infty}\phi\,\exp(-st)dt = s^2\phi^* \tag{2.1.9}$$

where the quiescent condition at the initial time is incorporated. To the last non-homogeneous term, the simple integration formula for the delta function,

$$\int_0^\infty \delta(t)\exp(-st)dt = 1 \tag{2.1.10}$$

is applied. Then, we have the ordinary differential equation for the function ϕ^* in the transformed domain,

$$\frac{d^2\phi^*}{dx^2} = (s/c)^2\phi^* - P\delta(x) \tag{2.1.11}$$

It is possible to obtain the exact solution for the above ordinary differential equation by the elementary method. However, in order to demonstrate the integral transform technique, we further apply Fourier transform to the ordinary differential equation (2.1.11). The Fourier transform defined by Eq. (2.1.5) is applied to Eq. (2.1.11),

$$\int_{-\infty}^{+\infty} \left(\frac{d^2\phi^*}{dx^2} = (s/c)^2\phi^* - P\delta(x)\right)\exp(+i\xi x)dx \tag{2.1.12}$$

The Fourier transform integral is applied to each term as

$$\int_{-\infty}^{+\infty} \frac{d^2\phi^*}{dx^2}\exp(+i\xi x)dx = (s/c)^2\int_{-\infty}^{+\infty} \phi^*\exp(+i\xi x)dx - P\int_{-\infty}^{+\infty} \delta(x)\exp(+i\xi x)dx \tag{2.1.13}$$

The convergence condition is also transformed, as

$$\int_0^\infty \left(\phi|_{x\to\pm\infty} = \left.\frac{\partial\phi}{\partial x}\right|_{x\to\pm\infty} = 0\right)\exp(-st)dt \quad \Rightarrow \quad \phi^*|_{x\to\pm\infty} = \left.\frac{d\phi^*}{dx}\right|_{x\to\pm\infty} = 0 \tag{2.1.14}$$

Defining the Fourier transform of the Laplace transformed unknown function,

$$\bar{\phi}^* = \int_{-\infty}^{+\infty} \phi^*\exp(+i\xi x)dx \tag{2.1.15}$$

the transform of the space-derivative in Eq. (2.1.13) is carried out with the aid of the convergence condition,

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{d^2\phi^*}{dx^2}\exp(+i\xi x)dx &= \left[\frac{d\phi^*}{dx}\exp(+i\xi x)\right]_{x\to-\infty}^{x\to+\infty} - i\xi[\phi^*\exp(+i\xi x)]_{x\to-\infty}^{x\to+\infty} - \xi^2\int_{-\infty}^{+\infty}\phi^*\exp(+i\xi x)dx \\ &= -\xi^2\tilde{\phi}^*\end{aligned} \tag{2.1.16}$$

The integration formula for the delta function

$$\int_{-\infty}^{+\infty} \delta(x)\exp(+i\xi x)dx = 1 \tag{2.1.17}$$

is also used for evaluating the last term. The Fourier transform of the ordinary differential equation (2.1.12) becomes a simple algebraic equation for the double transformed unknown function $\bar{\phi}^*$,

$$-\xi^2\bar{\phi}^* = \frac{s^2}{c^2}\bar{\phi}^* - P \tag{2.1.18}$$

Then we have the exact expression for the double transformed function $\bar{\phi}^*$,

$$\bar{\phi}^* = \frac{P}{\xi^2 + (s/c)^2} \tag{2.1.19}$$

The unknown function has just been determined explicitly in the transformed domain. We shall carry out two inverse transforms successively. As the first inversion, we apply the Fourier inversion integral which is defined by the second of Eq. (2.1.5). The formal Fourier inversion is given by the integral

$$\phi^* = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{P}{\xi^2 + (s/c)^2}\exp(-i\xi x)d\xi \tag{2.1.20}$$

Due to the symmetric nature of the integrand, the integral is reduced to the simpler semi-infinite integral,

$$\phi^* = \frac{P}{\pi}\int_0^{\infty} \frac{1}{\xi^2 + (s/c)^2}\cos(\xi x)d\xi \tag{2.1.21}$$

This is a simple integral and the integration formula (Erdélyi 1954, vol. I, pp. 8, 11),

$$\int_0^{+\infty} \frac{1}{x^2 + a^2} \cos(xy) dx = \frac{\pi}{2a} \exp(-a|y|) \tag{2.1.22}$$

can be applied. Then, Eq. (2.1.21) gives

$$\phi^* = \frac{P}{2(s/c)} \exp(-s|x|/c) \tag{2.1.23}$$

The next step is to carry out the Laplace inversion. The symbolical form of the Laplace inversion is given by

$$\phi = L^{-1}[\phi^*] = \frac{cP}{2} L^{-1}\left[\frac{1}{s}\exp(-s|x|/c)\right] \tag{2.1.24}$$

Fortunately, we have the Laplace inversion formula (Erdélyi 1954, vol. I, pp. 241, 1),

$$L^{-1}\left[\frac{1}{s}\exp(-as)\right] = \begin{cases} 0; & t < a \\ 1; & t > a \end{cases} = H(t-a) \tag{2.1.25}$$

where $H(.)$ is Heaviside's unit step function. Applying this formula to Eq. (2.1.24), the solution for the non-homogeneous 1D wave equation is obtained as

$$\phi = \frac{cP}{2} H(t - |x|/c) = \frac{cP}{2} H(ct - |x|) \tag{2.1.26}$$

This solution shows an expanding (or out-going) 1D wave with the uniform amplitude $cP/2$ and with the velocity c as shown in Fig. 2.1. Consequently, we get the Green's function for the 1D wave equation,

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - P\delta(x)\delta(t) \quad \Rightarrow \quad \phi(x,t) = \frac{cP}{2} H(ct - |x|) \tag{2.1.27}$$

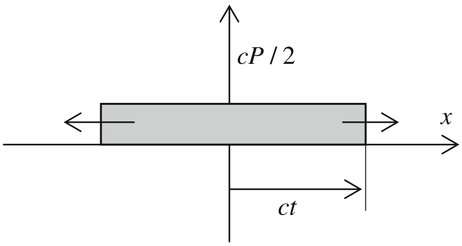

Fig. 2.1 1D expanding wave from a source point

2.2 1D Time-Harmonic Source

When the source is time-harmonic, the nonhomogeneous term in Eq. (2.1.1) is replaced with a harmonic function, but the source location is unchanged. Thus, the wave equation with a time-harmonic source is given by

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - Q\delta(x)\exp(+i\omega t) \tag{2.2.1}$$

where Q is the source magnitude and ω the frequency of the time-harmonic vibration. We assume that its solution satisfies the convergence condition at infinity, i.e.

$$\phi|_{x\to\pm\infty} = \left.\frac{\partial \phi}{\partial x}\right|_{x\to\pm\infty} = 0 \tag{2.2.2}$$

As the first step of the solution method, we assume that the solution is also time-harmonic,

$$\phi(x,t) = \phi^{\#}(x)\exp(+i\omega t) \tag{2.2.3}$$

where $\phi^{\#}$ is called the "amplitude function." Due to this assumption, the convergence condition (2.2.2) is rewritten for the amplitude function,

$$\phi^{\#}|_{x\to\pm\infty} = \left.\frac{d\phi^{\#}}{dx}\right|_{x\to\pm\infty} = 0 \tag{2.2.4}$$

Substituting the time-harmonic assumption of Eq. (2.2.3) into the wave equation (2.2.1), we have the ordinary differential equation for the amplitude function,

$$\frac{d^2\phi^{\#}}{dx^2} + (\omega/c)^2\phi^{\#} = -Q\delta(x) \tag{2.2.5}$$

The exact solution of this ordinary differential equation can also be obtained by the elementary method. However, in order to demonstrate the integral transform technique, we apply the Fourier transform, which is defined by Eq. (2.1.5), to the ordinary differential equation (2.2.5),

$$\int_{-\infty}^{+\infty} \left(\frac{d^2\phi^{\#}}{dx^2} + (\omega/c)^2\phi^{\#} = -Q\delta(x)\right)\exp(+i\xi x)dx \tag{2.2.6}$$

Defining the Fourier transform of the amplitude function as

$$\bar{\phi}^{\#} = \int_{-\infty}^{+\infty} \phi^{\#} \exp(+i\xi x)dx \tag{2.2.7}$$

the space derivative term in Eq. (2.2.6) is integrated by parts as

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{d^2\phi^{\#}}{dx^2}\exp(+i\xi x)dx =& \left[\frac{d\phi^{\#}}{dx}\exp(+i\xi x)\right]_{x\to-\infty}^{x\to+\infty} - i\xi\left[\phi^{\#}\exp(+i\xi x)\right]_{x\to-\infty}^{x\to+\infty} \\ &- \xi^2 \int_{-\infty}^{+\infty} \phi^{\#}\exp(+i\xi x)dx\end{aligned} \tag{2.2.8}$$

We apply the convergence condition of Eq. (2.2.4) and the definition of the Fourier transform (2.2.7) to the above equation. The Fourier transform of the double derivative is then reduced to

$$\int_{-\infty}^{+\infty} \frac{d^2\phi^{\#}}{dx^2}\exp(+i\xi x)dx = -\xi^2\bar{\phi}^{\#} \tag{2.2.9}$$

The nonhomogeneous term is evaluated as

$$-Q\int_{-\infty}^{+\infty} \delta(x)\exp(+i\xi x)dx = -Q \tag{2.2.10}$$

Then, Eq. (2.2.6) gives a simple algebraic equation for the Fourier transformed amplitude function,

$$-\xi^2\bar{\phi}^{\#} + (\omega/c)^2\bar{\phi}^{\#} = -Q \tag{2.2.11}$$

The Fourier transformed amplitude is determined completely,

$$\bar{\phi}^{\#} = \frac{Q}{\xi^2 - (\omega/c)^2} \tag{2.2.12}$$

Our next task is to invert the transformed amplitude. Applying the formal Fourier inversion integral to Eq. (2.2.12), we get

$$\phi^{\#} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{P}{\xi^2 - (\omega/c)^2}\exp(-i\xi x)d\xi \tag{2.2.13}$$

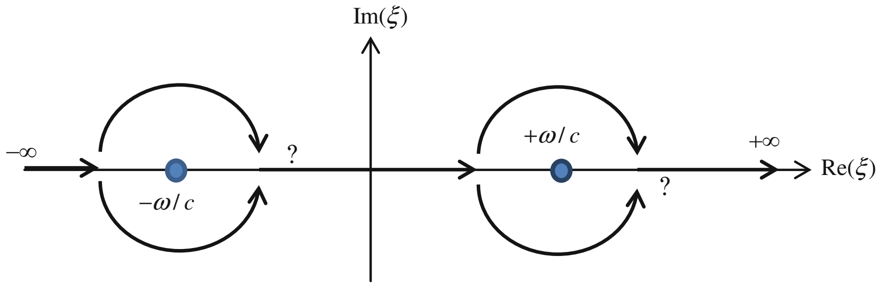

Fig. 2.2 Possible deformations of the integration path around the pole

Inspecting the integrand, we see that it has two simple poles at $\xi = \pm(\omega/c)$, that is to say, the poles are located on the integration path (real axis in the complex ξ-plane). Since the integration cannot be performed through these singular points, we have to distort the integration path around the poles. There are two ways of deforming the path. One is through an upper semi-circle, another is through a lower semi-circle as shown in Fig. 2.2. We have to determine which semi-circle is suitable. Discussing the nature of the initial wave equation (2.2.1), we learn that the wave will expand to the outer region from the source point, i.e. wave radiation from the source. Therefore, we have to choose the path so that the inversion integral results in a radiating (out-going) wave from the source.

Still, it is somewhat complicated to explain the path selection. To aid the understanding, two integrals with complex frequency are considered. After discussing the wave nature derived from each integral, we will determine and understand the path distortion.

Let us introduce and add a small imaginary number ε to the frequency in Eq. (2.2.13) so that the poles are shifted from the real axis and are not on the integration path. The complex frequency is considered in two ways, positive and negative imaginary parts, $\omega \to \omega \pm i\varepsilon$. Employing the complex frequency ϖ, we consider the complex integral,

$$\Phi = \frac{1}{2\pi}\int_C \frac{P}{\xi^2 - (\varpi/c)^2}\exp(-i\xi x)d\xi \tag{2.2.14}$$

where the integrand is the same as that in the Fourier inversion integral (2.2.13), but the frequency is complex, $\varpi = \omega \pm i\varepsilon$. The integration loop C for the two cases of complex frequency, with positive and negative imaginary parts, is discussed separately.

(1) Small positive imaginary part: $\varpi = \omega + i\varepsilon$

When the frequency has a small positive imaginary part, the poles are shifted from the real axis. The integration path C is chosen so that the integrand vanishes on the large semi-circle with infinite radius. We employ the lower closed loop $C^{(-)}$ in the case of positive x and the upper loop $C^{(+)}$ in that of negative x as shown in Fig. 2.3a.

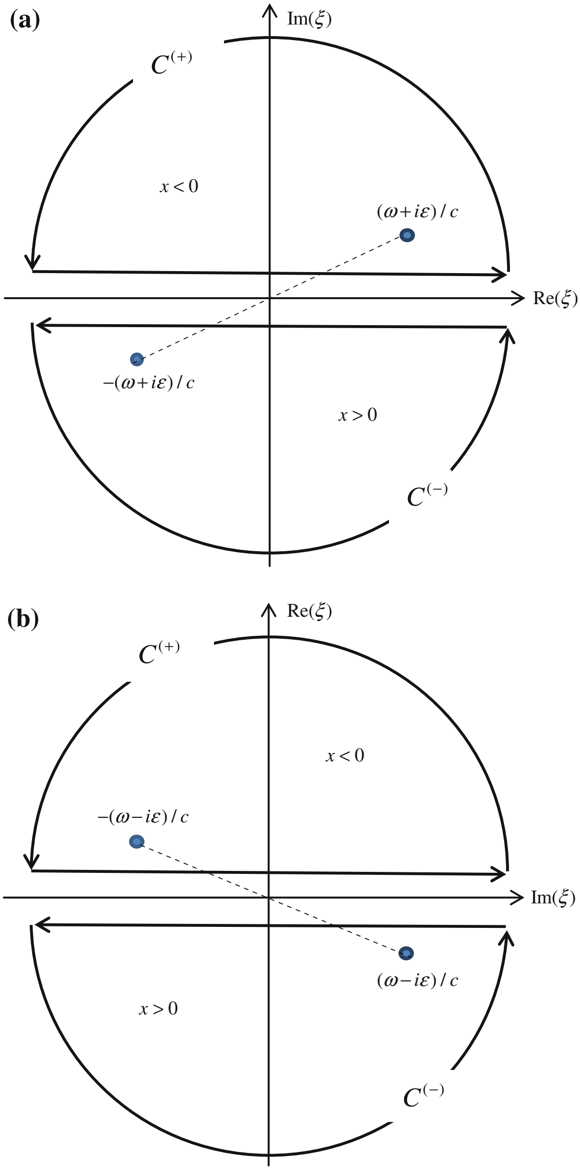

Fig. 2.3 Integration loop for the complex integral Φ in the case of **a** the positive imaginary part of the frequency, **b** the negative imaginary part of the frequency

Applying Cauchy's integral theorem (Jordan's lemma) to the complex integral Φ with the loop $C^{(-)}$ in Fig. 2.3a, the integral along the real axis is evaluated as the residue at the lower pole $\xi = -\varpi/c = -(\omega + i\varepsilon)/c$,

$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{P}{\xi^2-(\varpi/c)^2}\exp(-i\xi x)d\xi = \frac{2\pi i}{2\pi}\left[P\frac{\xi+\varpi/c}{\xi^2-(\varpi/c)^2}\exp(-i\xi x)\right]_{\xi=-\varpi/c} \quad (2.2.15)$$

Rewriting the above equation, and taking the limit $\varepsilon \to 0$, we have in the case of positive x,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{P}{\xi^2-(\omega/c)^2}\exp(-i\xi x)d\xi = \frac{iP}{2(\omega/c)}\exp(+i\omega x/c); \quad x > 0 \quad (2.2.16)$$

On the other hand, when $x < 0$, we employ the upper loop $C^{(+)}$ for the complex integral Φ and have

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{P}{\xi^2-(\varpi/c)^2}\exp(-i\xi x)d\xi = \frac{2\pi i}{2\pi}\left[P\frac{\xi-\varpi/c}{\xi^2-(\varpi/c)^2}\exp(-i\xi x)\right]_{\xi=\varpi/c} \quad (2.2.17)$$

Then, we take the limit $\varepsilon \to 0$,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{P}{\xi^2-(\omega/c)^2}\exp(-i\xi x)d\xi = \frac{iP}{2(\omega/c)}\exp(-i\omega x/c); \quad x<0 \quad (2.2.18)$$

Unifying the two Eqs. (2.2.16) and (2.2.18), we have for the Fourier inversion integral, where the positive imaginary part of the complex frequency tends to zero, i.e.

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{P}{\xi^2-(\omega/c)^2}\exp(-i\xi x)d\xi = \frac{iP}{2(\omega/c)}\exp(+i\omega|x|/c); \quad \omega + i\varepsilon|_{\varepsilon\to 0} \quad (2.2.19)$$

(2) Small negative imaginary part: $\varpi = \omega - i\varepsilon$

When the imaginary part of the complex frequency is negative, the poles are also shifted from the real axis as shown in Fig. 2.3b. In order to guarantee the convergence on the large semi-circle, the lower loop $C^{(-)}$ is employed in the case of positive x, and the upper $C^{(+)}$ in that of negative x.

When $x > 0$, we employ the loop $C^{(-)}$ and apply Jordan's lemma to the complex integral Φ in Eq. (2.2.14). The integral along the real axis is converted to the residue at the lower pole, $\xi = \varpi(= \omega - i\varepsilon)/c$,

$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{P}{\xi^2-(\varpi/c)^2}\exp(-i\xi x)d\xi = \frac{2\pi i}{2\pi}\left[P\frac{\xi-\varpi/c}{\xi^2-(\varpi/c)^2}\exp(-i\xi x)\right]_{\xi=\varpi/c} \quad (2.2.20)$$

Rewriting the above and taking the limit $\varepsilon \to 0$, we have in the case of positive x,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{P}{\xi^2-(\omega/c)^2}\exp(-i\xi x)d\xi = -\frac{iP}{2(\omega/c)}\exp(-i\omega x/c); \quad x > 0 \quad (2.2.21)$$

Similarly, when we employ the upper loop $C^{(+)}$ in the case of $x<0$,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{P}{\xi^2-(\varpi/c)^2}\exp(-i\xi x)d\xi = \frac{2\pi i}{2\pi}\left[P\frac{\xi+\varpi/c}{\xi^2-(\varpi/c)^2}\exp(-i\xi x)\right]_{\xi=-\varpi/c} \quad (2.2.22)$$

Taking the limit $\varepsilon \to 0$, the Fourier inversion integral is evaluated as

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{P}{\xi^2-(\omega/c)^2}\exp(-i\xi x)d\xi = -\frac{iP}{2(\omega/c)}\exp(+i\omega x/c); \quad x<0 \quad (2.2.23)$$

Unifying two Eqs. (2.2.21) and (2.2.23), we have the unified expression when the negative imaginary part of the frequency approached to zero,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{P}{\xi^2-(\omega/c)^2}\exp(-i\xi x)d\xi = -\frac{iP}{2(\omega/c)}\exp(-i\omega|x|/c); \quad \omega - i\varepsilon|_{\varepsilon\to 0} \quad (2.2.24)$$

(3) Selection of the imaginary part

Two expressions are obtained for the single Fourier inversion integral. They are Eqs. (2.2.19) and (2.2.24) and are summarized as

$$\phi^{\#} = \begin{cases} -\frac{iP}{2(\omega/c)}\exp(-i\omega|x|/c); & \omega - i\varepsilon|_{\varepsilon\to 0} \\ +\frac{iP}{2(\omega/c)}\exp(+i\omega|x|/c); & \omega + i\varepsilon|_{\varepsilon\to 0} \end{cases} \quad (2.2.25)$$

The wave nature of the two expressions is discussed by multiplying the time factor,

$$\phi = \begin{cases} -\frac{iP}{2(\omega/c)} \exp\{+i\omega(t - |x|/c)\}; & \omega - i\varepsilon|_{\varepsilon\to 0} \\ +\frac{iP}{2(\omega/c)} \exp\{+i\omega(t + |x|/c)\}; & \omega + i\varepsilon|_{\varepsilon\to 0} \end{cases} \tag{2.2.26}$$

Inspecting the argument of the exponential function in the above equation, the upper solution shows an out-going (radiation) wave from the source, since the argument of the exponential function is $t - |x|/c$. On the other hand, the lower includes $t + |x|/c$ and shows an incoming wave, coming from infinity. Since the suitable solution must be the out-going wave, we employ the upper formula in Eqs. (2.2.25) and (2.2.26). Thus, the complex frequency with the negative imaginary part is the right assumption. Then, the suitable integration contour for the Fourier inversion integral is that of Fig. 2.3b and the final result for the time-harmonic response is given by

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - Q\delta(x)\exp(+i\omega t) \quad \Rightarrow \quad \phi = -\frac{iQ}{2(\omega/c)}\exp\{+i\omega(t - |x|/c)\} \tag{2.2.27}$$

We have just learned that the right selection for the complex frequency is $\varpi = \omega - i\varepsilon$, i.e. negative imaginary part, and the suitable integration loop is $C^{(\pm)}$ in Fig. 2.3b. If we do not introduce the small imaginary part and keep the integration path on the real axis, the poles is on the real axis and the integration path around the pole must be deformed by a small semi-circle shown in Fig. 2.4a. This deformation is valid only in the case of the positive time factor, $\exp(+i\omega t)$. If we assume the negative time factor $\exp(-i\omega t)$, the integration path on the real axis is deformed as that shown in Fig. 2.4b. Thus, the selection of the deformed path around the pole depends on the sign of the imaginary part of the complex frequency. Therefore, we could answer to the initial question about the deformation of the integration path for the Fourier inversion integral.

2.3 2D Static Source

Let us consider Green's function for a typical partial differential equation, the so-called Laplace equation. The Laplace equation with a source S is the nonhomogeneous differential equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -S\delta(x)\delta(y) \tag{2.3.1}$$

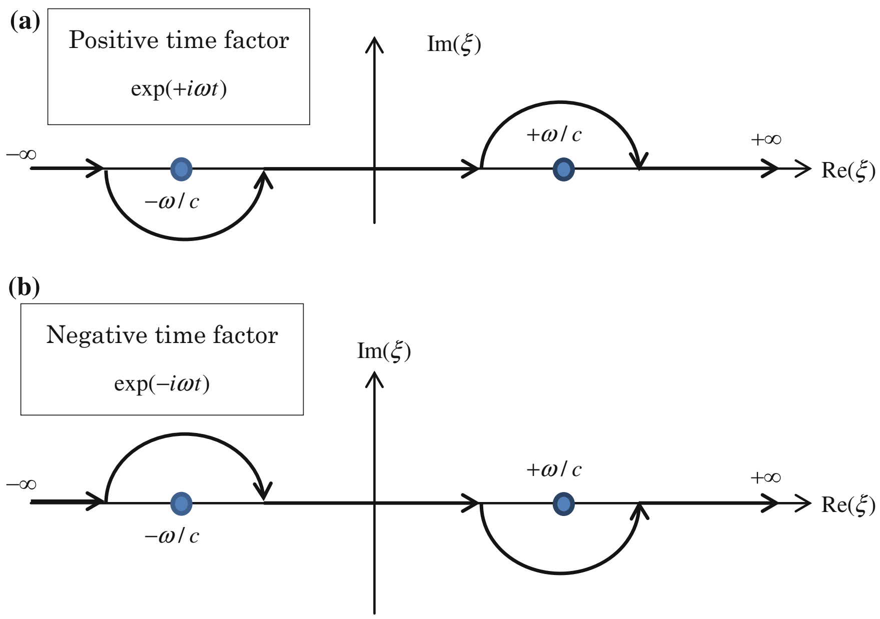

Fig. 2.4 Path deformation for the inversion integral. **a** Positive time factor, **b** Negative time factor

The product of two delta functions in the nonhomogeneous term shows the location of the source, i.e. the source S is placed at the coordinate origin (0, 0) in (x, y)-plane. The convergence condition at infinity,

$$\phi|_{\sqrt{x^2+y^2}\to\infty} = \left.\frac{\partial\phi}{\partial x}\right|_{\sqrt{x^2+y^2}\to\infty} = \left.\frac{\partial\phi}{\partial y}\right|_{\sqrt{x^2+y^2}\to\infty} = 0 \tag{2.3.2}$$

is also imposed.

Now, we apply the integral transforms to Eq. (2.3.1). Since the unknown function ϕ has two space variables, we apply the double Fourier transform defined by

$$\bar{\phi}(\xi) = \int_{-\infty}^{+\infty} \phi(x)\exp(+i\xi x)dx, \qquad \phi(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{\phi}(\xi)\exp(-i\xi x)d\xi \tag{2.3.3}$$

$$\tilde{\phi}(\eta) = \int_{-\infty}^{+\infty} \phi(y)\exp(+i\eta y)dy, \qquad \phi(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{\phi}(\eta)\exp(-i\eta y)d\eta \tag{2.3.4}$$

to the differential equation (2.3.1) successively, as

$$\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=-S\delta(x)\delta(y)\right)\exp(+i\xi x)dx\right)\exp(+i\eta y)dy \tag{2.3.5}$$

With the aid of the convergence condition (2.3.2), each term is transformed as follows:

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\frac{\partial^2\phi}{\partial x^2}\exp(+i\xi x)dx\right)\exp(+i\eta y)dy=-\xi^2\bar{\tilde{\phi}}\\
&\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\frac{\partial^2\phi}{\partial y^2}\exp(+i\xi x)dx\right)\exp(+i\eta y)dy=-\eta^2\bar{\tilde{\phi}}\\
&\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}S\delta(x)\delta(y)\exp(+i\xi x)dx\right)\exp(+i\eta y)dy=S
\end{aligned} \tag{2.3.6}$$

Then, we have the simple algebraic equation $-(\xi^2+\eta^2)\bar{\tilde{\phi}}=-S$ for the double transformed function and its solution is given by

$$\bar{\tilde{\phi}}=\frac{S}{\xi^2+\eta^2} \tag{2.3.7}$$

The reader will find that the partial differential equation (2.3.1) is transformed to the simple algebraic equation. There is thus no need of solving the differential equation directly. The subsequent inversion process is however crucial for the solution in the actual space. The formal Fourier inversion integral with respect to the parameter η,

$$\bar{\phi}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{S}{\xi^2+\eta^2}\exp(-i\eta y)d\eta=\frac{S}{\pi}\int_{0}^{\infty}\frac{1}{\xi^2+\eta^2}\cos(\eta y)d\eta \tag{2.3.8}$$

is evaluated with the aid of the formula (2.1.22) and yields

$$\bar{\phi}=\frac{S}{2|\xi|}\exp(-|\xi||y|) \tag{2.3.9}$$

The next inversion is to evaluate the Fourier inversion integral with respect to the parameter ξ,

$$\phi = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{S}{2|\xi|} \exp(-|\xi||y|) \exp(-i\xi x) d\xi \tag{2.3.10}$$

Inspecting the integrand, the singular point at $\xi = 0$ lies on the real axis, i.e. on the integration path. It is impossible to evaluate the integral in the regular sense. So, we have to deform the integration path around the pole as that in the previous section. But it was somewhat complicated to determine the path deformation. In order to avoid this troublesome work, we employ a simpler way for evaluating the integral.

Since the trouble stems from the singular point at $\xi = 0$, in order to avoid the trouble, we differentiate equation (2.3.10) with respect to the space variables, x and y, respectively,

$$\frac{\partial \phi}{\partial x} = -\frac{S}{4\pi} \int_{-\infty}^{+\infty} \frac{i\xi}{|\xi|} \exp(-|\xi||y|) \exp(-i\xi x) d\xi \tag{2.3.11}$$

$$\frac{\partial \phi}{\partial y} = -\frac{S}{4\pi} \operatorname{sgn}(y) \int_{-\infty}^{+\infty} \exp(-|\xi||y|) \exp(-i\xi x) d\xi \tag{2.3.12}$$

where sgn(.) is the sign function defined by

$$\operatorname{sgn}(y) = \begin{cases} +1; & y > 0 \\ -1; & y < 0 \end{cases} \tag{2.3.13}$$

Using the symmetry of the integrand in Eqs. (2.3.11) and (2.3.12), the integrals are reduced to the real valued semi-infinite integrals,

$$\frac{\partial \phi}{\partial x} = -\frac{S}{2\pi} \int_{0}^{\infty} \exp(-\xi|y|) \sin(\xi x) d\xi \tag{2.3.14}$$

$$\frac{\partial \phi}{\partial y} = -\frac{S}{2\pi} \operatorname{sgn}(y) \int_{0}^{\infty} \exp(-\xi|y|) \cos(\xi x) d\xi \tag{2.3.15}$$

The two integrals in the above equations are well-known from Calculus and we have the formulas,

$$\int_0^\infty \exp(-\xi y)\sin(\xi x)d\xi = \frac{x}{x^2+y^2}, \quad \int_0^\infty \exp(-\xi y)\cos(\xi x)d\xi = \frac{y}{x^2+y^2} \tag{2.3.16}$$

Then, Eqs. (2.3.14) and (2.3.15) are evaluated as

$$\frac{\partial\phi}{\partial x} = -\frac{S}{2\pi}\frac{x}{x^2+y^2}, \qquad \frac{\partial\phi}{\partial y} = -\frac{S}{2\pi}\frac{y}{x^2+y^2} \tag{2.3.17}$$

The derivative of ϕ is completely determined. We integrate the above two equations with respect to x and y, respectively. This integration leads to two expressions for the single function ϕ as

$$\begin{aligned} \frac{\partial\phi}{\partial x} \quad &\Rightarrow \quad \phi = -\frac{S}{4\pi}\log\left(x^2+y^2\right) + c'(y) \\ \frac{\partial\phi}{\partial y} \quad &\Rightarrow \quad \phi = -\frac{S}{4\pi}\log\left(x^2+y^2\right) + c''(x) \end{aligned} \tag{2.3.18}$$

Since the above two equations must be equal, two constant terms should be identical,

$$c'(y) = c''(x) \tag{2.3.19}$$

This condition is satisfied only when the two terms are pure constant and do not include any space variable:

$$c'(y) = c''(x) = \text{constant} \tag{2.3.20}$$

Then, we have Green's function for the Laplace equation,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -S\delta(x)\delta(y) \quad \Rightarrow \quad \phi = -\frac{S}{4\pi}\log\left(x^2+y^2\right) + \text{arbitrary constant} \tag{2.3.21}$$

This Green's function does not satisfy the convergence condition at infinity, since it has the constant. This is because we could not carry out the Fourier inversion integral of Eq. (2.3.10) directly.

2.4 2D Impulsive Source

The 2D wave equation with an impulsive source is given by

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - P\delta(x)\delta(y)\delta(t) \tag{2.4.1}$$

The nonhomogeneous term states that the source with magnitude P is placed at the coordinate origin and is impulsive in time. The quiescent condition at an initial time,

$$\phi|_{t=0} = \left.\frac{\partial \phi}{\partial t}\right|_{t=0} = 0 \tag{2.4.2}$$

and the convergence condition at infinity,

$$\phi|_{\sqrt{x^2+y^2}\to\infty} = \left.\frac{\partial \phi}{\partial x}\right|_{\sqrt{x^2+y^2}\to\infty} = \left.\frac{\partial \phi}{\partial y}\right|_{\sqrt{x^2+y^2}\to\infty} = 0 \tag{2.4.3}$$

are also employed. As the wave equation (2.4.1) has two space variables x and y, and one time variable t, the triple integral transform is applied to the differential equation (2.4.1): the Laplace transform with respect to the time variable,

$$\phi^*(s) = \int_0^\infty \phi(t)\exp(-st)dt \tag{2.4.4}$$

and the double Fourier transform with respect to the space variables,

$$\bar{\phi}(\xi) = \int_{-\infty}^{+\infty} \phi(x)\exp(+i\xi x)dx, \quad \phi(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{\phi}(\xi)\exp(-i\xi x)d\xi \tag{2.4.5}$$

$$\tilde{\phi}(\eta) = \int_{-\infty}^{+\infty} \phi(y)\exp(+i\eta y)dy, \quad \phi(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{\phi}(\eta)\exp(-i\eta y)d\eta \tag{2.4.6}$$

Applying this triple integral transform with the quiescent and convergence conditions, the original differential equation (2.4.1) is transformed to the simple algebraic equation for the unknown function $\tilde{\bar{\phi}}^*$,

$$-\xi^2\tilde{\bar{\phi}}^* - \eta^2\tilde{\bar{\phi}}^* = (s/c)^2\tilde{\bar{\phi}}^* - P \tag{2.4.7}$$

Then, the exact expression for $\tilde{\bar{\phi}}^*$ in the transformed domain is given by

$$\tilde{\bar{\phi}}^* = \frac{P}{\xi^2 + \eta^2 + (s/c)^2} \tag{2.4.8}$$

For the inversion, three inversion integrals must be carried out successively. The first one is the Fourier inversion integral with respect to the parameter η. This is reduced to the semi-infinite integral as

$$\bar{\phi}^* = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{P}{\xi^2 + \eta^2 + (s/c)^2}\exp(-i\eta y)d\eta = \frac{P}{\pi}\int_0^{\infty} \frac{\cos(\eta y)}{\eta^2 + \sqrt{\xi^2 + (s/c)^2}^2}d\eta \tag{2.4.9}$$

The integral on the far right side is easily evaluated by applying the formula (2.1.22). Then, the first Fourier inversion integral in Eq. (2.4.9) yields

$$\bar{\phi}^* = \frac{P}{2\sqrt{\xi^2 + (s/c)^2}}\exp\left\{-|y|\sqrt{\xi^2 + (s/c)^2}\right\} \tag{2.4.10}$$

Secondly, we apply the inversion integral with respect to the parameter ξ,

$$\phi^* = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{P}{2\sqrt{\xi^2 + (s/c)^2}}\exp\left\{-|y|\sqrt{\xi^2 + (s/c)^2}\right\}\exp(-i\xi x)d\xi \tag{2.4.11}$$

The above integral is also reduced to the semi-infinite integral, as

$$\phi^* = \frac{P}{2\pi}\int_0^{\infty} \frac{1}{\sqrt{\xi^2 + (s/c)^2}}\exp\left\{-|y|\sqrt{\xi^2 + (s/c)^2}\right\}\cos(\xi x)d\xi \tag{2.4.12}$$

and we apply the integration formula (Erdélyi 1954, vol. I, pp. 17, 27)

$$\int_0^{\infty} \frac{1}{\sqrt{x^2 + a^2}}\exp\left(-c\sqrt{x^2 + a^2}\right)\cos(bx)dx = K_0\left(a\sqrt{b^2 + c^2}\right) \tag{2.4.13}$$

where $K_0(.)$ is the zeroth order modified Bessel function of the second kind. Then, Eq. (2.4.12) takes the compact form

$$\phi^* = \frac{P}{2\pi}K_0\left(\frac{s}{c}\sqrt{x^2 + y^2}\right) \tag{2.4.14}$$

The last inversion is the Laplace inversion. The Laplace inversion is symbolically expressed as

$$\phi = \frac{P}{2\pi} L^{-1}\left[K_0\left(\frac{s}{c}\sqrt{x^2 + y^2}\right)\right] \tag{2.4.15}$$

We have the suitable inversion formula (Erdélyi 1954, vol. I, pp. 277, 8),

$$L^{-1}[K_0(as)] = \frac{H(t-a)}{\sqrt{t^2 - a^2}} = \begin{cases} 0; & t < a \\ \frac{1}{\sqrt{t^2-a^2}}; & t > a \end{cases} \tag{2.4.16}$$

Applying this formula to Eq. (2.4.15), the simple expression for ϕ is obtained as

$$\phi = \frac{cP}{2\pi} \frac{H(ct - r)}{\sqrt{(ct)^2 - r^2}} = \frac{cP}{2\pi} \begin{cases} 0; & ct < r \\ \frac{1}{\sqrt{(ct)^2 - r^2}}; & ct > r \end{cases} \tag{2.4.17}$$

where $H(.)$ is Heaviside's unit step function and the radial distance r from the source is defined by

$$r = \sqrt{x^2 + y^2} \tag{2.4.18}$$

Consequently, we have the exact expression for Green's function of the 2D wave equation as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - P\delta(x)\delta(y)\delta(t) \quad \Rightarrow \quad \phi = \frac{cP}{2\pi}\frac{H(ct - r)}{\sqrt{(ct)^2 - r^2}} \tag{2.4.19}$$

2.5 2D Time-Harmonic Source

When the source is vibrating harmonically, the nonhomogeneous 2D wave equation is given by

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - Q\delta(x)\delta(y)\exp(+i\omega t) \tag{2.5.1}$$

where Q and ω are the magnitude and the frequency of the source, respectively. We assume that its Green's function satisfies the convergence condition at infinity,

$$\phi|_{\sqrt{x^2+y^2}\to\infty} = \left.\frac{\partial \phi}{\partial x}\right|_{\sqrt{x^2+y^2}\to\infty} = \left.\frac{\partial \phi}{\partial y}\right|_{\sqrt{x^2+y^2}\to\infty} = 0 \tag{2.5.2}$$

As the standard technique for the time-harmonic response, the Green's function is assumed as the product

$$\phi(x,y,t) = \phi^{\#}(x,y)\exp(+i\omega t) \tag{2.5.3}$$

where $\phi^{\#}(x,y)$ is an amplitude function to be determined. Substituting this assumption into the nonhomogeneous wave equation (2.5.1), we have the reduced wave equation (so-called Helmholtz equation) for the amplitude function $\phi^{\#}(x,y)$,

$$\frac{\partial^2\phi^{\#}}{\partial x^2} + \frac{\partial^2\phi^{\#}}{\partial y^2} + (\omega/c)^2\phi^{\#} = -Q\delta(x)\delta(y) \tag{2.5.4}$$

The convergence condition (2.5.2) is also rewritten for the amplitude function as

$$\phi^{\#}\Big|_{\sqrt{x^2+y^2}\to\infty} = \frac{\partial\phi^{\#}}{\partial x}\Big|_{\sqrt{x^2+y^2}\to\infty} = \frac{\partial\phi^{\#}}{\partial y}\Big|_{\sqrt{x^2+y^2}\to\infty} = 0 \tag{2.5.5}$$

The double Fourier transform with respect to two space variables as defined by Eqs. (2.4.5) and (2.4.6) is applied to the nonhomogeneous Helmholtz equation (2.5.4),

$$\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\left(\frac{\partial^2\phi^{\#}}{\partial x^2} + \frac{\partial^2\phi^{\#}}{\partial y^2} = -(\omega/c)^2\phi^{\#} - Q\delta(x)\delta(y)\right)\exp(+i\xi x)dx\right]\exp(+i\eta y)dy \tag{2.5.6}$$

Defining the double transform of the amplitude function as

$$\tilde{\bar{\phi}}^{\#} = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\phi^{\#}(x,y)\exp(+i\xi x)dx\right]\exp(+i\eta y)dy \tag{2.5.7}$$

Equation (2.5.6) is transformed into the algebraic equation for the unknown amplitude function $\tilde{\bar{\phi}}^{\#}$,

$$-\xi^2\tilde{\bar{\phi}}^{\#} - \eta^2\tilde{\bar{\phi}}^{\#} = -(\omega/c)^2\tilde{\bar{\phi}}^{\#} - Q \tag{2.5.8}$$

Then, the explicit expression for the amplitude function in the transformed domain is given by

$$\tilde{\bar{\phi}}^{\#} = \frac{Q}{\xi^2 + \eta^2 - (\omega/c)^2} \tag{2.5.9}$$

As the first inversion, we apply the Fourier inversion integral with respect to the parameter η. Its formal inversion integral is simplified as

$$\bar{\phi}^{\#} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{Q}{\xi^2+\eta^2-(\omega/c)^2}\exp(-i\eta y)d\eta = \frac{Q}{\pi}\int_0^{\infty} \frac{\cos(\eta y)}{\eta^2+\sqrt{\xi^2-(\omega/c)^2}^2}d\eta \tag{2.5.10}$$

The far right integral is evaluated with the aid of the formula (2.1.22) and yields

$$\bar{\phi}^{\#} = \frac{Q}{2\sqrt{\xi^2-(\omega/c)^2}}\exp\left\{-|y|\sqrt{\xi^2-(\omega/c)^2}\right\} \tag{2.5.11}$$

where the real part of the square root function $\sqrt{\xi^2-(\omega/c)^2}$ must be positive in order to guarantee the convergence at infinity $|y| \to \infty$, i.e.

$$\mathrm{Re}\left(\sqrt{\xi^2-(\omega/c)^2}\right) \geq 0 \tag{2.5.12}$$

The second Fourier inversion integral with respect to the parameter ξ is given by

$$\phi^{\#} = \frac{Q}{4\pi}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\xi^2-(\omega/c)^2}}\exp\left\{-|y|\sqrt{\xi^2-(\omega/c)^2}\right\}\exp(-i\xi x)d\xi \tag{2.5.13}$$

The integrand in the above inversion integral has two branch points at $\xi = \pm\omega/c$ which are on the integration path, and the square root function (radical) $\sqrt{\xi^2-(\omega/c)^2}$ changes its sign around the branch point. Thus, we have to discuss the path deformation around the branch point as was done in the case of the 1D time-harmonic problem in Sect. 2.2. However, the singular point for the 1D problem was the simple pole, but that for the present 2D problem is the branch point for the square root function. Therefore, we have to discuss the introduction of the branch cut in the complex plane. As the detailed discussion for the introduction of branch cuts has been carried out in Sect. 1.3.2 in Chap. 1, we do not repeat, but, we cite here its result. If the reader needs more about the introduction of the branch cut, please return to Sect. 1.3 in Chap. 1.

In order to evaluate the inversion integral of Eq. (2.5.13), we define the integral I as

$$I = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\xi^2-(\omega/c)^2}}\exp\left\{-i\xi x-|y|\sqrt{\xi^2-(\omega/c)^2}\right\}d\xi \tag{2.5.14}$$

and consider the complex integral having the same integrand as that in the above equation,

$$\Phi = \frac{1}{2\pi}\int_C \frac{1}{\sqrt{\zeta^2-(\omega/c)^2}}\exp\left\{-i\zeta x-|y|\sqrt{\zeta^2-(\omega/c)^2}\right\}d\zeta \tag{2.5.15}$$

In the subsequent analysis, it should be understood that the real and imaginary parts of the complex variable ζ are ξ and η, i.e. $\zeta=\xi+i\eta$. The square root function $\sqrt{\zeta^2-(\omega/c)^2}$ has two branch points at $\zeta=\pm\omega/c$, and the corresponding two branch cuts must be introduced in the complex ζ-plane, in order to make the radical be single-valued and to guarantee the convergence condition,

$$\mathrm{Re}\left(\sqrt{\zeta^2-(\omega/c)^2}\right)\geq 0 \tag{2.5.16}$$

The discussion on the branch cut for the square root function in Sect. 1.3.2 shows two ways to consider the complex frequency: one is the complex frequency with a small **positive** imaginary part and the other with a small **negative** imaginary part. As the shape of the branch cut depends on the sign of the imaginary part, we evaluate the integral I for these two cases of the complex frequency, separately, and then determine the suitable imaginary part.

(1) Complex frequency with the negative imaginary part

If we add a small negative imaginary part to the frequency, the branch points are shifted from the real axis and then two branch cuts are introduced along the hyperbola in the second and the fourth quadrants in the complex ζ-plane as shown in Figs. 1.7 and 1.8 in Chap. 1. When the imaginary part of the frequency tends to zero as the limit, the branch cut lies on the real and imaginary axes as shown by the thick lines in Fig. 1.8a. Therefore, we take the closed loop C for our complex integral of Eq. (2.5.15) as shown in Fig. 2.5. We have two closed loops for C: one is the upper closed loop $C^{(+)}$ and the other is the lower loop $C^{(-)}$. The selection of the two loops depends on the convergence of the integrands at the infinity $|\zeta|\to\infty$. The lower loop is employed when the parameter (space variable) x is positive, and the upper loop is employed when $x<0$. The following evaluation is carried out for the case of the positive space variable, $x>0$.

When $x>0$, the loop for the complex integral is $C^{(-)}$ as shown in Fig. 2.5 where the integration path is the straight line $\overrightarrow{\mathrm{AOB}}$ on the real axis. In order to exclude the lower branch cut, the path along the branch cut is introduced. The path is composed of two lines, $\overline{\mathrm{CDE}}$ and $\overline{\mathrm{FGH}}$ along the branch cut, and a small circle $\overset{\frown}{\mathrm{EF}}$ around the branch point. It should be understood that the radius of the small circle vanishes as the limit. Two edges of the straight line $\overrightarrow{\mathrm{AOB}}$ are connected to the edges

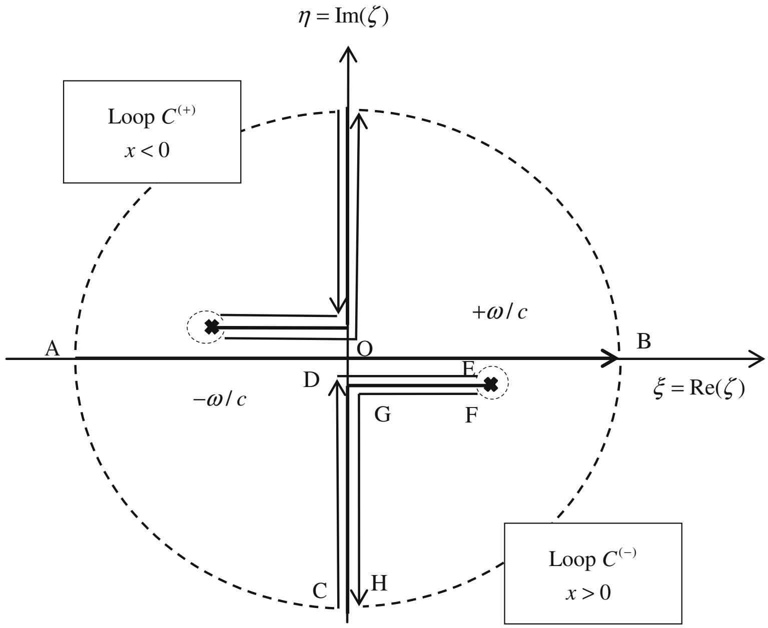

Fig. 2.5 Closed loop C for the complex integral Φ in Eq. (2.5.15) (the negative imaginary part of the complex frequency vanished)

of the path $\overline{\mathrm{CDE}}$ and $\overline{\mathrm{FGH}}$ through two quarter circles, $\overset{\frown}{\mathrm{AC}}$ and $\overset{\frown}{\mathrm{BH}}$, respectively. It is also understood that the radius of the quarter circle tends to infinity as the limit. Then, the closed loop $C^{(-)}$ is the sum of the integrals along these paths. In the clockwise direction, the closed loop is

$$C^{(-)}:\ \overrightarrow{\mathrm{AOB}} + \overset{\frown}{\mathrm{BH}} + \overrightarrow{\mathrm{HGF}} + \overset{\frown}{\mathrm{FE}} + \overrightarrow{\mathrm{EDC}} + \overset{\frown}{\mathrm{CA}} \tag{2.5.17}$$

The complex integral of Eq. (2.5.15) is decomposed into the integrals along the path segments as

$$\Phi = \frac{1}{2\pi}\left(\int_{\overrightarrow{\mathrm{AOB}}} + \int_{\overset{\frown}{\mathrm{BH}}} + \int_{\overrightarrow{\mathrm{HGF}}} + \int_{\overset{\frown}{\mathrm{FE}}} + \int_{\overrightarrow{\mathrm{EDC}}} + \int_{\overset{\frown}{\mathrm{CA}}}\right)\frac{1}{\sqrt{\zeta^2-(\omega/c)^2}}\exp\left\{-i\zeta x - |y|\sqrt{\zeta^2-(\omega/c)^2}\right\}d\zeta \tag{2.5.18}$$

The integral along the small circle $\overset{\frown}{\mathrm{FE}}$ around the branch point vanishes as its radius tends to zero. The two integrals along the large arcs $\overset{\frown}{\mathrm{BH}}$ and $\overset{\frown}{\mathrm{CA}}$ also vanish as the radius tends to infinity due to the convergence condition of Eq. (2.5.16). Thus, since no singular point is included in the closed loop and the total value of the complex integral vanishes, i.e. $\Phi = 0$, the integral along the real axis $\overrightarrow{\mathrm{AOB}}$ is converted to the integrals along the branch cut,

$$\begin{aligned}
&\frac{1}{2\pi}\int\limits_{\overrightarrow{\mathrm{AOB}}} \frac{1}{\sqrt{\zeta^2-(\omega/c)^2}}\exp\left\{-i\zeta x-|y|\sqrt{\zeta^2-(\omega/c)^2}\right\}d\zeta \\
&=\frac{1}{2\pi}\left(\int\limits_{\overrightarrow{\mathrm{CDE}}}+\int\limits_{\overrightarrow{\mathrm{FGH}}}\right)\frac{1}{\sqrt{\zeta^2-(\omega/c)^2}}\exp\left\{-i\zeta x-|y|\sqrt{\zeta^2-(\omega/c)^2}\right\}d\zeta
\end{aligned} \tag{2.5.19}$$

where the directions of paths $\overrightarrow{\mathrm{CDE}}$ and $\overrightarrow{\mathrm{FGH}}$ are inverse of the paths $\overrightarrow{\mathrm{EDC}}$ and $\overrightarrow{\mathrm{HGF}}$, respectively. Equation (2.5.19) states that it is enough to consider the integrals along the branch cut to evaluate the integral on the real axis. The line paths $\overrightarrow{\mathrm{CDE}}$ and $\overrightarrow{\mathrm{FGH}}$ are further decomposed into the line segments along the real and imaginary axes. They are

$$\overrightarrow{\mathrm{CDE}}=\overrightarrow{\mathrm{CD}}+\overrightarrow{\mathrm{DE}},\ \overrightarrow{\mathrm{FGH}}=\overrightarrow{\mathrm{FG}}+\overrightarrow{\mathrm{GH}} \tag{2.5.20}$$

The argument and value of the square root function have already been determined in Sect. 1.3.2(2) in Chap. 1 and are tabulated in Table 2.1 with the integration variable along each line segment. Applying these results, the integral along each path is given by

$$\begin{aligned}
&\frac{1}{2\pi}\int\limits_{\overrightarrow{\mathrm{AOB}}} \frac{1}{\sqrt{\zeta^2-(\omega/c)^2}}\exp\left\{-i\zeta x-|y|\sqrt{\zeta^2-(\omega/c)^2}\right\}d\zeta \\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\xi^2-(\omega/c)^2}}\exp\left\{-i\xi x-|y|\sqrt{\xi^2-(\omega/c)^2}\right\}d\xi
\end{aligned} \tag{2.5.21}$$

$$\begin{aligned}
&\frac{1}{2\pi}\int\limits_{\overrightarrow{\mathrm{CD}}} \frac{1}{\sqrt{\zeta^2-(\omega/c)^2}}\exp\left\{-i\zeta x-|y|\sqrt{\zeta^2-(\omega/c)^2}\right\}d\zeta \\
&=\frac{1}{2\pi}\int_{0}^{\infty}\frac{1}{\sqrt{\eta^2+(\omega/c)^2}}\exp\left\{-\eta x-i|y|\sqrt{\eta^2+(\omega/c)^2}\right\}d\eta
\end{aligned} \tag{2.5.22}$$

Table 2.1 Value and argument of the square root function on the integration path in Fig. 2.5

Path	$\sqrt{\zeta^2-(\omega/c)^2}$	Variable ζ and its range
$\overrightarrow{\mathrm{AOB}}$	$\sqrt{\xi^2-(\omega/c)^2}$	$\zeta=\xi$
		$-\infty<\xi<+\infty$
$\overrightarrow{\mathrm{CD}}$	$+i\sqrt{\eta^2+(\omega/c)^2}$	$\zeta=-i\eta$
		$0<\eta<\infty$
$\overrightarrow{\mathrm{DE}}$	$+i\sqrt{(\omega/c)^2-\xi^2}$	$\zeta=+\xi$
		$0<\xi<\omega/c$
$\overrightarrow{\mathrm{FG}}$	$-i\sqrt{(\omega/c)^2-\xi^2}$	$\zeta=+\xi$
		$0<\xi<\omega/c$
$\overrightarrow{\mathrm{GH}}$	$-i\sqrt{\eta^2+(\omega/c)^2}$	$\xi=-i\eta$
		$0<\eta<\infty$

ξ and η are positive real, except $\overrightarrow{\mathrm{BOA}}$

$$\begin{aligned}&\frac{1}{2\pi}\int\limits_{\overrightarrow{\mathrm{DE}}}\frac{1}{\sqrt{\zeta^2-(\omega/c)^2}}\exp\left\{-i\zeta x-|y|\sqrt{\zeta^2-(\omega/c)^2}\right\}d\zeta\\&=-\frac{i}{2\pi}\int\limits_0^{\omega/c}\frac{1}{\sqrt{(\omega/c)^2-\xi^2}}\exp\left\{-i\xi x-i|y|\sqrt{(\omega/c)^2-\xi^2}\right\}d\xi\end{aligned}\tag{2.5.23}$$

$$\begin{aligned}&\frac{1}{2\pi}\int\limits_{\overrightarrow{\mathrm{FG}}}\frac{1}{\sqrt{\zeta^2-(\omega/c)^2}}\exp\left\{-i\zeta x-|y|\sqrt{\zeta^2-(\omega/c)^2}\right\}d\zeta\\&=-\frac{i}{2\pi}\int\limits_0^{\omega/c}\frac{1}{\sqrt{(\omega/c)^2-\xi^2}}\exp\left\{-i\xi x+i|y|\sqrt{(\omega/c)^2-\xi^2}\right\}d\xi\end{aligned}\tag{2.5.24}$$

$$\begin{aligned}&\frac{1}{2\pi}\int\limits_{\overrightarrow{\mathrm{GH}}}\frac{1}{\sqrt{\zeta^2-(\omega/c)^2}}\exp\left\{-i\zeta x-|y|\sqrt{\zeta^2-(\omega/c)^2}\right\}d\zeta\\&=\frac{1}{2\pi}\int\limits_0^{\infty}\frac{1}{\sqrt{\eta^2+(\omega/c)^2}}\exp\left\{-\eta x+i|y|\sqrt{\eta^2+(\omega/c)^2}\right\}d\eta\end{aligned}\tag{2.5.25}$$

Substituting the above equations into Eq. (2.5.19) with the path decomposition of Eq. (2.5.20), we have for the integral I,

$$
\begin{aligned}
I &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\sqrt{\xi^2-(\omega/c)^2}} \exp\left\{-i\xi x - |y|\sqrt{\xi^2-(\omega/c)^2}\right\} d\xi \\
&= \frac{1}{\pi}\int_{0}^{\infty} \frac{1}{\sqrt{\eta^2+(\omega/c)^2}} \exp(-\eta x)\cos\left\{y\sqrt{\eta^2+(\omega/c)^2}\right\} d\eta \\
&\quad -\frac{1}{\pi}\int_{0}^{\omega/c} \frac{1}{\sqrt{(\omega/c)^2-\xi^2}} \sin(\xi x)\cos\left\{y\sqrt{(\omega/c)^2-\xi^2}\right\} d\xi \\
&\quad -\frac{i}{\pi}\int_{0}^{\omega/c} \frac{1}{\sqrt{(\omega/c)^2-\xi^2}} \cos(\xi x)\cos\left\{y\sqrt{(\omega/c)^2-\xi^2}\right\} d\xi
\end{aligned}
\tag{2.5.26}
$$

The Fourier inversion integral has just been converted to three real-valued integrals. Fortunately, we can evaluate these integrals with the aid of integration formulas. Two integration formulas (Gradshteyn and Ryzhik 1980, pp. 755, 6.677, No. 4)*

$$
\begin{aligned}
&\int_{0}^{\infty} \frac{1}{\sqrt{a^2+x^2}} \cos\left(b\sqrt{a^2+x^2}\right)\exp(-cx)dx - \int_{0}^{a} \frac{1}{\sqrt{a^2-x^2}} \cos\left(b\sqrt{a^2-x^2}\right)\sin(cx)dx \\
&= -\frac{\pi}{2} Y_0\left(a\sqrt{b^2+c^2}\right)
\end{aligned}
\tag{2.5.27}
$$

and (Erdélyi 1954, vol. I, pp. 28, 42)

$$
\int_{0}^{a} \frac{1}{\sqrt{a^2-x^2}} \cos\left(b\sqrt{a^2-x^2}\right)\cos(cx)dx = \frac{\pi}{2} J_0\left(a\sqrt{b^2+c^2}\right)
\tag{2.5.28}
$$

are applied to the right hand side of Eq. (2.5.26) and the Hankel function (Watson 1966, p. 73) is introduced. Then, Eq. (2.5.26) is simplified as

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\xi^2-(\omega/c)^2}} \exp\left\{-i\xi x - |y|\sqrt{\xi^2-(\omega/c)^2}\right\} d\xi \\
&= -\frac{1}{2} Y_0\left(\frac{\omega}{c}\sqrt{x^2+y^2}\right) - \frac{i}{2} J_0\left(\frac{\omega}{c}\sqrt{x^2+y^2}\right) \\
&= -\frac{i}{2} H_0^{(2)}\left(\frac{\omega}{c}\sqrt{x^2+y^2}\right)
\end{aligned}
\tag{2.5.29}
$$

where $J_0(.)$ and $Y_0(.)$ are Bessel functions of the first and second kind, respectively, and Hankel function of the second kind is defined by $H_0^{(2)}(.) = J_0(.) - iY_0(.)$. Finally, we have the simple expression for the Fourier inversion integral I. That is

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\xi^2 - (\omega/c)^2}} \exp\left\{-i\xi x - |y|\sqrt{\xi^2 - (\omega/c)^2}\right\} d\xi \\ &= -\frac{i}{2} H_0^{(2)}\left(\frac{\omega}{c}\sqrt{x^2 + y^2}\right) \end{aligned} \tag{2.5.30}$$

It is easily understood that the above equation is also valid for the negative space variable, $x < 0$, since Eq. (2.5.30) is the even function of the space variable x. So, if we perform the complex integral with the upper closed loop $C^{(+)}$ in Fig. 2.5, we can obtain the same expression for the negative space variable. However, the result in Eq. (2.5.30) is valid only when the negative imaginary part of the complex frequency tends to zero.

(2) Complex frequency with the positive imaginary part

If we assume that the imaginary part of the complex frequency approaches to zero from the positive, the branch cuts corresponding to two branch points are introduced in the first and the third quadrants in the complex plane as was discussed in Sect. 1.3.2 in Chap. 1 and are shown in Fig. 1.11 as its zero limit. Thus, the

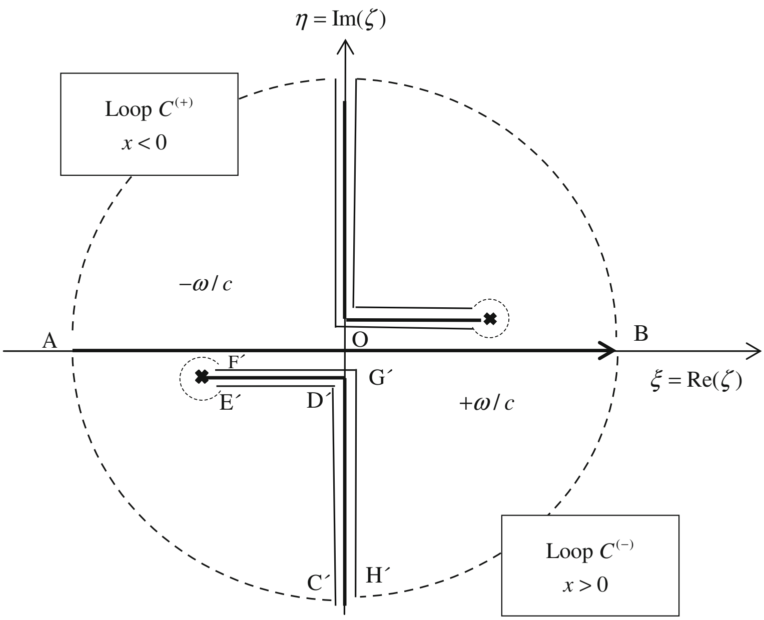

Fig. 2.6 Closed loop C for the complex integral Φ in Eq. (2.5.15) (the positive imaginary part of the complex frequency vanished)

Table 2.2 Value and argument of the root function along the integration path in Fig. 2.6

Path	$\sqrt{\zeta^2 - (\omega/c)^2}$	Variable ζ and its range
$\overrightarrow{\mathrm{AOB}}$	$\sqrt{\xi^2 - (\omega/c)^2}$	$\zeta = \xi$
		$-\infty < \xi < +\infty$
$\overrightarrow{\mathrm{C'D'}}$	$+i\sqrt{\eta^2 + (\omega/c)^2}$	$\zeta = -i\eta$
		$0 < \eta < \infty$
$\overrightarrow{\mathrm{D'E'}}$	$+i\sqrt{(\omega/c)^2 - \xi^2}$	$\zeta = -\xi$
		$0 < \xi < \omega/c$
$\overrightarrow{\mathrm{F'G'}}$	$-i\sqrt{(\omega/c)^2 - \xi^2}$	$\zeta = -\xi$
		$0 < \xi < \omega/c$
$\overrightarrow{\mathrm{G'H'}}$	$-i\sqrt{\eta^2 + (\omega/c)^2}$	$\xi = -i\eta$
		$0 < \eta < \infty$

ξ and η are positive real, except $\overrightarrow{\mathrm{BOA}}$

closed loop for the complex integral Φ is taken as $C^{(\pm)}$ in the complex ζ-plane as shown in Fig. 2.6 and the value and argument of the square root function are listed in Table 2.2. We can apply the same evaluation procedure as that in the previous subsection. The analysis is carried out for the case of the positive space variable, $x > 0$. The integral on the real axis, i.e. the integral I, is given by

$$\begin{aligned} I &= \frac{1}{2\pi} \int_{\overrightarrow{\mathrm{AOB}}} \frac{1}{\sqrt{\zeta^2 - (\omega/c)^2}} \exp\left\{ -i\zeta x - |y|\sqrt{\zeta^2 - (\omega/c)^2} \right\} d\zeta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\xi^2 - (\omega/c)^2}} \exp\left\{ -i\xi x - |y|\sqrt{\xi^2 - (\omega/c)^2} \right\} d\xi \end{aligned} \tag{2.5.31}$$

and the integrals along the branch cut are

$$\begin{aligned} &\frac{1}{2\pi} \int_{\overrightarrow{\mathrm{C'D'}}} \frac{1}{\sqrt{\zeta^2 - (\omega/c)^2}} \exp\left\{ -i\zeta x - |y|\sqrt{\zeta^2 - (\omega/c)^2} \right\} d\zeta \\ &= \frac{1}{2\pi} \int_{0}^{\infty} \frac{1}{\sqrt{\eta^2 + (\omega/c)^2}} \exp\left\{ -\eta x - i|y|\sqrt{\eta^2 + (\omega/c)^2} \right\} d\eta \end{aligned} \tag{2.5.32}$$

$$\begin{aligned} &\frac{1}{2\pi} \int_{\overrightarrow{\mathrm{D'E'}}} \frac{1}{\sqrt{\zeta^2 - (\omega/c)^2}} \exp\left\{ -i\zeta x - |y|\sqrt{\zeta^2 - (\omega/c)^2} \right\} d\zeta \\ &= \frac{i}{2\pi} \int_{0}^{\omega/c} \frac{1}{\sqrt{(\omega/c)^2 - \xi^2}} \exp\left\{ +i\xi x - i|y|\sqrt{(\omega/c)^2 - \xi^2} \right\} d\xi \end{aligned} \tag{2.5.33}$$

$$
\begin{aligned}
&\frac{1}{2\pi}\int\limits_{\overrightarrow{\mathrm{F'G'}}} \frac{1}{\sqrt{\zeta^2-(\omega/c)^2}}\exp\left\{-i\zeta x-|y|\sqrt{\zeta^2-(\omega/c)^2}\right\}d\zeta \\
&=\frac{i}{2\pi}\int\limits_0^{\omega/c} \frac{1}{\sqrt{(\omega/c)^2-\xi^2}}\exp\left\{+i\xi x+i|y|\sqrt{(\omega/c)^2-\xi^2}\right\}d\xi
\end{aligned}
\tag{2.5.34}
$$

$$
\begin{aligned}
&\frac{1}{2\pi}\int\limits_{\overrightarrow{\mathrm{G'H'}}} \frac{1}{\sqrt{\zeta^2-(\omega/c)^2}}\exp\left\{-i\zeta x-|y|\sqrt{\zeta^2-(\omega/c)^2}\right\}d\zeta \\
&=\frac{1}{2\pi}\int\limits_0^{\infty} \frac{1}{\sqrt{\eta^2+(\omega/c)^2}}\exp\left\{-\eta x+i|y|\sqrt{\eta^2+(\omega/c)^2}\right\}d\eta
\end{aligned}
\tag{2.5.35}
$$

Since no singular point is included in the closed loop, $\Phi = 0$. Thus, the integral I on the real axis is given by

$$
\begin{aligned}
I &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{\xi^2-(\omega/c)^2}}\exp\left\{-i\xi x-|y|\sqrt{\xi^2-(\omega/c)^2}\right\}d\xi \\
&= \frac{1}{\pi}\int\limits_0^{\infty} \frac{1}{\sqrt{\eta^2+(\omega/c)^2}}\exp(-\eta x)\cos\left\{y\sqrt{\eta^2+(\omega/c)^2}\right\}d\eta \\
&\quad -\frac{1}{\pi}\int\limits_0^{\omega/c} \frac{1}{\sqrt{(\omega/c)^2-\xi^2}}\sin(\xi x)\cos\left\{y\sqrt{(\omega/c)^2-\xi^2}\right\}d\xi \\
&\quad +\frac{i}{\pi}\int\limits_0^{\omega/c} \frac{1}{\sqrt{(\omega/c)^2-\xi^2}}\cos(\xi x)\cos\left\{y\sqrt{(\omega/c)^2-\xi^2}\right\}d\xi
\end{aligned}
\tag{2.5.36}
$$

The above equation can be simplified by applying the formulas (2.5.27) and (2.5.28). We have

$$
\begin{aligned}
&\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{\zeta^2-(\omega/c)^2}}\exp\left\{-i\xi x-|y|\sqrt{\xi^2-(\omega/c)^2}\right\}d\xi \\
&\quad = +\frac{i}{2}H_0^{(1)}\left(\frac{\omega}{c}\sqrt{x^2+y^2}\right)
\end{aligned}
\tag{2.5.37}
$$

where Hankel function of the first kind is defined by $H_0^{(1)}(.) = J_0(.) + iY_0(.)$.
*Derivation of formula (2.5.27).

We have the integration formula (Gradshteyn and Ryzhik 1980, pp. 755, 6.677),

$$\int_0^\infty Y_0\left(\alpha\sqrt{x^2+z^2}\right)\cos(\beta x)dx = \begin{cases} \dfrac{1}{\sqrt{\alpha^2-\beta^2}}\sin\left(z\sqrt{\alpha^2-\beta^2}\right); & \beta<\alpha \\ -\dfrac{1}{\sqrt{\beta^2-\alpha^2}}\exp\left(-z\sqrt{\beta^2-\alpha^2}\right); & \beta>\alpha \end{cases} \tag{2.5.38}$$

If we consider the above as the Fourier cosine transform with respect to the transform parameter β, its inverse cosine transform is given by

$$\begin{aligned} \frac{\pi}{2}Y_0\left(\alpha\sqrt{x^2+z^2}\right) &= \int_0^\alpha \frac{1}{\sqrt{\alpha^2-\beta^2}}\sin\left(z\sqrt{\alpha^2-\beta^2}\right)\cos(\beta x)d\beta \\ &\quad - \int_\alpha^\infty \frac{1}{\sqrt{\beta^2-\alpha^2}}\exp\left(-z\sqrt{\beta^2-\alpha^2}\right)\cos(\beta x)d\beta \end{aligned} \tag{2.5.39}$$

Making the change of variable, $u=\sqrt{\beta^2-\alpha^2}$, for the second integral in the right hand side, the formula (2.5.27) is obtained as

$$\begin{aligned} \frac{\pi}{2}Y_0\left(\alpha\sqrt{x^2+z^2}\right) &= \int_0^\alpha \frac{1}{\sqrt{\alpha^2-\beta^2}}\sin\left(z\sqrt{\alpha^2-\beta^2}\right)\cos(\beta x)d\beta \\ &\quad - \int_0^\infty \frac{1}{\sqrt{u^2+\alpha^2}}\exp(-zu)\cos\left(x\sqrt{u^2+\alpha^2}\right)du \end{aligned} \tag{2.5.40}$$

(3) Selection of the branch cut and the integration loop

The Fourier inversion integral I has just been evaluated in two ways of the branch cut and we get two expressions for the single integral I as

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\xi^2-(\omega/c)^2}}\exp\left\{-i\xi x-|y|\sqrt{\xi^2-(\omega/c)^2}\right\}d\xi \\ &= \begin{cases} -\dfrac{i}{2}H_0^{(2)}(\omega r/c); & \omega\equiv\omega-i\varepsilon|_{\varepsilon\to 0} \\ +\dfrac{i}{2}H_0^{(1)}(\omega r/c); & \omega\equiv\omega+i\varepsilon|_{\varepsilon\to 0} \end{cases} \end{aligned} \tag{2.5.41}$$

where the radial distance r from the source point is defined by

$$r = \sqrt{x^2 + y^2} \tag{2.5.42}$$

In order to determine the correct evaluation, we multiply the time factor $\exp(+i\omega t)$ and replace the Hankel functions with their asymptotic forms (Watson 1966, p. 198) as

$$\left.\begin{aligned} H_0^{(1)}(z) &\sim \sqrt{\frac{2}{\pi z}}\exp\{-i(z-\pi/4)\} \\ H_0^{(2)}(z) &\sim \sqrt{\frac{2}{\pi z}}\exp\{+i(z-\pi/4)\} \end{aligned}\right\}; \quad z \to \infty \tag{2.5.43}$$

The asymptotic form of the integral I yields to

$$\begin{aligned} &\frac{\exp(+i\omega t)}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{\xi^2-(\omega/c)^2}}\exp\left\{-i\xi x - |y|\sqrt{\xi^2-(\omega/c)^2}\right\}d\xi \\ &\underset{r\to\infty}{\approx}\begin{cases} -\dfrac{i}{2}\sqrt{\dfrac{2c}{\pi\omega r}}\exp(+\pi i/4)\exp\{+i\omega(t-r/c)\}; & \omega \equiv \omega - i\varepsilon|_{\varepsilon\to 0} \\ +\dfrac{i}{2}\sqrt{\dfrac{2c}{\pi\omega r}}\exp(-\pi i/4)\exp\{+i\omega(t+r/c)\}; & \omega \equiv \omega + i\varepsilon|_{\varepsilon\to 0} \end{cases} \end{aligned} \tag{2.5.44}$$

It is clear that the asymptotic form in the upper line shows the out-going wave from the source point $r = 0$, and that in the lower line does the in-coming wave from the infinity. Thus, the right evaluation for the integral I is

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{\xi^2-(\omega/c)^2}}\exp\left\{-i\xi x - |y|\sqrt{\xi^2-(\omega/c)^2}\right\}d\xi = -\frac{i}{2}H_0^{(2)}(\omega r/c) \tag{2.5.45}$$

and the correct introduction of the branch cut and the integration loop are shown in Fig. 2.5 for our positive time factor $\exp(+i\omega t)$. However, if we employ the negative time factor $\exp(-i\omega t)$, the branch cut and the loop in Fig. 2.6 are the right choices, needless to say.

Anyway, we could evaluate the infinite integral of the Fourier inversion and obtained the amplitude function in the closed form as

$$\phi^{\#} = -\frac{iQ}{4}H_0^{(2)}(\omega r/c); \quad r = \sqrt{x^2+y^2} \tag{2.5.46}$$

Thus, the 2D Green's function for the time-harmonic source is given by

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-Q\delta(x)\delta(y)\exp(+i\omega t)\;\Rightarrow\;\phi=-\frac{iQ}{4}H_0^{(2)}(\omega r/c)\exp(+i\omega t) \tag{2.5.47}$$

When the time factor is negative, we have

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-Q\delta(x)\delta(y)\exp(-i\omega t)\;\Rightarrow\;\phi=+\frac{iQ}{4}H_0^{(1)}(\omega r/c)\exp(-i\omega t) \tag{2.5.48}$$

(4) Convolution integral

The solution of Eq. (2.5.47) can be obtained directly by applying the convolution integral of the impulsive solution which is given by Eq. (2.4.19). Let us consider its convolution. The convolution integral for the Laplace transform is defined by

$$\phi(x,y,t)=\int_0^t \phi^{(implse)}(x,y,t')\exp\{+i\omega(t-t')\}dt' \tag{2.5.49}$$

where $\phi^{(implse)}$ is the solution for the impulsive source. We substitute the impulsive Green's function of Eq. (2.4.19) with the source magnitude Q into Eq. (2.5.49),

$$\phi(x,y,t)=\int_0^t \frac{cQ}{2\pi}\frac{H(ct'-r)}{\sqrt{(ct')^2-r^2}}\exp\{+i\omega(t-t')\}dt' \tag{2.5.50}$$

and examine the supporting region for the step function. We have

$$\phi(x,y,t)=\frac{cQ}{2\pi}\exp(+i\omega t)H(t-r/c)\int_{r/c}^t \frac{1}{\sqrt{(ct')^2-r^2}}\exp(-i\omega t')dt' \tag{2.5.51}$$

As we are considering the steady-state response, the time in the upper integration limit and that in the step function can be set to be infinite as

$$\phi(x,y,t)=\frac{cQ}{2\pi}\exp(+i\omega t)\lim_{t\to\infty}\left[H(t-r/c)\int_{r/c}^t \frac{1}{\sqrt{(ct')^2-r^2}}\exp(-i\omega t')dt'\right] \tag{2.5.52}$$

Taking the limit for the time, we have

$$\phi(x,y,t) = \frac{Q}{2\pi}\exp(+i\omega t)\int_r^\infty \frac{1}{\sqrt{u^2 - r^2}}\exp(-i\omega u/c)du \tag{2.5.53}$$

This is just the integral representation for the Hankel function of the second kind (Watson 1966, p. 170),

$$H_0^{(2)}(x) = \frac{2i}{\pi}\int_1^\infty \frac{1}{\sqrt{u^2 - 1}}\exp(-ixu)du \tag{2.5.54}$$

Thus, we have

$$\phi(x,y,t) = -\frac{iQ}{4}\exp(+i\omega t)H_0^{(2)}(r\omega/c) \tag{2.5.55}$$

(5) Direct evaluation

In the former subsection (1) and (2), the Fourier inversion is carried out by applying the Cauchy complex integral. It was lengthy troublesome work. Without applying the Cauchy theorem, we can evaluate the integral I,

$$I = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\xi^2 - (\omega/c)^2}}\exp\left\{-i\xi x - |y|\sqrt{\xi^2 - (\omega/c)^2}\right\}d\xi \tag{2.5.56}$$

Firstly, we reduce the infinite integral to the semi-infinite integral,

$$I = \frac{1}{\pi}\int_0^{+\infty} \frac{1}{\sqrt{\xi^2 - (\omega/c)^2}}\cos(\xi x)\exp\left\{-|y|\sqrt{\xi^2 - (\omega/c)^2}\right\}d\xi \tag{2.5.57}$$

In the complex ξ-plane, the integration path for this integral is just on the real axis and two branch cuts for the radical $\sqrt{\xi^2 - (\omega/c)^2}$ are introduced on the real and imaginary axes as was discussed in Sect. 1.3.2 in Chap. 1. The slightly shifted branch cuts are shown in Fig. 2.7 where the integration path for the semi-infinite integral is OP. The integration path in the region $0 < \xi < \omega/c$ is slightly up from the

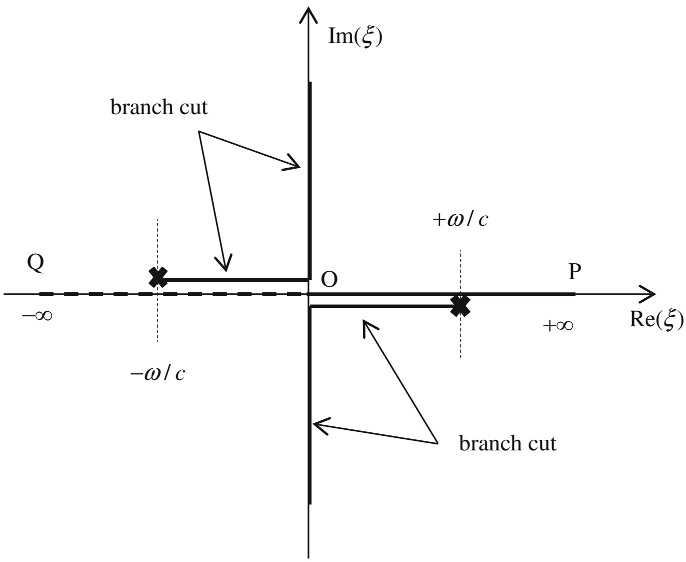

Fig. 2.7 Branch cuts and integration path for the inversion integral of Eq. (2.5.57)

lower branch cut. When the cut approaches to the real axis as was discussed in Sect. 1.3.2 (2), the argument of the square root function in the integrand yields to $+\pi/2$, i.e.

$$\sqrt{\xi^2 - (\omega/c)^2} = +i\sqrt{(\omega/c)^2 - \xi^2}; \quad |\xi| < \omega/c \tag{2.5.58}$$

In the other positive region $\xi > \omega/c$ on the real axis, the root function is positive real. Then, the semi-infinite integral is decomposed into two integrals in the regions, $0 < \xi < \omega/c$ and $\xi > \omega/c$, as

$$\begin{aligned} I = &\frac{1}{\pi}\int_0^{\omega/c} \frac{1}{(+i)\sqrt{(\omega/c)^2 - \xi^2}} \cos(\xi x) \exp\left\{-i|y|\sqrt{(\omega/c)^2 - \xi^2}\right\} d\xi \\ &+ \frac{1}{\pi}\int_{\omega/c}^{+\infty} \frac{1}{\sqrt{\xi^2 - (\omega/c)^2}} \cos(\xi x) \exp\left\{-|y|\sqrt{\xi^2 - (\omega/c)^2}\right\} d\xi \end{aligned} \tag{2.5.59}$$

The above integral I is further decomposed into the real and imaginary parts,

$$
\begin{aligned}
I = & -\frac{1}{\pi}\int\limits_{0}^{\omega/c} \frac{1}{\sqrt{(\omega/c)^2-\xi^2}}\cos(\xi x)\sin\left\{|y|\sqrt{(\omega/c)^2-\xi^2}\right\}d\xi \\
& +\frac{1}{\pi}\int\limits_{\omega/c}^{+\infty} \frac{1}{\sqrt{\xi^2-(\omega/c)^2}}\cos(\xi x)\exp\left\{-|y|\sqrt{\xi^2-(\omega/c)^2}\right\}d\xi \\
& -\frac{i}{\pi}\int\limits_{0}^{\omega/c} \frac{1}{\sqrt{(\omega/c)^2-\xi^2}}\cos(\xi x)\cos\left\{y\sqrt{(\omega/c)^2-\xi^2}\right\}d\xi
\end{aligned}
\tag{2.5.60}
$$

We make the change of variable for the first and second integrals in the above equation. The changes are defined by

$$
\eta=\sqrt{(\omega/c)^2-\xi^2},\quad \xi=\sqrt{(\omega/c)^2-\eta^2},\quad d\xi=-\frac{\eta d\eta}{\sqrt{(\omega/c)^2-\eta^2}}
\tag{2.5.61}
$$

for the first integral, and

$$
\eta=\sqrt{\xi^2-(\omega/c)^2},\quad \xi=\sqrt{\eta^2+(\omega/c)^2},\quad d\xi=\frac{\eta d\eta}{\sqrt{\eta^2+(\omega/c)^2}}
\tag{2.5.62}
$$

for the second integral. Equation (2.5.60) yields to

$$
\begin{aligned}
I = & -\frac{1}{\pi}\int\limits_{0}^{\omega/c} \frac{1}{\sqrt{(\omega/c)^2-\eta^2}}\cos\left\{x\sqrt{(\omega/c)^2-\eta^2}\right\}\sin(|y|\eta)d\eta \\
& +\frac{1}{\pi}\int\limits_{0}^{\infty} \frac{1}{\sqrt{\eta^2+(\omega/c)^2}}\cos\left\{x\sqrt{\eta^2+(\omega/c)^2}\right\}\exp(-|y|\eta)d\eta \\
& -\frac{i}{\pi}\int\limits_{0}^{\omega/c} \frac{1}{\sqrt{(\omega/c)^2-\xi^2}}\cos(\xi x)\cos\left\{y\sqrt{(\omega/c)^2-\xi^2}\right\}d\xi
\end{aligned}
\tag{2.5.63}
$$

Then, we apply the formula of Eq. (2.5.27) to the first two integrals and the formula of Eq. (2.5.28) to the third integral. The integral I is exactly evaluated as

$$
I=-\frac{1}{2}Y_0\left(\frac{\omega}{c}\sqrt{x^2+y^2}\right)-\frac{i}{2}J_0\left(\frac{\omega}{c}\sqrt{x^2+y^2}\right)=-\frac{i}{2}H_0^{(2)}\left(\frac{\omega}{c}\sqrt{x^2+y^2}\right)
\tag{2.5.64}
$$

This is just the same as Eqs. (2.5.30) and (2.5.45) for the integral I.

2.6 3D Static Source

The static 3D Green's function for the Laplace equation is a particular solution of the nonhomogeneous differential equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -S\delta(x)\delta(y)\delta(z) \tag{2.6.1}$$

where S is the magnitude of the source which is placed at the coordinate origin (0, 0, 0). The convergence condition at infinity,

$$\phi|_{\sqrt{x^2+y^2+z^2}\to\infty} = \left.\frac{\partial \phi}{\partial x}\right|_{\sqrt{x^2+y^2+z^2}\to\infty} = \left.\frac{\partial \phi}{\partial y}\right|_{\sqrt{x^2+y^2+z^2}\to\infty} = \left.\frac{\partial \phi}{\partial z}\right|_{\sqrt{x^2+y^2+z^2}\to\infty} = 0 \tag{2.6.2}$$

is applied to the Green's function ϕ.

The Green's function is obtained by the method of integral transform. For three space variables, the triple Fourier transform defined by

$$\bar{\phi}(\xi) = \int_{-\infty}^{+\infty} \phi(x)\exp(+i\xi x)dx, \quad \phi(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{\phi}(\xi)\exp(-i\xi x)d\xi \tag{2.6.3}$$

$$\tilde{\phi}(\eta) = \int_{-\infty}^{+\infty} \phi(y)\exp(+i\eta y)dy, \quad \phi(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{\phi}(\eta)\exp(-i\eta y)d\eta \tag{2.6.4}$$

$$\hat{\phi}(\zeta) = \int_{-\infty}^{+\infty} \phi(z)\exp(+i\zeta z)dz, \quad \phi(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{\phi}(\zeta)\exp(-i\zeta z)d\zeta \tag{2.6.5}$$

is applied to Eq. (2.6.1),

$$\int_{-\infty}^{+\infty}\left\langle\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\left\{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -S\delta(x)\delta(y)\delta(z)\right\}\exp(+i\xi x)dx\right]\exp(+i\eta y)dy\right\rangle\exp(+i\zeta z)dz \tag{2.6.6}$$

Applying the convergence condition (2.6.2), the above Eq. (2.6.6) is transformed to the algebraic equation for the triple transformed function $\hat{\tilde{\bar{\phi}}}$,

$$-(\xi^2 + \eta^2 + \zeta^2)\hat{\tilde{\bar{\phi}}} = -S \tag{2.6.7}$$

The transformed function is determined explicitly,

$$\bar{\tilde{\hat{\phi}}} = \frac{S}{\xi^2 + \eta^2 + \zeta^2} \tag{2.6.8}$$

The first Fourier inversion integral with respect to the parameter ζ is

$$\bar{\tilde{\phi}} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{S}{\xi^2 + \eta^2 + \zeta^2}\exp(-i\zeta z)d\zeta = \frac{S}{\pi}\int_{0}^{\infty} \frac{1}{\zeta^2 + \sqrt{\xi^2 + \eta^2}^2}\cos(\zeta z)d\zeta \tag{2.6.9}$$

The above integral is easily evaluated with the aid of the formula (2.1.22). It yields

$$\bar{\tilde{\phi}} = \frac{S}{2\sqrt{\xi^2 + \eta^2}}\exp\left(-|z|\sqrt{\xi^2 + \eta^2}\right) \tag{2.6.10}$$

The second inversion integral with respect to the parameter η is reduced to the semi-infinite integral as

$$\begin{aligned}\bar{\phi} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{S}{2\sqrt{\xi^2 + \eta^2}}\exp\left(-|z|\sqrt{\xi^2 + \eta^2}\right)\exp(-i\eta y)d\eta \\ &= \frac{S}{2\pi}\int_{0}^{\infty} \frac{1}{\sqrt{\xi^2 + \eta^2}}\exp\left(-|z|\sqrt{\xi^2 + \eta^2}\right)\cos(\eta y)d\eta\end{aligned} \tag{2.6.11}$$

The latter semi-infinite integral can be evaluated by using the formula (2.4.13). It follows that

$$\bar{\phi} = \frac{S}{2\pi}K_0\left(|\xi|\sqrt{y^2 + z^2}\right) \tag{2.6.12}$$

The last inversion integral with respect to the parameter ξ is also reduced to the semi-infinite integral,

$$\begin{aligned}\phi &= \frac{S}{4\pi^2}\int_{-\infty}^{+\infty} K_0\left(|\xi|\sqrt{y^2 + z^2}\right)\exp(-i\xi x)d\xi \\ &= \frac{S}{2\pi^2}\int_{0}^{\infty} K_0\left(\xi\sqrt{y^2 + z^2}\right)\cos(\xi x)d\xi\end{aligned} \tag{2.6.13}$$

Fortunately, we have the suitable integration formula, (Erdélyi 1954, vol. I, pp. 49, 40)

$$\int_0^\infty K_0(a\xi)\cos(b\xi)d\xi = \frac{\pi}{2\sqrt{a^2+b^2}} \tag{2.6.14}$$

Applying this formula to Eq. (2.6.13), the last inversion integral is exactly evaluated as

$$\phi = \frac{S}{4\pi\sqrt{x^2+y^2+z^2}} \tag{2.6.15}$$

Consequently, we have the static 3D Green's function for the Laplace equation,

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = -S\delta(x)\delta(y)\delta(z) \quad \Rightarrow \quad \phi = \frac{S}{4\pi R} \tag{2.6.16}$$

where R is the 3D radial distance from the source,

$$R = \sqrt{x^2+y^2+z^2} \tag{2.6.17}$$

2.7 3D Impulsive Source

Green's function for the 3D wave equation is discussed. The wave equation with an impulsive point source located at the coordinate origin is given by

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - P\delta(x)\delta(y)\delta(z)\delta(t) \tag{2.7.1}$$

where P is the magnitude of the source. The quiescent condition at an initial time,

$$\phi|_{t=0} = \left.\frac{\partial\phi}{\partial t}\right|_{t=0} = 0 \tag{2.7.2}$$

and the convergence condition at infinity,

$$\phi|_{\sqrt{x^2+y^2+z^2}\to\infty} = \left.\frac{\partial\phi}{\partial x}\right|_{\sqrt{x^2+y^2+z^2}\to\infty} = \left.\frac{\partial\phi}{\partial y}\right|_{\sqrt{x^2+y^2+z^2}\to\infty} = \left.\frac{\partial\phi}{\partial z}\right|_{\sqrt{x^2+y^2+z^2}\to\infty} = 0 \tag{2.7.3}$$

are also imposed.

Our dynamic Green's function has four variables: three space and one time variables. Laplace transform with respect to the time,

$$\phi^*(s) = \int_0^{+\infty} \phi(t)\exp(-st)dt \tag{2.7.4}$$

and the triple Fourier transform with respect to three space variables,

$$\bar{\phi}(\xi) = \int_{-\infty}^{+\infty} \phi(x)\exp(+i\xi x)dx, \quad \phi(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{\phi}(\xi)\exp(-i\xi x)d\xi \tag{2.7.5}$$

$$\tilde{\phi}(\eta) = \int_{-\infty}^{+\infty} \phi(y)\exp(+i\eta y)dy, \quad \phi(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{\phi}(\eta)\exp(-i\eta y)d\eta \tag{2.7.6}$$

$$\hat{\phi}(\zeta) = \int_{-\infty}^{+\infty} \phi(z)\exp(+i\zeta z)dz, \quad \phi(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{\phi}(\zeta)\exp(-i\zeta z)d\zeta \tag{2.7.7}$$

are applied to the nonhomogeneous wave equation (2.7.1). With the aid of the quiescent and convergence conditions, the wave equation is transformed to the simple algebraic equation for the multi-transformed unknown function $\hat{\tilde{\bar{\phi}}}^*$,

$$-\{\xi^2 + \eta^2 + \zeta^2 + (s/c)^2\}\hat{\tilde{\bar{\phi}}}^* = -P \tag{2.7.8}$$

The inversion starts from the transformed function,

$$\hat{\tilde{\bar{\phi}}}^* = \frac{P}{\xi^2 + \eta^2 + \zeta^2 + (s/c)^2} \tag{2.7.9}$$

As the first inversion, the Fourier inversion integral with respect to the parameter ζ

$$\begin{aligned} \tilde{\bar{\phi}}^* &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{P}{\xi^2 + \eta^2 + \zeta^2 + (s/c)^2}\exp(-i\zeta z)dz \\ &= \frac{P}{\pi}\int_0^{\infty} \frac{\cos(\zeta z)}{\zeta^2 + \sqrt{\xi^2 + \eta^2 + (s/c)^2}^2}dz \end{aligned} \tag{2.7.10}$$

is carried out by applying the formula (2.1.22). It follows that

$$\tilde{\bar{\phi}}^{*} = \frac{P}{2\sqrt{\xi^2 + \eta^2 + (s/c)^2}} \exp\left\{-|z|\sqrt{\xi^2 + \eta^2 + (s/c)^2}\right\} \tag{2.7.11}$$

The second inversion is the integral with respect to the parameter η,

$$\begin{aligned} \bar{\phi}^{*} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{P}{2\sqrt{\xi^2 + \eta^2 + (s/c)^2}} \exp\left\{-|z|\sqrt{\xi^2 + \eta^2 + (s/c)^2}\right\} \exp(-i\eta y)d\eta \\ &= \frac{P}{2\pi}\int_{0}^{\infty} \frac{1}{\sqrt{\xi^2 + \eta^2 + (s/c)^2}} \exp\left\{-|z|\sqrt{\xi^2 + \eta^2 + (s/c)^2}\right\} \cos(\eta y)d\eta \end{aligned} \tag{2.7.12}$$

The latter semi-infinite integral is also evaluated by applying the formula (2.4.13). It yields

$$\bar{\phi}^{*} = \frac{P}{2\pi} K_0\left(\sqrt{\xi^2 + (s/c)^2}\sqrt{y^2 + z^2}\right) \tag{2.7.13}$$

The third inversion integral with respect to the parameter ξ is

$$\begin{aligned} \phi^{*} &= \frac{P}{2\pi}\frac{1}{2\pi}\int_{-\infty}^{+\infty} K_0\left(\sqrt{\xi^2 + (s/c)^2}\sqrt{y^2 + z^2}\right) \exp(-i\xi x)d\xi \\ &= \frac{P}{2\pi^2}\int_{0}^{\infty} K_0\left(\sqrt{\xi^2 + (s/c)^2}\sqrt{y^2 + z^2}\right) \cos(\xi x)d\xi \end{aligned} \tag{2.7.14}$$

The integration formula (Erdélyi 1954, vol. I, pp. 56, 43)

$$\int_{0}^{\infty} K_0\left(a\sqrt{\xi^2 + b^2}\right) \cos(c\xi)d\xi = \frac{\pi}{2\sqrt{a^2 + c^2}} \exp\left(-b\sqrt{a^2 + c^2}\right) \tag{2.7.15}$$

is very helpful for our task. Then, applying the above formula to the last integral in Eq. (2.7.14), we have

$$\phi^{*} = \frac{P}{4\pi\sqrt{x^2 + y^2 + z^2}} \exp\left(-\frac{s}{c}\sqrt{x^2 + y^2 + z^2}\right) \tag{2.7.16}$$

The last is the Laplace inversion. Its symbolical form is

$$\phi = \frac{P}{4\pi\sqrt{x^2+y^2+z^2}} L^{-1}\left[\exp\left(-\frac{s}{c}\sqrt{x^2+y^2+z^2}\right)\right] \tag{2.7.17}$$

The transform parameter "s" is included only in the argument of the exponential function. We remember the simple Laplace inversion formula for the delta function,

$$L^{-1}[\exp(-as)] = \delta(t-a) \tag{2.7.18}$$

Thus, applying this inversion formula, Eq. (2.7.17) is fully inverted as

$$\phi = \frac{P}{4\pi R}\delta(t - R/c) \tag{2.7.19}$$

where the radial distance R from the source is defined by Eq. (2.6.17). Finally, we have the 3D Green's function for the wave equation with the impulsive point source,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - P\delta(x)\delta(y)\delta(z)\delta(t) \;\Rightarrow\; \phi = \frac{P}{4\pi R}\delta(t - R/c) \tag{2.7.20}$$

This is the very simple expression in spite of the 3D nature!

2.8 3D Time-Harmonic Source

This section derives the 3D Green's function for a time-harmonic source. It is the convolution integral of the impulsive Green's function obtained in the previous section. The wave equation with the time-harmonic source is given by

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - Q\delta(x)\delta(y)\delta(z)\exp(+i\omega t) \tag{2.8.1}$$

where Q is the magnitude of the source and ω the frequency of the time-harmonic vibration. The convergence condition at infinity,

$$\phi|_{\sqrt{x^2+y^2+z^2}\to\infty} = \left.\frac{\partial\phi}{\partial x}\right|_{\sqrt{x^2+y^2+z^2}\to\infty} = \left.\frac{\partial\phi}{\partial y}\right|_{\sqrt{x^2+y^2+z^2}\to\infty} = \left.\frac{\partial\phi}{\partial z}\right|_{\sqrt{x^2+y^2+z^2}\to\infty} = 0 \tag{2.8.2}$$

is also imposed.

The standard multiple integral transform technique is available for getting the time-harmonic Green's function. However, as shown in the case of 2D Green's function in Sect. 2.5 (4), we take a very simple way, i.e. the convolution integral of the impulsive Green's function.

In the previous section we get the Green's function for the impulsive source. Replacing the source magnitude with unit 1, the Green function is given by

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \delta(x)\delta(y)\delta(z)\delta(t) \Rightarrow \phi = \frac{1}{4\pi R}\delta(t - R/c) \quad (2.8.3)$$

Employing the time-harmonic source with the frequency ω and the magnitude Q

$$Q \exp\{+i\omega(t - t')\} \quad (2.8.4)$$

the convolution integral for the Laplace transform is given by

$$\phi = \int_0^t \frac{Q}{4\pi R}\delta(t' - R/c)\exp\{+i\omega(t - t')\}dt' \quad (2.8.5)$$

It is very easy to evaluate the above integral, since the integrand includes Dirac's delta function and we can apply the simple integration formula (1.2.3) in Sect. 1.2,

$$\int_a^b f(x)\delta(x - c)dx = \begin{cases} f(c); & a < c < b \\ 0; & c < a \text{ or } b < c \end{cases} \quad (2.8.6)$$

Then, we can evaluate the integral in Eq. (2.8.5) and have for ϕ

$$\phi = \frac{Q}{4\pi R}H(t - R/c)\exp\{+i\omega(t - R/c)\} \quad (2.8.7)$$

The step function ahead of the equation means that Eq. (2.8.7) is the transient response to the time-harmonic source and the disturbance starts from the wave arrival $t = R/c$. When sufficient long time has passed and the response becomes steady, the step function is meaningless. Then, we have the steady-state time-harmonic response as

$$\phi = \frac{Q}{4\pi R}\exp\{+i\omega(t - R/c)\} \quad (2.8.8)$$

Therefore, the 3D Green's function for the wave equation with the time-harmonic source is given by

Table 2.3 Green's function for Laplace and Wave equations

	Differential equation	Source	Green's function
1D	$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \text{Source}$	$P\delta(x)\delta(t)$	$\phi(x,t) = \frac{cP}{2}H(ct-\lvert x\rvert)$
		$Q\delta(x)\exp(\pm i\omega t)$	$\phi(x,t) = \mp\frac{iQ}{2(\omega/c)}\exp\{\pm i\omega(t-\lvert x\rvert/c)\}$
2D	$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\text{Source}$	$S\delta(x)\delta(y)$	$\phi(x,y) = -\frac{S}{4\pi}\log(r) + \text{arbitrary constant}; \quad r = \sqrt{x^2+y^2}$
	$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \text{Source}$	$P\delta(x)\delta(y)\delta(t)$	$\phi(x,y,t) = \frac{cP}{2\pi\sqrt{(ct)^2-r^2}}H(ct-r); \quad r = \sqrt{x^2+y^2}$
		$Q\delta(x)\delta(y)\exp(+i\omega t)$	$\phi(x,y,t) = -\frac{iQ}{4}H_0^{(2)}(\omega r/c)\exp(+i\omega t); \quad r = \sqrt{x^2+y^2}$
		$Q\delta(x)\delta(y)\exp(-i\omega t)$	$\phi(x,y,t) = +\frac{iQ}{4}H_0^{(1)}(\omega r/c)\exp(-i\omega t); \quad r = \sqrt{x^2+y^2}$
3D	$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -\text{Source}$	$S\delta(x)\delta(y)\delta(z)$	$\phi(x,y,z) = \frac{S}{4\pi R}; \quad R = \sqrt{x^2+y^2+z^2}$
	$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \text{Source}$	$P\delta(x)\delta(y)\delta(z)\delta(t)$	$\phi(x,y,z,t) = \frac{P}{4\pi R}\delta(t-R/c); \quad R = \sqrt{x^2+y^2+z^2}$
		$Q\delta(x)\delta(y)\delta(z)\exp(\pm i\omega t)$	$\phi(x,y,z,t) = \frac{Q}{4\pi R}\exp\{\pm i\omega(t-R/c)\}; \quad R = \sqrt{x^2+y^2+z^2}$

$$\begin{aligned}\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}&=\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-Q\delta(x)\delta(y)\delta(z)\exp(+i\omega t)\\ \Rightarrow \phi&=\frac{Q}{4\pi R}\exp\{+i\omega(t-R/c)\}\end{aligned} \tag{2.8.9}$$

Exercise

(2.1) Using the impulsive response (2.1.27) for 1D wave equation, derive the time-harmonic Green's function through the convolution integral and compare it with the time-harmonic Green's function (2.2.27).

Appendix

See Table 2.3.

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