# Chapter 1
# Period Mappings for Families of Complex Manifolds

## 1.1 Introduction

Consider a family of compact complex manifolds $f : X \to S$. Concretely, let $X$ and $S$ be complex manifolds and $f$ a proper, submersive holomorphic map between them. Then by Ehresmann's fibration theorem [1, (8.12), p. 84], $f : X \to S$ is a locally topologically trivial family (as a matter of fact, even a locally $\mathscr{C}^\infty$ trivial family). Recall that this means the following: for every point $s \in S$, there exists an open neighborhood $U$ of $s$ in $S$ as well as a homeomorphism (or $\mathscr{C}^\infty$ diffeomorphism)

$$h : X|_{f^{-1}(U)} \longrightarrow U \times f^{-1}(\{s\})$$

such that $h$, composed with the projection to the first factor (i.e., the projection to $U$), yields the restriction $f|_{f^{-1}(U)}$. In particular, for every natural number (or else integer) $n$, we infer that $\mathrm{R}^n f_*(\mathbf{C}_X)$ is a locally constant sheaf on the topological space $S_{\mathrm{top}}$, and that, for all $s \in S$, the topological base change map[1]

$$(\mathrm{R}^n f_*(\mathbf{C}_X))_s \longrightarrow \mathrm{H}^n(X_s, \mathbf{C}_{X_s}) =: \mathrm{H}^n(X_s, \mathbf{C})$$

is a bijection.

Assume that the complex manifold $S$ is simply connected. Then the sheaf $\mathrm{R}^n f_*(\mathbf{C}_X)$ is even a constant sheaf on $S_{\mathrm{top}}$, and, for all $s \in S$, the residue map from the set of global sections of $\mathrm{R}^n f_*(\mathbf{C}_X)$ to its stalk $(\mathrm{R}^n f_*(\mathbf{C}_X))_s$ at $s$ is one-to-one and onto. Thus, by passing through the appropriate base change maps as well as through

[1]Observe that the base of the family changes from $S$ to the one-point space $\{s\}$.



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the set of global sections of $\mathrm{R}^n f_*(\mathbf{C}_X)$, we obtain, for every two elements $s_0, s_1 \in S$, a bijection

$$\phi^n_{s_0,s_1} : \mathrm{H}^n(X_{s_0}, \mathbf{C}) \longrightarrow \mathrm{H}^n(X_{s_1}, \mathbf{C}).$$

Suppose that, for all $s \in S$, the complex manifold $X_s$ is of Kähler type [15, p. 188]. Moreover, fix an element $t \in S$. Then define $\mathcal{P}^{p,n}_t$, for any natural number (or else integer) $p$, to be the unique function on $S$ such that

$$\mathcal{P}^{p,n}_t(s) = \phi^n_{s,t}[\mathrm{F}^p\mathrm{H}^n(X_s)]$$

holds for all $s \in S$, where $\mathrm{F}^p\mathrm{H}^n(X_s)$ denotes the degree-$p$ piece of the Hodge filtration on the $n$th cohomology of $X_s$. For sake of clarity, I use square brackets to denote the set-theoretic image of a set under a function. $\mathcal{P}^{p,n}_t$ is called a *period mapping* for the family $f$. The following result is a variant of a theorem of P. Griffiths's [6, Theorem (1.1)].[2]

**Theorem 1.1.1** *Under the above hypotheses,* $\mathcal{P}^{p,n}_t$ *is a holomorphic mapping from* $S$ *to the Grassmannian* $\mathrm{Gr}(\mathrm{H}^n(X_t, \mathbf{C}))$.

Note that Theorem 1.1.1 comprises the fact that $\mathcal{P}^{p,n}$ is a continuous map from $S$ to the Grassmannian of $\mathrm{H}^n(X_t, \mathbf{C})$. Specifically, taking into account the topology of the Grassmannian, we infer that the complex vector spaces $\mathrm{F}^p\mathrm{H}^n(X_s)$ are of a constant finite dimension as $s$ varies through $S$—a fact which is, in its own right, not at all obvious to begin with.[3]

In addition to Theorem 1.1.1, I would like to recall another, closely related theorem of Griffiths [6, Proposition (1.20) or Theorem (1.22)]. To that end, put $q := n - p$ and denote by

$$\gamma : \mathrm{H}^1(X_t, \Theta_{X_t}) \longrightarrow \mathrm{Hom}(\mathrm{H}^q(X_t, \Omega^p_{X_t}), \mathrm{H}^{q+1}(X_t, \Omega^{p-1}_{X_t}))$$

the morphism of complex vector spaces which is obtained by means of tensor-hom adjunction from the composition

$$\mathrm{H}^1(X_t, \Theta_{X_t}) \otimes_{\mathbf{C}} \mathrm{H}^q(X_t, \Omega^p_{X_t}) \xrightarrow{\smile} \mathrm{H}^{q+1}(X_t, \Theta_{X_t} \otimes_{X_t} \Omega^p_{X_t}) \longrightarrow \mathrm{H}^{q+1}(X_t, \Omega^{p-1}_{X_t})$$

[2] When you consult Griffiths's source, you will notice several conceptual differences to the text at hand. Most prominently, Griffiths works with de Rham and Dolbeault cohomology where I work with abstract sheaf cohomology. Besides, in his construction of the period mapping, he employs a $\mathscr{C}^\infty$ diffeomorphism $X_t \to X_s$ directly in order to obtain the isomorphism $\phi^n_{s,t}$.

[3] Think about how you would prove it. What theorems do you have to invoke?

of the evident cup product morphism and the $\mathrm{H}^{q+1}(X_t, -)$ of the sheaf-theoretic contraction morphism

$$\Theta_{X_t} \otimes_{X_t} \Omega^p_{X_t} \longrightarrow \Omega^{p-1}_{X_t}.$$

Since $X_t$ is a compact, Kähler type complex manifold, the Frölicher spectral sequence of $X_t$ degenerates at sheet 1 and we have, for any integer $\nu$, an induced morphism of complex vector spaces

$$\psi^\nu : \mathrm{F}^\nu \mathrm{H}^n(X_t)/\mathrm{F}^{\nu+1}\mathrm{H}^n(X_t) \longrightarrow \mathrm{H}^{n-\nu}(X_t, \Omega^\nu_{X_t}).$$

In fact, $\psi^\nu$ an isomorphism for all $\nu$. Define $\alpha$ to be the composition of the quotient morphism

$$\mathrm{F}^p\mathrm{H}^n(X_t) \longrightarrow \mathrm{F}^p\mathrm{H}^n(X_t)/\mathrm{F}^{p+1}\mathrm{H}^n(X_t)$$

and $\psi^p$. Dually, define $\beta$ to be the composition of $(\psi^{p-1})^{-1}$ and the morphism

$$\mathrm{F}^{p-1}\mathrm{H}^n(X_t)/\mathrm{F}^p\mathrm{H}^n(X_t) \longrightarrow \mathrm{H}^n(X_t, \mathbf{C})/\mathrm{F}^p\mathrm{H}^n(X_t)$$

which is obtained from the inclusion of $\mathrm{F}^{p-1}\mathrm{H}^n(X_t)$ in $\mathrm{H}^n(X_t, \mathbf{C})$ by quotienting out $\mathrm{F}^p\mathrm{H}^n(X_t)$. Further, denote by

$$\mathrm{KS} : \mathrm{T}_S(t) \longrightarrow \mathrm{H}^1(X_t, \Theta_{X_t})$$

the Kodaira-Spencer map for the family $f$ with respect to the basepoint $t$ (see Notation 1.8.3 for an explanation), and write

$$\theta : \mathrm{T}_{\mathrm{Gr}(\mathrm{H}^n(X_t,\mathbf{C}))}(\mathrm{F}^p\mathrm{H}^n(X_t)) \longrightarrow \mathrm{Hom}(\mathrm{F}^p\mathrm{H}^n(X_t), \mathrm{H}^n(X_t, \mathbf{C})/\mathrm{F}^p\mathrm{H}^n(X_t))$$

for the isomorphism of complex vector spaces which is induced by the canonical open immersion

$$\mathrm{Hom}(\mathrm{F}^p\mathrm{H}^n(X_t), E) \longrightarrow \mathrm{Gr}(\mathrm{H}^n(X_t, \mathbf{C}))$$

of complex manifolds, where $E$ is a complex vector subspace of $\mathrm{H}^n(X_t, \mathbf{C})$ such that $\mathrm{H}^n(X_t, \mathbf{C}) = \mathrm{F}^p\mathrm{H}^n(X_t) \oplus E$ (see Construction 1.7.19). Then we are in the position to formulate the following theorem.

**Theorem 1.1.2** *Let $f : X \to S$, $n$, $p$, and $t$ be as above and define $\alpha$, $\beta$, $\gamma$,* KS, *and $\theta$ accordingly. Then the following diagram commutes in* $\mathrm{Mod}(\mathbf{C})$*:*

$$\begin{tikzcd}
\mathrm{T}_S(t) \arrow[r, "\mathrm{KS}"] \arrow[dd, "\mathrm{T}_t(\mathcal{P}_t^{p,n})"'] & \mathrm{H}^1(X_t, \Theta_{X_t}) \arrow[d, "\gamma"] \\
& \mathrm{Hom}(\mathrm{H}^q(X_t, \Omega^p_{X_t}), \mathrm{H}^{q+1}(X_t, \Omega^{p-1}_{X_t})) \arrow[d, "\mathrm{Hom}(\alpha,\beta)"] \\
\mathrm{T}_{\mathrm{Gr}(\mathrm{H}^n(X_t,\mathbf{C}))}(\mathrm{F}^p\mathrm{H}^n(X_t)) \arrow[r, "\theta"'] & \mathrm{Hom}(\mathrm{F}^p\mathrm{H}^n(X_t), \mathrm{H}^n(X_t,\mathbf{C})/\mathrm{F}^p\mathrm{H}^n(X_t))
\end{tikzcd} \tag{1.1}$$

The objective of this chapter is to state and prove a proposition analogous to Theorem 1.1.2 (possibly even, in a sense, generalizing Theorem 1.1.2) for families of not necessarily compact manifolds—that is, for submersive, yet possibly nonproper, morphisms of complex manifolds $f : X \to S$. Approaching this task in a naive way, you immediately encounter problems. First of all, together with the properness of $f$, you lose Ehresmann's fibration theorem. Hence, you lose the local topological triviality of $f$. In fact, when $f$ is not proper, the cohomology of the fibers of $f$ may jump wildly when passing from one fiber to the next—the cohomology might also be infinite dimensional. The sheaf $\mathrm{R}^n f_*(\mathbf{C}_X)$ will typically be far from locally constant on $S$. The cure, of course, is to impose conditions on $f$ that ensure some sort of local triviality. In view of my applications in Chap. 3, however, I do not want to impose that $f$ be locally topologically trivial. That would be too restrictive. Instead, I require that $f$ satisfy a certain local *cohomological triviality* which is affixed to a given cohomological degree.

Specifically, I am interested in a submersive morphism $f : X \to S$ of complex manifolds such that, for a fixed integer $n$, the relative algebraic de Rham cohomology sheaf $\mathscr{H}^n(f)$ (which is, per definitionem, $\mathrm{R}^n f_*(\Omega_f^\bullet)$ equipped with its canonical $\mathscr{O}_S$-module structure) is a locally finite free module on $S$ which is compatible with base change in the sense that, for all $s \in S$, the de Rham base change map

$$\phi^n_{f,s} : \mathbf{C} \otimes_{\mathscr{O}_{S,s}} (\mathscr{H}^n(f))_s \longrightarrow \mathscr{H}^n(X_s)$$

is an isomorphism of complex vector spaces. We will observe that the kernel $H$ of the Gauß-Manin connection

$$\nabla^n_{\mathrm{GM}}(f) : \mathscr{H}^n(f) \longrightarrow \Omega^1_S \otimes_S \mathscr{H}^n(f),$$

which I define in the spirit of Katz and Oda [11], makes up a locally constant sheaf of $\mathbf{C}_S$-modules on $S_{\mathrm{top}}$. Moreover, the stalks of $H$ are isomorphic to the $n$th de Rham cohomologies of the fibers of $f$ in virtue of the inclusion of $H$ in $\mathscr{H}^n(f)$ and the de Rham base change maps.

In this way, when the complex manifold $S$ is simply connected, I construct, for any integer $p$ and any basepoint $t \in S$, a period mapping $\mathcal{P}_t^{p,n}(f)$ by transporting, for varying $s$, the Hodge filtered piece $\mathrm{F}^p\mathscr{H}^n(X_s)$ from inside $\mathscr{H}^n(X_s)$ to $\mathscr{H}^n(X_t)$ along the global sections of $H$. When I require the relative Hodge filtered piece $\mathrm{F}^p\mathscr{H}^n(f)$ to be a vector subbundle of $\mathscr{H}^n(f)$ on $S$ which is compatible with base change (in an appropriate sense), the holomorphicity of the period mapping

$$\mathcal{P}_t^{p,n}(f) : S \longrightarrow \mathrm{Gr}(\mathscr{H}^n(X_t))$$

is basically automatic. Eventually, we learn that certain degeneration properties of the Frölicher spectral sequences of $f$ and the fibers of $f$ ensure the possibility to define morphisms $\alpha$ and $\beta$ such that a diagram similar to the one in Eq. (1.1)—namely, the one in Eq. (1.65)—commutes in $\mathrm{Mod}(\mathbf{C})$.

This chapter is organized as follows. My ultimate results are Theorem 1.8.8 as well as its, hopefully more accessible, corollary Theorem 1.8.10. The chapter's sections come in two groups: the final Sects. 1.7 and 1.8 deal with the concept period mappings, whereas the initial Sects. 1.2–1.6 don't. The upshot of Sects. 1.2–1.6, besides establishing constructions and notation—for instance, for the Gauß-Manin connection—is Theorem 1.6.14. Theorem 1.6.14 is, in turn, a special case of Theorem 1.5.14, whose proof Sect. 1.5 is consecrated to. Sects. 1.2–1.4 are preparatory for Sect. 1.5.

The overall point of view that I am adopting here is inspired by works of Nicholas Katz and Tadao Oda [10, 11]. As a matter of fact, my Sects. 1.2–1.6 are put together along the very lines of [10, Sect. 1]. I recommend that you compare Katz's and Oda's exposition to mine. My view on period mappings and relative connections, which you can find in Sect. 1.7, is furthermore inspired by Deligne's well-known lecture notes [2].

## 1.2 The $\Lambda^p$ Construction

*For the entire section, let $X$ be a ringed space [7, p. 35, 13, Tag 0091].*

In what follows, I introduce a construction which associates to a right exact triple of modules $t$ on $X$ (see Definitions 1.2.2 and 1.2.3), given some integer $p$, another right exact triple of modules on $X$, denoted by $\Lambda_X^p(t)$—see Construction 1.2.8. This "$\Lambda^p$ construction" will play a central role in this chapter at least up to (and including) Sect. 1.6.

The $\Lambda^p$ construction is closely related to and, in fact, essentially based upon the following notion of a "Koszul filtration"; cf. [10, (1.2.1.2)].

**Construction 1.2.1** Let $p$ be an integer. Moreover, let $\alpha : G \to H$ be a morphism of modules on $X$. We define a $\mathbf{Z}$-sequence $K$ by setting, for all $i \in \mathbf{Z}$,

$$K^i := \begin{cases} \operatorname{im}\Big(\wedge^{i,p-i}(H) \circ (\bigwedge^i \alpha \otimes \bigwedge^{p-i} \mathrm{id}_H)\Big) & \text{when } i \geq 0, \\ \bigwedge^p H & \text{when } i < 0. \end{cases} \tag{1.2}$$

Observe that we have

$$\bigwedge^i G \otimes \bigwedge^{p-i} H \xrightarrow{\bigwedge^i \alpha \otimes \bigwedge^{p-i} \mathrm{id}_H} \bigwedge^i H \otimes \bigwedge^{p-i} H \xrightarrow{\wedge^{i,p-i}(H)} \bigwedge^p H$$

in $\mathrm{Mod}(X)$; see Appendix A.6. We refer to $K$ as the *Koszul filtration* in degree $p$ induced by $\alpha$ on $X$.

Let us verify that $K$ is indeed a decreasing filtration of $\bigwedge^p H$ by submodules on $X$. Since $K^i$ is obviously a submodule of $\bigwedge^p H$ on $X$ for all integers $i$, it remains to show that, for all integers $i$ and $j$ with $i \leq j$, we have $K^j \subset K^i$. In case $i < 0$, this is clear as then, $K^i = \bigwedge^p H$. Similarly, when $j > p$, we know that $K^j$ is the zero submodule of $\bigwedge^p H$, so that $K^j \subset K^i$ is evident. We are left with the case where $0 \leq i \leq j \leq p$. To that end, denote by $\phi_i$ the composition of the following obvious morphisms in $\mathrm{Mod}(X)$:

$$\begin{aligned} \bigotimes^p (\overbrace{G, \dots, G}^{i}, \overbrace{H, \dots, H}^{p-i}) \longrightarrow \mathrm{T}^i(G) \otimes \mathrm{T}^{p-i}(H) \longrightarrow \bigwedge^i G \otimes \bigwedge^{p-i} H \\ \longrightarrow \bigwedge^i H \otimes \bigwedge^{p-i} H \longrightarrow \bigwedge^p H. \end{aligned} \tag{1.3}$$

Then $K^i = \operatorname{im}(\phi_i)$ since the first and second of the morphisms in Eq. (1.3) are an isomorphism and an epimorphism in $\mathrm{Mod}(X)$, respectively. The same holds for $i$ replaced by $j$ assuming that we define $\phi_j$ accordingly. Thus, our claim is implied by the easy to verify identity

$$\phi_j = \phi_i \circ \bigotimes^p (\underbrace{\mathrm{id}_G, \dots, \mathrm{id}_G}_{i}, \underbrace{\alpha, \dots, \alpha}_{j-i}, \underbrace{\mathrm{id}_H, \dots, \mathrm{id}_H}_{p-j}).$$

**Definition 1.2.2**

1. Let $\mathcal{C}$ be a category. Then a *triple* in $\mathcal{C}$ is a functor from $\mathbf{3}$ to $\mathcal{C}$.[4]
2. A *triple of modules* on $X$ is a triple in $\mathrm{Mod}(X)$ in the sense of item 1.

[4]For small categories (i.e., sets) $\mathcal{C}$ this definition is an actual definition, in the sense that there is a formula in the language of in the language of Zermelo-Fraenkel set theory expressing it. When $\mathcal{C}$ is large, however, the definition is rather a "definition scheme"—that is, it becomes an actual definition when spelled out for a particular instance of $\mathcal{C}$.

When we say that $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ is a triple in $\mathcal{C}$ (resp. a triple of modules on $X$), we mean that $t$ is a triple in $\mathcal{C}$ (resp. a triple of modules on $X$) and that $G = t0$, $H = t1$, $F = t2$, $\alpha = t_{0,1}$, and $\beta = t_{1,2}$.

**Definition 1.2.3**

1. Let $\mathcal{C}$ be an abelian category, $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ a triple in $\mathcal{C}$. Then we call $t$ *left exact* (resp. *right exact*) in $\mathcal{C}$ when

$$0 \longrightarrow G \xrightarrow{\alpha} H \xrightarrow{\beta} F \quad (\text{resp. } G \xrightarrow{\alpha} H \xrightarrow{\beta} F \longrightarrow 0)$$

is an exact sequence in $\mathcal{C}$. We call $t$ *short exact* in $\mathcal{C}$ when $t$ is both left exact in $\mathcal{C}$ and right exact in $\mathcal{C}$, or equivalently when $0 \to G \to H \to F \to 0$ is an exact sequence in $\mathcal{C}$.
2. Let $t$ be a triple of modules on $X$. Then we call $t$ *left exact* (resp. *right exact*, resp. *short exact*) on $X$ when $t$ is left exact (resp. right exact, resp. short exact) in $\mathrm{Mod}(X)$ in the sense of item 1.

The upcoming series of results is preparatory for Construction 1.2.8.

**Lemma 1.2.4** *Let $p$ be an integer, $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ a right exact triple of modules on $X$. Then the sequence*

$$G \otimes \bigwedge^{p-1} H \xrightarrow{\alpha'} \bigwedge^{p} H \xrightarrow{\beta'} \bigwedge^{p} F \longrightarrow 0, \tag{1.4}$$

*with*

$$\alpha' = \wedge^{1,p-1}(H) \circ (\alpha \otimes \bigwedge^{p-1} \mathrm{id}_H) \quad \textit{and} \quad \beta' = \bigwedge^{p} \beta,$$

*is exact in* $\mathrm{Mod}(X)$.

*Proof* When $p \leq 1$, the claim is basically evident. So, assume that $1 < p$. It is (more or less) obvious that $\beta' \circ \alpha' = 0$ in $\mathrm{Mod}(X)$. Therefore, $\beta'$ factors through $\mathrm{coker}(\alpha')$; that is, there exists a (unique) morphism $\phi : \mathrm{coker}(\alpha') \to \beta'$ in the undercategory ${}_{\wedge^p H/}\mathrm{Mod}(X)$. We are finished when we prove that $\phi$ is an isomorphism. For that matter, we construct an inverse of $\phi$ explicitly.

Here goes a preliminary observation. Let $U$, $V$, and $V'$ be open sets of $X$ and

$$f = (f_0, \dots, f_{p-1}) \in F(U) \times \cdots \times F(U) = (F \times \cdots \times F)(U),$$
$$h = (h_0, \dots, h_{p-1}) \in H(V) \times \cdots \times H(V),$$
$$h' = (h'_0, \dots, h'_{p-1}) \in H(V') \times \cdots \times H(V')$$

such that $V, V' \subset U$ and, for all $i < p$, we have $\beta_V(h_i) = f_i|_V$ and $\beta_{V'}(h'_i) = f_i|_{V'}$. Then, for all $i < p$, the difference $h'_i|_{V \cap V'} - h_i|_{V \cap V'}$ is sent to 0 in $F(V \cap V')$ by $\beta_{V \cap V'}$. Thus, since $\alpha$ maps surjectively onto the kernel of $\beta$, there exists an open cover $\mathfrak{V}$ of $V \cap V'$ such that, for all $W \in \mathfrak{V}$ and all $i < p$, there exists $g_i \in G(W)$ such that $\alpha_W(g_i) = h'_i|_W - h_i|_W$. On such a $W \in \mathfrak{V}$, we have (calculating in $\bigwedge^p H(W)$):

$$\begin{aligned} h'_0 \wedge \cdots \wedge h'_{p-1} &= ((h'_0 - h_0) + h_0) \wedge \cdots \wedge h'_{p-1} \\ &= (h'_0 - h_0) \wedge h'_1 \wedge \cdots \wedge h'_{p-1} + h_0 \wedge h'_1 \wedge \cdots \wedge h'_{p-1} \\ &= \ldots \\ &= \alpha_W(g_0) \wedge h'_1 \wedge \cdots \wedge h'_{p-1} + h_0 \wedge \alpha_W(g_1) \wedge \cdots \wedge h'_{p-1} + \ldots \\ &\quad + h_0 \wedge \cdots \wedge h_{p-2} \wedge \alpha_W(g_{p-1}) + h_0 \wedge \cdots \wedge h_{p-1}. \end{aligned}$$

Thus the difference $h'_0 \wedge \cdots \wedge h'_{p-1} - h_0 \wedge \cdots \wedge h_{p-1}$ (taken in $(\bigwedge^p H)(W)$) lies in the image of $\alpha'_W$. In turn, the images of $h'_0 \wedge \cdots \wedge h'_{p-1}$ and $h_0 \wedge \cdots \wedge h_{p-1}$ in the cokernel of $\alpha'$ agree on $W$. As $\mathfrak{V}$ covers $V \cap V'$, these images agree in fact on $V \cap V'$.

Now, we define a sheaf map $\psi : F \times \cdots \times F \to \operatorname{coker}(\alpha')$ as follows: For $U$ and $f$ as above we take $\psi_U(f)$ to be the unique element $c$ of $(\operatorname{coker}(\alpha'))(U)$ such that there exists an open cover $\mathfrak{U}$ of $U$ with the property that, for all $V \in \mathfrak{U}$, there exist $h_0, \ldots, h_{p-1} \in H(V)$ such that $\beta_V(h_i) = f_i|_V$ for all $i < p$ and the image of $h_0 \wedge \cdots \wedge h_{p-1}$ in $\operatorname{coker}(\alpha')$ is equal to $c|_V$. Assume we have two such elements $c$ and $c'$ with corresponding open covers $\mathfrak{U}$ and $\mathfrak{U}'$. Then by the preliminary observation, for all $V \in \mathfrak{U}$ and all $V' \in \mathfrak{U}'$, we have $c|_{V \cap V'} = c'|_{V \cap V'}$. Thus $c = c'$. This proves the uniqueness of $c$. For the existence of $c$ note that since $\beta : H \to F$ is surjective, there exists an open cover $\mathfrak{U}$ of $U$ such that, for all $V \in \mathfrak{U}$ and all $i < p$, there is an $h_i \in H(V)$ such that $\beta_V(h_i) = f_i|_V$. For all $V \in \mathfrak{U}$, any choice of $h_i$'s give rise to an element in $(\bigwedge^p H)(V)$ and thus to an element in $(\operatorname{coker}(\alpha'))(V)$. However, by the preliminary observation, any two choices of $h_i$'s define the same element in $\operatorname{coker}(\alpha')$. Moreover, for any two $V, V' \in \mathfrak{U}$, the corresponding elements agree on $V \cap V'$. Therefore, there exists $c$ with the desired property.

It is an easy matter to check that $\psi$ is actually a morphism of sheaves on $X_{\mathrm{top}}$. Moreover, it is straightforward to check that $\psi$ is $\mathscr{O}_X$-multilinear as well as alternating. Therefore, by the universal property of the alternating product, $\psi$ induces a morphism $\bar{\psi} : \bigwedge^p F \to \operatorname{coker}(\alpha')$ of modules on $X$. Finally, an easy argument shows that $\bar{\psi}$ makes up an inverse of $\phi$, which was to be demonstrated. □

**Proposition 1.2.5** *Let $p$ be an integer, $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ a right exact triple of modules on $X$. Write $K = (K^i)_{i \in \mathbf{Z}}$ for the Koszul filtration in degree $p$ induced by $\alpha : G \to H$ on $X$. Then the following sequence is exact in* $\mathrm{Mod}(X)$*:*

$$0 \longrightarrow K^1 \xrightarrow{\subset} \bigwedge^p H \xrightarrow{\bigwedge^p \beta} \bigwedge^p F \longrightarrow 0. \tag{1.5}$$

*Proof* By Lemma 1.2.4, the sequence in Eq. (1.4) is exact in Mod($X$). By the definition of the Koszul filtration, the inclusion morphism $K^1 \to \bigwedge^p H$ is an image in Mod($X$) of the morphism

$$\overset{1,p-1}{\wedge}(H) \circ (\alpha \otimes \bigwedge^{p-1} \mathrm{id}_H) : G \otimes \bigwedge^{p-1} H \longrightarrow \bigwedge^{p} H; \tag{1.6}$$

note that $\bigwedge^1$ equals the identity functor on Mod($X$) by definition. Hence our claim follows. □

**Corollary 1.2.6** *Let $p$ be an integer, $t : G \overset{\alpha}{\to} H \overset{\beta}{\to} F$ a right exact triple of modules on $X$. Denote by $K = (K^i)_{i \in \mathbf{Z}}$ the Koszul filtration in degree $p$ induced by $\alpha : G \to H$ on $X$.*

1. *There exists one, and only one, $\psi$ rendering commutative in* Mod($X$) *the following diagram:*

$$\begin{tikzcd} \bigwedge^p H \arrow[r, "\bigwedge^p \beta"] \arrow[d] & \bigwedge^p F \\ (\bigwedge^p H)/K^1 \arrow[ur, dotted, "\psi"'] & \end{tikzcd} \tag{1.7}$$

2. *Let $\psi$ be such that the diagram in Eq. (1.7) commutes in* Mod($X$)*. Then $\psi$ is an isomorphism in* Mod($X$).

*Proof* Both assertions are immediate consequences of Proposition 1.2.5. In order to obtain item 1, exploit the fact that the composition of the inclusion morphism $K^1 \to \bigwedge^p H$ and $\bigwedge^p \beta$ is a zero morphism in Mod($X$). In order to obtain item 2, make use of the fact that, by the exactness of the sequence in Eq. (1.5), $\bigwedge^p \beta$ is a cokernel in Mod($X$) of the inclusion morphism $K^1 \to \bigwedge^p H$. □

**Proposition 1.2.7** *Let $p$ be an integer, $t : G \overset{\alpha}{\to} H \overset{\beta}{\to} F$ a right exact triple of modules on $X$. Denote by $K = (K^i)_{i \in \mathbf{Z}}$ the Koszul filtration in degree $p$ induced by $\alpha : G \to H$ on $X$.*

1. *There exists a unique ordered pair $(\phi_0, \phi)$ such that the following diagram commutes in* Mod($X$)*:*

$$\begin{tikzcd} H \otimes \bigwedge^{p-1} H \arrow[d, "\wedge^{1,p-1}(H)"'] & G \otimes \bigwedge^{p-1} H \arrow[l, "\alpha \otimes \bigwedge^{p-1} \mathrm{id}_H"'] \arrow[r, "id_G \otimes \bigwedge^{p-1} \beta"] \arrow[d, dotted, "\phi_0"'] & G \otimes \bigwedge^{p-1} F \arrow[d, dotted, "\phi"] \\ \bigwedge^p H & K^1 \arrow[l, "\supset"] \arrow[r] & K^1/K^2 \end{tikzcd} \tag{1.8}$$

2. *Let* $(\phi_0, \phi)$ *be an ordered pair such that the diagram in Eq. (1.8) commutes in* $\mathrm{Mod}(X)$. *Then* $\phi$ *is an epimorphism in* $\mathrm{Mod}(X)$.

*Proof* Item 1. By the definition of the Koszul filtration, the inclusion morphism $K^1 \to \bigwedge^p H$ is an image in $\mathrm{Mod}(X)$ of the morphism in Eq. (1.6). Therefore, there exists one, and only one, morphism $\phi_0$ making the left-hand square of the diagram in Eq. (1.8) commute in $\mathrm{Mod}(X)$.

By Lemma 1.2.4, we know that the sequence in Eq. (1.4), where we replace $p$ by $p-1$ and define the arrows as indicated in the text of the lemma, is exact in $\mathrm{Mod}(X)$. Tensoring the latter sequence with $G$ on the left, we obtain yet another exact sequence in $\mathrm{Mod}(X)$:

$$G \otimes (G \otimes \bigwedge^{p-2} H) \longrightarrow G \otimes \bigwedge^{p-1} H \longrightarrow G \otimes \bigwedge^{p-1} F \longrightarrow 0. \tag{1.9}$$

The exactness of the sequence in Eq. (1.9) implies that the morphism

$$\mathrm{id}_G \otimes \bigwedge^{p-1} \beta : G \otimes \bigwedge^{p-1} H \longrightarrow G \otimes \bigwedge^{p-1} F$$

is a cokernel in $\mathrm{Mod}(X)$ of the morphism given by the first arrow in Eq. (1.9). Besides, the definition of the Koszul filtration implies that the composition

$$G \otimes (G \otimes \bigwedge^{p-2} H) \longrightarrow G \otimes \bigwedge^{p-1} H \longrightarrow K^1$$

of the first arrow in Eq. (1.9) with $\phi_0$ maps into $K^2 \subset K^1$, whence composing it further with the quotient morphism $K^1 \to K^1/K^2$ yields a zero morphism in $\mathrm{Mod}(X)$. Thus, by the universal property of the cokernel, there exists a unique $\phi$ rendering commutative in $\mathrm{Mod}(X)$ the right-hand square in Eq. (1.8).

Item 2. Observe that by the commutativity of the left-hand square in Eq. (1.8), $\phi_0$ is a coimage of the morphism in Eq. (1.6), whence an epimorphism in $\mathrm{Mod}(X)$. Moreover, the quotient morphism $K^1 \to K^1/K^2$ is an epimorphism in $\mathrm{Mod}(X)$. Thus, the composition of $\phi_0$ and $K^1 \to K^1/K^2$ is an epimorphism in $\mathrm{Mod}(X)$. By the commutativity of the right-hand square in Eq. (1.8), we see that $\phi$ is an epimorphism in $\mathrm{Mod}(X)$. □

**Construction 1.2.8** Let $p$ be an integer. Moreover, let $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ be a right exact triple of modules on $X$. Write $K = (K^i)_{i \in \mathbf{Z}}$ for the Koszul filtration in degree $p$ induced by $\alpha : G \to H$ on $X$. Recall that $K$ is a decreasing filtration of $\bigwedge^p H$ by submodules on $X$. We define a functor $\Lambda^p(t)$ from **3** to Mod($X$) by setting, in the first place,

$$(\Lambda^p(t))(0) := G \otimes \bigwedge^{p-1} F, \qquad (\Lambda^p(t))_{0,0} := \mathrm{id}_{G\otimes\bigwedge^{p-1}F},$$

$$(\Lambda^p(t))(1) := (\bigwedge^{p} H)/K^2, \qquad (\Lambda^p(t))_{1,1} := \mathrm{id}_{(\bigwedge^p H)/K^2},$$

$$(\Lambda^p(t))(2) := \bigwedge^{p} F, \qquad (\Lambda^p(t))_{2,2} := \mathrm{id}_{\bigwedge^p F}.$$

Now let $\iota$ and $\pi$ be the unique morphisms such that the following diagram commutes in Mod($X$):

$$\begin{tikzcd} K^2 \arrow[r, "\subset"] & K^1 \arrow[r, "\subset"] \arrow[d] & \bigwedge^p H \arrow[d] \arrow[dr] & \\ & K^1/K^2 \arrow[r, dotted, "\iota"'] & (\bigwedge^p H)/K^2 \arrow[r, dotted, "\pi"'] & (\bigwedge^p H)/K^1 \end{tikzcd} \tag{1.10}$$

By item 1 of Proposition 1.2.7 we know that there exists a unique ordered pair $(\phi_0, \phi)$ rendering commutative in Mod($X$) the diagram in Eq. (1.8). Likewise, by item 1 of Corollary 1.2.6, there exists a unique $\psi$ rendering commutative in Mod($X$) the diagram in Eq. (1.7). We complete our definition of $\Lambda^p(t)$ by setting:

$$(\Lambda^p(t))_{0,1} := \iota \circ \phi, \qquad (\Lambda^p(t))_{1,2} := \psi \circ \pi,$$

$$(\Lambda^p(t))_{0,2} := (\psi \circ \pi) \circ (\iota \circ \phi).$$

It is a straightforward matter to check that the so defined $\Lambda^p(t)$ is indeed a functor from **3** to Mod($X$) (i.e., a triple of modules on $X$).

I claim that $\Lambda^p(t)$ is even a right exact triple of modules on $X$. To see this, observe that firstly, the bottom row of the diagram in Eq. (1.10) makes up a short exact triple of modules on $X$, that secondly, $\psi$ is an isomorphism in Mod($X$) by item 2 of Corollary 1.2.6, and that thirdly, $\phi$ is an epimorphism in Mod($X$) by item 2 of Proposition 1.2.7. Naturally, the construction of $\Lambda^p(t)$ depends on the ringed space $X$. So, whenever I feel the need to make the reference to the ringed space $X$ explicit, I resort to writing $\Lambda^p_X(t)$ instead of $\Lambda^p(t)$.

Let us show that the $\Lambda^p$ construction is nicely compatible with the restriction to open subspaces.

**Proposition 1.2.9** *Let $U$ be an open subset of $X$, $p$ an integer, and $t : G \to H \to F$ a right exact triple of modules on $X$. Then*

$$t|_U := (-|_U) \circ t : G|_U \longrightarrow H|_U \longrightarrow F|_U$$

*is a right exact triple of modules on $X|_U$ and we have*

$$(\Lambda^p_X(t))|_U = \Lambda^p_{X|_U}(t|_U).^5 \tag{1.11}$$

*Proof* The fact that the triple $t|_U$ is right exact on $X|_U$ is clear since the restriction to an open subspace functor $-|_U : \mathrm{Mod}(X) \to \mathrm{Mod}(X|_U)$ is exact. Denote by $K = (K^i)_{i\in\mathbf{Z}}$ and $K' = (K'^i)_{i\in\mathbf{Z}}$ the Koszul filtrations in degree $p$ induced by $t_{0,1} : G \to H$ and $t_{0,1}|_U : G|_U \to H|_U$ on $X$ and $X|_U$, respectively. Then by the presheaf definitions of the wedge and tensor product, we see that $K^i|_U = K'^i$ for all integers $i$. Now define $\iota$ and $\pi$ just as in Construction 1.2.8. Similarly, define $\iota'$ and $\pi'$ using $K'$ instead of $K$ and $X|_U$ instead of $X$. Then by the presheaf definition of quotient sheaves, we see that the following diagram commutes in $\mathrm{Mod}(X|_U)$:

$$\begin{tikzcd}
(K^1/K^2)|_U \arrow[r, "\iota|_U"] \arrow[d, equal] & (K^0/K^2)|_U \arrow[r, "\pi|_U"] \arrow[d, equal] & (K^0/K^1)|_U \arrow[d, equal] \\
K'^1/K'^2 \arrow[r, "\iota'"'] & K'^0/K'^2 \arrow[r, "\pi'"'] & K'^0/K'^1
\end{tikzcd}$$

Defining $\phi$ and $\psi$ just as in Construction 1.2.8, and defining $\phi'$ and $\psi'$ analogously for $t|_U$ instead of $t$ and $X|_U$ instead of $X$, we deduce that $\phi|_U = \phi'$ and $\psi|_U = \psi'$. Hence, Eq. (1.11) holds according to the definitions given in Construction 1.2.8. □

The remainder of this section is devoted to the investigation of the $\Lambda^p$ construction when applied to a split exact triple of modules on $X$.

**Definition 1.2.10** Let $t$ be a triple of modules on $X$.

1. We say that $t$ is *split exact* on $X$ when there exist modules $F$ and $G$ on $X$ such that $t$ is isomorphic, in the functor category $\mathrm{Mod}(X)^{\mathbf{3}}$, to the triple

$$G \xrightarrow{\iota} G \oplus F \xrightarrow{\pi} F, \tag{1.12}$$

   where $\iota$ and $\pi$ stand respectively for the coprojection to the first summand and the projection to the second summand.
2. $\phi$ is called a *right splitting* of $t$ on $X$ when $\phi$ is a morphism of modules on $X$, $\phi : t2 \to t1$, such that we have $t_{1,2} \circ \phi = \mathrm{id}_{t2}$ in $\mathrm{Mod}(X)$.[6]

[5]Note that in order to get a real equality here, as opposed to only a "canonical isomorphism," you have to work with the correct sheafification functor.

[6]Thus, a right splitting of $t$ is nothing but a right *inverse* of the morphism $t_{1,2} : t1 \to t2$.

*Remark 1.2.11* Let $t$ be a triple of modules on $X$. Then the following are equivalent:

1. $t$ is split exact on $X$.
2. $t$ is short exact on $X$ and there exists a right splitting of $t$ on $X$.
3. $t$ is left exact on $X$ and there exists a right splitting of $t$ on $X$.

For the proof, assume item 1. Then $t$ is isomorphic to a triple as in Eq. (1.12). The fact that the triple in Eq. (1.12) is short exact on $X$ can be verified either by elementary means (i.e., using the concrete definition of the direct sum for sheaves of modules on $X$) or by an abstract argument which is valid in any Ab-enriched, or "preadditive," category [13, Tag 09QG]. Note that in order to prove that the triple in Eq. (1.12) is short exact on $X$, it suffices to prove that $\iota$ is a kernel of $\pi$ and that, conversely, $\pi$ is a cokernel of $\iota$. A right splitting of the triple in Eq. (1.12) is given by the coprojection $F \to G \oplus F$ to the second summand. Since the quality of being short exact on $X$ as well as the existence of a right splitting on $X$ are invariant under isomorphism in $\mathrm{Mod}(X)^{\mathbf{3}}$, we obtain item 2.

That item 2 implies item 3 is clear, for short exactness implies left exactness per definitionem. So, assume item 3. Write $t : G \to H \to F$ and $\alpha := t_{0,1}$. Denote by $s$ the triple in Eq. (1.12). We are going to fabricate a morphism $\psi : s \to t$ in $\mathrm{Mod}(X)^{\mathbf{3}}$. Note that $\psi$ should be a natural transformation of functors. We set $\psi 0 := \mathrm{id}_G$ and $\psi 2 := \mathrm{id}_F$. Recall that there exists a right splitting $\phi$ of $t$ on $X$. Given $\phi$, we define $\psi 1 := \alpha + \phi : G \oplus F \to H$. Then, the validation that $\psi$ is a natural transformation $s \to t$ of functors from $\mathbf{3}$ to $\mathrm{Mod}(X)$ is straightforward. $\psi : s \to t$ is an isomorphism in $\mathrm{Mod}(X)^{\mathbf{3}}$ due to the snake lemma [13, Tag 010H]. Thus, we infer item 1.

**Lemma 1.2.12** *Let $p$ be an integer, $\alpha : G \to H$ and $\phi : F \to H$ morphisms of modules on $X$. Assume that $\alpha + \phi : G \oplus F \to H$ is an isomorphism in* $\mathrm{Mod}(X)$.

1. *The morphism*

$$\sum_{\nu \in \mathbf{Z}} (\overset{\nu,p-\nu}{\wedge}(H) \circ (\bigwedge^{\nu} \alpha \otimes \bigwedge^{p-\nu} \phi)) : \bigoplus_{\nu \in \mathbf{Z}} (\bigwedge^{\nu} G \otimes \bigwedge^{p-\nu} F) \longrightarrow \bigwedge^{p} H$$

   *is an isomorphism in* $\mathrm{Mod}(X)$.
2. *Denote by $K = (K^i)_{i \in \mathbf{Z}}$ the Koszul filtration in degree $p$ induced by $\alpha$ on $X$. Then, for all integers $i$, we have*

$$K^i = \mathrm{im}\left( \sum_{\nu \geq i} (\overset{\nu,p-\nu}{\wedge}(H) \circ (\bigwedge^{\nu} \alpha \otimes \bigwedge^{p-\nu} \phi)) \right).$$

*Proof* Item 1. We take the assertion for granted in case $X$ is not a ringed space, but an ordinary ring [4, Proposition A2.2c]. Thus, for all open subsets $U$ of $X$, the map

$$\sum_{\nu \in \mathbf{Z}} (\overset{\nu,p-\nu}{\wedge}(H(U)) \circ (\bigwedge^{\nu} \alpha_U \otimes \bigwedge^{p-\nu} \phi_U)) : \bigoplus_{\nu \in \mathbf{Z}} (\bigwedge^{\nu} G(U) \otimes \bigwedge^{p-\nu} F(U)) \longrightarrow \bigwedge^{p} H(U)$$

is an isomorphism of $\mathscr{O}_X(U)$-modules. In other words, the morphism in item 1 is an isomorphism of presheaves of $\mathscr{O}_X$-modules on $X_{\mathrm{top}}$ when we replace $\bigwedge$, $\otimes$, $\wedge$, and $\bigoplus$ by their presheaf counterparts. Noting that the actual morphism in item 1 is isomorphic to the sheafification of its presheaf counterpart, we are finished.[7]

Item 2. Let $i$ be an integer. Then for all integers $\nu \geq i$, the sheaf morphism

$$\overset{\nu,p-\nu}{\wedge}(H) \circ (\bigwedge^{\nu} \alpha \otimes \bigwedge^{p-\nu} \phi) : \bigwedge^{\nu} G \otimes \bigwedge^{p-\nu} F \longrightarrow \bigwedge^{p} H$$

clearly maps into $K^i$; see Construction 1.2.1. Therefore, the sum over all of these morphisms maps into $K^i$ as well. Conversely, any section in $\bigwedge^p H$ coming from

$$\overset{i,p-i}{\wedge}(H) \circ (\bigwedge^{i} \alpha \otimes \bigwedge^{p-i} \mathrm{id}_H) : \bigwedge^{i} G \otimes \bigwedge^{p-i} H \longrightarrow \bigwedge^{p} H$$

comes from

$$\bigoplus_{\nu \geq i} (\bigwedge^{\nu} G \otimes \bigwedge^{p-\nu} F)$$

under the given morphism, as you see decomposing $\bigwedge^{p-i} H$ in the form

$$\bigoplus_{\mu \geq 0} (\bigwedge^{\mu} G \otimes \bigwedge^{p-i-\mu} F) \cong \bigwedge^{p-i} H$$

according to item 1 (where you replace $p$ by $p - i$). □

**Proposition 1.2.13** *Let $p$ be an integer and $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ a right exact triple of modules on $X$.*

1. *Let $\phi$ be a right splitting of $t$ on $X$. Denote by $K = (K^i)_{i \in \mathbf{Z}}$ the Koszul filtration in degree $p$ induced by $\alpha : G \to H$ on $X$ and write $\kappa : \bigwedge^p H \to (\bigwedge^p H)/K^2$ for the evident quotient morphism. Then the composition*

$$\kappa \circ \bigwedge^{p} \phi : \bigwedge^{p} F \longrightarrow (\bigwedge^{p} H)/K^2$$

   *is a right splitting of $\Lambda^p(t)$ on $X$.*
2. *When $t$ is split exact on $X$, then $\Lambda^p(t)$ is split exact on $X$.*

[7] In detail, what you have to prove is this: when $I$ and $J$ are two presheaves of modules on $X$, then the composition $I \mathbin{\bar{\otimes}} J \to I^{\#} \mathbin{\bar{\otimes}} J^{\#} \to I^{\#} \otimes J^{\#}$ is isomorphic to the sheafification of $I \mathbin{\bar{\otimes}} J$, where $\bar{\otimes}$ denotes the presheaf tensor product.

*Proof* Item 1. By Construction 1.2.8,

$$(\Lambda^p(t))_{1,2} : (\bigwedge^p H)/K^2 \longrightarrow \bigwedge^p F$$

is the unique morphism of modules on $X$ which, precomposed with the quotient morphism $\kappa$, yields $\bigwedge^p \beta : \bigwedge^p H \to \bigwedge^p F$. Therefore, we have

$$(\Lambda^p(t))_{1,2} \circ (\kappa \circ \bigwedge^p \phi) = \bigwedge^p \beta \circ \bigwedge^p \phi = \bigwedge^p (\beta \circ \phi) = \bigwedge^p \mathrm{id}_F = \mathrm{id}_{\bigwedge^p F}.$$

Item 2. Write $K = (K^i)_{i \in \mathbf{Z}}$ for the Koszul filtration in degree $p$ induced by $\alpha : G \to H$ on $X$. Then, by the definition of $(\Lambda^p(t))_{0,1}$ in Construction 1.2.8, the following diagram commutes in $\mathrm{Mod}(X)$:

$$\begin{tikzcd}
G \otimes \bigwedge^{p-1} H \arrow[r, "\mathrm{id}_G \otimes \bigwedge^{p-1} \beta"] \arrow[d, "\wedge^{1,p}(H) \circ (\alpha \otimes \bigwedge^{p-1} \mathrm{id}_H)"'] & G \otimes \bigwedge^{p-1} F \arrow[d, "(\Lambda^p(t))_{0,1}"] \\
\bigwedge^p H \arrow[r, "\kappa"'] & (\bigwedge^p H)/K^2
\end{tikzcd}$$

Since $t$ is a split exact triple of modules on $X$, there exists a right splitting $\phi$ of $t$ on $X$; see Remark 1.2.11. Using the commutativity of the diagram, we deduce that

$$(\Lambda^p(t))_{0,1} = \kappa \circ (\overset{1,p}{\wedge}(H) \circ (\alpha \otimes \bigwedge^{p-1} \phi)).$$

As a matter of fact, the diagram yields the latter equality when precomposed with $\mathrm{id}_G \otimes \bigwedge^{p-1} \beta$. You may cancel the term $\mathrm{id}_G \otimes \bigwedge^{p-1} \beta$ on the right, however, remarking that it is an epimorphism in $\mathrm{Mod}(X)$.

Now, by Lemma 1.2.12, we see that the sheaf map $(\Lambda^p(t))_{0,1}$ is injective. Taking into account that the triple $\Lambda^p(t)$ is right exact on $X$ (see Construction 1.2.8), we deduce that $\Lambda^p(t)$ is, in fact, short exact on $X$. Therefore, $\Lambda^p(t)$ is split exact on $X$ as by item 1, there exists a right splitting of $\Lambda^p(t)$ on $X$; see Remark 1.2.11 again. □

## 1.3 Locally Split Exact Triples and Their Extension Classes

*For the entire section, let X be a ringed space.*

Let $p$ be an integer. In the following, we are going to examine the $\Lambda^p$ construction (i.e., Construction 1.2.8) when applied to locally split exact triples of modules on $X$; see Definition 1.3.1. So, let $t$ be such a triple. Then, as it turns out, $\Lambda^p(t)$ is a locally split exact triple of modules on $X$ too. Now given that $t$ is in particular a short exact triple of modules on $X$, we may consider its extension class, which is an element of $\mathrm{Ext}^1(F, G)$ writing $t : G \to H \to F$. Similarly, the extension class of $\Lambda^p(t)$ is an element of $\mathrm{Ext}^1(\bigwedge^p F, G \otimes \bigwedge^{p-1} F)$.

The decisive result of Sect. 1.3 is Proposition 1.3.12, which tells us how to compute the extension class of $\Lambda^p(t)$ from the extension class of $t$ by means of an interior product morphism,

$$\iota^p(F, G) : \mathscr{H}om(F, G) \longrightarrow \mathscr{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F).$$

The latter is to be defined in the realm of Construction 1.3.11. In order to describe the relationship between the extension classes of $t$ and $\Lambda^p(t)$ conveniently, I introduce the device of "locally split extension classes"; see Construction 1.3.3.

First of all, however, let me state local versions of, respectively, Definition 1.2.10 and Proposition 1.2.13.

**Definition 1.3.1** Let $t$ be a triple of modules on $X$.

1. $t$ is called *locally split exact* on $X$ when there exists an open cover $\mathfrak{U}$ of $X_{\mathrm{top}}$ such that, for all $U \in \mathfrak{U}$, the triple $t|_U$ (i.e., the composition of $t$ with the restriction to an open subspace functor $-|_U : \mathrm{Mod}(X) \to \mathrm{Mod}(X|_U)$) is a split exact triple of modules on $X|_U$.
2. $\phi$ is called a *local right splitting* of $t$ on $X$ when $\phi$ is a function whose domain of definition, call it $\mathfrak{U}$, is an open cover of $X_{\mathrm{top}}$ such that $\phi(U)$ is a right splitting of $t|_U$ on $X|_U$ for all $U \in \mathfrak{U}$.

**Proposition 1.3.2** *Let $p$ be an integer and $t : G \xrightarrow{\alpha} H \to F$ a right exact triple of modules on $X$.*

1. *Let $\phi$ be a local right splitting of $t$ on $X$ and let $\phi'$ be a function on $\mathfrak{U} := \mathrm{dom}(\phi)$ such that, for all $U \in \mathfrak{U}$, we have*

$$\phi'(U) = \kappa|_U \circ \bigwedge^p(\phi(U)) : (\bigwedge^p F)|_U \longrightarrow ((\bigwedge^p H)/K^2)|_U,$$

   *where $\kappa$ denotes the quotient morphism $\bigwedge^p H \to (\bigwedge^p H)/K^2$ and $K = (K^i)_{i\in\mathbf{Z}}$ denotes the Koszul filtration in degree $p$ induced by $\alpha : G \to H$ on $X$. Then $\phi'$ is a local right splitting of $\Lambda^p(t)$ on $X$.*
2. *When $t$ is locally split exact on $X$, then $\Lambda^p(t)$ is locally split exact on $X$.*

*Proof* Item 1. Let $U \in \mathfrak{U}$. Then $\phi(U)$ is a right splitting of $t|_U$ on $X|_U$. Thus by item 1 of Proposition 1.2.13 we know that

$$\kappa' \circ \bigwedge^p(\phi(U)) : \bigwedge^p(F|_U) \longrightarrow (\bigwedge^p(H|_U))/K'^2$$

is a right splitting of $\Lambda^p_{X|_U}(t|_U)$ on $X|_U$, where

$$\kappa' : \bigwedge^p (H|_U) \longrightarrow (\bigwedge^p (H|_U))/K'^2$$

denotes the quotient morphism and $K' = (K'^i)_{i \in \mathbf{Z}}$ denotes the Koszul filtration in degree $p$ induced by $\alpha|_U : G|_U \to H|_U$ on $X|_U$. Since $(\bigwedge^p H)|_U = \bigwedge^p (H|_U)$ and $K^2|_U = K'^2$, we have $\kappa|_U = \kappa'$. Therefore, $\phi'(U)$ is a right splitting of $\Lambda^p_{X|_U}(t|_U)$ on $X|_U$. Given that $\Lambda^p_X(t)|_U = \Lambda^p_{X|_U}(t|_U)$, we deduce that $\phi'$ is a local right splitting of $\Lambda^p_X(t)$ on $X$.

Item 2. Since $t$ is a locally split exact triple of modules on $X$, there exists an open cover $\mathfrak{U}$ of $X_{\mathrm{top}}$ such that, for all $U \in \mathfrak{U}$, the triple $t|_U$ is split exact on $X|_U$. Therefore, by item 2 of Proposition 1.2.13, the triple $\Lambda^p_{X|_U}(t|_U)$ is split exact on $X|_U$ for all $U \in \mathfrak{U}$. As $(\Lambda^p_X(t))|_U = \Lambda^p_{X|_U}(t|_U)$ for all $U \in \mathfrak{U}$, we infer that $\Lambda^p_X(t)$ is a locally split exact triple of modules on $X$. □

**Construction 1.3.3** Let $t : G \to H \to F$ be a short exact triple of modules on $X$ with the property that

$$\mathscr{H}om(F,t) := \mathscr{H}om(F,-) \circ t : \mathscr{H}om(F,G) \longrightarrow \mathscr{H}om(F,H) \longrightarrow \mathscr{H}om(F,F)$$

is again a short exact triple of modules on $X$. Then we write $\xi_X(t)$ for the image of the identity sheaf morphism $\mathrm{id}_F$ under the composition of mappings

$$(\mathscr{H}om(F,F))(X) \xrightarrow{\text{can.}} \mathrm{H}^0(X, \mathscr{H}om(F,F)) \xrightarrow{\delta^0} \mathrm{H}^1(X, \mathscr{H}om(F,G)),$$

where $\delta^0$ stands for the connecting homomorphism in degree 0 associated to the triple $\mathscr{H}om(F,t)$ with respect to the functor

$$\Gamma(X,-) : \mathrm{Mod}(X) \longrightarrow \mathrm{Mod}(\mathbf{Z}).$$

Note that as $(\mathscr{H}om(F,F))(X) = \mathrm{Hom}(F,F)$, we have $\mathrm{id}_F \in (\mathscr{H}om(F,F))(X)$, so that the above definition makes indeed sense. We call $\xi_X(t)$ the *locally split extension class* of $t$ on $X$. As usual, I write $\xi(t)$ instead of $\xi_X(t)$ whenever I feel that the reference to the ringed space $X$ is clear from the context.

*Remark 1.3.4* Let $t : G \to H \xrightarrow{\beta} F$ be a locally split exact triple of modules on $X$. I claim that

$$\mathscr{H}om(F,t) : \mathscr{H}om(F,G) \longrightarrow \mathscr{H}om(F,H) \longrightarrow \mathscr{H}om(F,F)$$

is a locally split exact triple of modules on $X$, too. In fact, let $\phi$ be a local right splitting of $t$ on $X$. Put $\mathfrak{U} := \mathrm{dom}(\phi)$. Then, for all $U \in \mathfrak{U}$, we have

$$\beta|_U = t_{1,2}|_U = (t|_U)_{1,2} : H|_U \longrightarrow F|_U,$$

and thus

$$\begin{aligned}\mathscr{H}om(F|_U, \beta|_U) \circ \mathscr{H}om(F|_U, \phi(U)) &= \mathscr{H}om(F|_U, \beta|_U \circ \phi(U)) \\ &= \mathscr{H}om(F|_U, \mathrm{id}_{F|_U}) = \mathrm{id}_{\mathscr{H}om(F|_U, F|_U)}\end{aligned}$$

according to item 2 of Definition 1.3.1 and item 2 of Definition 1.2.10. Since

$$\mathscr{H}om_X(F, \beta)|_U = \mathscr{H}om_{X|_U}(F|_U, \beta|_U)$$

for all $U \in \mathfrak{U}$, the assignment $U \mapsto \mathscr{H}om(F|_U, \phi(U))$, for $U$ varying through $\mathfrak{U}$, constitutes a local right splitting of $\mathscr{H}om(F, t)$ on $X$. Moreover, since the functor

$$\mathscr{H}om(F, -) : \mathrm{Mod}(X) \longrightarrow \mathrm{Mod}(X)$$

is left exact and the triple $t$ is short exact on $X$, the triple $\mathscr{H}om(F, t)$ is left exact on $X$. In conclusion, we see that the triple $\mathscr{H}om(F, t)$ is locally split exact on $X$, as claimed.

Specifically, $\mathscr{H}om(F, t)$ is a short exact triple of modules on $X$. In view of Construction 1.3.3 this tells us that any locally split exact triple of modules on $X$ possesses a locally split extension class on $X$.

The following remark explains briefly how our newly coined notion of a locally split extension class relates to the customary extension class of a short exact triple (i.e., a short exact sequence) on $X$. Let me point out that, though interesting, the contents of Remark 1.3.5 are dispensable for the subsequent exposition.

*Remark 1.3.5* Let $t : G \to H \to F$ be a short exact triple of modules on $X$. Recall that the *extension class* of $t$ on $X$ is, by definition, the image of the identity sheaf morphism $\mathrm{id}_F$ under the composition of mappings

$$\mathrm{Hom}(F, F) \xrightarrow{\text{can.}} (\mathrm{R}^0\mathrm{Hom}(F, -))(F) \xrightarrow{\delta'^0} (\mathrm{R}^1\mathrm{Hom}(F, -))(G) =: \mathrm{Ext}^1(F, G),$$

where $\delta' = (\delta'^n)_{n \in \mathbf{Z}}$ stands for the sequence of connecting homomorphisms for the triple $t$ with respect to the right derived functor of

$$\mathrm{Hom}(F, -) : \mathrm{Mod}(X) \longrightarrow \mathrm{Mod}(\mathbf{Z}).$$

Observe that the following diagram of categories and functors commutes:

$$\begin{tikzcd} \mathrm{Mod}(X) \arrow[rr, "\mathscr{H}om(F,-)"] \arrow[dr, "\mathrm{Hom}(F,-)"'] & & \mathrm{Mod}(X) \arrow[dl, "\Gamma(X,-)"] \\ & \mathrm{Mod}(\mathbf{Z}) & \end{tikzcd}$$

Combined with the fact that, for all injective modules $I$ on $X$, the module $\mathscr{H}om(F, I)$ is a flasque sheaf on $X$, whence an acyclic object for the functor

$$\Gamma(X, -) : \mathrm{Mod}(X) \longrightarrow \mathrm{Mod}(\mathbf{Z}),$$

this induces a sequence $\tau = (\tau^q)_{q \in \mathbf{Z}}$ of natural transformations

$$\tau^q : \mathrm{H}^q(X, -) \circ \mathscr{H}om(F, -) \longrightarrow \mathrm{Ext}^q(F, -)$$

of functors from $\mathrm{Mod}(X)$ to $\mathrm{Mod}(\mathbf{Z})$. The sequence $\tau$ has the property that when $\mathscr{H}om(F, t)$ is a short exact triple of modules on $X$ and $\delta = (\delta^n)_{n \in \mathbf{Z}}$ denotes the sequence of connecting homomorphisms for the triple $\mathscr{H}om(F, t)$ with respect to the right derived functor of $\Gamma(X, -) : \mathrm{Mod}(X) \to \mathrm{Mod}(\mathbf{Z})$, then, for any integer $q$, the following diagram commutes in $\mathrm{Mod}(\mathbf{Z})$:

$$\begin{tikzcd}
\mathrm{H}^q(X, \mathscr{H}om(F, F)) \arrow[r, "\tau^q(F)"] \arrow[d, "\delta^q"'] & \mathrm{Ext}^q(F, F) \arrow[d, "\delta'^q"] \\
\mathrm{H}^{q+1}(X, \mathscr{H}om(F, G)) \arrow[r, "\tau^{q+1}(G)"'] & \mathrm{Ext}^{q+1}(F, G)
\end{tikzcd}$$

Moreover, the following diagram commutes in $\mathrm{Mod}(\mathbf{Z})$:

$$\begin{tikzcd}
(\mathscr{H}om(F, F))(X) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{can.}"'] & \mathrm{Hom}(F, F) \arrow[d, "\mathrm{can.}"] \\
\mathrm{H}^0(X, \mathscr{H}om(F, F)) \arrow[r, "\tau^0(F)"'] & \mathrm{Ext}^0(F, F)
\end{tikzcd}$$

Hence, we see that the function $\tau^1(G)$ maps the locally split extension class $\xi(t)$ of $t$ to the (usual) extension class of $t$. In addition, by means of general homological algebra—namely, the Grothendieck spectral sequence—you can show that the mapping $\tau^1(G)$ is one-to-one. Therefore, $\xi(t)$ is the unique element of $\mathrm{H}^1(X, \mathscr{H}om(F, G))$ which is mapped to the extension class of $t$ by the function $\tau^1(G)$.

I think that Remark 1.3.5 justifies our referring to $\xi(t)$ as the "locally split extension class" of $t$ on $X$.

The next couple of results are aimed at deriving, for $t$ a locally split exact triple of modules on $X$, from a local right splitting of $t$, a Čech representation of the locally split extension class $\xi(t)$. Since the definition of Čech cohomology tends to vary from source to source, let us settle once and for all on the following conventions.

*Remark 1.3.6* Let $\mathfrak{U}$ be an open cover of $X_{\text{top}}$ and $n$ a natural number. An *n-simplex* of $\mathfrak{U}$ is an ordered $(n+1)$-tuple of elements of $\mathfrak{U}$; that is,

$$u = (u_0, \ldots, u_n) \in \mathfrak{U}^{n+1}$$

such that $u_0 \cap \cdots \cap u_n \neq \emptyset$. When $u = (u_0, \ldots, u_n)$ is an $n$-simplex of $\mathfrak{U}$, the intersection $u_0 \cap \cdots \cap u_n$ is called the *support* of $u$ and denoted by $|u|$.

Let $F$ be a (pre)sheaf of modules on $X$. Then a *Čech n-cochain* of $\mathfrak{U}$ with coefficients in $F$ is a function $c$ defined on the set $S$ of $n$-simplices of $\mathfrak{U}$ such that, for all $u \in S$, we have $c(u) \in F(|u|)$; in other words, $c$ is an element of $\prod_{u \in S} F(|u|)$. We denote by

$$\check{\mathrm{C}}^n_X(\mathfrak{U}, -) : \mathrm{Mod}(X) \longrightarrow \mathrm{Mod}(\mathbf{Z})$$

the Čech $n$-cochain functor, so that $\check{\mathrm{C}}^n_X(\mathfrak{U}, F)$ is the set of Čech $n$-cochains of $\mathfrak{U}$ with coefficients in $F$ equipped with the obvious addition and $\mathbf{Z}$-scalar multiplication. Similarly, we denote by

$$\check{\mathrm{C}}^\bullet_X(\mathfrak{U}, -) : \mathrm{Mod}(X) \longrightarrow \mathrm{Com}(\mathbf{Z})$$

the Čech complex functor, so that $\check{\mathrm{C}}^\bullet_X(\mathfrak{U}, F)$ is the habitual Čech complex of $\mathfrak{U}$ with coefficients in $F$. We write

$$\check{\mathrm{Z}}^n_X(\mathfrak{U}, -), \check{\mathrm{B}}^n_X(\mathfrak{U}, -), \check{\mathrm{H}}^n_X(\mathfrak{U}, -) : \mathrm{Mod}(X) \longrightarrow \mathrm{Mod}(\mathbf{Z})$$

for the functors obtained by composing the functor $\check{\mathrm{C}}^\bullet_X(\mathfrak{U}, -)$ with the $n$-cocycle, -coboundary, and -cohomology functors for complexes over $\mathrm{Mod}(\mathbf{Z})$, respectively. In any of the expressions $\check{\mathrm{C}}^\bullet_X$, $\check{\mathrm{C}}^n_X$, $\check{\mathrm{Z}}^n_X$, $\check{\mathrm{B}}^n_X$, and $\check{\mathrm{H}}^n_X$, we suppress the subscript "$X$" whenever we feel that the correct ringed space can be guessed unambiguously from the context.

In Proposition 1.3.10 as well as in the proof of Proposition 1.3.12, we will make use of the familiar sequence $\tau = (\tau^q)_{q \in \mathbf{Z}}$ of natural transformations

$$\tau^q : \check{\mathrm{H}}^q(\mathfrak{U}, -) \longrightarrow \mathrm{H}^q(X, -)$$

of functors from $\mathrm{Mod}(X)$ to $\mathrm{Mod}(\mathbf{Z})$. The sequence $\tau$ is obtained considering the Čech resolution functors $\check{\mathscr{C}}^\bullet_X(\mathfrak{U}, -) : \mathrm{Mod}(X) \to \mathrm{Com}(X)$ together with Lemma A.3.6; in fact, the suggested construction yields a natural transformation

$$\check{\mathrm{C}}^\bullet(\mathfrak{U}, -) = \Gamma(X, -) \circ \check{\mathscr{C}}^\bullet(\mathfrak{U}, -) \longrightarrow \mathrm{R}\Gamma(X, -)$$

of functors from $\mathrm{Mod}(X)$ to $\mathrm{K}^+(\mathbf{Z})$, from which one derives $\tau^q$, for any $q \in \mathbf{Z}$, by applying the $q$th cohomology functor $\mathrm{H}^q : \mathrm{K}^+(\mathbf{Z}) \to \mathrm{Mod}(\mathbf{Z})$.

**Construction 1.3.7** Let $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ be a short exact triple of modules on $X$ and $\phi$ a local right splitting of $t$ on $X$. For the time being, fix a 1-simplex $u = (u_0, u_1)$ of $\mathfrak{U} := \mathrm{dom}(\phi)$. Set $v := |u|$ for better readability. Then, calculating in $\mathrm{Mod}(X|_v)$, we have:

$$\begin{aligned}
\beta|_v \circ (\phi(u_1)|_v - \phi(u_0)|_v) &= (\beta|_{u_1})|_v \circ \phi(u_1)|_v - (\beta|_{u_0})|_v \circ \phi(u_0)|_v \\
&= (\beta|_{u_1} \circ \phi(u_1))|_v - (\beta|_{u_0} \circ \phi(u_0))|_v \\
&= \mathrm{id}_{F|_{u_1}}|_v - \mathrm{id}_{F|_{u_0}}|_v \\
&= \mathrm{id}_{F|_v} - \mathrm{id}_{F|_v} = 0.
\end{aligned}$$

Since the triple $t$ is short exact on $X$, we deduce that $\alpha|_v : G|_v \to H|_v$ is a kernel of $\beta|_v : H|_v \to F|_v$ in $\mathrm{Mod}(X|_v)$. So, there exists one, and only one, morphism $c(u) : F|_v \to G|_v$ in $\mathrm{Mod}(X|_v)$ such that

$$\alpha|_v \circ c(u) = \phi(u_1)|_v - \phi(u_0)|_v.$$

Abandoning our fixation of $u$, we define $c$ to be the function on the set of 1-simplices of $\mathfrak{U}$ which is given by the assignment $u \mapsto c(u)$. We call $c$ the *right splitting Čech 1-cochain* of $(t, \phi)$ on $X$.

By definition, for all 1-simplices $u$ of $\mathfrak{U}$, we know that $c(u)$ is a morphism $F|_{|u|} \to G|_{|u|}$ of modules on $X|_{|u|}$; that is, $c(u) \in (\mathscr{H}om(F, G))(|u|)$. Thus,

$$c \in \check{\mathrm{C}}^1_X(\mathfrak{U}, \mathscr{H}om(F, G)).$$

In this regard, we define $\bar{c}$ to be the residue class of $c$ in the quotient module

$$\check{\mathrm{C}}^1_X(\mathfrak{U}, \mathscr{H}om(F, G))/\check{\mathrm{B}}^1_X(\mathfrak{U}, \mathscr{H}om(F, G)).$$

We call $\bar{c}$ the *right splitting Čech 1-class* of $(t, \phi)$ on $X$.

**Lemma 1.3.8** *Let $t : G \to H \xrightarrow{\beta} F$ be a short exact triple of modules on $X$ and $\mathfrak{U}$ an open cover of $X_{\mathrm{top}}$.*

1. *Let $n \in \mathbf{N}$. Then $\check{\mathrm{C}}^n(\mathfrak{U}, t)$ is a short exact triple in* $\mathrm{Mod}(\mathbf{Z})$ *if and only if, for all $n$-simplices $u$ of $\mathfrak{U}$, the mapping $\beta_{|u|} : H(|u|) \to F(|u|)$ is surjective.*
2. *$\check{\mathrm{C}}^\bullet(\mathfrak{U}, t)$ is a short exact triple in* $\mathrm{Com}(\mathbf{Z})$ *if and only if, for all nonempty, finite subsets $\mathfrak{V}$ of $\mathfrak{U}$ such that $V := \bigcap \mathfrak{V} \neq \emptyset$, the mapping $\beta_V : H(V) \to F(V)$ is surjective.*
3. *Let $\phi$ be a local right splitting of $t$ on $X$ such that $\mathfrak{U} = \mathrm{dom}(\phi)$. Then $\check{\mathrm{C}}^\bullet(\mathfrak{U}, t)$ is a short exact triple in* $\mathrm{Com}(\mathbf{Z})$.

*Proof* Item 1. We denote the set of $n$-simplices of $\mathfrak{U}$ by $S$ and write $\varGamma = (\varGamma_u)_{u\in S}$ for the family of section functors

$$\varGamma_u := \Gamma_X(|u|, -) : \mathrm{Mod}(X) \longrightarrow \mathrm{Mod}(\mathbf{Z}).$$

Then $\check{\mathrm{C}}^n(\mathfrak{U}, -)$ equals, by definition, the composition of functors

$$\mathrm{Mod}(X) \xrightarrow{\varDelta_S} \mathrm{Mod}(X)^S \xrightarrow{\prod \varGamma} \mathrm{Mod}(\mathbf{Z})^S \xrightarrow{\varPi_S} \mathrm{Mod}(\mathbf{Z}),$$

where $\varDelta_S$ denotes the $S$-fold diagonal functor associated to the category $\mathrm{Mod}(\mathbf{Z})$, $\varPi_S$ denotes the (canonical) $S$-fold product functor of $\mathrm{Mod}(\mathbf{Z})$, and $\prod \varGamma$ denotes the functor which arises as the (external) product of the family of functors $\varGamma$; more elaborately, $\prod \varGamma = \prod_{u\in S} \varGamma_u$.

We formulate a sublemma. Let $\mathcal{C}$ be any category of modules and $(M_i \to N_i \to P_i)_{i\in I}$ a family of triples in $\mathcal{C}$. Then the sequence $\prod M_i \to \prod N_i \to \prod P_i$ (taking the $I$-fold product within $\mathcal{C}$) is exact in $\mathcal{C}$ if and only if, for all $i \in I$, the sequence $M_i \to N_i \to P_i$ is exact $\mathcal{C}$. The proof of this sublemma is clear.

Employing the sublemma in our situation, we obtain that since, for all $u \in S$, the functor $\varGamma_u$ is left exact, the functor $\check{\mathrm{C}}^n(\mathfrak{U}, -)$ is left exact, too. Thus with $t$ being short exact, the triple $\check{\mathrm{C}}^n(\mathfrak{U}, t)$ is left exact. Hence, the triple $\check{\mathrm{C}}^n(\mathfrak{U}, t)$ is short exact if and only if $\check{\mathrm{C}}^n(\mathfrak{U}, H) \to \check{\mathrm{C}}^n(\mathfrak{U}, F) \to 0$ is exact. By the sublemma this is equivalent to saying that $\varGamma_u(H) \to \varGamma_u(F) \to 0$ is exact for all $u \in S$, but $\varGamma_u(H) \to \varGamma_u(F) \to 0$ is exact if and only if $H(|u|) \to F(|u|)$ is surjective.

Item 2. A triple of complexes of modules is short exact if and only if, for all integers $n$, the triple of modules in degree $n$ is short exact. Since the triple of complexes $\check{\mathrm{C}}^\bullet(\mathfrak{U}, t)$ is trivial in negative degrees, we see that $\check{\mathrm{C}}^\bullet(\mathfrak{U}, t)$ is a short exact triple in $\mathrm{Com}(\mathbf{Z})$ if and only if, for all $n \in \mathbf{N}$, the triple $\check{\mathrm{C}}^n(\mathfrak{U}, t)$ is short exact in $\mathrm{Mod}(\mathbf{Z})$, which by item 1 is the case if and only if, for all nonempty, finite subsets $\mathfrak{V} \subset \mathfrak{U}$ with $V := \bigcap \mathfrak{V} \neq \emptyset$, the mapping $H(V) \to F(V)$ is surjective.

Item 3. Let $\mathfrak{V}$ be a nonempty, finite subset of $\mathfrak{U}$ such that $V := \bigcap \mathfrak{V} \neq \emptyset$. Then there exists an element $U$ in $\mathfrak{V}$. By assumption, $\phi(U) : F|_U \to H|_U$ is a morphism of modules on $X|_U$ such that $\beta|_U \circ \phi(U) = \mathrm{id}_{F|_U}$. Thus, given that $V \subset U$, we have $\beta_V \circ \phi(U)_V = \mathrm{id}_{F(V)}$, which entails that $\beta_V : H(V) \to F(V)$ is surjective. Therefore, $\check{\mathrm{C}}^\bullet(\mathfrak{U}, t)$ is a short exact triple in $\mathrm{Com}(\mathbf{Z})$ by means of item 2. □

**Proposition 1.3.9** *Let $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ be a short exact triple of modules on $X$ and $\phi$ a local right splitting of $t$ on $X$. Put $\mathfrak{U} := \mathrm{dom}(\phi)$ and denote by $c$ (resp. $\bar{c}$) the right splitting Čech 1-cochain (resp. right splitting Čech 1-class) of $(t, \phi)$ on $X$. Then the following assertions hold:*

1. *The triple $\check{\mathrm{C}}^\bullet(\mathfrak{U}, \mathscr{H}om(F, t))$ :*

$$\check{\mathrm{C}}^\bullet(\mathfrak{U}, \mathscr{H}om(F, G)) \longrightarrow \check{\mathrm{C}}^\bullet(\mathfrak{U}, \mathscr{H}om(F, H)) \longrightarrow \check{\mathrm{C}}^\bullet(\mathfrak{U}, \mathscr{H}om(F, F)) \tag{1.13}$$

*is a short exact triple in* $\mathrm{Com}(\mathbf{Z})$.

2. *We have* $c \in \check{\mathrm{Z}}^1(\mathfrak{U}, \mathscr{H}om(F,G))$ *and* $\bar{c} \in \check{\mathrm{H}}^1(\mathfrak{U}, \mathscr{H}om(F,G))$.
3. *When* $\delta = (\delta^n)_{n\in\mathbf{Z}}$ *denotes the sequence of connecting homomorphisms associated to the triple* $\check{\mathrm{C}}^\bullet(\mathfrak{U}, \mathscr{H}om(F,t))$ *of complexes over* $\mathrm{Mod}(\mathbf{Z})$ *and* $\bar{e}$ *denotes the image of the identity sheaf map* $\mathrm{id}_F$ *under the canonical function*

$$(\mathscr{H}om(F,F))(X) \longrightarrow \check{\mathrm{H}}^0(\mathfrak{U}, \mathscr{H}om(F,F)),$$

*then* $\delta^0(\bar{e}) = \bar{c}$.

*Proof* Item 1. By Remark 1.3.4, the function on $\mathfrak{U}$ given by the assignment

$$\mathfrak{U} \ni U \longmapsto \mathscr{H}om(F|_U, \phi(U))$$

constitutes a local right splitting of $\mathscr{H}om(F,t)$ on $X$. Moreover, the triple $\mathscr{H}om(F,t)$ is a short exact triple of modules on $X$. Thus $\check{\mathrm{C}}^\bullet(\mathfrak{U}, \mathscr{H}om(F,t))$ is a short exact triple in $\mathrm{Com}(\mathbf{Z})$ by item 3 of Lemma 1.3.8.

Item 2. Observe that since $\phi$ is a local right splitting of $t$ on $X$ and $\mathfrak{U} = \mathrm{dom}(\phi)$, we have $\phi \in \check{\mathrm{C}}^0(\mathfrak{U}, \mathscr{H}om(F,H))$. Furthermore, writing $d = (d^n)_{n\in\mathbf{Z}}$ for the sequence of differentials of the complex $\check{\mathrm{C}}^\bullet(\mathfrak{U}, \mathscr{H}om(F,H))$, the mapping

$$\check{\mathrm{C}}^1(\mathfrak{U}, \mathscr{H}om(F,\alpha)) : \check{\mathrm{C}}^1(\mathscr{H}om(F,G)) \longrightarrow \check{\mathrm{C}}^1(\mathfrak{U}, \mathscr{H}om(F,H))$$

sends $c$ to $d^0(\phi)$ since, for all 1-simplices $u = (u_0, u_1)$ of $\mathfrak{U}$,

$$(d^0(\phi))(u) = \phi(u_1)|_{|u|} - \phi(u_0)|_{|u|}$$

and $c(u)$ is, by definition, the unique morphism $F|_{|u|} \to G|_{|u|}$ such that

$$\alpha|_{|u|} \circ c(u) = \phi(u_1)|_{|u|} - \phi(u_0)|_{|u|}.$$

Denote the sequence of differentials of the complex $\check{\mathrm{C}}^\bullet(\mathfrak{U}, \mathscr{H}om(F,G))$ by $d'' = (d''^n)_{n\in\mathbf{Z}}$. Then we have

$$\check{\mathrm{C}}^2(\mathfrak{U}, \mathscr{H}om(F,\alpha)) \circ d''^1 = d^1 \circ \check{\mathrm{C}}^1(\mathfrak{U}, \mathscr{H}om(F,\alpha)).$$

So, since the mapping $\check{\mathrm{C}}^2(\mathfrak{U}, \mathscr{H}om(F,\alpha))$ is one-to-one and $d^1(d^0(\phi)) = 0$, we see that $d''^1(c) = 0$, which implies that $c \in \check{\mathrm{Z}}^1(\mathfrak{U}, \mathscr{H}om(F,G))$ and, in turn, that $\bar{c} \in \check{\mathrm{H}}^1(\mathfrak{U}, \mathscr{H}om(F,G))$.

Item 3. Write $e$ for the image of $\mathrm{id}_F$ under the canonical function

$$(\mathscr{H}om(F,F))(X) \longrightarrow \check{\mathrm{C}}^0(\mathfrak{U}, \mathscr{H}om(F,F)).$$

Note that $\phi$ gives rise to an element $\tilde{\phi}$ of $\check{C}^0(\mathfrak{U}, \mathscr{H}om(F, H))$. Explicitly, $\tilde{\phi}$ is given requiring $\tilde{\phi}(u) = \phi(u_0)$ for all 0-simplices $u$ of $\mathfrak{U}$; observe that $|u| = u_0$ here. Since, for all $U \in \mathfrak{U}$, we have:

$$\mathscr{H}om(F, \beta)_U(\phi(U)) = \beta|_U \circ \phi(U) = \mathrm{id}_{F|_U} = e(U),$$

we see that $\tilde{\phi}$ is sent to $e$ by the mapping

$$\check{C}^0(\mathfrak{U}, \mathscr{H}om(F, \beta)) : \check{C}^0(\mathfrak{U}, \mathscr{H}om(F, H)) \longrightarrow \check{C}^0(\mathfrak{U}, \mathscr{H}om(F, F)).$$

Combined with the fact that $c$ is sent to $d^0(\tilde{\phi})$ by $\check{C}^1(\mathfrak{U}, \mathscr{H}om(F, \alpha))$, we find that $\delta^0(\bar{e}) = \bar{c}$ by the elementary definition of connecting homomorphisms for short exact triples of complexes of modules. □

**Proposition 1.3.10** *Let $t : G \to H \to F$ be a short exact triple of modules on $X$, $\phi$ a local right splitting of $t$ on $X$, and $\bar{c}$ the right splitting Čech 1-class of $(t, \phi)$ on $X$. Put* $\mathfrak{U} := \mathrm{dom}(\phi)$. *Then the canonical mapping*

$$\check{H}^1(\mathfrak{U}, \mathscr{H}om(F, G)) \longrightarrow H^1(X, \mathscr{H}om(F, G)) \tag{1.14}$$

*sends $\bar{c}$ to the locally split extension class of $t$ on $X$.*

*Proof* By item 1 of Proposition 1.3.9 the triple $\check{C}^\bullet(\mathfrak{U}, \mathscr{H}om(F, t))$ (see Eq. (1.13)) is a short exact triple in $\mathrm{Com}(\mathbf{Z})$. So, denote by $\delta = (\delta^n)_{n \in \mathbf{Z}}$ the associated sequence of connecting homomorphisms. Likewise, denote by $\delta' = (\delta'^n)_{n \in \mathbf{Z}}$ the sequence of connecting homomorphisms for the triple $\mathscr{H}om(F, t)$ with respect to the right derived functor of the functor $\Gamma(X, -) : \mathrm{Mod}(X) \to \mathrm{Mod}(\mathbf{Z})$ (note that this makes sense as $\mathscr{H}om(F, t)$ is a short exact triple of modules on $X$; see Remark 1.3.4). Then the following diagram commutes in $\mathrm{Mod}(\mathbf{Z})$, where the unlabeled arrows stand for the respective canonical morphisms:

$$\begin{tikzcd}
(\mathscr{H}om(F, F))(X) \arrow[r, "\mathrm{id}"] \arrow[d] & \Gamma(X, \mathscr{H}om(F, F)) \arrow[d] \\
\check{H}^0(\mathfrak{U}, \mathscr{H}om(F, F)) \arrow[r] \arrow[d, "\delta^0"'] & H^0(X, \mathscr{H}om(F, F)) \arrow[d, "\delta'^0"] \\
\check{H}^1(\mathfrak{U}, \mathscr{H}om(F, G)) \arrow[r] & H^1(X, \mathscr{H}om(F, G))
\end{tikzcd} \tag{1.15}$$

By item 3 of Proposition 1.3.9 the identity sheaf morphism $\mathrm{id}_F$ is sent to $\bar{c}$ by the composition of the two downwards arrows on the left in Eq. (1.15). Moreover, the identity sheaf morphism $\mathrm{id}_F$ is sent to $\xi(t)$ by the composition of the two downwards arrows on the right in Eq. (1.15); see Construction 1.3.3. Therefore, by the commutativity of the diagram in Eq. (1.15), $\bar{c}$ is sent to $\xi(t)$ by the canonical mapping in Eq. (1.14). □

**Construction 1.3.11** Let $p$ be an integer. Moreover, let $F$ and $G$ be modules on $X$. We define a morphism of modules on $X$,

$$\iota^p_X(F,G) : \mathscr{H}om(F,G) \longrightarrow \mathscr{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F),$$

called *interior product morphism* in degree $p$ for $F$ and $G$ on $X$, as follows: When $p \leq 0$, we define $\iota^p_X(F,G)$ to be the zero morphism (note that we do not actually have a choice here). Assume $p > 0$ now. Let $U$ be an open set of $X$ and $\phi$ an element of $(\mathscr{H}om(F,G))(U)$—that is, a morphism $F|_U \to G|_U$ of modules on $X|_U$. Then there is one, and only one, morphism

$$\psi : (\bigwedge^p F)|_U \longrightarrow (G \otimes \bigwedge^{p-1} F)|_U$$

of modules on $X|_U$ such that for all open sets $V$ of $X|_U$ and all $p$-tuples $(x_0, \ldots, x_{p-1})$ of elements of $F(V)$, we have:

$$\psi_V(x_0 \wedge \cdots \wedge x_{p-1}) = \sum_{\nu<p} (-1)^{\nu-1} \cdot \phi_V(x_\nu) \otimes (x_0 \wedge \cdots \wedge \widehat{x_\nu} \wedge \cdots \wedge x_{p-1}).$$

We let $(\iota^p_X(F,G))_U$ be the function on $(\mathscr{H}om(F,G))(U)$ given by the assignment $\phi \mapsto \psi$, where $\phi$ varies. We let $\iota^p_X(F,G)$ be the function on the set of open sets of $X$ obtained by varying $U$.

Then, as you readily verify, $\iota^p_X(F,G)$ is a morphism of modules on $X$ from $\mathscr{H}om(F,G)$ to $\mathscr{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F)$. As usual, I write $\iota^p(F,G)$ instead of $\iota^p_X(F,G)$ whenever I feel the ringed space $X$ is clear from the context.

**Proposition 1.3.12** *Let $t : G \to H \to F$ be a locally split exact triple of modules on $X$ and $p$ an integer. Then the map*

$$\mathrm{H}^1(X, \iota^p(F,G)) : \mathrm{H}^1(X, \mathscr{H}om(F,G)) \longrightarrow \mathrm{H}^1(X, \mathscr{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F))$$

*sends $\xi(t)$ to $\xi(\Lambda^p(t))$.*

*Proof* First of all, we note that since $t$ is a locally split exact triple of modules on $X$, the triple $t' := \Lambda^p(t)$ is a locally split exact triple of modules on $X$ by item 2 of Proposition 1.3.2, whence it makes sense to speak of $\xi(t')$. When $p \leq 0$, we know that $\mathscr{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F) \cong 0$ in $\mathrm{Mod}(X)$ and thus

$$\mathrm{H}^1(X, \mathscr{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F)) \cong 0$$

in $\mathrm{Mod}(\mathbf{Z})$, so that our assertion is true in this case.

So, from now on, we assume that $p$ is a natural number different from 0. As $t$ is a locally split exact triple of modules on $X$, there exists a local right splitting $\phi$ of $t$ on $X$. Put $\mathfrak{U} := \mathrm{dom}(\phi)$. Let $c \in \check{\mathrm{C}}^1(\mathfrak{U}, \mathscr{H}om(F, G))$ be the right splitting Čech 1-cochain associated to $(t, \phi)$ (see Construction 1.3.7) and denote by $K = (K^i)_{i \in \mathbf{Z}}$ the Koszul filtration in degree $p$ induced by $\alpha := t_{0,1} : G \to H$ on $X$. Define $\phi'$ to be the unique function on $\mathfrak{U}$ such that, for all $U \in \mathfrak{U}$, we have

$$\phi'(U) = \kappa|_U \circ \bigwedge^p(\phi(U)) : (\bigwedge^p F)|_U \longrightarrow ((\bigwedge^p H)/K^2)|_U,$$

where $\kappa$ denotes the quotient morphism of sheaves $\bigwedge^p H \to (\bigwedge^p H)/K^2$. Then $\phi'$ is a local right splitting of $t'$ by item 1 of Proposition 1.3.2. Write $c'$ for the right splitting Čech 1-cochain associated to $(t', \phi')$ and abbreviate $\iota_X^p(F, G)$ to $\iota$. I claim that $c$ is sent to $c'$ by the mapping

$$\check{\mathrm{C}}^1(\mathfrak{U}, \iota) : \check{\mathrm{C}}^1(\mathfrak{U}, \mathscr{H}om(F, G)) \longrightarrow \check{\mathrm{C}}^1(\mathfrak{U}, \mathscr{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F)).$$

In order to check this, let $u$ be a 1-simplex of $\mathfrak{U}$ and $V$ an open set of $X$ which is contained in $|u| = u_0 \cap u_1$. Observe that when $h_0, \dots, h_{p-1}$ are elements of $H(V)$ and $g_0$ and $g_1$ are elements of $G(V)$ such that $h_0 = \alpha_V(g_0)$ and $h_1 = \alpha_V(g_1)$ (specifically, $p > 1$), we have

$$\kappa_V(h_0 \wedge \cdots \wedge h_{p-1}) = 0 \tag{1.16}$$

in $((\bigwedge^p H)/K^2)(V)$ by the definition of the Koszul filtration. Further, observe that writing $\alpha'$ for $t'_{0,1}$ and $\beta$ for $t_{1,2}$, the following diagram commutes in $\mathrm{Mod}(X)$ by the definition of $\alpha'$ in the $\Lambda^p$ construction:

$$\begin{tikzcd} G \otimes \bigwedge^{p-1} H \arrow[r, "\mathrm{id}_G \otimes \bigwedge^{p-1} \beta"] \arrow[d, "\wedge^{1,p-1}(H) \circ (\alpha \otimes \bigwedge^{p-1} \mathrm{id}_H)"'] & G \otimes \bigwedge^{p-1} F \arrow[d, "\alpha'"] \\ \bigwedge^p H \arrow[r, "\kappa"'] & (\bigwedge^p H)/K^2 \end{tikzcd} \tag{1.17}$$

Let $f_0, \dots, f_{p-1}$ be elements of $F(V)$. Then, on the one hand, we have

$$\begin{aligned} \alpha'_V(c'(u)_V(f_0 \wedge \cdots \wedge f_{p-1})) &= (\alpha'|_{|u|} \circ c'(u))_V(f_0 \wedge \cdots \wedge f_{p-1}) \\ &= (\phi'(u_1)|_{|u|} - \phi'(u_0)|_{|u|})_V(f_0 \wedge \cdots \wedge f_{p-1}) \\ &= (\phi'(u_1)_V - \phi'(u_0)_V)(f_0 \wedge \cdots \wedge f_{p-1}) \\ &= \kappa_V\big(\phi(u_1)_V(f_0) \wedge \cdots \wedge \phi(u_1)_V(f_{p-1}) - \phi(u_0)_V(f_0) \wedge \cdots \wedge \phi(u_0)_V(f_{p-1})\big) \end{aligned}$$

$$
\begin{aligned}
&= \kappa_V\Big(\big(\phi(u_0)_V(f_0) + (\phi(u_1)_V - \phi(u_0)_V)(f_0)\big) \wedge \cdots \\
&\quad \wedge \big(\phi(u_0)_V(f_{p-1}) + (\phi(u_1)_V - \phi(u_0)_V)(f_{p-1})\big) \\
&\quad - \phi(u_0)_V(f_0) \wedge \cdots \wedge \phi(u_0)_V(f_{p-1})\Big) \\
&= \kappa_V\Big(\big(\phi(u_0)_V(f_0) + \alpha_V(c(u)_V(f_0))\big) \wedge \cdots \\
&\quad \wedge \big(\phi(u_0)_V(f_{p-1}) + \alpha_V(c(u)_V(f_{p-1}))\big) - \phi(u_0)_V(f_0) \wedge \cdots \wedge \phi(u_0)_V(f_{p-1})\Big) \\
&\overset{(1.16)}{=} \kappa_V\Bigg(\sum_{i<p}(-1)^i \alpha_V(c(u)_V(f_i)) \wedge \phi(u_0)_V(f_0) \wedge \cdots \\
&\quad \wedge \widehat{\phi(u_0)_V(f_i)} \wedge \cdots \wedge \phi(u_0)_V(f_{p-1})\Bigg) \\
&= \Big(\kappa \circ \wedge^{1,p-1}(H) \circ (\alpha \otimes \mathrm{id}_{\bigwedge^{p-1} H})\Big)_V \\
&\quad \left(\sum_{i<p}(-1)^i c(u)_V(f_i) \otimes (\phi(u_0)_V(f_0) \wedge \cdots \wedge \widehat{\phi(u_0)_V(f_i)} \wedge \cdots \wedge \phi(u_0)_V(f_{p-1}))\right) \\
&\overset{(1.17)}{=} \left(\alpha' \circ (\mathrm{id}_G \otimes \bigwedge^{p-1} \beta)\right)_V (\ldots) \\
&= \alpha'_V\left(\sum_{i<p}(-1)^i c(u)_V(f_i) \otimes (f_0 \wedge \cdots \wedge \widehat{f_i} \wedge \cdots \wedge f_{p-1})\right)
\end{aligned}
$$

On the other hand,

$$(\check{\mathrm{C}}^1(\mathfrak{U}, \iota)(c))(u) = \iota_{|u|}(c(u)),$$

meaning that

$$
\begin{aligned}
\Big((\check{\mathrm{C}}^1(\mathfrak{U}, \iota)(c))(u)\Big)_V (f_0 \wedge \cdots \wedge f_{p-1}) &= \big(\iota_{|u|}(c(u))\big)_V (f_0 \wedge \cdots \wedge f_{p-1}) \\
&= \sum_{i<p}(-1)^i (c(u))_V(f_i) \otimes (f_0 \wedge \cdots \wedge \widehat{f_i} \wedge \cdots \wedge f_{p-1}).
\end{aligned}
$$

Thus, using that the function $\alpha'_V$ is injective, we see that $(\check{\mathrm{C}}^1(\mathfrak{U}, \iota)(c))(u)$ and $c'(u)$ agree as sheaf morphisms

$$(\bigwedge^p F)|_{|u|} \longrightarrow (G \otimes \bigwedge^{p-1} F)|_{|u|},$$

whence as elements of $(\mathscr{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F))(|u|)$. In turn, as $u$ was an arbitrary 1-simplex of $\mathfrak{U}$, we have

$$(\check{\mathrm{C}}^1(\mathfrak{U}, \iota))(c) = c' \tag{1.18}$$

as claimed.

Write $t'$ as $t' : G' \to H' \to F'$. Then the following diagram, where the horizontal arrows altogether stand for the respective canonical morphisms, commutes in Mod($\mathbf{Z}$):

$$\begin{tikzcd}
\check{\mathrm{Z}}^1(\mathfrak{U}, \mathscr{H}om(F,G)) \arrow[r] \arrow[d, "\check{\mathrm{Z}}^1(\mathfrak{U},\iota)"'] & \check{\mathrm{H}}^1(\mathfrak{U}, \mathscr{H}om(F,G)) \arrow[r] \arrow[d, "\check{\mathrm{H}}^1(\mathfrak{U},\iota)"'] & \mathrm{H}^1(X, \mathscr{H}om(F,G)) \arrow[d, "\mathrm{H}^1(X,\iota)"] \\
\check{\mathrm{Z}}^1(\mathfrak{U}, \mathscr{H}om(F',G')) \arrow[r] & \check{\mathrm{H}}^1(\mathfrak{U}, \mathscr{H}om(F',G')) \arrow[r] & \mathrm{H}^1(X, \mathscr{H}om(F',G'))
\end{tikzcd} \tag{1.19}$$

By Proposition 1.3.10, we know that $c$ (resp. $c'$) is sent to $\xi(t)$ (resp. $\xi(t')$) by the composition of arrows in the upper (resp. lower) row of the diagram in Eq. (1.19). By Eq. (1.18), we have $(\check{\mathrm{Z}}^1(\mathfrak{U}, \iota))(c) = c'$. Hence,

$$(\mathrm{H}^1(X, \iota))(\xi(t)) = \xi(t')$$

by the commutativity of the diagram in Eq. (1.19). □

## 1.4 Connecting Homomorphisms

Let $f : X \to Y$ be a morphism of ringed spaces, $t$ a locally split exact triple of modules on $X$, and $p$ an integer. In the following, I intend to employ the results of Sect. 1.3—Proposition 1.3.12 specifically—in order to interpret the connecting homomorphisms for the triple $\Lambda^p(t)$ (which is short exact on $X$ by means of Proposition 1.3.2) with respect to the right derived functor of $f_*$.

The first pivotal outcome of this section will be Proposition 1.4.10. Observe that Proposition 1.3.12 (i.e., the upshot of Sect. 1.3) enters the proof of Proposition 1.4.10 via Corollary 1.4.4. The ultimate aim of Sect. 1.4, however, is Proposition 1.4.21, which interprets the connecting homomorphisms for $\Lambda^p(t)$ in terms of a “cup and contraction” with the Kodaira-Spencer class—at least, when the triple $t$ is of the form

$$t : f^*G \longrightarrow H \longrightarrow F$$

where $F$ and $G$ are locally finite free modules on $X$ and $Y$, respectively. The Kodaira-Spencer class I use here (see Construction 1.4.19) presents an abstract prototype of what will later—namely, in Sect. 1.6—become the familiar Kodaira-Spencer class.

To begin with, I introduce a relative version of the notion of a locally split extension class; see Construction 1.3.3.

**Construction 1.4.1** Let $f : X \to Y$ be a morphism of ringed spaces and $t : G \to H \to F$ a short exact triple of modules on $X$ such that the triple

$$\mathscr{H}om(F,t) = \mathscr{H}om(F,-) \circ t : \mathscr{H}om(F,G) \longrightarrow \mathscr{H}om(F,H) \longrightarrow \mathscr{H}om(F,F)$$

is again short exact on $X$. Write

$$\epsilon : \mathscr{O}_Y \longrightarrow f_*(\mathscr{H}om(F,F))$$

for the unique morphism of modules on $Y$ sending the 1 of $\mathscr{O}_Y(Y)$ to the identity sheaf map $\mathrm{id}_F : F \to F$, which is, as you note, an element of $(f_*(\mathscr{H}om(F,F)))\,(Y)$ since

$$(f_*(\mathscr{H}om(F,F)))(Y) = (\mathscr{H}om(F,F))(X) = \mathrm{Hom}(F,F).$$

Then, define $\xi_f(t)$ to be the composition of the following morphisms of modules on $Y$:

$$\mathscr{O}_Y \xrightarrow{\epsilon} f_*(\mathscr{H}om(F,F)) \xrightarrow{\text{can.}} \mathrm{R}^0 f_*(\mathscr{H}om(F,F)) \xrightarrow{\delta^0} \mathrm{R}^1 f_*(\mathscr{H}om(F,G)),$$

where $\delta^0 = \delta^0_f(\mathscr{H}om(F,t))$ denotes the connecting homomorphism in degree 0 for the triple $\mathscr{H}om(F,t)$ with respect to the right derived functor of $f_*$. We call $\xi_f(t)$ the *relative locally split extension class* of $t$ with respect to $f$.

*Remark 1.4.2* Say we are in the situation of Construction 1.4.1; that is, $f : X \to Y$ and $t$ be given. Then Construction 1.3.3 generates an element $\xi_X(t)$—namely, the locally split extension class of $t$ on $X$—in

$$\mathrm{H}^1(X,\mathscr{H}om(F,G)) = (\mathrm{R}^1\Gamma(X,-))(\mathscr{H}om(F,G)).$$

Observe that, in the sense of large functors, we have

$$\Gamma(X,-) = \Gamma(Y,-) \circ f_* : \mathrm{Mod}(X) \longrightarrow \mathrm{Mod}(\mathbf{Z}).$$

Hence, we dispose of a sequence $\tau = (\tau^q)_{q\in\mathbf{Z}}$ of natural transformations,

$$\tau^q : \mathrm{R}^q\Gamma(X,-) = \mathrm{R}^q(\Gamma(Y,-)\circ f_*) \longrightarrow \Gamma(Y,-)\circ \mathrm{R}^q f_*,$$

of functors going from $\mathrm{Mod}(X)$ to $\mathrm{Mod}(\mathbf{Z})$; see Constructions B.1.1 and B.1.3. Specifically, we obtain a mapping

$$\tau^1_{\mathscr{H}om(F,G)} : (\mathrm{R}^1\Gamma(X,-))(\mathscr{H}om(F,G)) \longrightarrow \Gamma(Y, \mathrm{R}^1 f_*(\mathscr{H}om(F,G))).$$

Comparing Constructions 1.3.3 and 1.4.1, you detect that

$$\xi_f(t) : \mathscr{O}_Y \longrightarrow \mathrm{R}^1 f_*(\mathscr{H}om(F,G))$$

is the unique morphism of modules on $Y$ such that the map $(\xi_f(t))_{|Y|}$ sends the 1 of the ring $\mathscr{O}_Y(|Y|)$ to $\xi_X(t)$.

**Proposition 1.4.3** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces and $t : G \to H \to F$ a short exact triple of modules on $X$ such that the triple $\mathscr{H}om(F,t)$ is again short exact on X. Then, setting $h := g \circ f$, the following diagram commutes in* $\mathrm{Mod}(Z)$*:*

$$\begin{tikzcd} \mathscr{O}_Z \arrow[r, "\xi_h(t)"] \arrow[d, "g^\sharp"'] & \mathrm{R}^1 h_*(\mathscr{H}om(F,G)) \arrow[d, "\mathrm{BC}^1"] \\ g_*(\mathscr{O}_Y) \arrow[r, "g_*(\xi_f(t))"'] & g_*\left(\mathrm{R}^1 f_*(\mathscr{H}om(F,G))\right) \end{tikzcd} \tag{1.20}$$

*Proof* Write $\epsilon : \mathscr{O}_Y \to f_*(\mathscr{H}om(F,F))$ for the unique morphism of modules on $Y$ sending the 1 of the ring $\mathscr{O}_Y(Y)$ to the identity sheaf map $\mathrm{id}_F$. Similarly, write $\zeta : \mathscr{O}_Z \to h_*(\mathscr{H}om(F,F))$ for the unique morphism of modules on $Z$ sending the 1 of the ring $\mathscr{O}_Z(Z)$ to the identity sheaf map $\mathrm{id}_F$. Then, clearly, the following diagram commutes in $\mathrm{Mod}(Z)$:

$$\begin{tikzcd} \mathscr{O}_Z \arrow[r, "\zeta"] \arrow[d, "g^\sharp"'] & h_*(\mathscr{H}om(F,F)) \arrow[d, "\mathrm{id}"] \\ g_*\mathscr{O}_Y \arrow[r, "g_*(\epsilon)"'] & g_* f_*(\mathscr{H}om(F,F)) \end{tikzcd}$$

Denote by $\delta^0$ and $\delta'^0$ the 0th connecting homomorphisms for the triple $\mathscr{H}om(F,t)$ with respect to the derived functors of $f_*$ and $h_*$, respectively. Then by the compatibility of the base change morphisms with the connecting homomorphisms and the compatibility of the base change morphisms in degree 0 with the natural transformations $h_* \to \mathrm{R}^0 h_*$ and $f_* \to \mathrm{R}^0 f_*$ of functors from, respectively, $\mathrm{Mod}(X)$

to Mod($Z$) and Mod($X$) to Mod($Y$), we see that the following diagram commutes in Mod($Z$):

$$\begin{tikzcd}
h_*(\mathscr{H}om(F,F)) \arrow[r, "\text{can.}"] \arrow[d, "\text{id}"'] & \mathrm{R}^0 h_*(\mathscr{H}om(F,F)) \arrow[r, "\delta'^0"] \arrow[d, "\mathrm{BC}^0"'] & \mathrm{R}^1 h_*(\mathscr{H}om(F,G)) \arrow[d, "\mathrm{BC}^1"] \\
g_* f_*(\mathscr{H}om(F,F)) \arrow[r, "g_*(\text{can.})"'] & g_*\mathrm{R}^0 f_*(\mathscr{H}om(F,F)) \arrow[r, "g_*(\delta^0)"'] & g_*\mathrm{R}^1 f_*(\mathscr{H}om(F,F))
\end{tikzcd}$$

Now, the commutativity of the diagram in Eq. (1.20) follows readily taking into account the definitions of $\xi_f(t)$ and $\xi_h(t)$; see Construction 1.4.1. □

**Corollary 1.4.4** *Let $f : X \to Y$ be a morphism of ringed spaces, $t : G \to H \to F$ a locally split exact triple of modules on X, and p an integer. Set $t' := \Lambda^p(t)$ and write $t'$ as $t' : G' \to H' \to F'$. Then the following diagram commutes in* Mod($Y$)*:*

$$\begin{tikzcd}
 & \mathscr{O}_Y \arrow[dl, "\xi_f(t)"'] \arrow[dr, "\xi_f(t')"] & \\
\mathrm{R}^1 f_*(\mathscr{H}om(F,G)) \arrow[rr, "\mathrm{R}^1 f_*(\iota^p(F,G))"'] & & \mathrm{R}^1 f_*(\mathscr{H}om(F',G'))
\end{tikzcd} \tag{1.21}$$

*Proof* Define $Z$ to be the distinguished terminal ringed space. Then there exists a unique morphism $g$ from $Y$ to $Z$. The composition $h := g \circ f$ is the unique morphism from $X$ to $Z$. Thus the commutativity of the diagram in Eq. (1.21) follows from Proposition 1.4.3 (applied twice, once for $t$, once for $t'$) in conjunction with Proposition 1.3.12. □

Many results of this section rely, in their formulation and proof, on the device of the cup product for higher direct image sheaves. For that matter, I curtly review this concept and state several of its properties.

**Construction 1.4.5** Let $f : X \to Y$ be a morphism of ringed spaces and $p$ and $q$ integers. Let $F$ and $G$ be modules on $X$. Then we denote by

$$\smile_f^{p,q}(F,G) : \mathrm{R}^p f_*(F) \otimes_Y \mathrm{R}^q f_*(G) \longrightarrow \mathrm{R}^{p+q} f_*(F \otimes_X G)$$

the *cup product morphism* in bidegree $(p,q)$ relative $f$ for $F$ and $G$.

For the definition of the cup product I suggest considering the Godement resolutions $\alpha : F \to L$ and $\beta : G \to M$ of $F$ and $G$, respectively, on $X$. Besides, let $\rho_F : F \to I_F$, $\rho_G : G \to I_G$, and $\rho_{F\otimes G} : F \otimes G \to I_{F\otimes G}$ be the canonical injective resolutions of $F$, $G$, and $F \otimes G$, respectively, on $X$. Then by Lemma A.3.6, there exists one, and only one, morphism $\zeta : L \to I_F$ (resp. $\eta : M \to I_G$) in K($X$) such that we have $\zeta \circ \alpha = \rho_F$ (resp. $\eta \circ \beta = \rho_G$) in K($X$). Since the Godement resolutions are flasque, whence acyclic for the functor $f_*$, we see that

$$\mathrm{H}^p(f_*\zeta) : \mathrm{H}^p(f_*L) \longrightarrow \mathrm{H}^p(f_*I_F),$$
$$\mathrm{H}^q(f_*\eta) : \mathrm{H}^q(f_*M) \longrightarrow \mathrm{H}^q(f_*I_G)$$

are isomorphisms in $\mathrm{Mod}(Y)$. Thus, we derive an isomorphism

$$\mathrm{H}^p(f_*L) \otimes \mathrm{H}^q(f_*M) \longrightarrow \mathrm{H}^p(f_*I_F) \otimes \mathrm{H}^q(f_*I_G). \tag{1.22}$$

Moreover, since the Godement resolutions are pointwise homotopically trivial,

$$\alpha \otimes \beta : F \otimes G \longrightarrow L \otimes M$$

is a resolution of $F \otimes G$ on $X$. So, again by Lemma A.3.6, there exists one, and only one, morphism $\theta : L \otimes M \to I_{F\otimes G}$ in $\mathrm{K}(X)$ such that we have $\theta \circ (\alpha \otimes \beta) = \rho_{F\otimes G}$ in $\mathrm{K}(X)$. Thus, by the compatibility of $f_*$ with the respective tensor products on $X$ and $Y$, we obtain the composition

$$f_*L \otimes f_*M \longrightarrow f_*(L \otimes M) \xrightarrow{f_*\theta} f_*I_{F\otimes G}$$

in $\mathrm{K}^+(Y)$, which in turn yields a composition

$$\mathrm{H}^p(f_*L) \otimes \mathrm{H}^q(f_*M) \longrightarrow \mathrm{H}^{p+q}(f_*L \otimes f_*M) \longrightarrow \mathrm{H}^{p+q}(f_*I_{F\otimes G}) \tag{1.23}$$

in $\mathrm{Mod}(Y)$. Now the composition of the inverse of Eq. (1.22) with the morphism in Eq. (1.23) is the cup product $\smile_f^{p,q}(F, G)$.

Note that the above construction is principally due to Godement [5, II, 6.6], although Godement restricts himself to applying global section functors (with supports) instead of the more general direct image functors. Also note that Grothendieck [8, (12.2.2)] defines his cup product in the relative situation $f : X \to Y$ by localizing Godement's construction over the base. Our cup product here agrees with Grothendieck's (though I won't prove this).

**Proposition 1.4.6** *Let $f : X \to Y$ be a morphism of ringed spaces and $p$, $q$, and $r$ integers.*

1. (Naturality) $\smile_f^{p,q}$ *is a natural transformation*

$$\smile_f^{p,q} : (- \otimes_Y -) \circ (\mathrm{R}^p f_* \times \mathrm{R}^q f_*) \longrightarrow \mathrm{R}^{p+q} f_* \circ (- \otimes_X -)$$

   *of functors from* $\mathrm{Mod}(X) \times \mathrm{Mod}(X)$ *to* $\mathrm{Mod}(Y)$.
2. (Connecting homomorphisms) *Let $t : F'' \to F \to F'$ be a short exact triple of modules on $X$ and $G$ a module on $X$ such that*

$$t \otimes G : F'' \otimes G \longrightarrow F \otimes G \longrightarrow F' \otimes G$$

*is again a short exact triple of modules on $X$. Then the following diagram commutes in* $\mathrm{Mod}(Y)$*:*

$$\begin{tikzcd}
\mathrm{R}^p f_*(F') \otimes \mathrm{R}^q f_*(G) \arrow[r, "\smile^{p,q}(F',G)"] \arrow[d, "\delta_f^p(t)\otimes \mathrm{R}^q f_*(G)"'] & \mathrm{R}^{p+q} f_*(F' \otimes G) \arrow[d, "\delta_f^{p+q}(t\otimes G)"] \\
\mathrm{R}^{p+1} f_*(F'') \otimes \mathrm{R}^q f_*(G) \arrow[r, "\smile^{p+1,q}(F'',G)"'] & \mathrm{R}^{p+q+1} f_*(F'' \otimes G)
\end{tikzcd}$$

3. (Units) *Let $G$ be a module on $X$. Then the following diagram commutes in* $\mathrm{Mod}(Y)$*, where $\phi$ denotes the canonical morphism of sheaves on $Y_{\mathrm{top}}$ from $f_*\mathscr{O}_X$ to* $\mathrm{R}^0 f_*(\mathscr{O}_X)$*:*

$$\begin{tikzcd}
\mathscr{O}_Y \otimes_Y \mathrm{R}^q f_*(G) \arrow[r, "\lambda_Y(\mathrm{R}^q f_*(G))"] \arrow[d, "(\phi\circ f^\sharp)\otimes \mathrm{id}"'] & \mathrm{R}^q f_*(G) \\
\mathrm{R}^0 f_*(\mathscr{O}_X) \otimes_Y \mathrm{R}^q f_*(G) \arrow[r, "\smile^{0,q}(\mathscr{O}_X,G)"'] & \mathrm{R}^q f_*(\mathscr{O}_X \otimes_X G) \arrow[u, "\mathrm{R}^q f_*(\lambda_X(G))"']
\end{tikzcd}$$

4. (Associativity) *Let $F$, $G$, and $H$ be modules on $X$. Then the following diagram commutes in* $\mathrm{Mod}(Y)$*:*

$$\begin{tikzcd}
\mathrm{R}^{p+q} f_*(F \otimes G) \otimes \mathrm{R}^r f_*(H) \arrow[r, "\smile^{p+q,r}(F\otimes G,H)"] & \mathrm{R}^{p+q+r} f_*((F \otimes G) \otimes H) \arrow[d, "\mathrm{R}^{p+q+r} f_*(\alpha_X)"] \\
(\mathrm{R}^p f_*(F) \otimes \mathrm{R}^q f_*(G)) \otimes \mathrm{R}^r f_*(H) \arrow[u, "\smile^{p+q}(F,G)\otimes \mathrm{id}"] \arrow[d, "\alpha_Y"'] & \mathrm{R}^{p+q+r} f_*(F \otimes (G \otimes H)) \\
\mathrm{R}^p f_*(F) \otimes (\mathrm{R}^q f_*(G) \otimes \mathrm{R}^r f_*(H)) \arrow[r, "\mathrm{id}\otimes \smile^{q,r}(G,H)"'] & \mathrm{R}^p f_*(F) \otimes \mathrm{R}^{q+r} f_*(G \otimes H) \arrow[u, "\smile^{p,q+r}(F,G\otimes H)"']
\end{tikzcd}$$

*Proof* I refrain from giving details here. Instead, let me refer you to Godement's summary of properties of the *cross* product [5, II, 6.5] and let me remark that these properties carry over to the *cup* product almost word by word—as, by the way, Godement [5, II, 6.6] himself points out. □

**Construction 1.4.7** Let $X$ be a ringed space. Let $F$ and $G$ be modules on $X$. Then we write

$$\epsilon_X(F, G) : \mathscr{H}om(F, G) \otimes F \longrightarrow G$$

for the familiar *evaluation morphism.*

When $U$ is an open set of $X$, and $\phi : F|_U \to G|_U$ is a morphism of sheaves of modules on $X|_U$ (i.e., an element of $(\mathscr{H}om(F,G))(U)$), and $s \in F(U)$, then

$$(\epsilon_X(F,G))_U(\phi \otimes s) = \phi_U(s).$$

Varying $G$, we may view $\epsilon_X(F,-)$ as a function on the class of modules on $X$. That way, $\epsilon_X(F,-)$ is a natural transformation

$$\epsilon_X(F,-) : (- \otimes F) \circ \mathscr{H}om(F,-) \longrightarrow \mathrm{id}_{\mathrm{Mod}(X)}$$

of endofunctors on $\mathrm{Mod}(X)$. We will write $\epsilon$ instead of $\epsilon_X$ when we feel that the ringed space $X$ is clear from the context.

**Proposition 1.4.8** *Let $f : X \to Y$ be a morphism of ringed spaces, $t : G \to H \to F$ a locally split exact triple of modules on $X$, and $q$ an integer. Then the following diagram commutes in* $\mathrm{Mod}(Y)$*:*

$$\begin{tikzcd} \mathrm{R}^q f_*(F) \arrow[r, "\delta_f^q(t)"] \arrow[d, "\xi_f(t)\otimes \mathrm{id}_{\mathrm{R}^q f_*(F)}"'] & \mathrm{R}^{q+1} f_*(G) \\ \mathrm{R}^1 f_*(\mathscr{H}om(F,G)) \otimes \mathrm{R}^q f_*(F) \arrow[r, "\smile^{1,q}(\mathscr{H}om(F,G),F)"'] & \mathrm{R}^{q+1} f_*(\mathscr{H}om(F,G)\otimes F) \arrow[u, "\mathrm{R}^{q+1} f_*(\epsilon(F,G))"'] \end{tikzcd} \tag{1.24}$$

*Proof* Since $t$ is a locally split exact triple of modules on $X$, we know that the triples

$$\mathscr{H}om(F,t) : \mathscr{H}om(F,G) \longrightarrow \mathscr{H}om(F,H) \longrightarrow \mathscr{H}om(F,F)$$

as well as

$$\mathscr{H}om(F,t) \otimes F : \mathscr{H}om(F,G) \otimes F \longrightarrow \mathscr{H}om(F,H) \otimes F \longrightarrow \mathscr{H}om(F,F) \otimes F$$

are locally split exact triples of modules on $X$; see Remark 1.3.4. Thus, by item 2 of Proposition 1.4.6 the following diagram commutes in $\mathrm{Mod}(Y)$:

$$\begin{tikzcd} \mathrm{R}^0 f_*(\mathscr{H}om(F,F)) \otimes \mathrm{R}^q f_*(F) \arrow[r, "\smile^{0,q}(\mathscr{H}om(F,F),F)"] \arrow[d, "\delta_f^0(\mathscr{H}om(F,t))\otimes \mathrm{R}^q f_*(F)"'] & \mathrm{R}^q f_*(\mathscr{H}om(F,F)\otimes F) \arrow[d, "\delta_f^q(\mathscr{H}om(F,t)\otimes F)"] \\ \mathrm{R}^1 f_*(\mathscr{H}om(F,G)) \otimes \mathrm{R}^q f_*(F) \arrow[r, "\smile^{1,q}(\mathscr{H}om(F,G),F)"'] & \mathrm{R}^{q+1} f_*(\mathscr{H}om(F,G)\otimes F) \end{tikzcd} \tag{1.25}$$

By the naturality of the evaluation morphism (see Construction 1.4.7) the composition $\epsilon(F,-) \circ t_0$—recall that $t_0$ denotes the object function of the functor $t$—is a morphism

$$\epsilon(F,-) \circ t_0 : \mathscr{H}om(F,t) \otimes F \longrightarrow t$$

of triples of modules on $X$ (i.e., a natural transformation of functors from **3** to $\mathrm{Mod}(X)$). In consequence, by the naturality of $\delta_f^q$, the following diagram commutes in $\mathrm{Mod}(Y)$:

$$\begin{tikzcd}
\mathrm{R}^q f_*(\mathscr{H}om(F,F) \otimes F) \arrow[r, "\mathrm{R}^q f_*(\epsilon(F,F))"] \arrow[d, "\delta_f^q(\mathscr{H}om(F,t)\otimes F)"'] & \mathrm{R}^q f_*(F) \arrow[d, "\delta_f^q(t)"] \\
\mathrm{R}^{q+1} f_*(\mathscr{H}om(F,G) \otimes F) \arrow[r, "\mathrm{R}^{q+1} f_*(\epsilon(F,G))"'] & \mathrm{R}^{q+1} f_*(G)
\end{tikzcd} \tag{1.26}$$

Denote by $\phi$ the composition

$$\mathscr{O}_Y \longrightarrow f_*(\mathscr{H}om(F,F)) \xrightarrow{\text{can.}} \mathrm{R}^0 f_*(\mathscr{H}om(F,F))$$

of morphisms in $\mathrm{Mod}(Y)$, where the first arrow stands for the unique morphism of modules on $Y$ which sends the 1 of $\mathscr{O}_Y(Y)$ to the identity sheaf map $\mathrm{id}_F$ in

$$(f_*(\mathscr{H}om(F,F)))\,(Y) = \mathrm{Hom}(F,F).$$

Then from the commutativity of the diagrams in Eqs. (1.25) and (1.26) as well as from the definition of $\xi_f(t)$ (see Construction 1.4.1), we deduce that

$$\begin{aligned}
&\mathrm{R}^{q+1} f_*(\epsilon(F,G)) \circ \overset{1,q}{\smile}(\mathscr{H}om(F,G),F) \circ (\xi_f(t) \otimes \mathrm{id}_{\mathrm{R}^q f_*(F)}) \\
&\quad = \delta_f^q(t) \circ \mathrm{R}^q f_*(\epsilon(F,F)) \circ \overset{0,q}{\smile}(\mathscr{H}om(F,F),F) \circ (\phi \otimes \mathrm{id}_{\mathrm{R}^q f_*(F)}) \circ \lambda(\mathrm{R}^q f_*(F))^{-1}
\end{aligned}$$

in $\mathrm{Mod}(Y)$. In addition, using item 3 of Proposition 1.4.6, you show that

$$\mathrm{R}^q f_*(\epsilon(F,F)) \circ \overset{0,q}{\smile}(\mathscr{H}om(F,F),F) \circ (\phi \otimes \mathrm{id}_{\mathrm{R}^q f_*(F)}) = \lambda(\mathrm{R}^q f_*(F)).$$

Hence, we see that the diagram in Eq. (1.24) commutes in $\mathrm{Mod}(Y)$. □

**Construction 1.4.9** Let $X$ be a ringed space. Let $p$ be an integer and $F$ and $G$ modules on $X$. Then by means of the adjunction between the functors $- \otimes \bigwedge^p F$

and $\mathscr{H}om(\bigwedge^p F, -)$, both going from $\mathrm{Mod}(X)$ to $\mathrm{Mod}(X)$, the interior product morphism

$$\iota^p_X(F,G) : \mathscr{H}om(F,G) \longrightarrow \mathscr{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F)$$

of Construction 1.3.11 corresponds to a morphism

$$\tilde{\iota}^p_X(F,G) : \mathscr{H}om(F,G) \otimes \bigwedge^p F \longrightarrow G \otimes \bigwedge^{p-1} F$$

of modules on $X$. The latter, I christen the *adjoint interior product* in degree $p$ for $F$ and $G$ on $X$. Explicitly, this means that $\tilde{\iota}^p_X(F,G)$ equals the composition

$$\epsilon(\bigwedge^p F, G \otimes \bigwedge^{p-1} F) \circ (\iota^p_X(F,G) \otimes \mathrm{id}_{\bigwedge^p F}) :$$
$$\mathscr{H}om(F,G) \otimes \bigwedge^p F \longrightarrow \mathscr{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F) \otimes \bigwedge^p F \longrightarrow G \otimes \bigwedge^{p-1} F.$$

Just as I did with $\iota^p_X(F,G)$, among others, I omit the subscript "$X$" in expressions like $\tilde{\iota}^p_X(F,G)$ whenever I feel this is expedient.

**Proposition 1.4.10** *Let $f : X \to Y$ be a morphism of ringed spaces, $t : G \to H \to F$ a locally split exact triple of modules on $X$, and $p$ and $q$ integers. Then the following diagram commutes in* $\mathrm{Mod}(Y)$*:*

$$\begin{tikzcd}
\mathrm{R}^q f_*(\bigwedge^p F) \arrow[r, "\delta^q_f(\Lambda^p(t))"] \arrow[d, "\xi_f(t) \otimes \mathrm{id}_{\mathrm{R}^q f_*(\bigwedge^p F)}"'] & \mathrm{R}^{q+1} f_*(G \otimes \bigwedge^{p-1} F) \\
\mathrm{R}^1 f_*(\mathscr{H}om(F,G)) \otimes \mathrm{R}^q f_*(\bigwedge^p F) \arrow[r, "\smile^{1,q}(\mathscr{H}om(F,G), \bigwedge^p F)"'] & \mathrm{R}^{q+1} f_*(\mathscr{H}om(F,G) \otimes \bigwedge^p F) \arrow[u, "\mathrm{R}^{q+1} f_*(\tilde{\iota}^p(F,G))"']
\end{tikzcd} \tag{1.27}$$

*Proof* Set $t' := \Lambda^p(t)$ and write $t'$ as

$$t' : G' \longrightarrow H' \longrightarrow F'.$$

By the definition of $\tilde{\iota}^p(F,G)$ via tensor-hom adjunction (see Construction 1.4.9) we know that

$$\tilde{\iota}^p(F,G) = \epsilon(F',G') \circ (\iota^p(F,G) \otimes \mathrm{id}_{F'}) \tag{1.28}$$

holds in $\mathrm{Mod}(X)$. Due to the naturality of the cup product relative $f$ (see item 1 of Proposition 1.4.6), the following diagram commutes in $\mathrm{Mod}(Y)$:

$$\begin{tikzcd}
\mathrm{R}^1f_*(\mathscr{H}om(F,G)) \otimes_Y \mathrm{R}^qf_*(F') \arrow[r, "\smile^{1,q}(\mathscr{H}om(F,G),F')"] \arrow[d, "\mathrm{R}^1f_*(\iota^p(F,G))\otimes \mathrm{R}^qf_*(\mathrm{id}_{F'})"'] & \mathrm{R}^{q+1}f_*(\mathscr{H}om(F,G) \otimes_X F') \arrow[d, "\mathrm{R}^{q+1}f_*(\iota^p(F,G)\otimes \mathrm{id}_{F'})"] \\
\mathrm{R}^1f_*(\mathscr{H}om(F',G')) \otimes_Y \mathrm{R}^qf_*(F') \arrow[r, "\smile^{1,q}(\mathscr{H}om(F',G'),F')"'] & \mathrm{R}^{q+1}f_*(\mathscr{H}om(F',G') \otimes_X F')
\end{tikzcd} \tag{1.29}$$

By Proposition 1.3.2, we know that since $t$ is a locally split exact triple of modules on $X$, the triple $t'$ is locally split exact on $X$, too. Thus, it makes sense to speak of the relative locally split extension class of $t'$ with respect to $f$; see Construction 1.4.1. By Corollary 1.4.4, the following diagram commutes in $\mathrm{Mod}(Y)$:

$$\begin{tikzcd}
& \mathscr{O}_Y \arrow[dl, "\xi_f(t)"'] \arrow[dr, "\xi_f(t')"] & \\
\mathrm{R}^1f_*(\mathscr{H}om(F,G)) \arrow[rr, "\mathrm{R}^1f_*(\iota^p(F,G))"'] & & \mathrm{R}^1f_*(\mathscr{H}om(F',G'))
\end{tikzcd} \tag{1.30}$$

By Proposition 1.4.8, the following diagram commutes in $\mathrm{Mod}(Y)$:

$$\begin{tikzcd}
\mathrm{R}^qf_*(F') \arrow[r, "\delta^q(t')"] \arrow[d, "\xi_f(t')\otimes \mathrm{id}_{\mathrm{R}^qf_*(F')}"'] & \mathrm{R}^{q+1}f_*(G') \\
\mathrm{R}^1f_*(\mathscr{H}om(F',G')) \otimes \mathrm{R}^qf_*(F') \arrow[r, "\smile^{1,q}(\mathscr{H}om(F',G'),F')"'] & \mathrm{R}^{q+1}f_*(\mathscr{H}om(F',G') \otimes F') \arrow[u, "\mathrm{R}^{q+1}f_*(\epsilon(F',G'))"']
\end{tikzcd} \tag{1.31}$$

All in all, we obtain

$$\begin{aligned}
\delta^q(t') &\overset{(1.31)}{=} \mathrm{R}^{q+1}f_*(\epsilon(F',G')) \circ \overset{1,q}{\smile}(\mathscr{H}om(F',G'),F') \circ (\xi_f(t') \otimes \mathrm{id}_{\mathrm{R}^qf_*(F')}) \\
&\overset{(1.30)}{=} \mathrm{R}^{q+1}f_*(\epsilon(F',G')) \circ \overset{1,q}{\smile}(\mathscr{H}om(F',G'),F') \\
&\qquad \circ (\mathrm{R}^1f_*(\iota^p(F,G)) \otimes \mathrm{R}^qf_*(\mathrm{id}_{F'})) \circ (\xi_f(t) \otimes \mathrm{id}_{\mathrm{R}^qf_*(F')})
\end{aligned}$$

$$\overset{(1.29)}{=} \mathrm{R}^{q+1}f_*(\epsilon(F', G')) \circ \mathrm{R}^{q+1}f_*(\iota^p(F, G) \otimes \mathrm{id}_{F'})$$

$$\circ \overset{1,q}{\smile}(\mathscr{H}om(F, G), F') \circ (\xi_f(t) \otimes \mathrm{id}_{\mathrm{R}^q f_*(F')})$$

$$\overset{(1.28)}{=} \mathrm{R}^{q+1}f_*(\tilde{\iota}^p(F, G)) \circ \overset{1,q}{\smile}(\mathscr{H}om(F, G), F') \circ (\xi_f(t) \otimes \mathrm{id}_{\mathrm{R}^q f_*(F')}).$$

This yields precisely the commutativity, in $\mathrm{Mod}(Y)$, of the diagram in Eq. (1.27). □

**Construction 1.4.11** Let $X$ be a ringed space, $F$ a module on $X$. We set

$$\gamma_X^p(F) := \lambda_X\left(\bigwedge^{p-1} F\right) \circ \tilde{\iota}_X^p(F, \mathscr{O}_X) : F^\vee \otimes \bigwedge^{p} F \longrightarrow \bigwedge^{p-1} F,$$

where we view $\mathscr{O}_X$ as a module on $X$. We call $\gamma_X^p(F)$ the *contraction morphism* in degree $p$ for $F$ on $X$.

**Construction 1.4.12** Let $X$ be a ringed space. Moreover, let $F$ and $G$ be modules on $X$. We define a morphism

$$\mu_X(F, G) : G \otimes F^\vee \longrightarrow \mathscr{H}om(F, G)$$

of modules on $X$ by requiring that, for all open sets $U$ of $X$, all $\theta \in (F^\vee)(U)$, and all $y \in G(U)$, the function $(\mu_X(F, G))_U$ send $y \otimes \theta \in (G \otimes F^\vee)(U)$ to the composition

$$\psi \circ \theta : F|_U \longrightarrow \mathscr{O}_X|_U \longrightarrow G|_U$$

of morphisms of modules on $X|_U$, where $\psi$ denotes the unique morphism of modules on $X|_U$ from $\mathscr{O}_X|_U$ to $G|_U$ mapping the 1 of $(\mathscr{O}_X|_U)(U)$ to $y \in G(U)$.

It is an easy matter to check that one, and only one, such morphism $\mu_X(F, G)$ exists. When the ringed space $X$ is clear from the context, we shall occasionally write $\mu$ instead of $\mu_X$.

**Proposition 1.4.13** *Let $X$ be a ringed space, $p$ an integer, and $F$ and $G$ modules on $X$. Then the following diagram commutes in* $\mathrm{Mod}(X)$*:*

$$\begin{tikzcd}
(G \otimes F^\vee) \otimes \bigwedge^p F \arrow[r, "\mu(F,G)\otimes \mathrm{id}_{\bigwedge^p F}"] \arrow[d, "\alpha(G,F^\vee,\bigwedge^p F)"'] & \mathscr{H}om(F, G) \otimes \bigwedge^p F \arrow[d, "\tilde{\iota}^p(F,G)"] \\
G \otimes (F^\vee \otimes \bigwedge^p F) \arrow[r, "\mathrm{id}_G \otimes \gamma^p(F)"'] & G \otimes \bigwedge^{p-1} F
\end{tikzcd}$$

*Proof* For $p \leq 0$ the assertion is clear since $G \otimes \bigwedge^{p-1} F \cong 0$ in $\mathrm{Mod}(X)$. So, assume that $p > 0$. Then for all open sets $U$ of $X$, all $p$-tuples $x = (x_0, \ldots, x_{p-1})$ of elements of $F(U)$, all morphisms $\theta : F|_U \to \mathscr{O}_X|_U$ of modules on $X|_U$ (i.e., $\theta \in (F^\vee)(U)$), and all elements $y \in G(U)$, you verify easily, given the definitions of $\mu$, $\tilde{\iota}^p$, and $\gamma^p$, that

$$(y \otimes \theta) \otimes (x_0 \wedge \cdots \wedge x_{p-1}) \in ((G \otimes F^\vee) \otimes \bigwedge^p F)(U)$$

is mapped to one and the same element of $(G \otimes \bigwedge^{p-1} F)(U)$ by either of the two paths from the upper left to the lower right corner in the above diagram. Therefore, the diagram commutes in $\mathrm{Mod}(X)$ by the universal property of the sheaf associated to a presheaf. □

**Construction 1.4.14** Let $n$ be an integer and $f : X \to Y$ a morphism of ringed spaces. Moreover, let $F$ and $G$ be modules on $X$ and $Y$, respectively. Then we define the $n$th *projection morphism* relative $f$ for $F$ and $G$, denoted

$$\pi_f^n(G, F) : G \otimes \mathrm{R}^n f_*(F) \longrightarrow \mathrm{R}^n f_*(f^* G \otimes F),$$

to be the morphism of modules on $Y$ which is obtained by first going along the composition

$$G \longrightarrow f_*(f^* G) \xrightarrow{\text{can.}} \mathrm{R}^0 f_*(f^* G)$$

(here the first arrow stands for the familiar adjunction morphism for $G$ with respect to $f$), tensored on the right with the identity of $\mathrm{R}^n f_*(F)$, and then applying the cup product morphism

$$\underset{f}{\overset{0,n}{\smile}}(f^* G, F) : \mathrm{R}^0 f_*(f^* G) \otimes \mathrm{R}^n f_*(F) \longrightarrow \mathrm{R}^n f_*(f^* G \otimes F).$$

Observe that this construction is suggested by Grothendieck [8, (12.2.3)].

Letting $F$ and $G$ vary, we may view $\pi_f^n$ as a function defined on the class of objects of the product category $\mathrm{Mod}(Y) \times \mathrm{Mod}(X)$. That way, it follows essentially from item 1 of Proposition 1.4.6—that is, the naturality of the cup product—that $\pi_f^n$ is a natural transformation

$$(- \otimes_Y -) \circ (\mathrm{id}_{\mathrm{Mod}(Y)} \times \mathrm{R}^n f_*) \longrightarrow \mathrm{R}^n f_* \circ (- \otimes_X -) \circ (f^* \times \mathrm{id}_{\mathrm{Mod}(X)})$$

of functors from $\mathrm{Mod}(Y) \times \mathrm{Mod}(X)$ to $\mathrm{Mod}(Y)$. As usual, I will write $\pi^n$ instead of $\pi_f^n$ when I think this is appropriate.

**Proposition 1.4.15** *Let $n$ be an integer, $f : X \to Y$ a morphism of ringed spaces, $F$ a module on $X$, and $G$ a locally finite free module on $Y$. Then the projection morphism*

$$\pi_f^n(G, F) : G \otimes \mathrm{R}^n f_*(F) \longrightarrow \mathrm{R}^n f_*(f^* G \otimes F)$$

*is an isomorphism in* $\mathrm{Mod}(Y)$.

*Proof* See [8, (12.2.3)]. □

**Proposition 1.4.16** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $q$ and $q'$ be integers, $F$ and $F'$ modules on $X$, and $G$ a module on $Y$. Then the following diagram commutes in* $\mathrm{Mod}(Y)$*:*

$$\begin{tikzcd}
\mathrm{R}^q f_*(f^*G \otimes F) \otimes \mathrm{R}^{q'} f_*(F') \arrow[r, "\smile_f^{q,q'}(f^*G\otimes F, F')"] & \mathrm{R}^{q+q'} f_*((f^*G \otimes F) \otimes F') \arrow[d, "\mathrm{R}^{q+q'}f_*(\alpha_X(f^*G,F,F'))"] \\
(G \otimes \mathrm{R}^q f_*(F)) \otimes \mathrm{R}^{q'} f_*(F') \arrow[u, "\pi_f^q(G,F)\otimes \mathrm{id}_{\mathrm{R}^{q'}f_*(F')}"] \arrow[d, "\alpha_Y(G,\mathrm{R}^q f_*(F),\mathrm{R}^{q'}f_*(F'))"'] & \mathrm{R}^{q+q'} f_*(f^*G \otimes (F \otimes F')) \\
G \otimes (\mathrm{R}^q f_*(F) \otimes \mathrm{R}^{q'} f_*(F')) \arrow[r, "\mathrm{id}_G \otimes \smile_f^{q,q'}(F,F')"'] & G \otimes \mathrm{R}^{q+q'} f_*(F \otimes F') \arrow[u, "\pi_f^{q+q'}(G,F\otimes F')"']
\end{tikzcd}$$

*Proof* This follows with ease from the associativity of the cup product as stated in item 4 of Proposition 1.4.6. □

*Remark 1.4.17* Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories and $S : \mathcal{C} \to \mathcal{E}$ and $T : \mathcal{D} \to \mathcal{E}$ functors. Then, the *fiber product category* of $\mathcal{C}$ and $\mathcal{D}$ over $\mathcal{E}$ with respect to $S$ and $T$—most of the time denoted ambiguously(!) by $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$—is by definition the subcategory of the ordinary product category $\mathcal{C} \times \mathcal{D}$ whose class of objects is given by the ordered pairs $(x, y)$ that satisfy $Sx = Ty$. Moreover, for two such ordered pairs $(x, y)$ and $(x', y')$ a $(x', y')$ a morphism

$$(\alpha, \beta) : (x, y) \longrightarrow (x', y')$$

in $\mathcal{C} \times \mathcal{D}$ is a morphism in $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ if and only if $S\alpha = T\beta$.

Two easy observations show that, for one, for all objects $(x, y)$ of $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$, the identity $\mathrm{id}_{(x,y)} : (x, y) \to (x, y)$ in $\mathcal{C} \times \mathcal{D}$ is a morphism in $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ and that, for another, for all objects $(x, y)$, $(x', y')$, and $(x'', y'')$ and morphisms

$$(\alpha, \beta) : (x, y) \longrightarrow (x', y'),$$
$$(\alpha', \beta') : (x', y') \longrightarrow (x'', y'')$$

of $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$, the composition

$$(\alpha', \beta') \circ (\alpha, \beta) : (x, y) \longrightarrow (x'', y'')$$

in $\mathcal{C} \times \mathcal{D}$ is again a morphism in $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$.

**Definition 1.4.18** Let $f : X \to Y$ be a morphism of ringed spaces, and consider the functors

$$f^* : \mathrm{Mod}(Y) \longrightarrow \mathrm{Mod}(X) \quad \text{and} \quad p_0 : \mathrm{Mod}(X)^{\mathbf{3}} \longrightarrow \mathrm{Mod}(X),$$

where $p_0$ stands for the "projection to 0"; that is, $p_0$ takes an object $t$ of $\mathrm{Mod}(X)^{\mathbf{3}}$ to $t(0)$ and a morphism $\alpha : t \to t'$ in $\mathrm{Mod}(X)^{\mathbf{3}}$ to $\alpha(0)$. Then, define $\mathrm{Trip}(f)$ to be the fiber product category of $\mathrm{Mod}(Y)$ and $\mathrm{Mod}(X)^{\mathbf{3}}$ over $\mathrm{Mod}(X)$ with respect to $f^*$ and $p_0$. In symbols,

$$\mathrm{Trip}(f) := \mathrm{Mod}(Y) \times_{\mathrm{Mod}(X)} \mathrm{Mod}(X)^{\mathbf{3}}.$$

**Construction 1.4.19** Let $f : X \to Y$ be a morphism of ringed spaces and $(G, t)$ an object of $\mathrm{Trip}(f)$ such that $t$ is a short exact triple of modules on $X$ and $F := t2$ and $G$ are locally finite free modules on $X$ and $Y$, respectively. We associate to $(G, t)$ a morphism

$$\xi_{\mathrm{KS},f}(G, t) : \mathscr{O}_Y \longrightarrow G \otimes \mathrm{R}^1 f_*(F^\vee)$$

in $\mathrm{Mod}(Y)$, henceforth called the *Kodaira-Spencer class* relative $f$ of $(G, t)$.

For the definition of $\xi_{\mathrm{KS},f}(G, t)$, we remark, to begin with, that since $F$ is a locally finite free module on $X$, the triple $t$ is not only short exact, but locally split exact on $X$. Thus, we may consider its relative locally split extension class with respect to $f$,

$$\xi_f(t) : \mathscr{O}_Y \longrightarrow \mathrm{R}^1 f_*(\mathscr{H}om(F, f^*G));$$

see Construction 1.4.1. Set

$$\mu := \mu_X(F, f^*G) : f^*G \otimes F^\vee \longrightarrow \mathscr{H}om(F, f^*G);$$

see Construction 1.4.12. Then, again by the local finite freeness of $F$ on $X$, we know that $\mu$ is an isomorphism in $\mathrm{Mod}(X)$. Given that $G$ is a locally finite free module on $Y$, the projection morphism

$$\pi := \pi_f^1(G, F^\vee) : G \otimes \mathrm{R}^1 f_*(F^\vee) \longrightarrow \mathrm{R}^1 f_*(f^*G \otimes F^\vee)$$

is an isomorphism in $\mathrm{Mod}(Y)$ by means of Proposition 1.4.15. Composing, we obtain an isomorphism in $\mathrm{Mod}(Y)$,

$$\mathrm{R}^1 f_*(\mu) \circ \pi : G \otimes \mathrm{R}^1 f_*(F^\vee) \longrightarrow \mathrm{R}^1 f_*(\mathscr{H}om(F, f^*G)).$$

Therefore, there exists a unique $\xi_{\mathrm{KS},f}(G,t)$ rendering commutative in $\mathrm{Mod}(Y)$ the following diagram:

$$\begin{tikzcd}
 & \mathscr{O}_Y \arrow[dl, "\xi_{\mathrm{KS},f}(G,t)"'] \arrow[dr, "\xi_f(t)"] & \\
G \otimes \mathrm{R}^1 f_*(F^\vee) \arrow[rr, "\mathrm{R}^1 f_*(\mu)\circ\pi"'] & & \mathrm{R}^1 f_*(\mathscr{H}om(F, f^*G))
\end{tikzcd}$$

**Construction 1.4.20** We proceed in the situation of Construction 1.4.19; that is, we assume that a morphism of ringed spaces $f : X \to Y$ as well as an object $(G,t)$ of $\mathrm{Trip}(f)$ be given such that $t$ is a short exact triple of modules on $X$ and $F := t2$ and $G$ are locally finite free modules on $X$ and $Y$, respectively. Additionally, let us fix two integers $p$ and $q$. Then we write $\gamma^{p,q}_{\mathrm{KS},f}(G,t)$ for the composition of the following morphisms in $\mathrm{Mod}(Y)$:

$$\begin{aligned}
\mathrm{R}^q f_*(\bigwedge^p F) &\xrightarrow{\lambda(\mathrm{R}^q f_*(\bigwedge^p F))^{-1}} \mathscr{O}_Y \otimes \mathrm{R}^q f_*(\bigwedge^p F) \\
&\xrightarrow{\xi_{\mathrm{KS},f}(G,t)\otimes \mathrm{id}_{\mathrm{R}^q f_*(\bigwedge^p F)}} (G \otimes \mathrm{R}^1 f_*(F^\vee)) \otimes \mathrm{R}^q f_*(\bigwedge^p F) \\
&\xrightarrow{\alpha(G, \mathrm{R}^1 f_*(F^\vee), \mathrm{R}^q f_*(\bigwedge^p F))} G \otimes (\mathrm{R}^1 f_*(F^\vee) \otimes \mathrm{R}^q f_*(\bigwedge^p F)) \\
&\xrightarrow{\mathrm{id}_G \otimes \smile^{1,q}(F^\vee, \bigwedge^p F)} G \otimes \mathrm{R}^{q+1} f_*(F^\vee \otimes \bigwedge^p F) \\
&\xrightarrow{\mathrm{id}_G \otimes \mathrm{R}^{q+1} f_*(\gamma^p(F))} G \otimes \mathrm{R}^{q+1} f_*(\bigwedge^{p-1} F).
\end{aligned}$$

The resulting morphism of modules on $Y$,

$$\gamma^{p,q}_{\mathrm{KS},f}(G,t) : \mathrm{R}^q f_*(\bigwedge^p F) \longrightarrow G \otimes \mathrm{R}^{q+1} f_*(\bigwedge^{p-1} F),$$

goes by the name of *cup and contraction with the Kodaira-Spencer class* in bidegree $(p,q)$ relative $f$ for $(G,t)$. The name should be self-explanatory looking at the definition of $\gamma^{p,q}_{\mathrm{KS},f}(G,t)$ above: first, we tensor on the left with the Kodaira-Spencer class $\xi_{\mathrm{KS},f}(G,t)$, then we take the cup product, then we contract.

**Proposition 1.4.21** *Let $f : X \to Y$ be a morphism of ringed spaces and $(G,t)$ an object of* $\mathrm{Trip}(f)$ *such that $t$ is a short exact triple of modules on $X$ and $F := t2$ and*

*$G$ are locally finite free modules on $X$ and $Y$, respectively. Moreover, let $p$ and $q$ be integers. Then we have*

$$\delta_f^q(\Lambda^p(t)) = \pi_f^{q+1}(G, \bigwedge^{p-1} F) \circ \gamma_{\mathrm{KS},f}^{p,q}(G,t). \tag{1.32}$$

*In other words, the following diagram commutes in* $\mathrm{Mod}(Y)$*:*

$$\begin{tikzcd}
\mathrm{R}^q f_*(\bigwedge^p F) \arrow[r, "\delta_f^q(\Lambda^p(t))"] \arrow[rr, bend right, "\gamma_{\mathrm{KS},f}^{p,q}(G,t)"'] & \mathrm{R}^{q+1} f_*(f^* G \otimes \bigwedge^{p-1} F) & G \otimes \mathrm{R}^{q+1} f_*(\bigwedge^{p-1} F) \arrow[l, "\pi_f^{q+1}(G,\bigwedge^{p-1} F)"']
\end{tikzcd}$$

*Proof* Set $\mu := \mu_X(F, f^*G)$ (see Construction 1.4.12) and consider the diagram in Fig. 1.1, where we have abstained from further specifying the cup products $\smile^{1,q}$, the associators $\alpha_X$ and $\alpha_Y$ for the tensor product, as well as the identity morphisms id. We show that the subdiagrams labeled ①–⑥ commute in $\mathrm{Mod}(Y)$ (this is indeed equivalent to saying that the diagram commutes as such in $\mathrm{Mod}(Y)$, but we will use merely the commutativity of the mentioned subdiagrams afterwards).

We know that the triple $t$ is locally split exact on $X$. Hence the commutativity of ① is implied by Proposition 1.4.10. The commutativity of ② follows immediately from the definition of the Kodaira-Spencer class $\xi_{\mathrm{KS},f}(G,t)$; see Construction 1.4.19.

$$\begin{tikzcd}
\mathrm{R}^q f_*(\bigwedge^p F) \arrow[r, "\delta^q(\Lambda^p(t))"] \arrow[d, "\xi(t)\otimes\mathrm{id}"] \arrow[ddd, bend right=60, "\xi_{\mathrm{KS}}(G,t)\otimes\mathrm{id}"'] \arrow[dr, phantom, "\textcircled{1}"] & \mathrm{R}^{q+1} f_*(f^* G \otimes \bigwedge^{p-1} F) & \\
\mathrm{R}^1 f_*(\mathscr{H}om(F,f^*G)) \otimes \mathrm{R}^q f_*(\bigwedge^p F) \arrow[r, "\smile^{1,q}"] \arrow[d, phantom, "\textcircled{2}"'] \arrow[dr, phantom, "\textcircled{3}"] & \mathrm{R}^{q+1} f_*(\mathscr{H}om(F,f^*G) \otimes \bigwedge^p F) \arrow[u, "\mathrm{R}^{q+1}f_*(\tilde{\iota}^p(F,f^*G))"'] \arrow[d, phantom, "\textcircled{4}"] & \\
\mathrm{R}^1 f_*(f^*G \otimes F^\vee) \otimes \mathrm{R}^q f_*(\bigwedge^p F) \arrow[u, "\mathrm{R}^1 f_*(\mu)\otimes \mathrm{R}^q f_*(\mathrm{id})"] \arrow[r, "\smile^{1,q}"] & \mathrm{R}^{q+1} f_*((f^*G \otimes F^\vee) \otimes \bigwedge^p F) \arrow[u, "\mathrm{R}^{q+1}f_*(\mu\otimes\mathrm{id})"'] \arrow[d, "\mathrm{R}^{q+1}f_*(\alpha_X)"] & G \otimes \mathrm{R}^{q+1} f_*(\bigwedge^{p-1} F) \arrow[uul, "\pi^{q+1}(G,\bigwedge^{p-1}F)"'] \\
(G \otimes \mathrm{R}^1 f_*(F^\vee)) \otimes \mathrm{R}^q f_*(\bigwedge^p F) \arrow[u, "\pi^1(G,F^\vee)\otimes\mathrm{id}"] \arrow[d, "\alpha_Y"] \arrow[r, phantom, "\textcircled{5}"] & \mathrm{R}^{q+1} f_*(f^*G \otimes (F^\vee \otimes \bigwedge^p F)) \arrow[uuu, bend right=60, "\mathrm{R}^{q+1}f_*(\mathrm{id}\otimes\gamma^p(F))"'] \arrow[r, phantom, "\textcircled{6}"] & \\
G \otimes (\mathrm{R}^1 f_*(F^\vee) \otimes \mathrm{R}^q f_*(\bigwedge^p F)) \arrow[r, "\mathrm{id}\otimes\smile^{1,q}"'] & G \otimes \mathrm{R}^{q+1} f_*(F^\vee \otimes \bigwedge^p F) \arrow[u, "\pi^{q+1}(G,F^\vee\otimes\bigwedge^p F)"] \arrow[uur, bend right, "\mathrm{id}\otimes\mathrm{R}^{q+1}f_*(\gamma^p(F))"'] &
\end{tikzcd}$$

**Fig. 1.1** A diagram for the proof of Proposition 1.4.21

③ commutes by the naturality of the cup product—that is, by item 1 of Proposition 1.4.6. The commutativity of ④ follows from Proposition 1.4.13 coupled with the fact that $\mathrm{R}^{q+1}f_*$ is a functor from $\mathrm{Mod}(X)$ to $\mathrm{Mod}(Y)$. ⑤ commutes due to Proposition 1.4.16. Last but not least, the commutativity of ⑥ follows from the fact that $\pi_f^{q+1}$ is a natural transformation

$$(- \otimes_Y -) \circ (\mathrm{id}_{\mathrm{Mod}(Y)} \times \mathrm{R}^{q+1}f_*) \longrightarrow \mathrm{R}^{q+1}f_* \circ (- \otimes_X -) \circ (f^* \times \mathrm{id}_{\mathrm{Mod}(X)})$$

of functors from $\mathrm{Mod}(Y) \times \mathrm{Mod}(X)$ to $\mathrm{Mod}(Y)$; see Construction 1.4.14.

Recalling the definition of the cup and contraction with the Kodaira-Spencer class from Construction 1.4.20, we see that the commutativity of ①–⑥ implies that Eq. (1.32) holds in $\mathrm{Mod}(Y)$. You simply go through the subdiagrams one by one in the given order. □

## 1.5 Frameworks for the Gauß-Manin Connection

This section makes up the technical heart of this chapter. In fact, Theorem 1.6.14 of Sect. 1.6, which turns out to be crucial in view of our aspired study of period mappings in Sect. 1.8, is a mere special case of Theorem 1.5.14 to be proven here. The results of Sect. 1.5 are all based upon Definition 1.5.3. Let me note that I am well aware that Definition 1.5.3 might seem odd at first sight. Yet, looking at Sect. 1.6, you will see that it is just the right thing to consider.

Throughout Sects. 1.5 and 1.6 we frequently encounter the situation where two modules, say $F$ and $G$, on a ringed space $X$ are given together with a sheaf map $\alpha : F \to G$ which is not, however, a morphism of modules on $X$. The map $\alpha$ usually satisfies a weaker linearity property. For instance, when $X$ is a complex space, $\alpha$ might not be $\mathscr{O}_X$-linear, but merely $\mathbf{C}_X$-linear, where the $\mathbf{C}_X$-module structures of $F$ and $G$ are obtained relaxing their $\mathscr{O}_X$-module structures along the morphism of sheaves of rings from $\mathbf{C}_X$ to $\mathscr{O}_X$ that the complex space is equipped with. In this regard, the following device comes in handy.

**Definition 1.5.1** Let $f : X \to S$ be a morphism of ringed spaces. Then we write $\mathrm{Mod}(f)$ for the *relative linear category* with respect to $f$.

By definition, the class of objects of $\mathrm{Mod}(f)$ is simply the class of modules on $X$; that is, the class of objects of $\mathrm{Mod}(f)$ agrees with the class of objects of $\mathrm{Mod}(X)$. For an ordered pair $(F, G)$ of modules on $X$, a morphism from $F$ to $G$ in $\mathrm{Mod}(f)$ is a morphism of sheaves of $f^{-1}\mathscr{O}_S$-modules on $X_{\mathrm{top}}$ from $\bar{F}$ to $\bar{G}$, where $\bar{F}$ and $\bar{G}$ stand respectively for the sheaves of $f^{-1}\mathscr{O}_S$-modules on $X_{\mathrm{top}}$ which are obtained from $F$ and $G$ by relaxing the scalar multiplications via the morphism of sheaves of rings $f^\sharp : f^{-1}\mathscr{O}_S \to \mathscr{O}_X$ on $X_{\mathrm{top}}$. The identity of an object $F$ of $\mathrm{Mod}(f)$ is just the identity sheaf map $\mathrm{id}_F$. The composition in $\mathrm{Mod}(f)$ is given by the composition of sheaf maps on $X_{\mathrm{top}}$. I omit the verification that $\mathrm{Mod}(f)$ is a (large) category.

Note that $\mathrm{Mod}(X)$ is a subcategory of $\mathrm{Mod}(f)$. In fact, the classes of objects of $\mathrm{Mod}(X)$ and $\mathrm{Mod}(f)$ agree. Yet, the hom-sets of $\mathrm{Mod}(f)$ are generally larger than the hom-sets of $\mathrm{Mod}(X)$.

Defining additions on the hom-sets of $\mathrm{Mod}(f)$ as usual, $\mathrm{Mod}(f)$ becomes an additive category. Thus, we can speak of complexes over $\mathrm{Mod}(f)$. We set

$$\mathrm{Com}(f) := \mathrm{Com}(\mathrm{Mod}(f)) \quad \text{and} \quad \mathrm{Com}^+(f) := \mathrm{Com}^+(\mathrm{Mod}(f)).$$

*Remark 1.5.2* Let $f : X \to S$ and $g : S \to T$ be two morphisms of ringed spaces, $h := g \circ f$. Then $\mathrm{Mod}(f)$ is a subcategory of $\mathrm{Mod}(h)$. Indeed, the classes of objects of $\mathrm{Mod}(f)$ and $\mathrm{Mod}(h)$ agree—namely, both with the class of modules on $X$. For two modules $F$ and $G$ on $X$, I contend that

$$(\mathrm{Mod}(f))_1(F, G) \subset (\mathrm{Mod}(h))_1(F, G).$$

As a matter of fact, you observe that the morphism of sheaves of rings $h^\sharp$ going from $h^{-1}\mathscr{O}_T$ to $\mathscr{O}_X$ factors over the morphism of sheaves of rings $f^\sharp : f^{-1}\mathscr{O}_S \to \mathscr{O}_X$ on $X_{\mathrm{top}}$; that is, there is a morphism $\phi : h^{-1}\mathscr{O}_T \to f^{-1}\mathscr{O}_S$ such that $h^\sharp = f^\sharp \circ \phi$. Therefore, the linearity of a sheaf map $\alpha : F \to G$ with respect to $f^{-1}\mathscr{O}_S$ will always imply the linearity of $\alpha$ with respect to $h^{-1}\mathscr{O}_T$. Hence, my claim follows as the composition in $\mathrm{Mod}(h)$ restricts to the composition of $\mathrm{Mod}(f)$, and the identities in $\mathrm{Mod}(f)$ are the identities in $\mathrm{Mod}(h)$.

Now here goes the main notion of this section.

**Definition 1.5.3** A *framework for the Gauß-Manin connection* is a quintuple $(f, g, G, t, l)$ such that the following assertions hold.

1. $(f, g)$ is a composable pair in the category of ringed spaces.
2. $(G, t)$ is an object of $\mathrm{Trip}(f)$ (see Definition 1.4.18) such that the triple $t$ is short exact on $X := \mathrm{dom}(f)$ and $t2$ and $G$ are locally finite free modules on $X$ and $S := \mathrm{cod}(f)$, respectively.
3. $l$ is a triple in $\mathrm{Com}(h)$, where $h := g \circ f$, such that $K := l2$ and $L := l0$ are objects of $\mathrm{Com}(f)$ and, for all integers $p$, we have
$$l^p = \Lambda^p_X(t), \tag{1.33}$$
where $l^p$ stands for the triple in $\mathrm{Mod}(h)$ that is obtained extracting the degree-$p$ part from the triple of complexes $l$.
Note that $\mathrm{Mod}(f)$ is a subcategory of $\mathrm{Mod}(h)$ by Remark 1.5.2. Morally, requiring that Eq. (1.33) holds for all integers $p$ means that the only new information when passing from $(G, t)$ to $l$ lies in the differentials of the complexes $l0 = L$, $l1$, and $l2 = K$.
4. The sequence $\gamma := (\gamma^p)_{p\in\mathbf{Z}}$ is a morphism of complexes of modules on $\bar{X}$,
$$\gamma : \bar{f}^*G \otimes_{\bar{X}} (\bar{K}[-1]) \longrightarrow \bar{L},$$

where we employ the following notation.

a. $\bar{X}$ denotes the ringed space $(X_{\mathrm{top}}, f^{-1}\mathscr{O}_S)$.
b. $\bar{f} : \bar{X} \to S$ denotes the morphism of ringed spaces which is given by $f_{\mathrm{top}}$ on topological spaces and by the adjunction morphism from $\mathscr{O}_S$ to $(f_{\mathrm{top}})_* f^{-1}\mathscr{O}_S$ on structure sheaves.
c. $\bar{K}$ and $\bar{L}$ denote the complexes of modules on $\bar{X}$ that are obtained, respectively, by relaxing the module multiplication of the terms of the complexes $K$ and $L$ via the morphism of sheaves of rings $f^\sharp : f^{-1}\mathscr{O}_S \to \mathscr{O}_X$ on $X_{\mathrm{top}}$.
d. For any integer $p$, the map $\gamma^p$ is the $f^{-1}\mathscr{O}_S$-linear sheaf map

$$f^{-1}G \otimes_{f^{-1}\mathscr{O}_S} \bar{K}^{p-1} \longrightarrow (\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_S} f^{-1}G) \otimes_{\mathscr{O}_X} K^{p-1}$$

that sends $\sigma \otimes \tau$ to $(1 \otimes \sigma) \otimes \tau$, precomposed, in the first factor of the tensor product, with the map

$$\bar{f}^* G = f^{-1}\mathscr{O}_S \otimes_{f^{-1}\mathscr{O}_S} f^{-1}G \longrightarrow f^{-1}G,$$

which is induced by the $f^{-1}\mathscr{O}_S$-scalar multiplication of $f^{-1}G$.

Note that, for all integers $p$, we have

$$\begin{aligned} L^p &= (l0)^p = l^p(0) = (\Lambda^p_X(t))(0) = f^*G \otimes_X (\Lambda^{p-1}_X(t))(2) \\ &= f^*G \otimes_X l^{p-1}(2) = f^*G \otimes (l2)^{p-1} = f^*G \otimes_X K^{p-1} \end{aligned} \tag{1.34}$$

on the account of items 2 and 3 and the definition of the $\Lambda^p$ construction (see Construction 1.2.8).

In order to formulate Lemma 1.5.7, I introduce the auxiliary device of what I have christened "augmented triples." These are triples of bounded below complexes of modules equipped with a little extra information. They come about with special "augmented" connecting homomorphisms.

**Definition 1.5.4** Let $b : \bar{X} \to X'$ be a morphism of ringed spaces. Temporarily, denote by $\mathcal{D}$ the category of short exact triples of bounded below complexes of modules on $X'$. Note that $\mathcal{D}$ is a full subcategory of the functor category $(\mathrm{Com}^+(X'))^{\mathbf{3}}$. Consider the following diagram of categories and functors:

$$\mathrm{Com}^+(\bar{X}) \times \mathrm{Com}^+(\bar{X}) \xrightarrow{b_* \times b_*} \mathrm{Com}^+(X') \times \mathrm{Com}^+(X') \xleftarrow{p_2 \times p_0} \mathcal{D}, \tag{1.35}$$

where $p_2$ signifies the "projection to 2"—that is, $p_2(t) = t(2)$ for all objects $t$ of $\mathcal{D}$ and $(p_2(t,t'))(\alpha) = \alpha(2)$ for all morphisms $\alpha : t \to t'$ in $\mathcal{D}$. Similarly, $p_0$ signifies the "projection to 0." Now, define $\mathrm{Aug}(b)$ to be the fiber product category over the diagram in Eq. (1.35)—that is,

$$\mathrm{Aug}(b) := (\mathrm{Com}^+(\bar{X}) \times \mathrm{Com}^+(\bar{X})) \times_{\mathrm{Com}^+(X') \times \mathrm{Com}^+(X')} \mathcal{D};$$

see Remark 1.4.17.

We refer to $\mathrm{Aug}(b)$ as the *category of augmented triples* with respect to $b$. An object $l_+$ of $\mathrm{Aug}(b)$ is called an *augmented triple* with respect to $b$. Note that an augmented triple with respect to $b$ can always be written in the form $((\bar{K}, \bar{L}), l')$, where $\bar{K}$ and $\bar{L}$ are bounded below complexes of modules on $X$ and $l'$ is a triple of bounded below complexes of modules on $X'$ such that $l'(0) = b_*(\bar{L})$ and $l'(2) = b_*(\bar{K})$.

**Construction 1.5.5** Suppose we are given a commutative square in the category of ringed spaces:

$$\begin{tikzcd} \bar{X} \arrow[r, "b"] \arrow[d, "\bar{f}"'] & X' \arrow[d, "f'"] \\ S \arrow[r, "c"'] & S' \end{tikzcd} \tag{1.36}$$

Assume that $b_{\mathrm{top}} = \mathrm{id}_{\bar{X}_{\mathrm{top}}}$ and $c_{\mathrm{top}} = \mathrm{id}_{S_{\mathrm{top}}}$—in particular, this means that $(X')_{\mathrm{top}} = \bar{X}_{\mathrm{top}}$ and $(S')_{\mathrm{top}} = S_{\mathrm{top}}$. Then the functors $b_*$ and $c_*$ are exact. Fix an integer $n$, and denote by

$$\kappa'^n : \mathrm{R}^n f'_* \circ b_* \longrightarrow c_* \circ \mathrm{R}^n \bar{f}_*$$

the natural transformation of functors from $\mathrm{Com}^+(\bar{X})$ to $\mathrm{Mod}(S')$ which we have associated to the square in Eq. (1.36) in virtue of Construction B.1.4. Since the functors $b_*$ and $c_*$ are exact, we know that, for all $F \in \mathrm{Com}^+(\bar{X})$, the morphism

$$\kappa'^n(F) : \mathrm{R}^n f'_*(b_*(F)) \longrightarrow c_*(\mathrm{R}^n \bar{f}_*(F))$$

is an isomorphism in $\mathrm{Mod}(S')$. That is, $\kappa'^n$ is a natural equivalence between the aforementioned functors.

Let $l_+ = ((\bar{K}, \bar{L}), l')$ be an augmented triple with respect to $b$ (i.e., an object of $\mathrm{Aug}(b)$; see Definition 1.5.4). We define $\delta^n_+(l_+)$ to be the composition of the following morphisms of modules on $S'$:

$$\begin{aligned} c_*(\mathrm{R}^n \bar{f}_*(\bar{K})) \xrightarrow{\kappa'^n(\bar{K})} \mathrm{R}^n f'_*(b_*(\bar{K})) &\xrightarrow{\delta^n_{f'}(l')} \mathrm{R}^{n+1} f'_*(b_*(\bar{L})) \\ &\xrightarrow{(\kappa'^{n+1}(\bar{L}))^{-1}} c_*(\mathrm{R}^{n+1} \bar{f}_*(\bar{L})). \end{aligned} \tag{1.37}$$

Note that $l'(2) = b_*(\bar{K})$ and $l'(0) = b_*(\bar{L})$, so that the composition in Eq. (1.37) makes indeed sense. Thus, $\delta^n_+(l_+)$ is a morphism

$$\delta^n_+(l_+) : \mathrm{R}^n \bar{f}_*(\bar{K}) \longrightarrow \mathrm{R}^{n+1} \bar{f}_*(\bar{L})$$

in the relative linear category $\mathrm{Mod}(c)$; see Definition 1.5.1.

Letting $l_+$ vary, we obtain a function $\delta^n_+$ (in the class sense) defined on the class of objects of Aug($b$). We call $\delta^n_+$ the $n$th *augmented connecting homomorphism* associated to the square in Eq. (1.36). Since $\kappa'^n$, $\delta^n_{f'}$, and $(\kappa'^{n+1})^{-1}$ are altogether natural transformations (of appropriate functors between appropriate categories), we infer that $\delta^n_+$ is a natural transformation

$$\delta^n_+ : \mathrm{R}^n\bar{f}_* \circ q_0 \longrightarrow \mathrm{R}^{n+1}\bar{f}_* \circ q_1$$

of functors from Aug($b$) to Mod($c$), where $q_0$ (resp. $q_1$) stands for the functor from Aug($b$) to $\mathrm{Com}^+(X)$ which is given as the composition of the projection from Aug($b$) to $\mathrm{Com}^+(X) \times \mathrm{Com}^+(X)$, coming from the definition of Aug($b$) as a fiber product category, and the projection from $\mathrm{Com}^+(X) \times \mathrm{Com}^+(X)$ to the first (resp. second) factor.

**Notation 1.5.6** Let $\mathfrak{F}$ be a framework for the Gauß-Manin connection. Denote the components of $\mathfrak{F}$ by $f$, $g$, $G$, $t$, and $l$. Moreover, define $X$, $S$, $K$, $L$, $\bar{X}$, $\bar{K}$, $\bar{L}$, and $\gamma$ as in Definition 1.5.3.

Observe that due to item 3 of Definition 1.5.3, $\bar{K}$ and $\bar{L}$ are bounded below complexes of modules on $\bar{X}$. For any integer $p$, define

$$\gamma^{\geq p} : \bar{f}^*G \otimes_{\bar{X}} ((\sigma^{\geq p-1}\bar{K})[-1]) \longrightarrow \sigma^{\geq p}\bar{L}$$

to be the morphism in $\mathrm{Com}^+(\bar{X})$ which is given by $\gamma^{p'}$ in degree $p'$ for all integers $p' \geq p$ and by the zero morphism in degrees $< p$. Similarly, for any integer $p$, define

$$\gamma^{=p} : \bar{f}^*G \otimes_{\bar{X}} ((\sigma^{=p-1}\bar{K})[-1]) \longrightarrow \sigma^{=p}\bar{L}$$

to be the morphism in $\mathrm{Com}^+(\bar{X})$ which is given by $\gamma^p$ in degree $p$ and the zero morphism in degrees different from $p$.

Put $T := \mathrm{cod}(g)$. Moreover, put

$$X' := (X_{\mathrm{top}}, f^{-1}(g^{-1}\mathscr{O}_T)), \qquad b := (\mathrm{id}_{|X|}, f^{-1}(g^\sharp) : f^{-1}(g^{-1}\mathscr{O}_T) \longrightarrow f^{-1}\mathscr{O}_S),$$
$$S' := (S_{\mathrm{top}}, g^{-1}\mathscr{O}_T), \qquad c := (\mathrm{id}_{|S|}, g^\sharp : g^{-1}\mathscr{O}_T \longrightarrow \mathscr{O}_S),$$

and

$$f' := \big(|f|, \eta_{g^{-1}\mathscr{O}_T} : g^{-1}\mathscr{O}_T \longrightarrow (f_{\mathrm{top}})_* f^{-1}(g^{-1}\mathscr{O}_T)\big),$$

where $\eta$ denotes the adjunction morphism for sheaves of sets on $S_{\mathrm{top}}$ with respect to $f_{\mathrm{top}}$. Then the diagram in Eq. (1.36) commutes in the category of ringed spaces and we have $b_{\mathrm{top}} = \mathrm{id}_{\bar{X}_{\mathrm{top}}}$ as well as $c_{\mathrm{top}} = \mathrm{id}_{S_{\mathrm{top}}}$. For any integer $n$, we let $\delta^n_+$ signify the $n$th augmented connecting homomorphism with respect to the square in Eq. (1.36), as disposed of in Construction 1.5.5.

Define $l'$ to be the triple of complexes of modules on $X'$ which is obtained by relaxing the module multiplication of the terms of $l$ via the composition

$$f^\sharp \circ f^{-1}(g^\sharp) : f^{-1}(g^{-1}\mathscr{O}_T) \longrightarrow f^{-1}\mathscr{O}_S \longrightarrow \mathscr{O}_X$$

of morphisms of sheaves of rings on $X_{\mathrm{top}}$. Furthermore, set $l_+ := ((\bar{K}, \bar{L}), l')$. Observe that, for all integers $p$, the triple $l^p$ is short exact on $X$ by item 3 of Definition 1.5.3 and item 2 of Proposition 1.3.2. Moreover, for $p < 0$, the triple of modules $l^p$ is trivial. Thus, $l'$ is a short exact triple of bounded below complexes of modules on $X'$. Therefore, $l_+$ is an augmented triple with respect to $b$ (i.e., an object of $\mathrm{Aug}(b)$).

Finally, denote by $u$ the morphism of ringed spaces from $X$ to $\bar{X}$ which is given by the identity on topological spaces and by $f^\sharp$ on structure sheaves. Then the diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & \bar{X} \arrow[d, "\bar{f}"] \\ S \arrow[r, "\mathrm{id}_S"'] & S \end{tikzcd} \tag{1.38}$$

commutes in the category of ringed spaces. For any integer $n$, we denote by

$$\kappa^n : \mathrm{R}^n\bar{f}_* \circ u_* \longrightarrow (\mathrm{id}_S)_* \circ \mathrm{R}^n f_* = \mathrm{R}^n f_*$$

the natural transformation of functors from $\mathrm{Com}^+(X)$ to $\mathrm{Mod}(S)$ which we have associated to the square in Eq. (1.38) in virtue of Construction B.1.4. The definition of $\kappa$ here is similar to the definition of $\kappa'$ in Construction 1.5.5.

The following lemma is the key step towards proving Theorem 1.5.14.

**Lemma 1.5.7** *Let $\mathfrak{F}$ be a framework for the Gauß-Manin connection. Adopt Notation 1.5.6. Then, for all integers n and p, the diagram in Fig. 1.2 commutes in* $\mathrm{Mod}(g)$.

*Proof* Fix $n, p \in \mathbf{Z}$. The commutativity of the diagram in Fig. 1.2 is equivalent to the commutativity of its subdiagrams ①–⑧. We treat the subdiagrams case by case.

The subdiagrams ① and ② commute, for

$$\delta_+^n : \mathrm{R}^n f_* \circ q_0 \longrightarrow \mathrm{R}^{n+1} f_* \circ q_1$$

is a natural transformation of functors from $\mathrm{Aug}(b)$ to $\mathrm{Mod}(c)$; see Construction 1.5.5. Additionally, one should point out that the projection functors $q_0$ and $q_1$

$$
\begin{tikzcd}[column sep=large]
\mathrm{R}^n\bar{f}_*(\bar{K}) \arrow[r, "\delta^n_+(l_+)"] & \mathrm{R}^{n+1}\bar{f}_*(\bar{L}) & \mathrm{R}^n\bar{f}_*(\bar{f}^*G\otimes\bar{K}) \arrow[l, "\mathrm{R}^{n+1}\bar{f}_*(\gamma)"'] & G\otimes\mathrm{R}^n\bar{f}_*(\bar{K}) \arrow[l, "\pi^n_{\bar{f}}(G,\bar{K})"'] \\
\mathrm{R}^n\bar{f}_*(\sigma^{\geq p}\bar{K}) \arrow[u, "\mathrm{R}^n\bar{f}_*(i^{\geq p}\bar{K})"] \arrow[d, "\mathrm{R}^n\bar{f}_*(j^{\leq p}(\sigma^{\geq p}\bar{K}))"'] \arrow[r, "\delta^n_+(\sigma^{\geq p}l_+)"] \arrow[ur, phantom, "①"] \arrow[dr, phantom, "②"] & \mathrm{R}^{n+1}\bar{f}_*(\sigma^{\geq p}\bar{L}) \arrow[u, "\mathrm{R}^{n+1}\bar{f}_*(i^{\geq p}\bar{L})"] \arrow[d, "\mathrm{R}^{n+1}\bar{f}_*(j^{\leq p}(\sigma^{\geq p}\bar{L}))"] \arrow[ur, phantom, "③"] \arrow[dr, phantom, "④"] & \mathrm{R}^n\bar{f}_*(\bar{f}^*G\otimes\sigma^{\geq p-1}\bar{K}) \arrow[l, "\mathrm{R}^{n+1}\bar{f}_*(\gamma^{\geq p})"'] \arrow[u, "\mathrm{R}^n\bar{f}_*(\bar{f}^*\mathrm{id}_G\otimes i^{\geq p-1}\bar{K})"] \arrow[d, "\mathrm{R}^n\bar{f}_*(\bar{f}^*\mathrm{id}_G\otimes j^{\leq p-1}(\sigma^{\geq p-1}\bar{K}))"] \arrow[ur, phantom, "⑤"] \arrow[dr, phantom, "⑥"] & G\otimes\mathrm{R}^n\bar{f}_*(\sigma^{\geq p-1}\bar{K}) \arrow[l, "\pi^n_{\bar{f}}(G,\sigma^{\geq p-1}\bar{K})"'] \arrow[u, "\mathrm{id}_G\otimes\mathrm{R}^n\bar{f}_*(i^{\geq p-1}\bar{K})"'] \arrow[d, "\mathrm{id}_G\otimes\mathrm{R}^n\bar{f}_*(j^{\leq p-1}(\sigma^{\geq p-1}\bar{K}))"] \\
\mathrm{R}^n\bar{f}_*(\sigma^{=p}\bar{K}) \arrow[r, "\delta^n_+(\sigma^{=p}l_+)"] \arrow[d, "\kappa^n(\sigma^{=p}K)"'] \arrow[dr, phantom, "⑦"] & \mathrm{R}^{n+1}\bar{f}_*(\sigma^{=p}\bar{L}) \arrow[d, "\kappa^{n+1}(\sigma^{=p}L)"] & \mathrm{R}^n\bar{f}_*(\bar{f}^*G\otimes\sigma^{=p-1}\bar{K}) \arrow[l, "\mathrm{R}^{n+1}\bar{f}_*(\gamma^{=p})"'] \arrow[d, phantom, "⑧"] & G\otimes\mathrm{R}^n\bar{f}_*(\sigma^{=p-1}\bar{K}) \arrow[l, "\pi^n_{\bar{f}}(G,\sigma^{=p-1}\bar{K})"'] \arrow[d, "\mathrm{id}_G\otimes\kappa^n(\sigma^{=p-1}K)"] \\
\mathrm{R}^{n-p}f_*(K^p) \arrow[r, "\delta^{n-p}_f(l^p)"'] & \mathrm{R}^{n-p+1}f_*(L^p) & & G\otimes\mathrm{R}^{n-p+1}f_*(K^{p-1}) \arrow[ll, "\pi^{n-p+1}_f(G,K^{p-1})"]
\end{tikzcd}
$$

**Fig. 1.2** The diagram whose commutativity in Mod($g$) is asserted by Lemma 1.5.7

commute with the stupid filtration functors $\sigma^{\geq p}$ and $\sigma^{\leq p}$ as well as with the natural transformations $i^{\geq p}$ and $j^{\leq p}$. In particular, we have

$$q_0(\sigma^{\geq p}(l_+)) = \sigma^{\geq p}(q_0(l_+)) = \sigma^{\geq p}\bar{K}, \quad q_0(i^{\geq p}(l_+)) = i^{\geq p}(q_0(l_+)) = i^{\geq p}(\bar{K}),$$
$$q_1(\sigma^{\geq p}(l_+)) = \sigma^{\geq p}(q_1(l_+)) = \sigma^{\geq p}\bar{L}, \quad q_1(i^{\geq p}(l_+)) = i^{\geq p}(q_1(l_+)) = i^{\geq p}(\bar{L}),$$

and

$$q_0(\sigma^{=p}(l_+)) = q_0(\sigma^{\leq p}\sigma^{\geq p}(l_+)) = \sigma^{\leq p}\sigma^{\geq p}(q_0(l_+)) = \sigma^{=p}\bar{K},$$
$$q_1(\sigma^{=p}(l_+)) = q_1(\sigma^{\leq p}\sigma^{\geq p}(l_+)) = \sigma^{\leq p}\sigma^{\geq p}(q_1(l_+)) = \sigma^{=p}\bar{L},$$

and

$$q_0(j^{\leq p}(\sigma^{\geq p}l_+)) = j^{\leq p}(q_0(\sigma^{\geq p}l_+)) = j^{\leq p}(\sigma^{\geq p}(q_0(l_+))) = j^{\leq p}(\sigma^{\geq p}\bar{K}),$$
$$q_1(j^{\leq p}(\sigma^{\geq p}l_+)) = j^{\leq p}(q_1(\sigma^{\geq p}l_+)) = j^{\leq p}(\sigma^{\geq p}(q_1(l_+))) = j^{\leq p}(\sigma^{\geq p}\bar{L}).$$

The commutativity of ③ follows now from the identity

$$i^{\geq p}(\bar{L}) \circ \gamma^{\geq p} = \gamma \circ (\bar{f}^* \mathrm{id}_G \otimes i^{\geq p-1}(\bar{K})[-1])$$

in $\mathrm{Mod}(\bar{X})$, which is checked degree-wise, and the fact that $\mathrm{R}^{n+1}\bar{f}_*$ is a functor from $\mathrm{Com}^+(\bar{X})$ to $\mathrm{Mod}(S)$. Similarly, the commutativity of ④ follows from the identity

$$j^{\leq p}(\sigma^{\geq p}(\bar{L})) \circ \gamma^{\geq p} = \gamma^{=p} \circ (\bar{f}^* \mathrm{id}_G \otimes j^{\leq p-1}(\sigma^{\geq p-1}(\bar{K}))[-1]).$$

Let me note that for all objects $F$ and all morphisms $\alpha$ of $\mathrm{Com}^+(\bar{X})$, we have

$$\mathrm{R}^{n+1}\bar{f}_*(F[-1]) = \mathrm{R}^n\bar{f}_*(F),$$
$$\mathrm{R}^{n+1}\bar{f}_*(\alpha[-1]) = \mathrm{R}^n\bar{f}_*(\alpha).$$

The subdiagrams ⑤ and ⑥ commute as

$$\pi^n_{\bar{f}} : (- \otimes_S -) \circ (\mathrm{id}_{\mathrm{Mod}(S)} \times \mathrm{R}^n\bar{f}_*) \longrightarrow \mathrm{R}^n\bar{f}_* \circ (- \otimes_{\bar{X}} -) \circ (\bar{f}^* \times \mathrm{id}_{\mathrm{Com}^+(\bar{X})})$$

is a natural transformation of functors from $\mathrm{Mod}(S) \times \mathrm{Com}^+(\bar{X})$ to $\mathrm{Mod}(S)$; see Construction 1.4.14.

Moving on to subdiagram ⑦, we first remark that

$$\mathrm{R}^{n-p}f_*(K^p) = \mathrm{R}^{n-p}f_*(K^p[0]) = \mathrm{R}^n f_*((K^p[0])[-p]) = \mathrm{R}^n f_*(\sigma^{=p}K)$$

and, in a similar fashion,

$$\mathrm{R}^{n-p+1}f_*(L^p) = \mathrm{R}^{n+1}f_*(\sigma^{=p}L).$$

Moreover, $\sigma^{=p}K$ (resp. $\sigma^{=p}L$) is an object of $\mathrm{Com}^+(X)$ and we have $u_*(\sigma^{=p}K) = \sigma^{=p}\bar{K}$ (resp. $u_*(\sigma^{=p}L) = \sigma^{=p}(\bar{L})$). Thus, we see that the domains and codomains which are given for the morphisms $\kappa^n(\sigma^{=p}K)$ and $\kappa^{n+1}(\sigma^{=p}L)$ in the diagram are the correct ones. We need two additional pieces of notation. For one, define $\kappa'$ as in Construction 1.5.5. For another, define $\kappa''$ for the commutative square

$$\begin{tikzcd}
X \arrow[r, "b\circ u"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\
S \arrow[r, "c\circ\mathrm{id}_S"'] & S'
\end{tikzcd}$$

just as $\kappa'$ was defined for the square in Eq. (1.36). Notice that the latter square is obtained by composing the squares in Eqs. (1.36) and (1.38) horizontally. I contend that the following diagram commutes in $\mathrm{Mod}(S')$:

$$\begin{tikzcd}
\mathrm{R}^n f'_*(b_*(\sigma^{=p}\bar{K})) \arrow[rr, "\kappa''^n(\sigma^{=p}K)"] \arrow[dr, "\kappa'^n(\sigma^{=p}\bar{K})"] \arrow[dd, "\delta^n_{f'}(\sigma^{=p}l')"'] & & c_*(\mathrm{R}^n f_*(\sigma^{=p}K)) \arrow[dd, "c_*(\delta^n_f(\sigma^{=p}l))"] \\
& c_*(\mathrm{R}^n \bar{f}_*(\sigma^{=p}\bar{K})) \arrow[ur, "c_*(\kappa^n(\sigma^{=p}K))"] \arrow[dd, "c_*(\delta^n_+(\sigma^{=p}l_+))"] & \\
\mathrm{R}^{n+1} f'_*(b_*(\sigma^{=p}\bar{L})) \arrow[rr, "\kappa''^{n+1}(\sigma^{=p}L)"] \arrow[dr, "\kappa'^{n+1}(\sigma^{=p}\bar{L})"'] & & c_*(\mathrm{R}^{n+1} f_*(\sigma^{=p}L)) \\
& c_*(\mathrm{R}^{n+1} \bar{f}_*(\sigma^{=p}\bar{L})) \arrow[ur, "c_*(\kappa^{n+1}(\sigma^{=p}L))"'] &
\end{tikzcd} \tag{1.39}$$

Indeed, the upper and lower triangles in Eq. (1.39) commute due to Proposition B.1.6—that is, due to the functoriality of Construction B.1.4, out of which $\kappa$, $\kappa'$, and $\kappa''$ arise. The background rectangle in Eq. (1.39) commutes due to the compatibility of the base change $\kappa''$ with the connecting homomorphisms noting that

$$(b \circ u)_*(\sigma^{=p}l) = \sigma^{=p}l'.$$

The left foreground rectangle (or "parallelogram") in Eq. (1.39) commutes by the very definition of the augmented connecting homomorphism $\delta^n_+$ noticing that the triple underlying $\sigma^{=p} l_+$ is nothing but $\sigma^{=p} l'$; see Construction 1.5.5. Since $\kappa'^n(\sigma^{=p}\bar{L})$ is a monomorphism—in fact, it is an isomorphism—we obtain the commutativity of the right foreground rectangle (or "parallelogram") of Eq. (1.39). This, in turn, implies the commutativity of ⑦ in $\mathrm{Mod}(S)$ since the functor $c_*$ from $\mathrm{Mod}(S)$ to $\mathrm{Mod}(S')$ is faithful and we have $\sigma^{=p} l = (l^p[0])[-p]$, whence

$$\delta^n_f(\sigma^{=p} l) = \delta^n_f((l^p[0])[-p]) = \delta^{n-p}_f(l^p[0]) = \delta^{n-p}_f(l^p).$$

We are left with ⑧. To this end, define

$$\phi : f^* G \otimes (\sigma^{=p-1} K) \longrightarrow (\sigma^{=p} L)[1]$$

to be the morphism in $\mathrm{Com}^+(S)$ which is given by the identity in degree $p-1$ (recall Eq. (1.34)) and by the zero morphism in all other degrees. In addition, define

$$\psi : \bar{f}_* G \otimes (\sigma^{=p-1} \bar{K}) = \bar{f}^* G \otimes u_*(\sigma^{=p-1} K) \longrightarrow u_*(f^* G \otimes (\sigma^{=p-1} K))$$

to be the evident base extension morphism in $\mathrm{Com}^+(\bar{X})$. We consider the auxiliary diagram in Fig. 1.3. By the definition of the projection morphism (see Construction 1.4.14) we have

$$\begin{aligned}
\pi^{n-p+1}_f(G, K^{p-1}) &= \mathrm{R}^{n-p+1} f_*(\phi[p-1]) \circ \pi^{n-p+1}_f(G, K^{p-1}[0]) \\
&= \mathrm{R}^n f_*(\phi) \circ \pi^n_f(G, (K^{p-1}[0])[-(p-1)]) \\
&= \mathrm{R}^n f_*(\phi) \circ \pi^n_f(G, \sigma^{=p-1} K).
\end{aligned}$$

$$\begin{tikzcd}
\mathrm{R}^{n+1}\bar{f}_*(\sigma^{=p}\bar{L}) \arrow[dd, "\kappa^{n+1}(\sigma^{=p}L)"'] & & \mathrm{R}^n\bar{f}_*(\bar{f}^*G \otimes \sigma^{=p-1}\bar{K}) \arrow[ll, "\mathrm{R}^{n+1}\bar{f}_*(\gamma^{=p})"'] \arrow[dl, "\mathrm{R}^n\bar{f}_*(\psi)"'] & G \otimes \mathrm{R}^n\bar{f}_*(\sigma^{=p-1}\bar{K}) \arrow[l, "\pi^n_{\bar{f}}(G,\sigma^{=p-1}\bar{K})"'] \arrow[dd, "\mathrm{id}_G \otimes \kappa^n(\sigma^{=p-1}K)"] \\
& \mathrm{R}^n\bar{f}_*(u_*(f^*G \otimes \sigma^{=p-1}K)) \arrow[ul, "\mathrm{R}^n\bar{f}_*(u_*(\phi))"'] \arrow[dd, "\kappa^n(f^*G\otimes(\sigma^{=p-1}K))"'] & & \\
\mathrm{R}^{n-p+1}f_*(L^p) & & & G \otimes \mathrm{R}^{n-p+1}f_*(K^{p-1}) \arrow[lll, "\pi^{n-p+1}_f(G,K^{p-1})"] \arrow[dll, "\pi^n_f(G,\sigma^{=p-1}K)"] \\
& \mathrm{R}^n f_*(f^*G \otimes \sigma^{=p-1}K) \arrow[ul, "\mathrm{R}^n f_*(\phi)"] & &
\end{tikzcd}$$

**Fig. 1.3** An auxiliary diagram for the proof of Lemma 1.5.7

Hence, the bottom triangle of the diagram in Fig. 1.3 commutes. Taking into account that $\kappa^{n+1}(\sigma^{=p}L) = \kappa^n((\sigma^{=p}L)[1])$, we see that the left foreground rectangle (or "parallelogram") commutes, for

$$\kappa^n : \mathrm{R}^n\bar{f}_* \circ u_* \longrightarrow (\mathrm{id}_S)_* \circ \mathrm{R}^n f_* = \mathrm{R}^n f_*$$

is a natural transformation of functors going from $\mathrm{Com}^+(X)$ to $\mathrm{Mod}(S)$. The pentagon in the right foreground commutes by the compatibility of the projection morphisms with base change. The top triangle commutes since firstly, we have

$$\gamma^{=p}[1] = u_*(\phi) \circ \psi,$$

in $\mathrm{Com}^+(\bar{X})$, as is easily checked degree-wise, secondly, $\mathrm{R}^n\bar{f}_*$ is a functor going from $\mathrm{Com}(\bar{X})$ to $\mathrm{Mod}(S)$, and thirdly, $\mathrm{R}^{n+1}\bar{f}_*(\gamma^{=p}) = \mathrm{R}^n\bar{f}_*(\gamma^{=p}[1])$. Therefore, we have established the commutativity of rectangle in the background of Fig. 1.3, which is, however, nothing but ⑧. □

**Proposition 1.5.8** *Let $\theta : B \to A$ be a morphism of rings, $M$ and $N$ modules over $A$ and $B$, respectively. Then the $B$-linear map*

$$N \otimes_B M \longrightarrow (A \otimes_B N) \otimes_A M$$

*sending $y \otimes x$ to $(1 \otimes y) \otimes x$ is an isomorphism of modules over $B$.*

*Proof* See [4, Proposition A2.1d]. □

**Notation 1.5.9** Let $\mathfrak{F}$ be a framework for the Gauß-Manin connection. Adopt Notation 1.5.6. Then, for all integers $p$,

$$\gamma^p : \bar{f}^*G \otimes_{\bar{X}} \bar{K}^{p-1} \longrightarrow \bar{L}^p$$

is an isomorphism in $\mathrm{Mod}(\bar{X})$ due to Proposition 1.5.8 (for the definition of $\gamma^p$ see item 4d of Definition 1.5.3). In turn,

$$\gamma : \bar{f}^*G \otimes_{\bar{X}} (\bar{K}[-1]) \longrightarrow \bar{L}$$

is an isomorphism in $\mathrm{Com}^+(\bar{X})$ and, for all integers $p$,

$$\gamma^{\geq p} : \bar{f}^*G \otimes_{\bar{X}} ((\sigma^{\geq p-1}\bar{K})[-1]) \longrightarrow \sigma^{\geq p}\bar{L},$$
$$\gamma^{=p} : \bar{f}^*G \otimes_{\bar{X}} ((\sigma^{=p-1}\bar{K})[-1]) \longrightarrow \sigma^{=p}\bar{L}$$

are isomorphisms in $\mathrm{Com}^+(\bar{X})$. Furthermore, as $G$ is a locally finite free module on $S$, Proposition 1.4.15 implies that, for all integers $n$,

$$\pi^n_{\bar{f}}(G, -) : (G \otimes -) \circ \mathrm{R}^n\bar{f}_* \longrightarrow \mathrm{R}^n\bar{f}_* \circ (\bar{f}^*G \otimes -)$$

is a natural equivalence of functors from $\mathrm{Com}^+(\bar{X})$ to $\mathrm{Mod}(S)$ . Thus, for all integers $n$ and $p$, it makes sense to define

$$\nabla^n := (\pi^n_{\bar{f}}(G, \bar{K}))^{-1} \circ (\mathrm{R}^{n+1}\bar{f}_*(\gamma))^{-1} \circ \delta^n_+(l_+), \tag{1.40}$$

$$\nabla^{\geq p,n} := (\pi^n_{\bar{f}}(G, \sigma^{\geq p-1}\bar{K}))^{-1} \circ (\mathrm{R}^{n+1}\bar{f}_*(\gamma^{\geq p}))^{-1} \circ \delta^n_+(\sigma^{\geq p} l_+), \tag{1.41}$$

and

$$\nabla^{=p,n} := (\pi^n_{\bar{f}}(G, \sigma^{=p-1}\bar{K}))^{-1} \circ (\mathrm{R}^{n+1}\bar{f}_*(\gamma^{=p}))^{-1} \circ \delta^n_+(\sigma^{=p} l_+), \tag{1.42}$$

where we compose in $\mathrm{Mod}(g) = \mathrm{Mod}(c)$. Observe that Eqs. (1.40)–(1.42) correspond to the first, second, and third left-to-right horizontal row of arrows in the diagram in Fig. 1.2, respectively.

**Notation 1.5.10** Let $\bar{f} : \bar{X} \to S$ be a morphism of ringed spaces, $\bar{K}$ an object of $\mathrm{Com}^+(\bar{X})$. For integers $n$ and $p$ we set

$$F^{p,n} := \mathrm{im}_S(\mathrm{R}^n\bar{f}_*(i^{\geq p}\bar{K}) : \mathrm{R}^n\bar{f}_*(\sigma^{\geq p}\bar{K}) \longrightarrow \mathrm{R}^n\bar{f}_*(\bar{K})).$$

Moreover, we write

$$\iota^n(p) : F^{p,n} \longrightarrow \mathrm{R}^n\bar{f}_*(\bar{K})$$

for the corresponding inclusion morphism of sheaves on $S_{\mathrm{top}}$, and we write $\lambda^n(p)$ for the unique morphism such that the following diagram commutes in $\mathrm{Mod}(S)$:

$$\begin{tikzcd}
\mathrm{R}^n\bar{f}_*(\sigma^{\geq p}\bar{K}) \arrow[rr, "\mathrm{R}^n\bar{f}_*(i^{\geq p}\bar{K})"] \arrow[dr, dotted, "\lambda^n(p)"'] & & \mathrm{R}^n\bar{f}_*(\bar{K}) \\
& F^{p,n} \arrow[ur, "\iota^n(p)"'] &
\end{tikzcd}$$

For all $n \in \mathbf{Z}$, the sequence $(F^{p,n})_{p\in\mathbf{Z}}$ clearly constitutes a descending sequence of submodules of $\mathrm{R}^n\bar{f}_*(\bar{K})$ on $S$. In more formal terms, we may express this observation by saying that, for all integers $n$, $p$, and $p'$ such that $p \leq p'$, there exists a unique morphism $\iota^n(p, p')$ in $\mathrm{Mod}(S)$ such that the following diagram commutes in $\mathrm{Mod}(S)$:

$$\begin{tikzcd}
& \mathrm{R}^n\bar{f}_*(\bar{K}) & \\
F^{p,n} \arrow[ur, "\iota^n(p)"] & & F^{p',n} \arrow[ul, "\iota^n(p')"'] \arrow[ll, dotted, "\iota^n(p,p')"]
\end{tikzcd}$$

**Proposition 1.5.11** *Let $\mathfrak{F}$ be a framework for the Gauß-Manin connection and adopt Notations 1.5.6, 1.5.9, and 1.5.10. Let n and p be integers. Then there exists one, and only one, $\zeta$ such that the following diagram commutes in* Mod($g$)*:*

$$\begin{tikzcd} \mathrm{R}^n\bar{f}_*(\bar{K}) \arrow[r, "\nabla^n"] & G \otimes \mathrm{R}^n\bar{f}_*(\bar{K}) \\ F^{p,n} \arrow[u, "\iota^n(p)"] \arrow[r, dotted, "\zeta"'] & G \otimes F^{p-1,n} \arrow[u, "\mathrm{id}_G \otimes \iota^n(p-1)"'] \end{tikzcd} \tag{1.43}$$

*Proof* Comparing Eqs. (1.40) and (1.41) with the diagram in Fig. 1.2, we detect that Lemma 1.5.7 implies the following identity in Mod($g$):

$$\nabla^n \circ \mathrm{R}^n\bar{f}_*(i^{\geq p}\bar{K}) = (\mathrm{id}_G \otimes \mathrm{R}^n\bar{f}_*(i^{\geq p-1}\bar{K})) \circ \nabla^{\geq p,n}. \tag{1.44}$$

Now since $G$ is a locally finite free, and hence flat, module on $S$, the functor

$$G \otimes - : \mathrm{Mod}(S) \longrightarrow \mathrm{Mod}(S)$$

is exact. Thus, in particular, it transforms images into images. Specifically, the morphism $\mathrm{id}_G \otimes \iota^n(p-1)$ is an image of $\mathrm{id}_G \otimes \mathrm{R}^n\bar{f}_*(i^{\geq p-1}\bar{K})$ in Mod($S$). This in mind, the claim follows readily from Eq. (1.44). □

**Proposition 1.5.12** *Let $\mathfrak{F}$ be a framework for the Gauß-Manin connection and adopt Notations 1.5.6, 1.5.9, and 1.5.10. Let n and p be integers, $\zeta$ such that the diagram in Eq. (1.43) commutes in* Mod($g$)*. Then there exists one, and only one, $\bar{\zeta}$ rendering commutative in* Mod($g$) *the following diagram:*

$$\begin{tikzcd} F^{p,n} \arrow[r, "\zeta"] \arrow[d, "\mathrm{coker}(\iota^n(p,p+1))"'] & G \otimes F^{p-1,n} \arrow[d, "id_G \otimes \mathrm{coker}(\iota^n(p-1,p))"] \\ F^{p,n}/F^{p+1,n} \arrow[r, dotted, "\bar{\zeta}"'] & G \otimes (F^{p-1,n}/F^{p,n}) \end{tikzcd} \tag{1.45}$$

*Proof* By Proposition 1.5.11, there exists $\zeta'$ such that the upper foreground trapezoid in the following diagram commutes in Mod($g$):

$$\begin{tikzcd}
 & \mathrm{R}^n\bar{f}_*(\bar{K}) \arrow[r, "\nabla^n"] & G \otimes \mathrm{R}^n\bar{f}_*(\bar{K}) & \\
F^{p+1,n} \arrow[ur, "\iota^n(p+1)"] \arrow[dr, "\iota^n(p,p+1)"'] \arrow[rrr, dotted, "\zeta'"] & & & G \otimes F^{p,n} \arrow[ul, "\mathrm{id}_G \otimes \iota^n(p)"'] \arrow[dl, "\mathrm{id}_G \otimes \iota^n(p-1,p)"] \\
 & F^{p,n} \arrow[uu, "\iota^n(p)"'] \arrow[r, "\zeta"'] & G \otimes F^{p-1,n} \arrow[uu, "\mathrm{id}_G \otimes \iota^n(p-1)"] &
\end{tikzcd} \tag{1.46}$$

I claim that the diagram in Eq. (1.46) commutes in $\mathrm{Mod}(g)$ as such. In fact, the left and right triangles commute by the very definitions of $\iota^n(p, p+1)$ and $\iota^n(p-1, p)$, respectively. The background square commutes by our assumption on $\zeta$; see Eq. (1.43). The lower trapezoid commutes as a consequence of the already established commutativities taking into account that $\mathrm{id}_G \otimes \iota^n(p-1)$ is a monomorphism, which is due to the flatness of $G$.

Using the commutativity of the lower trapezoid in Eq. (1.46), we obtain

$$\begin{aligned}
&((\mathrm{id}_G \otimes \mathrm{coker}(\iota^n(p-1,p))) \circ \zeta) \circ \iota^n(p,p+1) \\
&= (\mathrm{id}_G \otimes \mathrm{coker}(\iota^n(p-1,p))) \circ (\mathrm{id}_G \otimes \iota^n(p-1,p)) \circ \zeta' = 0.
\end{aligned}$$

Hence, there exists one, and only one, $\bar{\zeta}$ rendering the diagram in Eq. (1.45) commutative in $\mathrm{Mod}(g)$. ☐

**Proposition 1.5.13** *Let $\mathfrak{F}$ be a framework for the Gauß-Manin connection and adopt Notations 1.5.6, 1.5.9, and 1.5.10. Let $n$ and $p$ be integers, $\zeta$ and $\bar{\zeta}$ such that the diagrams in Eqs. (1.43) and (1.45) commute in* $\mathrm{Mod}(g)$, *and $\psi^p$ and $\psi^{p-1}$ such that the following diagram commutes in* $\mathrm{Mod}(S)$ *for $\nu \in \{p, p-1\}$:*

$$\begin{tikzcd}
F^{\nu,n} \arrow[d, "\mathrm{coker}(\iota^n(\nu,\nu+1))"'] & \mathrm{R}^n\bar{f}_*(\sigma^{\geq\nu}\bar{K}) \arrow[l, "\lambda^n(\nu)"'] \arrow[d, "\mathrm{R}^n\bar{f}_*(j^{\leq\nu}(\sigma^{\geq\nu}\bar{K}))"] \\
F^{\nu,n}/F^{\nu+1,n} \arrow[r, dotted, "\psi^\nu"'] & \mathrm{R}^n\bar{f}_*(\sigma^{=\nu}\bar{K})
\end{tikzcd} \tag{1.47}$$

*Then the following diagram commutes in* $\mathrm{Mod}(g)$*:*

$$
\begin{tikzcd}
F^{p,n}/F^{p+1,n} \arrow[r, "\bar{\zeta}"] \arrow[d, "\psi^p"'] & G \otimes (F^{p-1,n}/F^{p,n}) \arrow[d, "id_G \otimes \psi^{p-1}"] \\
\mathrm{R}^n \bar{f}_*(\sigma^{=p} \bar{K}) \arrow[r, "\nabla^{=p,n}"'] & G \otimes \mathrm{R}^n \bar{f}_*(\sigma^{=p-1} \bar{K})
\end{tikzcd}
\tag{1.48}
$$

*Proof* We proceed in three steps. In each step we derive the commutativity of a certain square-shaped (or maybe better "trapezoid-shaped") diagram by means of what I call a "prism diagram argument." To begin with, consider the following diagram ("prism"):

$$
\begin{tikzcd}
 & \mathrm{R}^n \bar{f}_*(\bar{K}) \arrow[r, "\nabla^n"] & G \otimes \mathrm{R}^n \bar{f}_*(\bar{K}) & \\
F^{p,n} \arrow[ur, "\iota^n(p)"] \arrow[rrr, "\zeta"] & & & G \otimes F^{p-1,n} \arrow[ul, "\mathrm{id}_G \otimes \iota^n(p-1)"'] \\
 & \mathrm{R}^n \bar{f}_*(\sigma^{\geq p} \bar{K}) \arrow[r, "\nabla^{\geq p,n}"'] \arrow[ul, "\lambda^n(p)"] \arrow[uu, "\mathrm{R}^n \bar{f}_*(i^{\geq p} \bar{K})"] & G \otimes \mathrm{R}^n \bar{f}_*(\sigma^{\geq p-1} \bar{K}) \arrow[ur, "\mathrm{id}_G \otimes \lambda^n(p-1)"'] \arrow[uu, "\mathrm{id}_G \otimes \mathrm{R}^n \bar{f}_*(i^{\geq p-1} \bar{K})"] &
\end{tikzcd}
\tag{1.49}
$$

The above diagram is, in fact, commutative: the left and right triangles commute according to the definitions of $\lambda^n(p)$ and $\lambda^n(p-1)$, respectively. The back square (or rectangle) commutes due to Lemma 1.5.7. The upper trapezoid commutes by our assumption on $\zeta$; see Eq. (1.43). Therefore, the lower trapezoid commutes too, taking into account that $\mathrm{id}_G \otimes \iota^n(p-1)$ is a monomorphism, which is due to the fact that $G$ is a flat module on $S$.

Next, I claim that the diagram

$$
\begin{tikzcd}
 & \mathrm{R}^n \bar{f}_*(\sigma^{\geq p} \bar{K}) \arrow[r, "\nabla^{\geq p,n}"] \arrow[dl, "\lambda^n(p)"'] \arrow[dd, "\mathrm{R}^n \bar{f}_*(j^{\leq p}(\sigma^{\geq p} \bar{K}))"] & G \otimes \mathrm{R}^n \bar{f}_*(\sigma^{\geq p-1} \bar{K}) \arrow[dr, "\mathrm{id}_G \otimes \lambda^n(p-1)"] \arrow[dd, "\mathrm{id}_G \otimes \mathrm{R}^n \bar{f}_*(j^{\leq p-1}(\sigma^{\geq p-1} \bar{K}))"] & \\
F^{p,n} \arrow[rrr, "\zeta"] \arrow[dr, "\phi^p"'] & & & G \otimes F^{p-1,n} \arrow[dl, "\mathrm{id}_G \otimes \phi^{p-1}"] \\
 & \mathrm{R}^n \bar{f}_*(\sigma^{=p} \bar{K}) \arrow[r, "\nabla^{=p,n}"'] & G \otimes \mathrm{R}^n \bar{f}_*(\sigma^{=p-1} \bar{K}) &
\end{tikzcd}
\tag{1.50}
$$

commutes in $\mathrm{Mod}(g)$, where

$$\phi^{\nu} := \psi^{\nu} \circ \mathrm{coker}(\iota^{n}(\nu, \nu + 1))$$

for $\nu \in \{p-1, p\}$. The left and right triangles commute according to the commutativity of Eq. (1.47) for $\nu = p$ and $\nu = p-1$, respectively. The back square commutes by means of Lemma 1.5.7. The upper trapezoid commutes by the commutativity of the lower trapezoid of the diagram in Eq. (1.49). Therefore, the lower trapezoid of the diagram in Eq. (1.50) commutes taking into account that $\lambda^{n}(p)$ is an epimorphism.

Finally, we deduce the commutativity of:

$$\begin{tikzcd}
 & F^{p,n} \arrow[rr, "\zeta"] \arrow[dl, "\mathrm{coker}(\iota^{n}(p,p+1))"'] \arrow[dd, "\phi^{p}"] & & G \otimes F^{p-1,n} \arrow[dr, "\mathrm{id}_G \otimes \mathrm{coker}(\iota^{n}(p,p-1))"] \arrow[dd, "\mathrm{id}_G \otimes \phi^{p-1}"] & \\
F^{p,n}/F^{p+1,n} \arrow[rrrr, "\bar{\zeta}"] \arrow[dr, "\psi^{p}"'] & & & & G \otimes F^{p-1,n}/F^{p,n} \arrow[dl, "\mathrm{id}_G \otimes \psi^{p-1}"] \\
 & \mathrm{R}^{n}\bar{f}_{*}(\sigma^{=p}\bar{K}) \arrow[rr, "\nabla^{=p,n}"'] & & G \otimes \mathrm{R}^{n}\bar{f}_{*}(\sigma^{=p-1}\bar{K}) &
\end{tikzcd} \tag{1.51}$$

Here, the left and right triangles commute by the definitions of $\phi^{p}$ and $\phi^{p-1}$, respectively. The back square commutes by the commutativity of the lower trapezoid of the diagram in Eq. (1.50). The upper trapezoid of the diagram in Eq. (1.51) commutes by our assumption on $\bar{\zeta}$; see Eq. (1.45). Therefore, the lower trapezoid, which is nothing but Eq. (1.48), commutes taking into account that $\mathrm{coker}(\iota^{n}(p, p+1))$ is an epimorphism. □

**Theorem 1.5.14** *Let $\mathfrak{F}$ be a framework for the Gauß-Manin connection and adopt Notations 1.5.6, 1.5.9, and 1.5.10. Let $n$ and $p$ be integers, $\zeta$ and $\bar{\zeta}$ such that the diagrams in Eqs. (1.43) and 1.48) commute in* $\mathrm{Mod}(g)$*, and $\psi^{p}$ and $\psi^{p-1}$ such that the diagram in Eq. (1.47) commutes in* $\mathrm{Mod}(S)$ *for $\nu \in \{p, p-1\}$. Then the following diagram commutes in* $\mathrm{Mod}(g)$*:*

$$\begin{tikzcd}
F^{p,n}/F^{p+1,n} \arrow[r, "\bar{\zeta}"] \arrow[d, "\kappa^{n}(\sigma^{=p}K)\circ\psi^{p}"'] & G \otimes (F^{p-1,n}/F^{p,n}) \arrow[d, "\mathrm{id}_G \otimes (\kappa^{n}(\sigma^{=p-1}K)\circ\psi^{p-1})"] \\
\mathrm{R}^{n-p}f_{*}(K^{p}) \arrow[r, "\gamma^{p,n-p}_{\mathrm{KS},f}(G,t)"'] & G \otimes \mathrm{R}^{n-p+1}f_{*}(K^{p-1})
\end{tikzcd} \tag{1.52}$$

*Proof* By Proposition 1.5.13, we know that the diagram in Eq. (1.48) commutes in $\mathrm{Mod}(g)$; that is,

$$(\mathrm{id}_G \otimes \psi^{p-1}) \circ \bar{\xi} = \nabla^{=p,n} \circ \psi^p.$$

We too know that

$$\kappa^n : \mathrm{R}^n \bar{f}_* \circ u_* \longrightarrow (\mathrm{id}_S)_* \circ \mathrm{R}^n f_* = \mathrm{R}^n f_*$$

is a natural equivalence of functors from $\mathrm{Com}^+(X)$ to $\mathrm{Mod}(S)$. Proposition 1.4.15 implies that the projection morphism $\pi_f^{n-p+1}(G, K^{p-1})$ is an isomorphism in $\mathrm{Mod}(S)$. Hence, by Lemma 1.5.7, specifically the commutativity of subdiagram ⑧ in Fig. 1.2, we have

$$(\mathrm{id}_G \otimes \kappa^n(\sigma^{=p-1}K)) \circ \nabla^{=p,n} = (\pi_f^{n-p+1}(G, K^{p-1}))^{-1} \circ \delta_f^{n-p}(l^p) \circ \kappa^n(\sigma^{=p}(K)).$$

By Proposition 1.4.21, recalling that $l^p = \Lambda_X^p(t)$, the following identity holds in $\mathrm{Mod}(S)$:

$$\gamma_{\mathrm{KS},f}^{p,n-p}(G,t) = (\pi_f^{n-p+1}(G, K^{p-1}))^{-1} \circ \delta_f^{n-p}(l^p).$$

In conclusion, we obtain the chain of equalities

$$\begin{aligned}
&\big(\mathrm{id}_G \otimes (\kappa^n(\sigma^{=p-1}K) \circ \psi^{p-1})\big) \circ \bar{\xi} \\
&= (\mathrm{id}_G \otimes \kappa^n(\sigma^{=p-1}K)) \circ (\mathrm{id}_G \otimes \psi^{p-1}) \circ \bar{\xi} \\
&= (\mathrm{id}_G \otimes \kappa^n(\sigma^{=p-1}K)) \circ \nabla^{=p,n} \circ \psi^p \\
&= (\pi_f^{n-p+1}(G, K^{p-1}))^{-1} \circ \delta_f^{n-p}(l^p) \circ \kappa^n(\sigma^{=p}(K)) \circ \psi^p \\
&= \gamma_{\mathrm{KS},f}^{p,n-p}(G,t) \circ (\kappa^n(\sigma^{=p}K) \circ \psi^p),
\end{aligned}$$

which was to be demonstrated. □

## 1.6 The Gauß-Manin Connection

In what follows, we basically apply the results established in Sect. 1.5 in a more concrete and geometric situation. Our predominant goal is to prove Theorem 1.6.14, which corresponds to the former Theorem 1.5.14.

"Geometric," for one thing, means the we are dealing with complex spaces.

**Notation 1.6.1** Let $(f, g)$ be a composable pair in the category of complex spaces—that is, an ordered pair of morphisms such that the codomain of $f$ equals the domain of $g$. Put $h := g \circ f$. Then we denote by $\Omega^1(f, g)$ the *triple of* 1-*differentials*,

$$f^*\Omega^1_g \xrightarrow{\alpha} \Omega^1_h \xrightarrow{\beta} \Omega^1_f,$$

which we have associated to the pair $(f, g)$ [9, Sect. 2]. Observe that $\Omega^1(f, g)$ is a triple of modules on $X := \mathrm{dom}(f)$. Moreover, observe that the pair $(\Omega^1_g, \Omega^1(f, g))$ is an object of $\mathrm{Trip}(f)$; see Definition 1.4.18.

We know that the triple $\Omega^1(f, g)$ is right exact on $X$ [9, Corollaire 4.5]. Furthermore, $\Omega^1(f, g)$ is short exact on $X$ whenever the morphism $f$ is submersive [9, Remarque 4.6].

**Definition 1.6.2** When $f : X \to S$ is a morphism of complex spaces, we denote by $\Theta_f$ the (relative) *tangent sheaf* of $f$; that is,

$$\Theta_f := \mathscr{H}om_X(\Omega^1_f, \mathscr{O}_X).$$

As a special case we set $\Theta_X := \Theta_{a_X}$, where $a_X : X \to \mathbf{e}$ denotes the unique morphism of complex spaces from $X$ to the distinguished one-point complex space.

**Definition 1.6.3** Let $(f, g)$ be a composable pair in the category of submersive complex spaces. Write $f : X \to S$. Then $(\Omega^1_g, \Omega^1(f, g))$ is an object of $\mathrm{Trip}(f)$ and $\Omega^1(f, g)$ is a short exact triple of modules on $X$. Besides, $(\Omega^1(f, g))(2) = \Omega^1_f$ and $\Omega^1_g$ are locally finite free modules on $X$ and $S$, respectively. Therefore, it makes sense to define

$$\xi_{\mathrm{KS}}(f, g) := \xi_{\mathrm{KS},f}(\Omega^1_g, \Omega^1(f, g)),$$

where the right-hand side is understood in the sense of Construction 1.4.19. Observe that, by definition, $\xi_{\mathrm{KS}}(f, g)$ is a morphism

$$\xi_{\mathrm{KS}}(f, g) : \mathscr{O}_S \longrightarrow \Omega^1_g \otimes_S \mathrm{R}^1 f_*(\Theta_f)$$

of modules on $S$.

Furthermore, in case a single submersive morphism of complex spaces $f$ with smooth $S = \mathrm{cod}(f)$ is given, we set $\xi_{\mathrm{KS}}(f) := \xi_{\mathrm{KS}}(f, a_S)$, where $a_S : S \to \mathbf{e}$ denotes the unique morphism from $S$ to the distinguished one-point complex space and the $\xi_{\mathrm{KS}}$ on the right-hand side is understood in the already defined sense. Observe that the previous definition can be applied since, with $S$ being smooth, the morphism of complex spaces $a_S$ is submersive. We call $\xi_{\mathrm{KS}}(f, g)$ (resp. $\xi_{\mathrm{KS}}(f)$) the *Kodaira-Spencer class* of $(f, g)$ (resp. $f$).

**Definition 1.6.4** Let again $(f, g)$ be a composable pair in the category of submersive complex spaces. Moreover, let $p$ and $q$ be integers. Then we define

$$\gamma^{p,q}_{\mathrm{KS}}(f, g) := \gamma^{p,q}_{\mathrm{KS},f}(\Omega^1_g, \Omega^1(f, g)),$$

where we interpret the right-hand side in the sense of Construction 1.4.20. Thus, $\gamma^{p,q}_{\mathrm{KS}}(f,g)$ is a morphism

$$\gamma^{p,q}_{\mathrm{KS}}(f,g) : \mathrm{R}^q f_*(\Omega^p_f) \longrightarrow \Omega^1_g \otimes_S \mathrm{R}^{q+1} f_*(\Omega^{p-1}_f)$$

of modules on $S := \mathrm{cod}(f) = \mathrm{dom}(g)$.

Just like in Definition 1.6.3, we define $\gamma^{p,q}_{\mathrm{KS}}(f) := \gamma^{p,q}_{\mathrm{KS}}(f, a_S)$ as a special case when only a single submersive morphism of complex spaces $f$ with smooth codomain is given. We call $\gamma^{p,q}_{\mathrm{KS}}(f,g)$ (resp. $\gamma^{p,q}_{\mathrm{KS}}(f)$) the *cup and contraction with the Kodaira-Spencer class* in bidegree $(p,q)$ for $(f,g)$ (resp. $f$).

**Proposition 1.6.5** *Let $(f,g)$ be a composable pair in the category of submersive complex spaces, $p$ and $q$ integers. Then the following identity holds in* $\mathrm{Mod}(S)$*, where we write $f : X \to S$:*

$$\delta^q_f(\Lambda^p_X(\Omega^1(f,g))) = \pi^{q+1}_f(\Omega^1_g, \Omega^{p-1}_f) \circ \gamma^{p,q}_{\mathrm{KS}}(f,g).$$

*Proof* Apply Proposition 1.4.21 to the morphism of ringed spaces $f$ and the object $(\Omega^1_g, \Omega^1(f,g))$ of $\mathrm{Trip}(f)$. □

**Construction 1.6.6** Let $f : X \to S$ and $g : S \to T$ be morphisms of complex spaces. Set $h := g \circ f$. I intend to construct a functor

$$\Omega^\bullet(f,g) : \mathbf{3} \longrightarrow \mathrm{Com}^+(h)$$

(i.e., a triple of bounded below complexes over $\mathrm{Mod}(h)$), which we call the *triple of de Rham complexes* associated to $(f,g)$. In order to simplify the notation, we shorten $\Omega^\bullet(f,g)$ to $\Omega^\bullet$ in what follows.

To begin with, we define the object function of the functor $\Omega^\bullet$. Recall that the set of objects of the category $\mathbf{3}$ is the set $3 = \{0,1,2\}$. We define $\Omega^\bullet(0)$ to be the unique complex over $\mathrm{Mod}(f)$ such that, for all integers $p$, firstly, we have

$$(\Omega^\bullet(0))^p = f^*\Omega^1_g \otimes_X \Omega^{p-1}_f,$$

and, secondly, the following diagram commutes in $\mathrm{Mod}(\bar{X})$:

$$\begin{tikzcd}
\bar{f}^*\Omega^1_g \otimes_{\bar{X}} \bar{\Omega}^{p-1}_f \arrow[r, "\mathrm{id}_{\bar{f}^*\Omega^1_g} \otimes \mathrm{d}^{p-1}_f"] \arrow[d, "\gamma^p"'] & \bar{f}^*\Omega^1_g \otimes_{\bar{X}} \bar{\Omega}^p_f \arrow[d, "\gamma^{p+1}"] \\
u_*(f^*\Omega^1_g \otimes_X \Omega^{p-1}_f) \arrow[r, dotted, "\mathrm{d}^p_{\Omega^\bullet(0)}"'] & u_*(f^*\Omega^1_g \otimes_X \Omega^p_f)
\end{tikzcd}$$

Here, $\bar{X}$ denotes the ringed space $(X_{\text{top}}, f^{-1}\mathscr{O}_S)$, and $\bar{f} : \bar{X} \to S$ denotes the morphism of ringed spaces that is given by $f_{\text{top}}$ on topological spaces and by the adjunction morphism from $\mathscr{O}_S$ to $(f_{\text{top}})_* f^{-1}\mathscr{O}_S$ on structure sheaves. $u : X \to \bar{X}$ stands for the morphism of ringed spaces that is given by $\mathrm{id}_{X_{\text{top}}}$ on topological spaces and by

$$f^\sharp : f^{-1}\mathscr{O}_S \longrightarrow \mathscr{O}_X = (\mathrm{id}_{X_{\text{top}}})_* \mathscr{O}_X$$

on structure sheaves. Moreover, $\bar{\Omega}_f^\nu := u_*(\Omega_f^\nu)$ for all integers $\nu$. Observe that since $u$ is the identity on topological spaces, $\bar{\Omega}_f^\nu$ and $\Omega_f^\nu$ agree as abelian sheaves—only their module structure differs. In fact, the module structure of $\bar{\Omega}_f^\nu$ is obtained from the module structure of $\Omega_f^\nu$ by relaxing the latter via the morphism of sheaves of rings $u^\sharp = f^\sharp : f^{-1}\mathscr{O}_S \to \mathscr{O}_X$. Finally, $\gamma^\nu$, for any integer $\nu$, signifies the composition of the following morphisms in $\mathrm{Mod}(\bar{X})$:

$$\begin{aligned} \bar{f}^*\Omega_g^1 \otimes_{\bar{X}} \bar{\Omega}_f^{\nu-1} &\longrightarrow u_*(u^*\bar{f}^*\Omega_g^1) \otimes_{\bar{X}} u_*(\Omega_f^{\nu-1}) \\ &\longrightarrow u_*(u^*\bar{f}^*\Omega_g^1 \otimes_X \Omega_f^{\nu-1}) \longrightarrow u_*(f^*\Omega_g^1 \otimes_X \Omega_f^{\nu-1}). \end{aligned}$$

In order to define $\Omega^\bullet(1)$, denote by $K^p = (K^{p,i})_{i \in \mathbf{Z}}$ the Koszul filtration in degree $p$ induced by

$$(\Omega^1(f,g))_{0,1} : f^*\Omega_g^1 \longrightarrow \Omega_h^1$$

on $X$. Then, for all integers $p$ and $i$, you verify easily that the differential $\mathrm{d}_h^p$ of the complex $\Omega_h^\bullet$ maps $K^{p,i}$ into $K^{p+1,i}$. Thus, we dispose of a quotient complex

$$\Omega^\bullet(1) := \Omega_h^\bullet / K^{\bullet,2}.$$

To finish the definition of the object function of $\Omega^\bullet$, we set

$$\Omega^\bullet(2) := \Omega_f^\bullet.$$

The morphism function of $\Omega^\bullet$ is defined so that, for all ordered pairs $(x, y)$ of objects of $\mathbf{3}$ (i.e., all elements $(x, y) \in 3 \times 3$) with $x \leq y$, we have

$$((\Omega^\bullet)_{x,y})^p = \left(\Lambda_X^p(\Omega^1(f,g))\right)_{x,y}$$

for all integers $p$. To verify that the so defined $\Omega^\bullet$ is a functor from $\mathbf{3}$ to $\mathrm{Com}^+(h)$, you have to check essentially that $(\Omega^\bullet)_{0,1}$ (resp. $(\Omega^\bullet)_{1,2}$) constitutes a morphism of complexes over $\mathrm{Mod}(h)$ from $\Omega^\bullet(0)$ to $\Omega^\bullet(1)$ (resp. from $\Omega^\bullet(1)$ to $\Omega^\bullet(2)$). This amounts to checking that the morphisms defined by the $\Lambda^p$ construction commute with the differentials of the respective complexes $\Omega^\bullet(x)$, for $x \in 3$, that we have introduced here. In case of $(\Omega^\bullet)_{1,2}$ the desired commutativity is rather obvious since the wedge powers of the morphism

$$(\Omega^1(f,g))_{1,2} : \Omega^1_h \longrightarrow \Omega^1_f$$

form a morphism of complexes over $\mathrm{Mod}(h)$ from $\Omega^\bullet_h$ to $\Omega^\bullet_f$. In case of $(\Omega^\bullet)_{0,1}$ the compatibility is harder to establish as the definition of $(\Lambda^p(t))_{0,1}$, for some right exact triple $t$ of modules on $X$, is more involved (see Construction 1.2.8). Nevertheless, I omit these details.

**Proposition 1.6.7** *Let $(f,g)$ be a composable pair in the category of submersive complex spaces. Then*

$$\mathfrak{F}(f,g) := (f, g, \Omega^1_g, \Omega^1(f,g), \Omega^\bullet(f,g))$$

*is a framework for the Gauß-Manin connection, where the $f$ and $g$ in the first and second component of the quintuple stand for the morphisms of ringed spaces obtained from the original $f$ and $g$ applying the forgetful functor* $\mathbf{An} \to \mathbf{Sp}$.

*Proof* Let $f : X \to S$ and $g : S \to T$ be submersive morphisms of complex spaces. By abuse of notation, we denote by $f$ and $g$, too, the morphisms of ringed spaces obtained respectively from $f$ and $g$ applying the forgetful functor from the category of complex spaces to the category of ringed spaces. Set $G := \Omega^1_g$ and $t := \Omega^1(f,g)$; see Notation 1.6.1. Then clearly, $(G,t)$ is an object of $\mathrm{Trip}(f)$; see Definition 1.4.18. As $f$ is a submersive morphism of complex spaces, we know that $t2 = \Omega^1_f$ is a locally finite free module on $X$ and $t$ is a short exact triple of modules on $X$. Since $g$ is a submersive morphism of complex spaces, $G$ is a locally finite free module on $S$.

Set $l := \Omega^\bullet(f,g)$ (see Construction 1.6.6). Then $l : L \to M \to K$ is a triple in $\mathrm{Com}^+(h)$, where $h := g \circ f$, and $K$ and $L$ are objects of $\mathrm{Com}^+(f)$. Moreover, for all integers $p$, we have $l^p = \Lambda^p_X(t)$, where $l^p$ stands for the triple in $\mathrm{Mod}(h)$ which is obtained extracting the degree-$p$ part from the triple of complexes $l$. Define $\gamma$, $\bar{f} : \bar{X} \to S$, $\bar{K}$, and $\bar{L}$ just as in Definition 1.5.3. Then $\gamma$ is a morphism in $\mathrm{Com}^+(\bar{X})$,

$$\gamma : \bar{f}^* G \otimes_{\bar{X}} (\bar{K}[-1]) \longrightarrow \bar{L},$$

by the very definition of the differentials of the complex $L = l(0) = (\Omega^\bullet(f,g))(0)$; see Construction 1.6.6. All in all, we see that with the morphisms of ringed spaces $f$ and $g$, with $(G,t)$, and with $l$, we are in the situation of Definition 1.5.3. $\square$

**Definition 1.6.8**

1. Let $(f,g)$ be a composable pair in the category of submersive complex spaces, $n$ an integer. Then we define

$$\nabla^n_{\mathrm{GM}}(f,g) := \nabla^n,$$

with $\nabla^n$, for $\mathfrak{F} = \mathfrak{F}(f, g)$, as in Notation 1.5.9. Note that this makes sense due to Proposition 1.6.7. We call $\nabla^n_{\mathrm{GM}}(f, g)$ the $n$th *Gauß-Manin connection* of $(f, g)$.

2. Let $f : X \to S$ be a submersive morphism of complex spaces such that the complex space $S$ is smooth. Let $n$ be an integer. Then we set

$$\nabla^n_{\mathrm{GM}}(f) := \nabla^n_{\mathrm{GM}}(f, a_S),$$

where $a_S : S \to \mathbf{e}$ denotes the unique morphism of complex spaces from $S$ to the distinguished one-point complex space. Observe that it makes sense to employ item 1 on the right-hand side since, given that the complex space $S$ is smooth, the morphism of complex spaces $a_S$ is submersive. We call $\nabla^n_{\mathrm{GM}}(f)$ the $n$th *Gauß-Manin connection* of $f$.

**Definition 1.6.9** Let $f$ be a morphism of complex spaces, $n$ an integer. Then we define

$$\mathscr{H}^n(f) := \mathrm{R}^n\bar{f}_*(\bar{\Omega}^\bullet_f),$$

where $\bar{f}$ and $\bar{\Omega}^\bullet_f$ have the same meaning as in Construction 1.6.6. We call $\mathscr{H}^n(f)$ the $n$th *algebraic de Rham module* of $f$.

**Construction 1.6.10** Let $f : X \to S$ be a morphism of complex spaces and $n$ an integer. Then, for any integer $p$, we set

$$\mathrm{F}^p\mathscr{H}^n(f) := \mathrm{im}_S(\mathrm{R}^n\bar{f}_*(i^{\geq p}\bar{\Omega}^\bullet_f) : \mathrm{R}^n\bar{f}_*(\sigma^{\geq p}\bar{\Omega}^\bullet_f) \longrightarrow \mathrm{R}^n\bar{f}_*(\bar{\Omega}^\bullet_f)),$$

in the sense that $\mathrm{F}^p\mathscr{H}^n(f)$ is a submodule of $\mathscr{H}^n(f)$ on $S$. Moreover, we write

$$\iota^n_f(p) : \mathrm{F}^p\mathscr{H}^n(f) \longrightarrow \mathscr{H}^n(f)$$

for the corresponding inclusion morphism of sheaves on $S_{\mathrm{top}}$ (note that $\mathscr{H}^n(f) = \mathrm{R}^n\bar{f}_*(\bar{\Omega}^\bullet_f)$ according to Definition 1.6.9). Furthermore, for any integer $p$, we denote by $\lambda^n_f(p)$ the unique morphism such that the following diagram commutes in $\mathrm{Mod}(S)$:

$$\begin{tikzcd}
\mathrm{R}^n\bar{f}_*(\sigma^{\geq p}\bar{\Omega}^\bullet_f) \arrow[rr, "\mathrm{R}^n\bar{f}_*(i^{\geq p}\bar{\Omega}^\bullet_f)"] \arrow[dr, dotted, "\lambda^n_f(p)"'] & & \mathrm{R}^n\bar{f}_*(\bar{\Omega}^\bullet_f) \\
& \mathrm{F}^p\mathscr{H}^n(f) \arrow[ur, "\iota^n_f(p)"'] &
\end{tikzcd}$$

Obviously, the sequence $(\mathrm{F}^p\mathscr{H}^n(f))_{p\in\mathbf{Z}}$ makes up a descending sequence of submodules of $\mathscr{H}^n(f)$ on $S$. In more formal terms, we may express this observation by saying that, for all integers $p, p'$ such that $p \leq p'$, there exists a unique morphism

$\iota_f^n(p, p')$ such that the following diagram commutes in $\mathrm{Mod}(S)$:

$$\begin{tikzcd}
 & \mathscr{H}^n(f) & \\
\mathrm{F}^p \mathscr{H}^n(f) \arrow[ur, "\iota_f^n(p)"] & & \mathrm{F}^{p'} \mathscr{H}^n(f) \arrow[ul, "\iota_f^n(p')"'] \arrow[ll, dotted, "\iota_f^n(p,p')"]
\end{tikzcd}$$

**Proposition 1.6.11** *Let $(f, g)$ be a composable pair in the category of submersive complex spaces, $n$ and $p$ integers. Then there exists one, and only one, ordered pair $(\zeta, \bar{\zeta})$ such that, abbreviating $\mathrm{F}^\nu \mathscr{H}^n(f)$ to $F^\nu$ for integers $\nu$, the following diagram commutes in* $\mathrm{Mod}(g)$*:*

$$\begin{tikzcd}
\mathscr{H}^n(f) \arrow[r, "\nabla^n_{\mathrm{GM}}(f,g)"] & \Omega^1_g \otimes \mathscr{H}^n(f) \\
F^p \arrow[u, "\iota_f^n(p)"] \arrow[r, dotted, "\zeta"] \arrow[d, "\mathrm{coker}(\iota_f^n(p,p+1))"'] & \Omega^1_g \otimes F^{p-1} \arrow[u, "\mathrm{id}_{\Omega^1_g} \otimes \iota_f^n(p-1)"'] \arrow[d, "id_{\Omega^1_g} \otimes \mathrm{coker}(\iota_f^n(p-1,p))"] \\
F^p / F^{p+1} \arrow[r, dotted, "\bar{\zeta}"'] & \Omega^1_g \otimes (F^{p-1}/F^p)
\end{tikzcd} \tag{1.53}$$

*Proof* This is an immediate consequence of Propositions 1.6.7, 1.5.11, and 1.5.12. □

**Definition 1.6.12** Let $(f, g)$ be a composable pair in the category of submersive complex spaces, $n$ and $p$ integers. Then we define

$$\overline{\nabla}^{p,n}_{\mathrm{GM}}(f, g) := \bar{\zeta},$$

where $(\zeta, \bar{\zeta})$ is the unique ordered pair such that the diagram in Eq. (1.53)—we abbreviate $\mathrm{F}^\nu \mathscr{H}^n(f)$ to $F^\nu$ for integers $\nu$ again—commutes in $\mathrm{Mod}(g)$. Note that this definition makes sense due to Proposition 1.6.11.

**Definition 1.6.13** Let $f$ be a morphism of complex spaces, $p$ and $q$ integers. Then we put

$$\mathscr{H}^{p,q}(f) := \mathrm{R}^q f_*(\Omega^p_f).$$

We call $\mathscr{H}^{p,q}(f)$ the *Hodge module* in bidegree $(p, q)$ of $f$.

**Theorem 1.6.14** *Let $(f, g)$ be a composable pair in the category of submersive complex spaces, $n$ and $p$ integers, $F^\mu = \mathrm{F}^\mu \mathscr{H}^n(f)$ for all integers $\mu$. Let $\psi^p$ and $\psi^{p-1}$ be such that the following diagram commutes in* $\mathrm{Mod}(S)$ *for $\nu \in \{p, p-1\}$:*

$$\begin{tikzcd}
\mathrm{R}^n\bar{f}_*(\sigma^{\geq\nu}\bar{\Omega}^\bullet_f) \arrow[r, "\lambda^n_f(\nu)"] \arrow[d, "\mathrm{R}^n\bar{f}_*(j^{\leq\nu}(\sigma^{\geq\nu}\bar{\Omega}^\bullet_f))"'] & F^\nu \arrow[d, "\mathrm{coker}(\iota^n_f(\nu,\nu+1))"] \\
\mathrm{R}^n\bar{f}_*(\sigma^{=\nu}\bar{\Omega}^\bullet_f) & F^\nu/F^{\nu+1} \arrow[l, dotted, "\psi^\nu"]
\end{tikzcd} \tag{1.54}$$

*Then the following diagram commutes in* $\mathrm{Mod}(S)$*:*

$$\begin{tikzcd}
F^p/F^{p+1} \arrow[r, "\overline{\nabla}^{p,n}_{\mathrm{GM}}(f,g)"] \arrow[d, "\kappa^n_f(\sigma^{=p}\Omega^\bullet_f)\circ\psi^p"'] & \Omega^1_g \otimes (F^{p-1}/F^p) \arrow[d, "\mathrm{id}_{\Omega^1_g}\otimes(\kappa^n_f(\sigma^{=p-1}\Omega^\bullet_f)\circ\psi^{p-1})"] \\
\mathscr{H}^{p,n-p}(f) \arrow[r, "\gamma^{p,n-p}_{\mathrm{KS}}(f,g)"'] & \Omega^1_g \otimes \mathscr{H}^{p-1,n-p+1}(f)
\end{tikzcd} \tag{1.55}$$

*Proof* According to Proposition 1.6.7, $\mathfrak{F}(f, g)$ is a framework for the Gauß-Manin connection. By Proposition 1.6.11, there exists an ordered pair $(\zeta, \bar{\zeta})$ such that the diagram in Eq. (1.53) commutes in $\mathrm{Mod}(g)$. Therefore, Theorem 1.5.14 implies that the diagram in Eq. (1.55) commutes in $\mathrm{Mod}(g)$, for we have $\overline{\nabla}^{p,n}_{\mathrm{GM}}(f, g) = \bar{\zeta}$ in virtue of Definition 1.6.12. □

## 1.7 Generalities on Period Mappings

This section, as well as the next, are devoted to the study of period mappings—namely, in a very broad sense of the word. Construction 1.7.3 captures the common basis for any sort of period mapping that we are going consider.

I like the idea of defining a period map in the situation where a representation

$$\rho : \Pi(X) \longrightarrow \mathrm{Mod}(A)$$

of the fundamental groupoid of some topological space $X$ is given, $A$ being some ring. Typically, people define period mappings in the situation where an $A$-local system on $X$ (i.e., a certain locally constant sheaf of $A_X$-modules on $X$) is given. The reason for my preferring representations over locally constant sheaves is of technical nature. When working with local systems in the sense of sheaves, you are bound to use a stalk of the given sheaf as the reference space for the period mapping.

When working with representations of the fundamental groupoid, however, you are at liberty to choose the reference space at will (where "at will" means up to isomorphism). The more familiar setting of working with locally constant sheaves becomes a special case of the representation setting by Construction 1.7.4 and Remark 1.7.5.

Eventually, I am interested in holomorphic period mappings. These arise from Construction 1.7.11, where the underlying local system comes about as the module of horizontal sections associated to a flat vector bundle. Lemma 1.7.20 will give a preliminary conceptual interpretation of the tangent morphism of such a period mapping. This interpretation will be exploited in Sect. 1.8 in order to derive, from Theorem 1.6.14, the concluding theorems of this chapter.

**Construction 1.7.1** Let $X$ be a topological space. Then we denote by $\Pi(X)$ the *fundamental groupoid* of $X$ [12, Chap. 2, §5]. Recall that $\Pi(X)$ is a category satisfying the following properties.

1. The set of objects of $\Pi(X)$ is $|X|$—that is, the set underlying $X$.
2. For any two elements $x$ and $y$ of $X$ (i.e., of $|X|$), the set of morphisms from $x$ to $y$ in $\Pi(X)$ is the set of continuous maps $\gamma : I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, modulo endpoint preserving homotopy. Here, $I$ signifies the unit interval $[0, 1]$ endowed with its Euclidean topology.
3. When $[\gamma] : x \to y$ and $[\delta] : y \to z$ are morphisms in $\Pi(X)$, then we have $[\delta] \circ [\gamma] = [\delta * \gamma]$ for the composition in $\Pi(X)$, where $\delta * \gamma$ stands for the habitual concatenation of paths.
4. The identities in $\Pi(X)$ are the residue classes of the constant maps to $X$.

**Definition 1.7.2** Let $A$ be a ring and $G$ a groupoid (or just any category for that matter).

1. We say that $\rho$ is an *$A$-representation* of $G$ when $\rho$ is a functor from $G$ to $\mathrm{Mod}(A)$.
2. Let $\rho$ be an $A$-representation of $G$. Then $F$ is called an *$A$-distribution* in $\rho$ when $F$ is a function whose domain of definition equals $\mathrm{dom}(\rho_0)$ (which, in turn, equals $G_0$—that is, the set of objects of the category $G$) such that $F(s)$ is an $A$-submodule of $\rho_0(s)$ for all $s \in \mathrm{dom}(\rho_0)$.

**Construction 1.7.3** Let $A$ be a ring, $S$ a simply connected topological space, $\rho$ an $A$-representation of $\Pi(S)$, $F$ an $A$-distribution in $\rho$, and $t \in S$. Since $S$ is simply connected, we know that, for all $s \in S$, there exists a unique morphism $a_{s,t}$ from $s$ to $t$ in $\Pi(S)$; that is, $a_{s,t}$ is the unique element of $(\Pi(S))_1(s, t)$. We define $\mathcal{P}^A_t(S, \rho, F)$ to be the unique function on $|S|$ such that, for all $s \in S$, we have

$$(\mathcal{P}^A_t(S, \rho, F))(s) = ((\rho_1(s, t))(a_{s,t}))\,[F(s)],$$

where I use square brackets to refer to the set-theoretic image of a set under a function. $\mathcal{P}^A_t(S, \rho, F)$ is called the *$A$-period mapping* on $S$ with basepoint $t$ associated to $\rho$ and $F$.

To make this definition somewhat clearer, observe the following. For all $s \in S$, we know that $\rho_1(s,t)(a_{s,t})$ is an element of $\mathrm{Hom}_A(\rho(s), \rho(t))$—that is, an $A$-linear map from $\rho(s)$ to $\rho(t)$. As a matter of fact, since $a_{s,t}$ is an isomorphism in $\Pi(S)$, the mapping $\rho_1(s,t)(a_{s,t})$ constitutes an isomorphism from $\rho(s)$ to $\rho(t)$ in $\mathrm{Mod}(A)$. Moreover, $F(s)$ is nothing but an $A$-linear subset of $\rho(s)$. Hence, the value of the period mapping at $s$ is nothing but an isomorphic image of $F(s)$ in the reference module $\rho(t)$ sitting over the basepoint.

**Construction 1.7.4** Let $A$ be a ring and $X$ a connected topological space. Let $F$ be a constant sheaf of $A_X$-modules on $X$. We define a functor

$$\rho : \Pi(X) \longrightarrow \mathrm{Mod}(A)$$

as follows: In the first place, we let $\rho_0$ be the unique function on $(\Pi(X))_0$ $(= |X|)$ such that, for all $x \in X$, we have

$$\rho_0(x) = F_x,$$

where the stalk $F_x$ is understood to be equipped with its induced $A$-module structure. In the second place, we observe that, for all $x \in X$, the residue map

$$\theta_x : F(X) \longrightarrow F_x$$

is a bijection since $F$ is a constant sheaf on $X$ and the topological space $X$ is connected. For all ordered pairs $(x, y)$ of elements of $|X|$, we define $\rho_1(x, y)$ to be the constant function on $(\Pi(X))_1(x, y)$ with value $\theta_y \circ (\theta_x)^{-1}$; that is, for all morphisms $a : x \to y$ in $\Pi(X)$ we have

$$(\rho_1(x, y))(a) = \theta_y \circ (\theta_x)^{-1}.$$

Observe that the latter is an $A$-linear map from $F_x$ to $F_y$. Now set $\rho := (\rho_0, \rho_1)$. Then clearly, $\rho$ is a functor from $\Pi(X)$ to $\mathrm{Mod}(A)$.

*Remark 1.7.5* Construction 1.7.4 is a special case of a more general construction which associates—given a ring $A$ and an arbitrary topological space $X$—to a locally constant sheaf $F$ of $A_X$-modules on $X$ an $A$-representation $\rho$ of the fundamental groupoid of $X$. I briefly sketch how this can be achieved.

The object function of $\rho$ is defined just as before; that is, we set $\rho_0(x) := F_x$ for all $x \in X$. The morphism function of $\rho$, however, is harder to define when $F$ is not a constant sheaf but only a locally constant sheaf on $X$. Let $x, y \in X$ and $a \in (\Pi(X))_1(x, y)$. Let $\gamma \in a$; that is, $\gamma : I \to X$ is a path in $X$ representing $a$, where $I$ stands for the Euclidean topologized unit interval $[0, 1]$. Then $\gamma^*F$ is a constant sheaf on $I$. Therefore, one obtains a mapping $(\gamma^*F)_0 \to (\gamma^*F)_1$ in the same fashion as in Construction 1.7.4 by passing through the set of global sections of $\gamma^*F$ on $I$. Plugging in the canonical bijections $(\gamma^*F)_0 \to F_x$ and $(\gamma^*F)_1 \to F_y$, we arrive at a function $F_x \to F_y$. After checking that the latter function $F_x \to F_y$ is independent of the choice $\gamma$ in $a$, we may define $(\rho_1(x, y))(a)$ accordingly. As you might imagine, verifying that $(\rho_1(x, y))(a)$ is independent of $\gamma$ is a little tedious, hence I omit it.

Next, you have to verify that the so defined $\rho$ is a functor from $\Pi(X)$ to $\mathrm{Mod}(A)$. This, again, turns out to be a little less obvious than in the "baby case" of Construction 1.7.4. Finally, you should convince yourself that in case $F$ is a constant sheaf on $X$ and $X$ is a connected topological space, the $\rho$ defined here agrees with the $\rho$ of Construction 1.7.4.

**Definition 1.7.6** Let $S$ be a complex space and $\mathscr{H}$ a module on $S$.

1. Let $g : S \to T$ be a morphism of complex spaces. Then $\nabla$ is called a *$g$-connection* on $\mathscr{H}$ when $\nabla$ is a morphism in $\mathrm{Mod}(g)$,
$$\nabla : \mathscr{H} \longrightarrow \Omega^1_g \otimes_S \mathscr{H},$$
such that for all open sets $U$ of $S$, all $\lambda \in \mathscr{O}_S(U)$, and all $\sigma \in \mathscr{H}(U)$ the Leibniz rule holds:
$$\nabla_U(\lambda \cdot \sigma) = (\mathrm{d}_g)_U(\lambda) \otimes \sigma + \lambda \cdot \nabla_U(\sigma).$$
2. $\nabla$ is called an *$S$-connection* on $\mathscr{H}$ when $\nabla$ is a $a_S$-connection on $\mathscr{H}$ in the sense of item 1, where $a_S : S \to \mathbf{e}$ denotes the unique morphism of complex spaces from $S$ to the distinguished one-point complex space.

**Construction 1.7.7** Let $g : S \to T$ be a morphism of complex spaces, $\mathscr{H}$ a module on $S$, and $\nabla$ a $g$-connection on $\mathscr{H}$. Put $S' := (S_{\mathrm{top}}, g^{-1}\mathscr{O}_T)$ and let $c : S \to S'$ be the morphism of ringed spaces given by
$$(\mathrm{id}_{|S|}, g^\sharp : g^{-1}\mathscr{O}_T \longrightarrow \mathscr{O}_S).$$
Then $\nabla$ is a morphism of modules on $S'$ from $c_*(\mathscr{H})$ to $c_*(\Omega^1_g \otimes_S \mathscr{H})$. Thus, it makes sense to set
$$\mathrm{Hor}_g(\mathscr{H}, \nabla) := \ker_{S'}(\nabla : c_*(\mathscr{H}) \longrightarrow c_*(\Omega^1_g \otimes_S \mathscr{H})).$$
Note that by definition, $\mathrm{Hor}_g(\mathscr{H}, \nabla)$ is a module on $S'$. We call $\mathrm{Hor}_g(\mathscr{H}, \nabla)$ the *module of horizontal sections* of $(\mathscr{H}, \nabla)$ relative $g$.

When instead of $g : S \to T$ merely a single complex space $S$ is given, and $\nabla$ is an $S$-connection on $\mathscr{H}$, we set
$$\mathrm{Hor}_S(\mathscr{H}, \nabla) := \mathrm{Hor}_{a_S}(\mathscr{H}, \nabla),$$
where the right-hand side is understood in the already defined sense. $\mathrm{Hor}_S(\mathscr{H}, \nabla)$ is then called the *module of horizontal sections* of $(\mathscr{H}, \nabla)$ on $S$.

**Definition 1.7.8** Let $g : S \to T$ be a morphism of complex spaces, $\mathscr{H}$ a module on $S$, and $\nabla$ a $g$-connection on $\mathscr{H}$. Let $p$ be a natural number. Then there exists a unique morphism

$$\nabla^p : \Omega_g^p \otimes \mathscr{H} \longrightarrow \Omega_g^{p+1} \otimes \mathscr{H}$$

in $\mathrm{Mod}(g)$ such that for all open sets $U$ of $S$, all $\alpha \in \Omega_g^p(U)$, and all $\sigma \in \mathscr{H}(U)$, we have

$$(\nabla^p)_U(\alpha \otimes \sigma) = (\mathrm{d}_g^p)_U(\alpha) \otimes \sigma + (-1)^p \Lambda_U(\alpha \otimes \nabla_U(\sigma)),$$

where $\Lambda$ stands for the composition of the following morphisms in $\mathrm{Mod}(S)$:

$$\Omega_g^p \otimes (\Omega_g^1 \otimes \mathscr{H}) \xrightarrow{\alpha^{-1}} (\Omega_g^p \otimes \Omega_g^1) \otimes \mathscr{H} \xrightarrow{\wedge^{p,1}(\Omega_g^1) \otimes \mathrm{id}_{\mathscr{H}}} \Omega_g^{p+1} \otimes \mathscr{H}.$$

The existence of $\nabla^p$ is not completely obvious [2, 2.10], yet we take it for granted here. We say that $\nabla$ is *flat* as a $g$-connection on $\mathscr{H}$ when the composition

$$\nabla^1 \circ \nabla : \mathscr{H} \longrightarrow \Omega_g^2 \otimes \mathscr{H}$$

is a zero morphism in $\mathrm{Mod}(g)$.

**Definition 1.7.9** Let $S$ be a complex space.

1. By a *vector bundle* on $S$ we understand a locally finite free module on $S$.
2. A *flat vector bundle* on $S$ is an ordered pair $(\mathscr{H}, \nabla)$ such that $\mathscr{H}$ is a vector bundle on $S$ and $\nabla$ is a flat $S$-connection on $\mathscr{H}$.
3. Let $\mathscr{H}$ be a vector bundle on $S$. Then $\mathscr{F}$ is a *vector subbundle* of $\mathscr{H}$ on $S$ when $\mathscr{F}$ is a locally finite free submodule of $\mathscr{H}$ on $S$ such that for all $s \in S$ the function

$$\iota(s) : \mathscr{F}(s) \longrightarrow \mathscr{H}(s)$$

   is one-to-one, where $\iota : \mathscr{F} \to \mathscr{H}$ denotes the inclusion morphism.

**Proposition 1.7.10** *Let $S$ be a complex manifold and $(\mathscr{H}, \nabla)$ a flat vector bundle on $S$. Then:*

1. *$H := \mathrm{Hor}_S(\mathscr{H}, \nabla)$ is a locally constant sheaf of $\mathbf{C}_S$-modules on $S_{\mathrm{top}}$.*
2. *The sheaf map*

$$\mathscr{O}_S \otimes_{\mathbf{C}_S} H \longrightarrow \mathscr{O}_S \otimes_{\mathbf{C}_S} \mathscr{H} \longrightarrow \mathscr{H}$$

   *induced by the inclusion $H \subset \mathscr{H}$ and the $\mathscr{O}_S$-scalar multiplication of $\mathscr{H}$ is an isomorphism of modules on $S$.*

*Proof* This is implied by Deligne [2, Théorème 2.17]. □

**Construction 1.7.11** Let $S$ be a simply connected complex manifold, $(\mathscr{H}, \nabla)$ a flat vector bundle on $S$, $\mathscr{F}$ a submodule of $\mathscr{H}$ on $S$, and $t \in S$. Put $H := \mathrm{Hor}_S(\mathscr{H}, \nabla)$.

Then by Proposition 1.7.10, $H$ is a locally constant sheaf of $\mathbf{C}_S$-modules on $S_{\mathrm{top}}$. As the topological space $S_{\mathrm{top}}$ is simply connected, $H$ is even a constant sheaf of $\mathbf{C}_S$-modules on $S_{\mathrm{top}}$. Thus, by means of Construction 1.7.4, we obtain a $\mathbf{C}$-representation $\rho$ of $\Pi(S)$:

$$\rho : \Pi(S) \longrightarrow \mathrm{Mod}(\mathbf{C}).$$

For all $s \in S$, we set $\mathscr{H}(s) := \mathbf{C} \otimes_{\mathscr{O}_{S,s}} \mathscr{H}_s$ (considered as a $\mathbf{C}$-module) and denote by

$$\psi_s : H_s \longrightarrow \mathscr{H}(s)$$

the evident morphism of $\mathbf{C}$-modules. We define a functor

$$\rho' : \Pi(S) \longrightarrow \mathrm{Mod}(\mathbf{C})$$

by composing $\rho$ with the family $(\psi_s)_{s\in S}$; explicitly, this means we set

$$\rho'_0(s) := \mathscr{H}(s)$$

for all $s \in S$ and

$$(\rho'{}_1(x, y))(a) := \psi_y \circ (\rho_1(x, y))(a) \circ (\psi_x)^{-1}$$

for all $x, y \in S$ and all morphisms $a : x \to y$ in $\Pi(S)$. You will validate without effort that the so declared $\rho'$ is in fact a functor from $\Pi(S)$ to $\mathrm{Mod}(\mathbf{C})$.

Next, define $F$ to be the unique function on $|S|$ such that, for all $s \in S$, we have

$$F(s) = \mathrm{im}(\iota(s) : \mathscr{F}(s) \longrightarrow \mathscr{H}(s)),$$

where $\mathscr{F}(s) := \mathbf{C} \otimes_{\mathscr{O}_{S,s}} \mathscr{F}_s$ and $\iota(s)$ stands for the morphism derived from the inclusion morphism $\mathscr{F} \to \mathscr{H}$. Then clearly, $F$ is a $\mathbf{C}$-distribution in $\rho'$ (see item 2 of Definition 1.7.2). Therefore, it makes sense to set

$$\mathcal{P}_t(S, (\mathscr{H}, \nabla), \mathscr{F}) := \mathcal{P}_t^{\mathbf{C}}(S_{\mathrm{top}}, \rho', F),$$

where the right-hand side is to be understood in the sense of Construction 1.7.3.

**Notation 1.7.12** Let $V$ be a finite dimensional $\mathbf{C}$-vector space. Then $\mathrm{Gr}(V)$ denotes the *Grassmannian* of $V$, regarded as a complex space. Let me amplify this a little.

First of all, set-theoretically, $\mathrm{Gr}(V)$ is nothing but the set of all $\mathbf{C}$-linear subsets (i.e., $\mathbf{C}$-vector subspaces) of $V$; that is,

$$|\mathrm{Gr}(V)| = \{W : W \text{ is a } \mathbf{C}\text{-linear subset of } V\}.$$

Note that many authors look only at subspaces $W$ of $V$ which are of a certain prescribed dimension—I look at subspaces of all dimensions at once. Secondly, you define a topology as well as a complex structure on $|\mathrm{Gr}(V)|$ by means of charts [14, Proposition 10.5]. I refrain from explaining the details. Lastly, as a technicality, you transform the obtained manifold into a complex space by means of the standard procedure; that is, $\mathscr{O}_{\mathrm{Gr}(V)}$ is the sheaf of holomorphic (in the chart sense) functions, and the morphism of ringed spaces $\mathrm{Gr}(V) \to \mathbf{e}$ identifies the constant maps from $|\mathrm{Gr}(V)|$ to $\mathbf{C}$.

**Proposition 1.7.13** *Let $S$ be a simply connected complex manifold, $(\mathscr{H}, \nabla)$ a flat vector bundle on $S$, $\mathscr{F}$ a vector subbundle of $\mathscr{H}$ on $S$, and $t \in S$. Then $\mathcal{P} := \mathcal{P}_t(S, (\mathscr{H}, \nabla), \mathscr{F})$ is a holomorphic map from $S$ to $\mathrm{Gr}(\mathscr{H}(t))$.*

*Proof* Set $H := \mathrm{Hor}_S(\mathscr{H}, \nabla)$. Then $H$ is a locally constant sheaf of $\mathbf{C}_S$-modules on $S$ by item 1 of Proposition 1.7.10. Since $S$ is simply connected, there exists thus a natural number $r$ as well as an isomorphism $(\mathbf{C}_S)^{\oplus r} \to H$ of $\mathbf{C}_S$-modules on $S_{\mathrm{top}}$. Denote by $e = (e_0, \ldots, e_{r-1})$ the thereby induced ordered $\mathbf{C}$-basis of $H(S)$. Let $s_0$ be an arbitrary element of $S$. Then, as $\mathscr{F}$ is a locally finite free module on $S$, there exist an open neighborhood $U$ of $s_0$ in $S$, a natural number $d$, as well as an isomorphism

$$\phi : (\mathscr{O}_S|_U)^{\oplus d} \longrightarrow \mathscr{F}|_U$$

of modules on $S|_U$. Denote, for any $j < d$, by $\sigma_j$ the image of the $j$th unit vector in $((\mathscr{O}_S|_U)^{\oplus d})(U) = (\mathscr{O}_S(U))^{\oplus d}$ under the function $\phi_U$. Then exploiting the fact that, by item 2 of Proposition 1.7.10, the canonical sheaf map

$$\mathscr{O}_S \otimes_{\mathbf{C}_S} H \longrightarrow \mathscr{H}$$

is an isomorphism of modules on $S$, we see that there exists an $r \times d$-matrix $\lambda = (\lambda_{ij})$ with values in $\mathscr{O}_S(U)$ such that, for all $j < d$, we have

$$\sigma_j = \sum_{i<r} \lambda_{ij} \cdot (e_i|_U),$$

where we add and multiply in the $\mathscr{O}_S(U)$-module $\mathscr{H}(U)$. Clearly, for all $s \in U$, the $d$-tuple $(\sigma_0(s), \ldots, \sigma_{d-1}(s))$ makes up an ordered $\mathbf{C}$-basis for $\mathscr{F}(s)$. Since $\mathscr{F}$ is a vector subbundle of $\mathscr{H}$ on $S$, we know that, for all $s \in S$, the map

$$\iota(s) : \mathscr{F}(s) \longrightarrow \mathscr{H}(s)$$

is one-to-one, where $\iota : \mathscr{F} \to \mathscr{H}$ stands for the inclusion morphism. Thus, for all $s \in U$, the $d$-tuple given by the association

$$j \longmapsto \sum_{i<r} \lambda_{ij}(s) \cdot e_i(s)$$

constitutes a **C**-basis of

$$F(s) := \mathrm{im}(\iota(s) : \mathscr{F}(s) \longrightarrow \mathscr{H}(s)).$$

Define

$$L : U \longrightarrow \mathbf{C}^{r\times d}, \quad L(s) = (\lambda_{ij}(s))_{i<r, j<c}.$$

Then, for all $s \in U$, the columns of the matrix $L(s)$ are linearly independent. In particular, without loss of generality, we may assume that the matrix $L(s_0)|_{d\times d}$ is invertible. Since the functions $s \mapsto \lambda_{ij}(s)$ are altogether continuous (from $U$ to $\mathbf{C}$), the set $U'$ of elements $s$ of $U$ such that $L(s)|_{d\times d}$ is invertible, is an open neighborhood of $s_0$ in $S$. We define

$$L' : U' \longrightarrow \mathbf{C}^{r\times d}, \quad L'(s) = L(s) \cdot (L(s)|_{d\times d})^{-1}.$$

Then, for all $s \in U'$, the space $\mathcal{P}(s)$ is the linear span in $\mathscr{H}(t)$ of the elements

$$e_j(t) + \sum_{d\leq i<r} (L'(s))_{ij} \cdot e_i(t), \quad j < d.$$

In other words, setting $c := r - d$ and

$$L'' : U' \longrightarrow \mathbf{C}^{c\times d}, \quad (L''(s))_{ij} = (L'(s))_{i+d,j},$$

when $h$ signifies the mapping which associates to a matrix $M \in \mathbf{C}^{c\times d}$ the linear span in $\mathscr{H}(t)$ of the elements

$$e_j(t) + \sum_{i<c} M_{ij} \cdot e_{i+d}(t), \quad j < d,$$

the following diagram commutes in the category of sets:

$$\begin{tikzcd} & U' \arrow[dl, "L''"'] \arrow[dr, "\mathcal{P}|_{U'}"] & \\ \mathbf{C}^{c\times d} \arrow[rr, "h"'] & & \mathrm{Gr}(\mathscr{H}(t)) \end{tikzcd}$$

Since the tuple $(e_0(t), \ldots, e_{r-1}(t))$ forms a **C**-basis of $\mathscr{H}(t)$, we find that $h$ is one-to-one, and $h^{-1}$ composed with the canonical function $\mathbf{C}^{c\times d} \to \mathbf{C}^{cd}$ is a holomorphic chart on the Grassmannian $\mathrm{Gr}(\mathscr{H}(t))$ (see Notation 1.7.12). Moreover, the components of $L''$ are holomorphic functions on $S|_{U'}$. This shows that $\mathcal{P}|_{U'}$ is a holomorphic map from $S|_{U'}$ to $\mathrm{Gr}(\mathscr{H}(t))$. Since $s_0$ was an arbitrary element of $S$, we infer that $\mathcal{P}$ is a holomorphic map from $S$ to $\mathrm{Gr}(\mathscr{H}(t))$. $\square$

*Remark 1.7.14* Let $S$ be a simply connected complex manifold, $(\mathscr{H}, \nabla)$ a flat vector bundle on $S$, $\mathscr{F}$ a vector subbundle of $\mathscr{H}$ on $S$, and $t \in S$. Then Proposition 1.7.13 implies that $\mathcal{P}_t(S, (\mathscr{H}, \nabla), \mathscr{F})$ is a holomorphic map from $S$ to $\mathrm{Gr}(\mathscr{H}(t))$. Therefore, since the complex space $S$ is reduced, there exists one, and only one, morphism of complex spaces

$$\mathcal{P}^+ : S \longrightarrow \mathrm{Gr}(\mathscr{H}(t))$$

such that the function underlying $\mathcal{P}^+$ is precisely $\mathcal{P}_t(S, (\mathscr{H}, \nabla), \mathscr{F})$. We agree on denoting $\mathcal{P}^+$ again by $\mathcal{P}_t(S, (\mathscr{H}, \nabla), \mathscr{F})$. Observe that, in view of Construction 1.7.11, this notation is somewhat ambiguous. In fact, $\mathcal{P}_t(S, (\mathscr{H}, \nabla), \mathscr{F})$ may refer to a morphism of complex spaces as well as to its underlying function now. I am confident, however, that you are not irritated by this sloppiness.

**Construction 1.7.15** Let $S$ be a complex space and $t \in S$. Moreover, let $F$ and $H$ be two modules on $S$. We intend to fabricate a mapping

$$\eta_{S,t}(F, H) : \mathrm{Hom}_S(F, \Omega^1_S \otimes H) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(\mathrm{T}_S(t), \mathrm{Hom}(F(t), H(t))).$$

For that matter, let

$$\phi : F \longrightarrow \Omega^1_S \otimes H$$

be a morphism of modules on $S$. Then consider the composition

$$\Theta_S \otimes F \xrightarrow{\mathrm{id}_{\Theta_S} \otimes \phi} \Theta_S \otimes (\Omega^1_S \otimes H) \xrightarrow{\alpha^{-1}} (\Theta_S \otimes \Omega^1_S) \otimes H \xrightarrow{\epsilon(\Omega^1_S, \mathscr{O}_S) \otimes \mathrm{id}_H} \mathscr{O}_S \otimes H \xrightarrow{\lambda(H)} H$$

in $\mathrm{Mod}(S)$. By means of tensor-hom adjunction on $S$ (with respect to the modules $\Theta_S$, $F$, and $H$) the latter morphism corresponds to a morphism

$$\Theta_S \longrightarrow \mathscr{H}\!om(F, H)$$

in $\mathrm{Mod}(S)$. Evaluating at $t$ and composing the result with the canonical map

$$(\mathscr{H}\!om(F, H))(t) \longrightarrow \mathrm{Hom}(F(t), H(t)),$$

we obtain a morphism of complex vector spaces

$$\Theta_S(t) \longrightarrow \mathrm{Hom}(F(t), H(t)).$$

Precomposing the latter with the inverse of the canonical isomorphism

$$\Theta_S(t) \longrightarrow \mathrm{T}_S(t),$$

we end up with a morphism

$$\mathrm{T}_S(t) \longrightarrow \mathrm{Hom}(F(t), H(t))$$

in Mod($\mathbf{C}$). The latter, we define to be the image of $\phi$ under $\eta_{S,t}(F, H)$. This yields our aspired function $\eta_{S,t}(F, H)$. Letting $(F, H)$ vary, we may view $\eta_{S,t}$ as a function, in the class sense, defined on the class of pairs of modules on $S$.

**Proposition 1.7.16** *Let $S$ be a complex space, $t \in S$. Then*

$$\eta_{S,t} : \mathrm{Hom}_S(-, \Omega^1_S \otimes -) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(\mathrm{T}_S(t), \mathrm{Hom}(-(t), -(t))).$$

*is a natural transformation of functors from* $\mathrm{Mod}(S)^{\mathrm{op}} \times \mathrm{Mod}(S)$ *to* **Set**.

*Proof* You verify that the individual steps taken in Construction 1.7.15 are altogether natural transformations between appropriate functors from $\mathrm{Mod}(S)^{\mathrm{op}} \times \mathrm{Mod}(S)$ to **Set**. I dare omit the details. □

*Remark 1.7.17* I would like to give a more down-to-earth interpretation of Proposition 1.7.16. So, let $S$ be a complex space and $t \in S$. Let $(F, H)$ and $(F', H')$ be two ordered pairs of modules on $S$, and let

$$(\alpha, \gamma) : (F, H) \longrightarrow (F', H')$$

be a morphism in $\mathrm{Mod}(S)^{\mathrm{op}} \times \mathrm{Mod}(S)$; that is, $\alpha : F' \to F$ and $\gamma : H \to H'$ are morphisms in Mod($S$). Moreover, let $\phi$ and $\phi'$ be such that the following diagram commutes in Mod($S$):

$$\begin{tikzcd} F \arrow[r, "\phi"] & \Omega^1_S \otimes H \arrow[d, "\mathrm{id}_{\Omega^1_S} \otimes \gamma"] \\ F' \arrow[u, "\alpha"] \arrow[r, "\phi'"'] & \Omega^1_S \otimes H' \end{tikzcd}$$

Then $\phi'$ is the image of $\phi$ under the function

$$\mathrm{Hom}_S(\alpha, \mathrm{id}_{\Omega^1_S} \otimes \gamma) : \mathrm{Hom}_S(F, \Omega^1_S \otimes H) \longrightarrow \mathrm{Hom}_S(F', \Omega^1_S \otimes H').$$

Therefore, by Proposition 1.7.16, $(\eta_{S,t}(F', H'))(\phi')$ is the image of $(\eta_{S,t}(F, H))(\phi)$ under the function

$$\mathrm{Hom}_{\mathbf{C}}(\mathrm{T}_S(t), \mathrm{Hom}(\alpha(t), \gamma(t))) :$$
$$\mathrm{Hom}_{\mathbf{C}}(\mathrm{T}_S(t), \mathrm{Hom}(F(t), H(t))) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(\mathrm{T}_S(t), \mathrm{Hom}(F'(t), H'(t))).$$

This in turn translates as the commutativity, in $\mathrm{Mod}(\mathbf{C})$, of the following diagram:

$$\begin{tikzcd}
\mathrm{T}_S(t) \arrow[r, "\mathrm{id}_{\mathrm{T}_S(t)}"] \arrow[d, "(\eta_{S,t}(F,H))(\phi)"'] & \mathrm{T}_S(t) \arrow[d, "(\eta_{S,t}(F',H'))(\phi')"] \\
\mathrm{Hom}(F(t), H(t)) \arrow[r, "\mathrm{Hom}(\alpha(t),\gamma(t))"'] & \mathrm{Hom}(F'(t), H'(t))
\end{tikzcd}$$

This line of reasoning will be exploited heavily in the proof of Proposition 1.8.7 in the upcoming Sect. 1.8.

**Proposition 1.7.18** *Let $S$ be a complex space, $\iota : \mathscr{F} \to \mathscr{H}$ a morphism of modules on $S$, and $\nabla$ an $S$-connection on $\mathscr{H}$. Then*

$$\nabla_\iota := (\mathrm{id}_{\Omega^1_S} \otimes \operatorname{coker}(\iota)) \circ \nabla \circ \iota \tag{1.56}$$

*is a morphism*

$$\nabla_\iota : \mathscr{F} \longrightarrow \Omega^1_S \otimes (\mathscr{H}/\mathscr{F})$$

*in* $\mathrm{Mod}(S)$.

*Proof* To begin with, $\nabla_\iota$ is a morphism from $\mathscr{F}$ to $\Omega^1_S \otimes (\mathscr{H}/\mathscr{F})$ in $\mathrm{Mod}(a_S)$. That $\nabla_\iota$ is a morphism in $\mathrm{Mod}(S)$ is equivalent to saying that it is compatible with the $\mathscr{O}_S$-scalar multiplications of $\mathscr{F}$ and $\Omega^1_S \otimes (\mathscr{H}/\mathscr{F})$. Let $U$ be an open set of $S$, $\sigma \in \mathscr{F}(U)$, and $\lambda \in \mathscr{O}_S(U)$. Then we have

$$(\nabla \circ \iota)_U(\lambda \cdot \sigma) = \nabla_U(\lambda \cdot \iota_U(\sigma)) = \mathrm{d}_U(\lambda) \otimes \iota_U(\sigma) + \lambda \cdot \nabla_U(\iota_U(\sigma)),$$

where d is short for the differential $\mathrm{d}^0_S : \mathscr{O}_S \to \Omega^1_S$. Thus,

$$(\nabla_\iota)_U(\lambda \cdot \sigma) = \lambda \cdot (\nabla_\iota)_U(\sigma),$$

which proves our claim. □

**Construction 1.7.19** Let $V$ be a finite dimensional $\mathbf{C}$-vector space and $F$ a $\mathbf{C}$-linear subset of $V$ (i.e., $F \in \mathrm{Gr}(V)$). Then we write

$$\theta(V, F) : \mathrm{T}_{\mathrm{Gr}(V)}(F) \longrightarrow \mathrm{Hom}(F, V/F)$$

for the typical interpretation of the tangent space of the Grassmannian [14, Lemme 10.7].

Let me indicate how you define $\theta(V, F)$. For that matter, let $E$ be a $\mathbf{C}$-vector subspace of $V$ such that $V = F \oplus E$. Then there exists a morphism of complex spaces

$$g_E : \mathrm{Hom}(F, E) \longrightarrow \mathrm{Gr}(V)$$

with the property that $g_E$ sends a homomorphism $\alpha : F \to E$ to the image of the function $\alpha' : F \to V$, $\alpha'(x) := x + \alpha(x)$. In fact, $g_E$ is an open immersion which maps the 0 of $\mathrm{Hom}(F, E)$ to $F$ in $\mathrm{Gr}(V)$. Hence, the tangent map

$$\mathrm{T}_0(g_E) : \mathrm{T}_{\mathrm{Hom}(F,E)}(0) \longrightarrow \mathrm{T}_{\mathrm{Gr}(V)}(F)$$

is an isomorphism in $\mathrm{Mod}(\mathbf{C})$. Moreover, we have a canonical isomorphism

$$\mathrm{T}_{\mathrm{Hom}(F,E)}(0) \longrightarrow \mathrm{Hom}(F, E),$$

as well as the morphism

$$\mathrm{Hom}(F, E) \longrightarrow \mathrm{Hom}(F, V/F)$$

which is induced by the restriction of the quotient mapping $V \to V/F$ to $E$.

**Lemma 1.7.20** *Let $S$ be a simply connected complex manifold, $(\mathscr{H}, \nabla)$ a flat vector bundle on $S$, $\mathscr{F}$ a vector subbundle of $\mathscr{H}$ on $S$, and $t \in S$. Set*

$$\mathcal{P} := \mathcal{P}_t(S, (\mathscr{H}, \nabla), \mathscr{F})$$

*and define $\nabla_\iota$ by Eq. (1.56), where $\iota : \mathscr{F} \to \mathscr{H}$ denotes the inclusion morphism. Moreover, set*

$$F(t) := \mathrm{im}(\iota(t) : \mathscr{F}(t) \longrightarrow \mathscr{H}(t)),$$

*and write*

$$\sigma : \mathscr{F}(t) \longrightarrow F(t),$$
$$\tau : \mathscr{H}(t)/F(t) \longrightarrow (\mathscr{H}/\mathscr{F})(t)$$

*for the evident mappings. Then $\sigma$ and $\tau$ are isomorphisms in* $\mathrm{Mod}(\mathbf{C})$ *and the following diagram commutes in* $\mathrm{Mod}(\mathbf{C})$*:*

$$\begin{tikzcd}
\mathrm{T}_S(t) \arrow[r, "\eta_{S,t}(\mathscr{F},\mathscr{H}/\mathscr{F})(\nabla_\iota)"] \arrow[d, "\mathrm{T}_t(\mathcal{P})"'] & \mathrm{Hom}(\mathscr{F}(t), (\mathscr{H}/\mathscr{F})(t)) \\
\mathrm{T}_{\mathrm{Gr}(\mathscr{H}(t))}(F(t)) \arrow[r, "\theta(\mathscr{H}(t),F(t))"'] & \mathrm{Hom}(F(t), \mathscr{H}(t)/F(t)) \arrow[u, "\mathrm{Hom}(\sigma,\tau)"']
\end{tikzcd} \tag{1.57}$$

*Proof* The fact that $\sigma$ and $\tau$ are isomorphisms is pretty obvious: Since $\mathscr{F}$ is a vector subbundle of $\mathscr{H}$ on $S$, we know that $\sigma$ is injective. $\sigma$ is surjective by the definition of $F(t)$. $\tau$ is an isomorphism since both $\mathscr{H}(t) \to \mathscr{H}(t)/F(t)$ and $\mathscr{H}(t) \to (\mathscr{H}/\mathscr{F})(t)$ are cokernels in $\mathrm{Mod}(\mathbf{C})$ of $\iota(t) : \mathscr{F}(t) \to \mathscr{H}(t)$, where we take into account in particular the right exactness of the evaluation functor "$-(t)$".

Set $H := \mathrm{Hor}_S(\mathscr{H}, \nabla)$. Then by the same method as in the proof of Proposition 1.7.13, we deduce that there exist an ordered $\mathbf{C}$-basis $e' = (e'_0, \dots, e'_{r-1})$ for $H(S)$, an open neighborhood $U$ of $t$ in $S$, as well as a $c \times d$-matrix $\lambda'$ with values in $\mathscr{O}_S(U)$ such that the $d$-tuple $\alpha = (\alpha_0, \dots, \alpha_{d-1})$ given by

$$\alpha_j = e'_j|_U + \sum_{i<c} \lambda'_{ij} \cdot (e'_{d+i}|_U)$$

for all $j < d$, where we sum and multiply within the $\mathscr{O}_S(U)$-module $\mathscr{H}(U)$, trivializes the module $\mathscr{F}$ on $S$ over $U$—that is, the unique morphism $(\mathscr{O}_S|_U)^{\oplus d} \to \mathscr{F}|_U$ of modules on $S|_U$ which sends the standard basis of $(\mathscr{O}_S(U))^{\oplus d}$ to $\alpha$ is an isomorphism. Define a $c$-tuple $e$ and a $d$-tuple $f$ by setting

$$e_i := e'_{d+i},$$
$$f_j := e'_j + \sum_{i<c} \lambda'_{ij}(t) \cdot e'_{d+i}$$

for all $i < c$ and $j < d$, respectively. Moreover, define a $c \times d$-matrix $\lambda$ by

$$\lambda_{ij} := \lambda'_{ij} - \lambda'_{ij}(t)$$

for all $(i, j) \in c \times d$. Then the concatenated tuple

$$(f_0, \dots, f_{d-1}, e_0, \dots, e_{c-1})$$

is an ordered $\mathbf{C}$-basis for $H(S)$, and

$$\alpha_j = f_j|_U + \sum_{i<c} \lambda_{ij} \cdot (e_i|_U)$$

for all $j < d$. Thus, for all $s \in U$, the space $\mathcal{P}(s)$ equals the $\mathbf{C}$-linear span of the elements

$$f_j(t) + \sum_{i<c} \lambda_{ij}(s) \cdot e_i(t), \quad j < d,$$

in $\mathscr{H}(t)$. Specifically, as $\lambda_{ij}(t) = 0$ for all $(i,j) \in c \times d$, we see that $F(t)$ equals the $\mathbf{C}$-linear span of $f_0(t), \dots, f_{d-1}(t)$ in $\mathscr{H}(t)$. Define $E$ to be the $\mathbf{C}$-linear span of $e_0(t), \dots, e_{c-1}(t)$ in $\mathscr{H}(t)$. Let

$$g : \mathrm{Hom}(F(t), E) \longrightarrow \mathrm{Gr}(\mathscr{H}(t))$$

be the morphism of complex spaces which sends an element $\phi \in \mathrm{Hom}(F(t), E)$ to the range of the homomorphism $\mathrm{id}_{F(t)} + \phi : F(t) \to \mathscr{H}(t)$. Let

$$\bar{\mathcal{P}} : S|_U \longrightarrow \mathrm{Hom}(F(t), E)$$

be the morphism of complex spaces which sends $s$ to the homomorphism $F(t) \to E$ which is represented by the $c \times d$-matrix

$$(i,j) \longmapsto \lambda_{ij}(s)$$

with respect to the bases $(f_0(t), \dots, f_{d-1}(t))$ and $(e_0(t), \dots, e_{c-1}(t))$ of $F(t)$ and $E$, respectively. Then the following diagram commutes in the category of complex spaces:

$$\begin{tikzcd} S|_U \arrow[rr, "\mathcal{P}|_U"] \arrow[dr, "\bar{\mathcal{P}}"'] & & \mathrm{Gr}(\mathscr{H}(t)) \\ & \mathrm{Hom}(F(t), E) \arrow[ur, "g"'] & \end{tikzcd}$$

Let $v$ be an arbitrary element of $\mathrm{T}_S(t)$. Then by the explicit description of $\bar{\mathcal{P}}$, we see that the image of $v$ under the composition

$$\mathrm{can.} \circ \mathrm{T}_t(\bar{\mathcal{P}}) : \mathrm{T}_S(t) \longrightarrow \mathrm{T}_{\mathrm{Hom}(F(t),E)}(0) \longrightarrow \mathrm{Hom}(F(t), E)$$

is represented by the matrix

$$c \times d \ni (i,j) \longmapsto v \lhd (\mathrm{d}_S)_U(\lambda_{ij}) \tag{1.58}$$

with respect to the bases $(f_0(t), \dots, f_{d-1}(t))$ and $(e_0(t), \dots, e_{c-1}(t))$, where, for any $\omega \in \Omega^1_S(U)$,

$$v \lhd \omega := v(\omega(t)).$$

Note that $v$ is a $\mathbf{C}$-linear functional on $\Omega^1_S(t) = \mathbf{C} \otimes_{\mathcal{O}_{S,s}} \Omega^1_{S,s}$. By the definition of $\theta$ in Construction 1.7.19, when $\pi : \mathscr{H}(t) \to \mathscr{H}(t)/F(t)$ denotes the residue class mapping, the following diagram commutes in $\mathrm{Mod}(\mathbf{C})$:

$$\begin{tikzcd} \mathrm{T}_{\mathrm{Hom}(F(t),E)}(0) \arrow[r, "\text{can.}"] \arrow[d, "\mathrm{T}_0(g)"'] & \mathrm{Hom}(F(t),E) \arrow[d, "\mathrm{Hom}(\mathrm{id}_{F(t)},\pi|_E)"] \\ \mathrm{T}_{\mathrm{Gr}(\mathscr{H}(t))}(F(t)) \arrow[r, "\theta(\mathscr{H}(t),F(t))"'] & \mathrm{Hom}(F(t),\mathscr{H}(t)/F(t)) \end{tikzcd}$$

Hence, the image of $v$ under the composition

$$\theta(\mathscr{H}(t), F(t)) \circ \mathrm{T}_t(\mathcal{P})$$

is represented by the matrix in Eq. (1.58) with respect to the bases $(f_0(t), \ldots, f_{d-1}(t))$ and $(\pi(e_0(t)), \ldots, \pi(e_{c-1}(t)))$. On the other hand, for all $j < d$, we have

$$\nabla_U(\alpha_j) = \sum_{i<c} (\mathrm{d}_S)_U(\lambda_{ij}) \otimes (e_i|_U),$$

whence

$$(\nabla_\iota)_U(\alpha_j) = \sum_{i<c} (\mathrm{d}_S)_U(\lambda_{ij}) \otimes \bar{e}_i,$$

where $\bar{e}_i$ denotes the image of $e_i|_U$ under the mapping $\mathscr{H}(U) \to (\mathscr{H}/\mathscr{F})(U)$. Put $A := \eta_{S,t}(\mathscr{F}, \mathscr{H}/\mathscr{F})(\nabla_\iota)$. Then by Construction 1.7.15, we have

$$Av(\alpha_j(t)) = \sum_{i<c} (v \lhd (\mathrm{d}_S)_U(\lambda_{ij})) \cdot \bar{e}_i(t).$$

Evidently, for all $j < d$, the mapping $\iota(t) : \mathscr{F}(t) \to \mathscr{H}(t)$ sends $\alpha_j(t)$ (evaluation in $\mathscr{F}$ here) to

$$\alpha_j(t) = f_j(t) + \sum_{i<c} \lambda_{ij}(t) \cdot e_i(t) = f_j(t)$$

(evaluation in $\mathscr{H}$). Thus, $\sigma(\alpha_j(t)) = f_j(t)$. Likewise, for all $i < c$, the mapping $(\operatorname{coker}\iota)(t) : \mathscr{H}(t) \to (\mathscr{H}/\mathscr{F})(t)$ sends $e_i(t)$ to $\bar{e}_i(t)$. Thus, $\tau(\pi(e_i(t))) = \bar{e}_i(t)$. This proves the commutativity of the diagram in Eq. (1.57). □

## 1.8 Period Mappings of Hodge-de Rham Type

After the ground-laying work of the Sect. 1.7, we are now in the position to analyze period mappings of "Hodge-de Rham type"; the concept will be made precise in the realm of Notation 1.8.2 below. As a preparation we need the following result.

**Proposition 1.8.1** *Let $n$ be an integer.*

1. *Let $(f, g)$ be a composable pair in the category of submersive complex spaces. Then $\nabla^n_{\mathrm{GM}}(f, g)$ is a flat $g$-connection on $\mathscr{H}^n(f)$.*
2. *Let $f : X \to S$ be a submersive morphism of complex spaces such that $S$ is a complex manifold. Then $\nabla^n_{\mathrm{GM}}(f)$ is a flat $S$-connection on $\mathscr{H}^n(f)$.*

*Proof* Clearly, item 2 follows from item 1 letting $g = a_S$. As to item 1, the Leibniz rule and the flatness have been established by Katz and Oda [11, Sect. 2]. □

**Notation 1.8.2** Let $f : X \to S$ be a submersive morphism of complex spaces such that $S$ is a simply connected complex manifold. Let $n$ and $p$ be integers and $t \in S$. Assume that $\mathscr{H}^n(f)$ is a locally finite free module on $S$ and $\mathrm{F}^p\mathscr{H}^n(f)$ is a vector subbundle of $\mathscr{H}^n(f)$ on $S$.

1. We put

$$\mathcal{P}'^{p,n}_t(f) := \mathcal{P}_t(S, (\mathscr{H}^n(f), \nabla^n_{\mathrm{GM}}(f)), \mathrm{F}^p\mathscr{H}^n(f)), \tag{1.59}$$

where the right-hand side is to be interpreted in the sense of Construction 1.7.11. Note that Eq. (1.59) makes sense in particular because by item 2 of Proposition 1.8.1, $\nabla^n_{\mathrm{GM}}(f)$ is a flat $S$-connection on $\mathscr{H}^n(f)$, whence $(\mathscr{H}^n(f), \nabla^n_{\mathrm{GM}}(f))$ is a flat vector bundle on $S$. Note that by means of Proposition 1.7.13 we may regard $\mathcal{P}'^{p,n}_t(f)$ as a morphism of complex spaces,

$$\mathcal{P}'^{p,n}_t(f) : S \longrightarrow \mathrm{Gr}((\mathscr{H}^n(f))(t)).$$

2. Assume that for all $s \in S$ the base change maps

$$\phi^n_{f,s} : (\mathscr{H}^n(f))(s) \longrightarrow \mathscr{H}^n(X_s)$$
$$\phi^{p,n}_{f,s} : (\mathrm{F}^p\mathscr{H}^n(f))(s) \longrightarrow \mathrm{F}^p\mathscr{H}^n(X_s)$$

are isomorphisms in $\mathrm{Mod}(\mathbf{C})$. Write $\rho'$ for the $\mathbf{C}$-representation of the fundamental groupoid of $S$ which is defined for

$$(\mathscr{H}, \nabla) := (\mathscr{H}^n(f), \nabla^n_{\mathrm{GM}}(f))$$

in Construction 1.7.11. Let

$$\rho : \Pi(S) \longrightarrow \mathrm{Mod}(\mathbf{C})$$

be the functor which is obtained by "composing" $\rho'$ with the family of isomorphisms $\phi := (\phi^n_{f,s})_{s \in S}$. Define $F$ to be the unique function on $S$ such that, for all $s \in S$, we have

$$F(s) = \mathrm{F}^p\mathscr{H}^n(X_s).$$

Then clearly, $F$ is a $\mathbf{C}$-distribution in $\rho$. We set

$$\mathcal{P}_t^{p,n}(f) := \mathcal{P}_t^{\mathbf{C}}(S, \rho, F);$$

see Construction 1.7.3. Note that $\phi$ is an isomorphism of functors from $\Pi(S)$ to $\mathrm{Mod}(\mathbf{C})$ from $\rho'$ to $\rho$. Moreover, when $F'$ denotes the unique function on $S$ such that, for all $s \in S$, we have

$$F'(s) = \mathrm{im}((\iota_f^n(p))(s) : (\mathrm{F}^p\mathscr{H}^n(f))(s) \longrightarrow (\mathscr{H}^n(f))(s)),$$

then

$$\phi_{f,s}^n[F'(s)] = F(s)$$

for all $s \in S$. Therefore, the following diagram commutes in the category of sets:

$$\begin{tikzcd}
S \arrow[r, "\mathrm{id}_S"] \arrow[d, "\mathcal{P}_t'^{p,n}(f)"'] & S \arrow[d, "\mathcal{P}_t^{p,n}(f)"] \\
\mathrm{Gr}((\mathscr{H}^n(f))(t)) \arrow[r, "\mathrm{Gr}(\phi_{f,t}^n)"'] & \mathrm{Gr}(\mathscr{H}^n(X_t))
\end{tikzcd}$$

Since $\mathcal{P}_t'^{p,n}(f)$ is a holomorphic map from $S$ to $\mathrm{Gr}((\mathscr{H}^n(f))(t))$ by Proposition 1.7.13 and since $\mathrm{Gr}(\phi_{f,t}^n)$ is an isomorphism of complex spaces (as $\phi_{f,t}^n$ is an isomorphisms of $\mathbf{C}$-vector spaces), we may view $\mathcal{P}_t^{p,n}(f)$ as a morphism of complex spaces from $S$ to $\mathrm{Gr}(\mathscr{H}^n(X_t))$. We call $\mathcal{P}_t^{p,n}(f)$ the *Hodge-de Rham period mapping* in bidegree $(p, n)$ of $f$ with basepoint $t$.

Next, I introduce the classical concept of Kodaira-Spencer maps. My definition shows how to construct these maps out of the Kodaira-Spencer class given by Definition 1.6.3 and Construction 1.4.19. As an auxiliary means, I also introduce "Kodaira-Spencer maps without base change".

**Notation 1.8.3** Let $f : X \to S$ be a submersive morphism of complex spaces such that $S$ is a complex manifold. Then, by means of Definition 1.6.3, we may speak of the Kodaira-Spencer class of $f$, written $\xi_{\mathrm{KS}}(f)$, which is a morphism

$$\xi_{\mathrm{KS}}(f) : \mathscr{O}_S \longrightarrow \Omega_S^1 \otimes \mathrm{R}^1 f_*(\Theta_f)$$

of modules on $S$. We write $\mathrm{KS}_f$ for the composition of the following morphisms in $\mathrm{Mod}(S)$:

$$\Theta_S \xrightarrow{\rho(\Theta_S)^{-1}} \Theta_S \otimes \mathscr{O}_S \xrightarrow{\mathrm{id}_{\Theta_S} \otimes \xi_{\mathrm{KS}}(f)} \Theta_S \otimes (\Omega_S^1 \otimes \mathrm{R}^1 f_*(\Theta_f))$$

$$\xrightarrow{\alpha(\Theta_S, \Omega^1_S, \mathrm{R}^1 f_*(\Theta_f))^{-1}} (\Theta_S \otimes \Omega^1_S) \otimes \mathrm{R}^1 f_*(\Theta_f)$$

$$\xrightarrow{\gamma^1(\Omega^1_S) \otimes \mathrm{id}_{\mathrm{R}^1 f_*(\Theta_f)}} \mathscr{O}_S \otimes \mathrm{R}^1 f_*(\Theta_f) \xrightarrow{\lambda(\mathrm{R}^1 f_*(\Theta_f))} \mathrm{R}^1 f_*(\Theta_f).$$

Let $t \in S$. Then define

$$\mathrm{KS}'_{f,t} : \mathrm{T}_S(t) \longrightarrow (\mathrm{R}^1 f_*(\Theta_f))(t)$$

to be the composition of the inverse of the canonical isomorphism $\Theta_S(t) \to \mathrm{T}_S(t)$ with $\mathrm{KS}_f(t) : \Theta_S(t) \to (\mathrm{R}^1 f_*(\Theta_f))(t)$. We call $\mathrm{KS}'_{f,t}$ the *Kodaira-Spencer map without base change* of $f$ at $t$.

Furthermore, define

$$\mathrm{KS}_{f,t} : \mathrm{T}_S(t) \longrightarrow \mathrm{H}^1(X_t, \Theta_{X_t})$$

to be the composition of $\mathrm{KS}'_{f,t}$ with the evident base change morphism

$$\beta^1_{f,t} : (\mathrm{R}^1 f_*(\Theta_f))(t) \longrightarrow \mathrm{H}^1(X_t, \Theta_{X_t}).$$

We call $\mathrm{KS}_{f,t}$ the *Kodaira-Spencer map* of $f$ at $t$.

**Construction 1.8.4** Let $f : X \to S$ be an arbitrary morphism of complex spaces. Let $p$ and $q$ be integers. We define

$$\gamma_f^{p,q} : \mathrm{R}^1 f_*(\Theta_f) \otimes \mathscr{H}^{p,q}(f) \longrightarrow \mathscr{H}^{p-1,q+1}(f)$$

to be the composition, in $\mathrm{Mod}(S)$, of the cup product morphism

$$\smile_f^{1,q}(\Theta_f, \Omega^p_f) : \mathrm{R}^1 f_*(\Theta_f) \otimes \mathrm{R}^q f_*(\Omega^p_f) \longrightarrow \mathrm{R}^{q+1} f_*(\Theta_f \otimes \Omega^p_f)$$

and the $\mathrm{R}^{q+1} f_*(-)$ of the contraction morphism

$$\gamma^p_X(\Omega^1_f) : \Theta_f \otimes \Omega^p_f \longrightarrow \Omega^{p-1}_f;$$

see Construction 1.4.11. $\gamma_f^{p,q}$ is called the *cup and contraction* in bidegree $(p, q)$ for $f$. As a shorthand, we write $\gamma_X^{p,q}$ for $\gamma_{a_X}^{p,q}$.

By means of tensor-hom adjunction on $S$ (with respect to the modules $\mathrm{R}^1 f_*(\Theta_f)$, $\mathscr{H}^{p,q}(f)$, and $\mathscr{H}^{p-1,q+1}(f)$), the morphism $\gamma_f^{p,q}$ corresponds to a morphism

$$\mathrm{R}^1 f_*(\Theta_f) \longrightarrow \mathscr{H}om_S(\mathscr{H}^{p,q}(f), \mathscr{H}^{p-1,q+1}(f))$$

in $\mathrm{Mod}(S)$. Let $t \in S$. Then evaluating the latter morphism at $t$, and composing the result in $\mathrm{Mod}(\mathbf{C})$ with the canonical morphism

$$\big(\mathscr{H}om(\mathscr{H}^{p,q}(f), \mathscr{H}^{p-1,q+1}(f))\big)(t) \longrightarrow \mathrm{Hom}\big((\mathscr{H}^{p,q}(f))(t), (\mathscr{H}^{p-1,q+1}(f))(t)\big),$$

yields

$$\gamma'^{p,q}_{f,t} : (\mathrm{R}^1 f_*(\Theta_f))(t) \longrightarrow \mathrm{Hom}\big((\mathscr{H}^{p,q}(f))(t), (\mathscr{H}^{p-1,q+1}(f))(t)\big).$$

We refer to $\gamma'^{p,q}_{f,t}$ as the *cup and contraction without base change* in bidegree $(p, q)$ of $f$ at $t$.

The following two easy lemmata pave the way for the first essential statement of Sect. 1.8—namely, Proposition 1.8.7.

**Lemma 1.8.5** *Let $f : X \to S$ be a submersive morphism of complex spaces such that $S$ is a complex manifold. Let $p$ and $q$ be integers and $t \in S$. Then the following identity holds in* $\mathrm{Mod}(\mathbf{C})$*:*

$$\eta_{S,t}(\mathscr{H}^{p,q}(f), \mathscr{H}^{p-1,q+1}(f))(\gamma^{p,q}_{\mathrm{KS}}(f)) = \gamma'^{p,q}_{f,t} \circ \mathrm{KS}'_{f,t}. \tag{1.60}$$

*Proof* We argue in several steps. To begin with, observe that the following diagram commutes in $\mathrm{Mod}(S)$:

$$\begin{tikzcd}
\Theta_S \otimes \mathscr{H}^{p,q}(f) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{KS}_f \otimes \mathrm{id}"'] & \Theta_S \otimes \mathscr{H}^{p,q}(f) \arrow[d, "\gamma^{p,q}_{\mathrm{KS}}(f)"] \\
\mathrm{R}^1 f_*(\Theta_f) \otimes \mathscr{H}^{p,q}(f) \arrow[r, "\gamma^{p,q}_f"'] & \mathscr{H}^{p-1,q+1}(f)
\end{tikzcd}$$

Thus, by the naturality of the tensor-hom adjunction, the next diagram commutes in $\mathrm{Mod}(S)$, too:

$$\begin{tikzcd}
\Theta_S \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{KS}_f"'] & \Theta_S \arrow[d] \\
\mathrm{R}^1 f_*(\Theta_f) \arrow[r] & \mathscr{H}om(\mathscr{H}^{p,q}(f), \mathscr{H}^{p-1,q+1}(f))
\end{tikzcd}$$

Evaluating at $t$, we deduce that the diagram

$$\begin{tikzcd}
\Theta_S(t) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{KS}_f(t)"'] & \Theta_S(t) \arrow[d] \\
(\mathrm{R}^1 f_*(\Theta_f))(t) \arrow[r, "\gamma'^{p,q}_{f,t}"'] & \mathrm{Hom}((\mathscr{H}^{p,q}(f))(t), (\mathscr{H}^{p-1,q+1}(f))(t))
\end{tikzcd}$$

commutes in $\mathrm{Mod}(\mathbf{C})$. Plugging in the inverse of the canonical isomorphism from $\Theta_S(t)$ to $\mathrm{T}_S(t)$, and taking into account the definitions of $\eta_{S,t}$ and $\mathrm{KS}'_{f,t}$, we infer Eq. (1.60). □

**Lemma 1.8.6** *Let $n$ and $p$ be integers and $f : X \to S$ be a submersive morphism of complex spaces such that $S$ is a complex manifold. Put $\mathscr{H} := \mathscr{H}^n(f)$ and, for any integer $\nu$, $\mathscr{F}^\nu := \mathrm{F}^\nu \mathscr{H}^n(f)$. Denote by*

$$\bar{\iota} : \mathscr{F}^{p-1}/\mathscr{F}^p \longrightarrow \mathscr{H}/\mathscr{F}^p$$

*the morphism in* $\mathrm{Mod}(S)$ *obtained from $\iota^n_f(p-1) : \mathscr{F}^{p-1} \to \mathscr{H}$ by quotienting out $\mathscr{F}^p$. Moreover, set*

$$\nabla_\iota := (\mathrm{id}_{\Omega^1_S} \otimes \mathrm{coker}(\iota^n_f(p))) \circ \nabla^n_{\mathrm{GM}}(f) \circ \iota^n_f(p).$$

*Then the following diagram commutes in* $\mathrm{Mod}(S)$*:*

$$\begin{tikzcd}
\mathscr{F}^p \arrow[r, "\nabla_\iota"] \arrow[d, "\mathrm{coker}(\iota^n_f(p,p+1))"'] & \Omega^1_S \otimes \mathscr{H}/\mathscr{F}^p \\
\mathscr{F}^p/\mathscr{F}^{p+1} \arrow[r, "\overline{\nabla}^{p,n}_{\mathrm{GM}}(f)"'] & \Omega^1_S \otimes \mathscr{F}^{p-1}/\mathscr{F}^p \arrow[u, "\mathrm{id}_{\Omega^1_S} \otimes \bar{\iota}"']
\end{tikzcd} \tag{1.61}$$

*Proof* By Proposition 1.6.11, there exists an ordered pair $(\zeta, \bar{\zeta})$ of morphisms in $\mathrm{Mod}(a_S)$ such that the following two identities hold in $\mathrm{Mod}(a_S)$:

$$\nabla^n_{\mathrm{GM}}(f) \circ \iota^n_f(p) = (\mathrm{id}_{\Omega^1_S} \otimes \iota^n_f(p-1)) \circ \zeta,$$

$$\left(\mathrm{id}_{\Omega^1_S} \otimes \mathrm{coker}\left(\iota^n_f(p-1,p)\right)\right) \circ \zeta = \bar{\zeta} \circ \mathrm{coker}(\iota^n_f(p,p+1)).$$

From this we deduce

$$
\begin{aligned}
\nabla_\iota &= \left(\mathrm{id}_{\Omega^1_S} \otimes \mathrm{coker}\left(\iota^n_f(p)\right)\right) \circ \nabla^n_{\mathrm{GM}}(f) \circ \iota^n_f(p) \\
&= \left(\mathrm{id}_{\Omega^1_S} \otimes \mathrm{coker}\left(\iota^n_f(p)\right)\right) \circ (\mathrm{id}_{\Omega^1_S} \otimes \iota^n_f(p-1)) \circ \zeta \\
&= \left(\mathrm{id}_{\Omega^1_S} \otimes \left(\mathrm{coker}(\iota^n_f(p)) \circ \iota^n_f(p-1)\right)\right) \circ \zeta \\
&= \left(\mathrm{id}_{\Omega^1_S} \otimes \left(\bar{\iota} \circ \mathrm{coker}(\iota^n_f(p-1,p))\right)\right) \circ \zeta \\
&= (\mathrm{id}_{\Omega^1_S} \otimes \bar{\iota}) \circ \left(\mathrm{id}_{\Omega^1_S} \otimes \mathrm{coker}\left(\iota^n_f(p-1,p)\right)\right) \circ \zeta \\
&= (\mathrm{id}_{\Omega^1_S} \otimes \bar{\iota}) \circ \bar{\zeta} \circ \mathrm{coker}(\iota^n_f(p,p+1)).
\end{aligned}
$$

Taking into account that, by Definition 1.6.12, $\overline{\nabla}^{p,n}_{\mathrm{GM}}(f) = \bar{\zeta}$, we are finished. □

**Proposition 1.8.7** *Let $f : X \to S$ be a submersive morphism of complex spaces such that $S$ is a simply connected complex manifold. Let $n$ and $p$ be integers and $t \in S$. In addition, let $\psi^p$ and $\psi^{p-1}$ be such that the following diagram commutes in* $\mathrm{Mod}(S)$ *for $\nu = p, p-1$:*

$$
\begin{tikzcd}
\mathrm{R}^n\bar{f}_*(\sigma^{\geq \nu}\bar{\Omega}^\bullet_f) \arrow[r, "\lambda^n_f(\nu)"] \arrow[d, "\mathrm{R}^n\bar{f}_*(j^{\leq \nu}(\sigma^{\geq \nu}\bar{\Omega}^\bullet_f))"'] & \mathrm{F}^\nu \mathscr{H}^n(f) \arrow[d, "\mathrm{coker}(\iota^n_f(\nu,\nu+1))"] \\
\mathrm{R}^n\bar{f}_*(\sigma^{=\nu}\bar{\Omega}^\bullet_f) & \mathrm{F}^\nu \mathscr{H}^n(f)/\mathrm{F}^{\nu+1}\mathscr{H}^n(f) \arrow[l, dotted, "\psi^\nu"]
\end{tikzcd}
\tag{1.62}
$$

*Let $\omega^{p-1}$ be a left inverse of $\psi^{p-1}$ in* $\mathrm{Mod}(S)$. *Assume that $\mathscr{H}^n(f)$ is a locally finite free module on $S$ and $\mathrm{F}^p\mathscr{H}^n(f)$ is a vector subbundle of $\mathscr{H}^n(f)$ on $S$. Put*

$$
\begin{aligned}
\alpha' &:= \kappa^n_f(\sigma^{=p}\Omega^\bullet_f) \circ \psi^p \circ \mathrm{coker}(\iota^n_f(p,p+1)), \\
\beta' &:= (\iota^n_f(p-1)/\mathrm{F}^p\mathscr{H}^n(f)) \circ \omega^{p-1} \circ (\kappa^n_f(\sigma^{=p-1}\Omega^\bullet_f))^{-1}.
\end{aligned}
$$

*Moreover, set*

$$
F'(t) := \mathrm{im}((\iota^n_f(p))(t) : (\mathrm{F}^p\mathscr{H}^n(f))(t) \longrightarrow (\mathscr{H}^n(f))(t)),
$$

*and write*

$$
\begin{aligned}
\sigma &: (\mathrm{F}^p\mathscr{H}^n(f))(t) \longrightarrow F'(t), \\
\tau &: (\mathscr{H}^n(f))(t)/F'(t) \longrightarrow (\mathscr{H}^n(f)/\mathrm{F}^p\mathscr{H}^n(f))(t)
\end{aligned}
$$

*for the evident morphisms. Then $\sigma$ and $\tau$ are isomorphisms in* Mod($\mathbf{C}$). *Besides, the following diagram commutes in* Mod($\mathbf{C}$)*:*

$$
\begin{tikzcd}
\mathrm{T}_S(t) \arrow[rr, "\mathrm{KS}'_{f,t}"] \arrow[dd, "\mathrm{T}_t(\mathcal{P}'^{p,n}_t(f))"'] & & (\mathrm{R}^1 f_*(\Theta_f))(t) \arrow[d, "\gamma'^{p,n-p}_{f,t}"] \\
& & \mathrm{Hom}((\mathscr{H}^{p,n-p}(f))(t), (\mathscr{H}^{p-1,n-p+1}(f))(t)) \arrow[d, "\mathrm{Hom}(\alpha'(t)\circ\sigma^{-1}, \tau^{-1}\circ\beta'(t))"] \\
\mathrm{T}_{\mathrm{Gr}((\mathscr{H}^n(f))(t))}(F'(t)) \arrow[rr, "\theta((\mathscr{H}^n(f))(t), F'(t))"'] & & \mathrm{Hom}(F'(t), (\mathscr{H}^n(f))(t)/F'(t))
\end{tikzcd}
\tag{1.63}
$$

*Proof* Introduce the following notational shorthands:

$$
\mathscr{H} := \mathscr{H}^n(f), \qquad \theta := \theta(\mathscr{H}(t), F'(t)),
$$
$$
\mathscr{F}^* := \mathrm{F}^*\mathscr{H}^n(f).
$$

Furthermore, set

$$
\nabla_\iota := (\mathrm{id}_{\Omega^1_S} \otimes \mathrm{coker}(\iota^n_f(p))) \circ \nabla^n_{\mathrm{GM}}(f) \circ \iota^n_f(p).
$$

Then by Lemma 1.7.20, $\sigma$ and $\tau$ are isomorphisms in Mod($\mathbf{C}$) and the following diagram commutes in Mod($\mathbf{C}$):

$$
\begin{tikzcd}
\mathrm{T}_S(t) \arrow[r, "\eta_{S,t}(\mathscr{F}^p, \mathscr{H}/\mathscr{F}^p)(\nabla_\iota)"] \arrow[d, "\mathrm{T}_t(\mathcal{P}'^{p,n}_t(f))"'] & \mathrm{Hom}(\mathscr{F}^p(t), (\mathscr{H}/\mathscr{F}^p)(t)) \arrow[d, "\mathrm{Hom}(\sigma^{-1}, \tau^{-1})"] \\
\mathrm{T}_{\mathrm{Gr}(\mathscr{H}(t))}(F'(t)) \arrow[r, "\theta"'] & \mathrm{Hom}(F'(t), \mathscr{H}(t)/F'(t))
\end{tikzcd}
$$

By Lemma 1.8.6, setting

$$
\bar{\iota} := \iota^n_f(p-1)/\mathscr{F}^p : \mathscr{F}^{p-1}/\mathscr{F}^p \longrightarrow \mathscr{H}/\mathscr{F}^p,
$$

the diagram in Eq. (1.61) commutes in Mod($S$). Therefore, with

$$
\bar{\bar{\iota}} := \mathrm{coker}(\iota^n_f(p, p+1)) : \mathscr{F}^p \longrightarrow \mathscr{F}^p/\mathscr{F}^{p+1},
$$

we have

$$\begin{aligned}&\eta_{S,t}(\mathscr{F}^p, \mathscr{H}/\mathscr{F}^p)(\nabla_\iota)\\&= \operatorname{Hom}(\bar{\bar{\iota}}(t), \bar{\iota}(t)) \circ \eta_{S,t}(\mathscr{F}^p/\mathscr{F}^{p+1}, \mathscr{F}^{p-1}/\mathscr{F}^p)(\overline{\nabla}^{p,n}_{\mathrm{GM}}(f))\end{aligned}$$

according to Remark 1.7.17. Hence, the following diagram commutes in $\mathrm{Mod}(\mathbf{C})$:

$$\begin{tikzcd}
\mathrm{T}_S(t) \arrow[r, "\eta_{S,t}(\mathscr{F}^p/\mathscr{F}^{p+1}{,}\mathscr{F}^{p-1}/\mathscr{F}^p)(\overline{\nabla}^{p,n}_{\mathrm{GM}}(f))"] \arrow[d, "\mathrm{T}_t(\mathcal{P}'^{p,n}_t(f))"'] & \operatorname{Hom}((\mathscr{F}^p/\mathscr{F}^{p+1})(t), (\mathscr{F}^{p-1}/\mathscr{F}^p)(t)) \arrow[d, "\operatorname{Hom}(\bar{\bar{\iota}}(t)\circ\sigma^{-1}{,}\tau^{-1}\circ\bar{\iota}(t))"] \\
\mathrm{T}_{\mathrm{Gr}(\mathscr{H}(t))}(F'(t)) \arrow[r, "\theta"'] & \operatorname{Hom}(F'(t), (\mathscr{H}^n(f))(t)/F'(t))
\end{tikzcd}$$

By Theorem 1.6.14, the following diagram commutes in $\mathrm{Mod}(S)$:

$$\begin{tikzcd}
\mathscr{F}^p/\mathscr{F}^{p+1} \arrow[r, "\overline{\nabla}^{p,n}_{\mathrm{GM}}(f)"] \arrow[d, "\kappa^n_f(\sigma^{=p}\Omega^\bullet_f)\circ\psi^p"'] & \Omega^1_S \otimes \mathscr{F}^{p-1}/\mathscr{F}^p \arrow[d, "\mathrm{id}_{\Omega^1_S}\otimes(\kappa^n_f(\sigma^{=p-1}\Omega^\bullet_f)\circ\psi^{p-1})"] \\
\mathscr{H}^{p,n-p} \arrow[r, "\gamma^{p,n-p}_{\mathrm{KS}}(f)"'] & \Omega^1_S \otimes \mathscr{H}^{p-1,n-p+1}
\end{tikzcd}$$

Thus, making use of Remark 1.7.17 again, we obtain

$$\begin{aligned}&\eta_{S,t}(\mathscr{F}^p/\mathscr{F}^{p+1}, \mathscr{F}^{p-1}/\mathscr{F}^p)(\overline{\nabla}^{p,n}_{\mathrm{GM}}(f))\\&\quad= \operatorname{Hom}\big((\kappa^n_f(\sigma^{=p}\Omega^\bullet_f)\circ\psi^p)(t), (\omega^{p-1}\circ(\kappa^n_f(\sigma^{=p-1}\Omega^\bullet_f))^{-1})(t)\big)\\&\qquad\qquad\circ\, \eta_{S,t}(\mathscr{H}^{p,n-p}, \mathscr{H}^{p-1,n-p+1})(\gamma^{p,n-p}_{\mathrm{KS}}(f)).\end{aligned}$$

Hence, this next diagram commutes in $\mathrm{Mod}(\mathbf{C})$:

$$\begin{tikzcd}
\mathrm{T}_S(t) \arrow[r, "\eta_{S,t}(\mathscr{H}^{p,n-p}{,}\mathscr{H}^{p-1,n-p+1})(\gamma^{p,n-p}_{\mathrm{KS}}(f))"] \arrow[d, "\mathrm{T}_t(\mathcal{P}'^{p,n}_t(f))"'] & \operatorname{Hom}(\mathscr{H}^{p,n-p}(t), \mathscr{H}^{p-1,n-p+1}(t)) \arrow[d, "\operatorname{Hom}(\alpha(t)\circ\sigma^{-1}{,}\tau^{-1}\circ\beta(t))"] \\
\mathrm{T}_{\mathrm{Gr}(\mathscr{H}(t))}(F'(t)) \arrow[r, "\theta"'] & \operatorname{Hom}(F'(t), (\mathscr{H}^n(f))(t)/F'(t))
\end{tikzcd}$$

Employing Lemma 1.8.5, we infer the commutativity of the diagram in Eq. (1.63). □

The next theorem is basically a variant of Proposition 1.8.7 that incorporates base changes.

**Theorem 1.8.8** *Let $f : X \to S$ be a submersive morphism of complex spaces such that $S$ is a simply connected complex manifold. Let $n$ and $p$ be integers and $t \in S$. Let $\psi^p_{X_t}$ and $\psi^{p-1}_{X_t}$ be such that the following diagram commutes in* $\mathrm{Mod}(\mathbf{C})$ *for $\nu = p, p-1$:*

$$\begin{tikzcd}
\mathrm{R}^n\overline{a_{X_t}}_*(\sigma^{\geq \nu}\bar{\Omega}^\bullet_{X_t}) \arrow[r, "\lambda^n_{X_t}(\nu)"] \arrow[d, "\mathrm{R}^n\bar{a}_{X_t *}(j^{\leq \nu}(\sigma^{\geq \nu}\bar{\Omega}^\bullet_{X_t}))"'] & \mathrm{F}^\nu \mathscr{H}^n(X_t) \arrow[d, "\mathrm{coker}(\iota^n_{X_t}(\nu,\nu+1))"] \\
\mathrm{R}^n\overline{a_{X_t}}_*(\sigma^{=\nu}\bar{\Omega}^\bullet_{X_t}) & \mathrm{F}^\nu \mathscr{H}^n(X_t)/\mathrm{F}^{\nu+1}\mathscr{H}^n(X_t) \arrow[l, dotted, "\psi^\nu_{X_t}"]
\end{tikzcd} \tag{1.64}$$

*Let $\omega^{p-1}_{X_t}$ be a left inverse of $\psi^{p-1}_{X_t}$ in* $\mathrm{Mod}(\mathbf{C})$. *Assume that $\mathscr{H}^n(f)$ is a locally finite free module on $S$, that $\mathrm{F}^p\mathscr{H}^n(f)$ is a vector subbundle of $\mathscr{H}^n(f)$ on $S$, and that the base change morphisms*

$$\phi^n_{f,s} : (\mathscr{H}^n(f))(s) \longrightarrow \mathscr{H}^n(X_s),$$
$$\phi^{p,n}_{f,s} : (\mathrm{F}^p\mathscr{H}^n(f))(s) \longrightarrow \mathrm{F}^p\mathscr{H}^n(X_s)$$

*are isomorphisms in* $\mathrm{Mod}(\mathbf{C})$ *for all $s \in S$. Assume there exist $\psi^p$, $\psi^{p-1}$, and $\omega^{p-1}$ such that firstly, the diagram in Eq. (1.62) commutes in* $\mathrm{Mod}(S)$ *for $\nu = p, p-1$ and secondly, $\omega^{p-1}$ is a left inverse of $\psi^{p-1}$ in* $\mathrm{Mod}(S)$. *Moreover, assume that the Hodge base change map*

$$\beta^{p,n-p}_{f,t} : (\mathscr{H}^{p,n-p}(f))(t) \longrightarrow \mathscr{H}^{p,n-p}(X_t)$$

*is an isomorphism in* $\mathrm{Mod}(\mathbf{C})$. *Then, setting*

$$\alpha := \kappa^n_{X_t}(\sigma^{=p}\Omega^\bullet_{X_t}) \circ \psi^p_{X_t} \circ \mathrm{coker}(\iota^n_{X_t}(p, p+1)),$$
$$\beta := (\iota^n_{X_t}(p-1)/\mathrm{F}^p\mathscr{H}^n(X_t)) \circ \omega^{p-1}_{X_t} \circ (\kappa^n_{X_t}(\sigma^{=p-1}\Omega^\bullet_{X_t}))^{-1},$$

*the following diagram commutes in* $\mathrm{Mod}(\mathbf{C})$*:*

$$
\begin{tikzcd}
\mathrm{T}_S(t) \arrow[r, "\mathrm{KS}_{f,t}"] \arrow[dd, "\mathrm{T}_t(\mathcal{P}_t^{p,n}(f))"'] & \mathrm{H}^1(X_t, \Theta_{X_t}) \arrow[d, "\gamma_{X_t}^{p,n-p}"] \\
 & \mathrm{Hom}(\mathscr{H}^{p,n-p}(X_t), \mathscr{H}^{p-1,n-p+1}(X_t)) \arrow[d, "\mathrm{Hom}(\alpha,\beta)"] \\
\mathrm{T}_{\mathrm{Gr}(\mathscr{H}^n(X_t))}(\mathrm{F}^p\mathscr{H}^n(X_t)) \arrow[r, "\theta(\mathscr{H}^n(X_t),\mathrm{F}^p\mathscr{H}^n(X_t))"'] & \mathrm{Hom}(\mathrm{F}^p\mathscr{H}^n(X_t), \mathscr{H}^n(X_t)/\mathrm{F}^p\mathscr{H}^n(X_t))
\end{tikzcd}
\tag{1.65}
$$

*Proof* We set

$$\phi := \phi_{f,t}^n : (\mathscr{H}^n(f))(t) \longrightarrow \mathscr{H}^n(X_t).$$

Then by Notation 1.8.2, the following diagram commutes in the category of complex spaces:

$$
\begin{tikzcd}
S \arrow[r, "\mathrm{id}_S"] \arrow[d, "\mathcal{P}_t^{\prime p,n}(f)"'] & S \arrow[d, "\mathcal{P}_t^{p,n}(f)"] \\
\mathrm{Gr}((\mathscr{H}^n(f))(t)) \arrow[r, "\mathrm{Gr}(\phi)"'] & \mathrm{Gr}(\mathscr{H}^n(X_t))
\end{tikzcd}
$$

In consequence, letting

$$F'(t) := \mathrm{im}((\iota_f^n(p))(t) : (\mathrm{F}^p\mathscr{H}^n(f))(t) \longrightarrow (\mathscr{H}^n(f))(t)),$$

the following diagram commutes in $\mathrm{Mod}(\mathbf{C})$:

$$
\begin{tikzcd}
\mathrm{T}_t(S) \arrow[r, "\mathrm{id}_{\mathrm{T}_t(S)}"] \arrow[d, "\mathrm{T}_t(\mathcal{P}_t^{\prime p,n}(f))"'] & \mathrm{T}_t(S) \arrow[d, "\mathrm{T}_t(\mathcal{P}_t^{p,n}(f))"] \\
\mathrm{T}_{\mathrm{Gr}((\mathscr{H}^n(f))(t))}(F'(t)) \arrow[r, "\mathrm{T}_{F'(t)}(\mathrm{Gr}(\phi))"'] & \mathrm{T}_{\mathrm{Gr}(\mathscr{H}^n(X_t))}(\mathrm{F}^p\mathscr{H}^n(X_t))
\end{tikzcd}
\tag{1.66}
$$

Note that

$$(\mathrm{Gr}(\phi))(F'(t)) = \phi[F'(t)] = \mathrm{F}^p\mathscr{H}^n(X_t)$$

due to the commutativity of the diagram

$$\begin{tikzcd}
(\mathrm{F}^p\mathscr{H}^n(f))(t) \arrow[r, "\phi^{p,n}_{f,t}"] \arrow[d, "(\iota^n_f(p))(t)"'] & \mathrm{F}^p\mathscr{H}^n(X_t) \arrow[d, "\iota^n_{X_t}(p)"] \\
(\mathscr{H}^n(f))(t) \arrow[r, "\phi"'] & \mathscr{H}^n(X_t)
\end{tikzcd}$$

in $\mathrm{Mod}(\mathbf{C})$ and the fact that $\phi^{p,n}_{f,t}$ is an isomorphism. So, when we denote by

$$\bar{\phi} : (\mathscr{H}^n(f))(t)/F'(t) \longrightarrow \mathscr{H}^n(X_t)/\mathrm{F}^p\mathscr{H}^n(X_t)$$

the morphism which is induced by $\phi$ the obvious way, the following diagram commutes in $\mathrm{Mod}(\mathbf{C})$ in virtue of the naturality of $\theta$:

$$\begin{tikzcd}
\mathrm{T}_{\mathrm{Gr}((\mathscr{H}^n(f))(t))}(F'(t)) \arrow[r, "\mathrm{T}_{F'(t)}(\mathrm{Gr}(\phi))"] \arrow[d, "\theta((\mathscr{H}^n(f))(t),F'(t))"'] & \mathrm{T}_{\mathrm{Gr}(\mathscr{H}^n(X_t))}(\mathrm{F}^p\mathscr{H}^n(X_t)) \arrow[d, "\theta(\mathscr{H}^n(X_t),\mathrm{F}^p\mathscr{H}^n(X_t))"] \\
\mathrm{Hom}(F'(t), (\mathscr{H}^n(f))(t)/F'(t)) \arrow[r, "\mathrm{Hom}((\phi|_{F'(t)})^{-1},\bar{\phi})"'] & \mathrm{Hom}(\mathrm{F}^p\mathscr{H}^n(X_t), \mathscr{H}^n(X_t)/\mathrm{F}^p\mathscr{H}^n(X_t))
\end{tikzcd} \tag{1.67}$$

Define

$$\alpha' := \kappa^n_f(\sigma^{=p}\Omega^\bullet_f) \circ \psi^p \circ \mathrm{coker}(\iota^n_f(p, p+1)),$$
$$\beta' := (\iota^n_f(p-1)/\mathrm{F}^p\mathscr{H}^n(f)) \circ \omega^{p-1} \circ (\kappa^n_f(\sigma^{=p-1}\Omega^\bullet_f))^{-1},$$

and introduce the evident morphisms

$$\sigma : (\mathrm{F}^p\mathscr{H}^n(f))(t) \longrightarrow F'(t),$$
$$\tau : (\mathscr{H}^n(f))(t)/F'(t) \longrightarrow (\mathscr{H}^n(f)/\mathrm{F}^p\mathscr{H}^n(f))(t).$$

Then by Proposition 1.8.7, $\sigma$ and $\tau$ are isomorphisms in $\mathrm{Mod}(\mathbf{C})$, and the diagram in Eq. (1.63) commutes in $\mathrm{Mod}(\mathbf{C})$.

Let

$$\bar{\phi}_1 : (\mathscr{H}^n(f)/\mathrm{F}^p\mathscr{H}^n(f))(t) \longrightarrow \mathscr{H}^n(X_t)/\mathrm{F}^p\mathscr{H}^n(X_t)$$

be the morphism which is naturally induced by $\phi$, similar to $\bar{\phi}$ above, using the fact that

$$\left(\operatorname{coker}(\iota_f^n(p))\right)(t) : (\mathscr{H}^n(f))(t) \longrightarrow (\mathscr{H}^n(f)/\mathrm{F}^p\mathscr{H}^n(f))(t)$$

is a cokernel for $(\iota_f^n(p))(t)$ in $\mathrm{Mod}(\mathbf{C})$. The latter follows as the evaluation functor "$-(t)$" is a right exact functor from $\mathrm{Mod}(S)$ to $\mathrm{Mod}(\mathbf{C})$. Then we obtain the identities

$$\phi_{f,t}^{p,n} = (\phi|_{F'(t)}) \circ \sigma \quad \text{and} \quad \bar{\phi} = \bar{\phi}_1 \circ \tau. \tag{1.68}$$

Moreover, comparing $\alpha'$ and $\alpha$, and setting $q := n-p$, we see that this next diagram commutes in $\mathrm{Mod}(\mathbf{C})$:

$$\begin{tikzcd}
(\mathrm{F}^p\mathscr{H}^n(f))(t) \arrow[r, "\phi_{f,t}^{p,n}"] \arrow[d, "\alpha'(t)"'] & \mathrm{F}^p\mathscr{H}^n(X_t) \arrow[d, "\alpha"] \\
(\mathscr{H}^{p,q}(f))(t) \arrow[r, "\beta_{f,t}^{p,q}"'] & \mathscr{H}^{p,q}(X_t)
\end{tikzcd} \tag{1.69}$$

Similarly, comparing $\beta'$ and $\beta$, we see that

$$\begin{tikzcd}
(\mathscr{H}^{p-1,q+1}(f))(t) \arrow[r, "\beta_{f,t}^{p-1,q+1}"] \arrow[d, "\beta'(t)"'] & \mathscr{H}^{p-1,q+1}(X_t) \arrow[d, "\beta"] \\
(\mathscr{H}^n(f)/\mathrm{F}^p\mathscr{H}^n(f))(t) \arrow[r, "\bar{\phi}_1"'] & \mathscr{H}^n(X_t)/\mathrm{F}^p\mathscr{H}^n(X_t)
\end{tikzcd} \tag{1.70}$$

commutes in $\mathrm{Mod}(\mathbf{C})$. Write

$$\beta_{f,t}^1 : (\mathrm{R}^1 f_*(\Theta_f))(t) \longrightarrow \mathrm{H}^1(X_t, \Theta_{X_t})$$

for the evident base change morphism. Then, since the cup product morphisms $\smile^{1,q}$ as well as the contraction morphisms $\gamma^p$ are compatible with base change, the following diagram commutes in $\mathrm{Mod}(\mathbf{C})$:

$$\begin{tikzcd}
(\mathrm{R}^1 f_*(\Theta_f) \otimes_S \mathscr{H}^{p,q}(f))(t) \arrow[r, "\gamma_f^{p,q}(t)"] \arrow[d, "\text{can.}"'] & (\mathscr{H}^{p-1,q+1}(f))(t) \arrow[dd, "\beta_{f,t}^{p-1,q+1}"] \\
(\mathrm{R}^1 f_*(\Theta_f))(t) \otimes_{\mathbf{C}} (\mathscr{H}^{p,q}(f))(t) \arrow[d, "\beta_{f,t}^1 \otimes \beta_{f,t}^{p,q}"'] & \\
\mathrm{H}^1(X_t, \Theta_{X_t}) \otimes_{\mathbf{C}} \mathscr{H}^{p,q}(X_t) \arrow[r, "\gamma_{X_t}^{p,q}"'] & \mathscr{H}^{p-1,q+1}(X_t)
\end{tikzcd}$$

Therefore, given that $\beta_{f,t}^{p,q}$ is an isomorphism by assumption, the following diagram commutes in $\mathrm{Mod}(\mathbf{C})$ also:

$$\begin{tikzcd}
(\mathrm{R}^1 f_*(\Theta_f))(t) \arrow[r, "\beta_{f,t}^1"] \arrow[d, "\gamma_{f,t}^{\prime p,q}"'] & \mathrm{H}^1(X_t, \Theta_{X_t}) \arrow[d, "\gamma_{X_t}^{p,q}"] \\
\mathrm{Hom}((\mathscr{H}^{p,q}(f))(t), (\mathscr{H}^{p-1,q+1}(f))(t)) \arrow[r, "{\mathrm{Hom}((\beta_{f,t}^{p,q})^{-1}, \beta_{f,t}^{p-1,q+1})}"'] & \mathrm{Hom}(\mathscr{H}^{p,q}(X_t), \mathscr{H}^{p-1,q+1}(X_t))
\end{tikzcd} \tag{1.71}$$

According to the definition of the Kodaira-Spencer map $\mathrm{KS}_{f,t}$ in Notation 1.8.3, the following diagram commutes in $\mathrm{Mod}(\mathbf{C})$:

$$\begin{tikzcd}
& \mathrm{T}_t(S) \arrow[dl, "\mathrm{KS}'_{f,t}"'] \arrow[dr, "\mathrm{KS}_{f,t}"] & \\
(\mathrm{R}^1 f_*(\Theta_f))(t) \arrow[rr, "\beta_{f,t}^1"'] & & \mathrm{H}^1(X_t, \Theta_{X_t})
\end{tikzcd} \tag{1.72}$$

Taking all our previous considerations into account, we obtain

$$\begin{aligned}
&\theta(\mathscr{H}^n(X_t), \mathrm{F}^p \mathscr{H}^n(X_t)) \circ \mathrm{T}_t(\mathcal{P}_t^{p,n}(f)) \\
&\overset{(1.66)}{=} \theta(\mathscr{H}^n(X_t), \mathrm{F}^p \mathscr{H}^n(X_t)) \circ \mathrm{T}_{F'(t)}(\mathrm{Gr}(\phi)) \circ \mathrm{T}_t(\mathcal{P}_t^{\prime p,n}(f)) \\
&\overset{(1.67)}{=} \mathrm{Hom}((\phi|_{F'(t)})^{-1}, \bar{\phi}) \circ \theta((\mathscr{H}^n(f))(t), F'(t)) \circ \mathrm{T}_t(\mathcal{P}_t^{\prime p,n}(f)) \\
&\overset{(1.63)}{=} \mathrm{Hom}((\phi|_{F'(t)})^{-1}, \bar{\phi}) \circ \mathrm{Hom}(\alpha'(t) \circ \sigma^{-1}, \tau^{-1} \circ \beta'(t)) \circ \gamma_{f,t}^{\prime p,n-p} \circ \mathrm{KS}'_{f,t} \\
&\overset{(1.68)}{=} \mathrm{Hom}(\alpha'(t) \circ (\phi_{f,t}^{p,n})^{-1}, \bar{\phi}_1 \circ \beta'(t)) \circ \gamma_{f,t}^{\prime p,n-p} \circ \mathrm{KS}'_{f,t}
\end{aligned}$$

$$\overset{(1.69)}{\overset{(1.70)}{=}} \operatorname{Hom}(\alpha, \beta) \circ \operatorname{Hom}((\beta_{f,t}^{p,q})^{-1}, \beta_{f,t}^{p-1,q+1}) \circ \gamma'^{p,q}_{f,t} \circ \mathrm{KS}'_{f,t}$$

$$\overset{(1.71)}{=} \operatorname{Hom}(\alpha, \beta) \circ \gamma_{X_t}^{p,q} \circ \beta_{f,t}^{1} \circ \mathrm{KS}'_{f,t}$$

$$\overset{(1.72)}{=} \operatorname{Hom}(\alpha, \beta) \circ \gamma_{X_t}^{p,n-p} \circ \mathrm{KS}_{f,t},$$

which implies precisely the commutativity of the diagram in Eq. (1.65). □

When it comes to applying Theorem 1.8.8, you are faced with the problem of deciding whether there exist morphisms $\psi^\nu$ (resp. $\psi^\nu_{X_t}$) rendering commutative in $\mathrm{Mod}(S)$ (resp. $\mathrm{Mod}(\mathbf{C})$) the diagram in Eq. (1.62) (resp. Eq. (1.64)). Let me formulate two tangible criteria.

**Proposition 1.8.9** *Let $n$ and $\nu$ be integers and $f : X \to S$ an arbitrary morphism of complex spaces. Denote by $E$ the Frölicher spectral sequence of $f$.*

1. *The following are equivalent:*
   a. *$E$ degenerates from behind in the entry $(\nu, n-\nu)$ at sheet 1 in $\mathrm{Mod}(S)$;*
   b. *there exists $\psi^\nu$ rendering commutative in $\mathrm{Mod}(S)$ the diagram in Eq. (1.62).*
2. *The following are equivalent:*
   a. *$E$ degenerates in the entry $(\nu, n-\nu)$ at sheet 1 in $\mathrm{Mod}(S)$;*
   b. *there exists an isomorphism $\psi^\nu$ rendering commutative in $\mathrm{Mod}(S)$ the diagram in Eq. (1.62).*

*Proof* Items 1 and 2 are special cases of standard interpretations of the degeneration of a spectral sequence associated to a filtered complex; cf. [3, §1]. □

**Theorem 1.8.10** *Let $n$ be an integer and $f : X \to S$ a submersive morphism of complex spaces such that $S$ is a simply connected complex manifold. Assume that*

1. *the Frölicher spectral sequence of $f$ degenerates in the entries*

$$I := \{(p, q) \in \mathbf{Z} \times \mathbf{Z} : p + q = n\}$$

   *at sheet 1 in $\mathrm{Mod}(S)$;*
2. *for all $(p, q) \in I$, the module $\mathscr{H}^{p,q}(f)$ is locally finite free on $S$;*
3. *for all $s \in S$, the Frölicher spectral sequence of $X_s$ degenerates in the entries $I$ at sheet 1 in $\mathrm{Mod}(\mathbf{C})$;*
4. *for all $s \in S$ and all $(p, q) \in I$, the Hodge base change map*

$$\beta_{f,s}^{p,q} : (\mathscr{H}^{p,q}(f))(s) \longrightarrow \mathscr{H}^{p,q}(X_s)$$

   *is an isomorphism in $\mathrm{Mod}(\mathbf{C})$.*

*Let $t \in S$. Then there exists a sequence $(\tilde{\psi}^{\nu})_{\nu \in \mathbf{Z}}$ of isomorphisms in* $\mathrm{Mod}(\mathbf{C})$*,*

$$\tilde{\psi}^{\nu} : \mathrm{F}^{\nu}\mathscr{H}^n(X_t)/\mathrm{F}^{\nu+1}\mathscr{H}^n(X_t) \longrightarrow \mathscr{H}^{\nu,n-\nu}(X_t),$$

*such that, for all $p \in \mathbf{Z}$, the diagram in Eq. (1.65) commutes in* $\mathrm{Mod}(\mathbf{C})$*, where we set*

$$\begin{aligned} \alpha &:= \tilde{\psi}^{p} \circ \mathrm{coker}(\iota^n_{X_t}(p, p+1)), \\ \beta &:= (\iota^n_{X_t}(p-1)/\mathrm{F}^p\mathscr{H}^n(X_t)) \circ (\tilde{\psi}^{p-1})^{-1}. \end{aligned} \tag{1.73}$$

*Proof* Using Proposition 1.8.9, item 1 tells us that, for all integers $\nu$, there exists one, and only one, $\psi^{\nu}$ such that the diagram in Eq. (1.62) commutes in $\mathrm{Mod}(S)$ (note that the uniqueness of $\psi^{\nu}$ follows from the fact that both $\lambda^n_f(\nu)$ and $\mathrm{coker}(\iota^n_f(\nu, \nu + 1))$, and whence their composition, are epimorphisms in $\mathrm{Mod}(S)$). Moreover, $\psi^{\nu}$ is an isomorphism. Furthermore, for all integers $\nu$,

$$\kappa^n_f(\sigma^{=\nu}\Omega^{\bullet}_f) : \mathrm{R}^n\bar{f}_*(\sigma^{=\nu}\bar{\Omega}^{\bullet}_f) \longrightarrow \mathrm{R}^nf_*(\sigma^{=\nu}\Omega^{\bullet}_f) = \mathrm{R}^{n-p}f_*(\Omega^p_f)$$

is an isomorphism. Thus, for all $\nu \in \mathbf{Z}$, there exists an isomorphism

$$\mathrm{F}^{\nu}\mathscr{H}^n(f)/\mathrm{F}^{\nu+1}\mathscr{H}^n(f) \longrightarrow \mathscr{H}^{\nu,n-\nu}(f)$$

in $\mathrm{Mod}(S)$. Now since, for all integers $\nu \geq n+1$, the module $\mathrm{F}^{\nu}\mathscr{H}^n(f)$ is zero on $S$, and in particular locally finite free, we conclude by descending induction on $\nu$ starting at $\nu = n+1$ that, for all integers $\nu$, the module $\mathrm{F}^{\nu}\mathscr{H}^n(f)$ is locally finite free on $S$. Along the way we make use of item 2. Specifically, since $\mathrm{F}^0\mathscr{H}^n(f) = \mathscr{H}^n(f)$, we see that $\mathscr{H}^n(f)$ is a locally finite free module (i.e., in the terminology of Definition 1.7.9, a vector bundle) on $S$. Moreover, for all integers $\mu$ and $\nu$ such that $\mu \leq \nu$, there exists a short exact sequence

$$0 \longrightarrow \mathrm{F}^{\mu}/\mathrm{F}^{\nu} \longrightarrow \mathrm{F}^{\mu-1}/\mathrm{F}^{\nu} \longrightarrow \mathrm{F}^{\mu-1}/\mathrm{F}^{\mu} \longrightarrow 0,$$

where we write $\mathrm{F}^*$ as a shorthand for $\mathrm{F}^*\mathscr{H}^n(f)$. Therefore, we see, using descending induction on $\mu$, that for all integers $\nu$ and all integers $\mu$ such that $\mu \leq \nu$ the quotient $\mathrm{F}^{\mu}/\mathrm{F}^{\nu}$ is a locally finite free module on $S$. Specifically, we see that for all integers $\nu$, the quotient $\mathscr{H}^n(f)/\mathrm{F}^{\nu}\mathscr{H}^n(f)$ is a locally finite free module on $S$. Thus, we conclude that $\mathrm{F}^{\nu}\mathscr{H}^n(f)$ is a vector subbundle of $\mathscr{H}^n(f)$ on $S$ for all integers $\nu$.

For the time being, fix an arbitrary element $s$ of $S$. Then by item 3 and Proposition 1.8.9 we deduce that, for all integers $\nu$, there exists a (unique) isomorphism $\psi^{\nu}_{X_s}$ such that the diagram in Eq. (1.64), where we replace $t$ by $s$, commutes in $\mathrm{Mod}(\mathbf{C})$.

As the base change commutes with taking stupid filtrations, the following diagram has exact rows and commutes in $\mathrm{Mod}(\mathbf{C})$ for all integers $\nu$:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{F}^{\nu}(s) \arrow[r, "(\iota_f^n(\nu-1,\nu))(s)"] \arrow[d, "\phi_{f,s}^{\nu,n}"'] & \mathrm{F}^{\nu-1}(s) \arrow[r] \arrow[d, "\phi_{f,s}^{\nu-1,n}"'] & \mathscr{H}^{\nu,n-\nu}(s) \arrow[r] \arrow[d, "\beta_{f,s}^{\nu,n-p}"] & 0 \\
0 \arrow[r] & \mathrm{F}^{\nu}\mathscr{H}^n(X_s) \arrow[r, "\iota_{X_s}^n(\nu-1,\nu)"'] & \mathrm{F}^{\nu-1}\mathscr{H}^n(X_s) \arrow[r] & \mathscr{H}^{\nu,n-\nu}(X_s) \arrow[r] & 0
\end{tikzcd}$$

Therefore, using a descending induction on $\nu$ starting at $\nu = n + 1$ together with item 4 and the "short five lemma," we infer that, for all $\nu \in \mathbf{Z}$, the base change map $\phi_{f,s}^{\nu,n}$ is an isomorphism in $\mathrm{Mod}(\mathbf{C})$. Specifically, since $\phi_{f,s}^{0,n} = \phi_{f,s}^{n}$, we see that the de Rham base change map $\phi_{f,s}^{n}$ is an isomorphism in $\mathrm{Mod}(\mathbf{C})$.

Abandon the fixation of $s$ and define a $\mathbf{Z}$-sequence $\tilde{\psi}$ by putting, for any $\nu \in \mathbf{Z}$,

$$\tilde{\psi}^{\nu} := \kappa_{X_t}^{n}(\sigma^{=\nu}\Omega_{X_t}^{\bullet}) \circ \psi_{X_t}^{\nu}.$$

Let $p$ be an integer. Then defining $\alpha$ and $\beta$ according to Eq. (1.73), the commutativity of the diagram in Eq. (1.65) is implied by Theorem 1.8.8. $\square$

## References


1. T. Bröcker, K. Jänich, *Introduction to Differential Topology* (Cambridge University Press, New York, 1982), pp. vii+160
2. P. Deligne, *Equations Différentielles à Points Singuliers Réguliers*. Lecture Notes in Mathematics, vol. 163 (Springer, Heidelberg, 1970)
3. P. Deligne, Théorie de Hodge: II. Publications Mathématiques de l'I.H.É.S. **40**(1), 5–57 (1971)
4. D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*. Graduate Texts in Mathematics, vol. 150 (Springer, Heidelberg, 1995)
5. R. Godement, *Topologie Algébrique et Théorie des Faisceaux*, 3rd rev. Actualités Scientifiques et Industrielles, vol. 1252 (Hermann, Paris, 1973)
6. P.A. Griffiths, Periods of integrals on algebraic manifolds, II. (Local study of the period mapping.) Am. J. Math. **90**(3), 805–865 (1968)
7. A. Grothendieck, Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): I. Le langage des schémas. Publications Mathématiques de l'I.H.É.S. **4**, 5–228 (1960)
8. A. Grothendieck, Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): III. Étude cohomologique des faisceaux cohérents, Première partie. Publications Mathématiques de l'I.H.É.S. **11**, 5–167 (1961)
9. A. Grothendieck, Techniques de construction en géométrie analytique. VII. Étude locale des morphismes: éléments de calcul infinitésimal. Séminaire Henri Cartan **13**(2), 1–27 (1960–1961)
10. N.M. Katz, Algebraic solutions of differential equations (p-curvature and the Hodge filtration). Invent. Math. **18**(1–2), 1–118 (1972). doi:10.1007/BF01389714

11. N.M. Katz, T. Oda, On the differentiation of de Rham cohomology classes with respect to parameters. J. Math. Kyoto Univ. **8**, 199–213 (1968)
12. J.P. May, *A Concise Course in Algebraic Topology*. Chigaco Lectures in Mathematics (The University of Chicago Press, Chicago, 1999)
13. The Stacks Project Authors, *Stacks Project* (2014). http://stacks.math.columbia.edu
14. C. Voisin, *Théorie de Hodge et géométrie algébrique complexe*. Cours Spécialisées, vol. 10 (Société Mathématique de France, Paris, 2002)
15. R.O. Wells Jr., *Differential Analysis on Complex Manifolds*, 3rd ed. Graduate Texts in Mathematics, vol. 65 (Springer, New York, 2008), pp. xiv+299. doi:10.1007/978-0-387-73892-5

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