

Preface

Through this book, I intend to deliver to readers three chapters of state-of-the-art mathematics positioned at the crossroad of algebraic geometry—more precisely, the homological algebra of sheaves of modules on ringed spaces—and the theory of complex analytic spaces. From my perspective as the author, the focal point of the text is Chap. 1.

The first chapter explores the territory surrounding Nicholas Katz’s and Tadao Oda’s conception of a Gauß-Manin connection defined on the relative algebraic de Rham cohomology sheaf. Here, I attempt to employ Katz’s and Oda’s idea in the realm of complex analytic spaces—to my knowledge for the first time in the literature. Note that Katz and Oda work with schemes rather than with analytic spaces. From the Gauß-Manin connection, I fabricate period mappings, and the nice thing, I hope the reader will agree, is that properties of the Gauß-Manin connection thus translate as properties of the period mappings. All of these are explained here in great detail.

For the first chapter, I have a broad audience in mind. So, if you are a complex geometer, then this will most likely meet your needs. From the point where period mappings make their appearance (Sect. 1.7), the geometric objects (i.e., the ringed spaces) under investigation are complex spaces, or even complex manifolds. In case you have seen period mappings for families of compact complex manifolds of Kähler type, or if you have encountered a variation of Hodge structure, most of my results will have a familiar ring. Keep in mind, though, that the assumptions I make in Chap. 1 are certainly very much different from what you are likely to be used to. Specifically, I study families of complex manifolds that are neither compact nor of Kähler type.

If you are not interested in complex manifolds, but you are interested in algebraic schemes, or in analytic spaces over a valued field other than that of complex numbers, then more than half of Chap. 1 still may interest you. As a matter of fact, I have conceived the central Sect. 1.5 of Chap. 1 so that it applies to arbitrary ringed spaces. You might even pursue this suggested setup further—namely, to ringed topoi—if you so wish. Also, the exposition is fully intended to be accessible to you, even if you are just getting acquainted with algebraic geometry. Accordingly,

I have put a lot of effort into making my exposition as elementary as possible. The prerequisites are kept to a bare minimum. Readers are assumed to be familiar with sheaves of rings and sheaves of modules on topological spaces, as well as with their morphisms and basic operations—that is, the tensor product, the sheaf hom, the pushforward, and the pullback. I use homological algebra to the extent that we use such terms as complexes of modules, cohomology, and right derived functors. Last but not least, readers are also assumed to understand the basic jargon of category theory—for instance, in the case of commutative diagrams.

Moving from Chap. 1 to Chap. 2 to Chap. 3, the geometric objects and ideas become increasingly concrete and specific. Already in Chap. 2, general ringed spaces have been replaced completely by complex analytic spaces. More to the point, geometric notions such as the Kähler property, analytic subsets, dimension, or the flatness of a morphism come into play. In Chap. 3, which treats the applications material hinted at in the book title, the geometry becomes even more special as we look at symplectic complex spaces.

A nice feature of this book is that its three chapters are almost entirely independent of one another. In fact, the chapters are really designed to be read as stand-alone. When you open up Chap. 2, you will find a problem set, which is entirely disjoint from Chap. 1. Specifically, not a single result established in Chap. 1 is needed for Chap. 2, although I recycle a couple of notations. While I think that Chap. 2 is of great interest in its own right, you should read the introductory Sect. 2.1 and see for yourself. With the geometric objects narrowing, the target audience of the text is bound to shift somewhat as it progresses. I hope that Chap. 2, as intended, hits the core of what “algebraic methods in the global theory of complex spaces” are all about. As far as the prerequisites go, I assume a certain reader familiarity in working with complex spaces. In particular, I assume that the reader knows what is the Frölicher spectral sequence of a complex space, or better of a morphism of complex spaces. Other than that, I have tried consistently to explain all the techniques that I employ—especially that of local cohomology and that of formal completions. Readers should be able to follow the proofs even though they haven’t previously seen the mentioned techniques in action before.

If you feel that Chap. 1 contains a lot of general nonsense, no offense is taken, and if you find Sect. 2.1 only mildly intriguing, maybe Chap. 3 can yet convince you of the contrary. Symplectic complex spaces have been, perhaps for the past two to three decades, and are, highly fashionable objects of research in complex geometry. The interest in them stems, naturally, from their connection with the all overshadowing hyperkähler manifolds. For context, let me refer you to the individual sections of Chap. 3. Clearly, symplectic complex manifolds—more precisely, irreducible symplectic complex manifolds—dominate the business. Singular symplectic complex spaces have, nevertheless, been present in the theory from very early on, albeit only implicitly at first. In 1983, for instance, Beauville already describes his Hilbert scheme of points on a K3 surface as a resolution of singularities of what is called a singular symplectic space today.

Chapter 3 of this book serves a twofold purpose. First, it provides you with a systematic introduction to the global theory of symplectic complex spaces. An entire section alone is devoted to the definition and the elementary properties of the Beauville-Bogomolov-Fujiki quadratic form in the possibly singular context. This is, to my knowledge, novel in the literature. Second, it seeks to prove two very contemporary theorems about what might be called “irreducible symplectic complex spaces”—namely, a local Torelli theorem and the so-called Fujiki relation. In order to deduce the local Torelli theorem, I invoke the conclusions of Chaps. 1 and 2. Apart from that, Chap. 3 is, again, fully independent of its predecessors. Besides, in order to understand the proof of my local Torelli theorem, an understanding of Chaps. 1 and 2 isn’t strictly necessary. One can simply take the final results of Chaps. 1 and 2 for granted and go from there.

The way it is presented here, the Fujiki relation is a consequence of the local Torelli theorem. So, when you think that symplectic complex spaces are great, you have your motivation to tackle what I do in Chaps. 1 and 2. On the one hand, I may suggest that, for singular spaces, my line of argument, which relies heavily on the results of Chaps. 1 and 2, is the only argument for the Fujiki relation yet proposed. On the other hand, the Fujiki relation is an important tool, which has proven vital for a multitude of applications in the context of manifolds.

Personally, I think that the proof of the Fujiki relation for singular symplectic spaces affords a prime motivation for the more abstract theory in Chaps. 1 and 2. I believe, however, that Chaps. 1 and 2—combined as in Chap. 3, or each one on its own—may hold a far greater potential, a potential that goes beyond the notion of symplectic spaces. So, my hope would be that you can find new, interesting applications of these ideas. On that note, enjoy the book and happy researching!

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