Chapter 2
Geometric Action Functionals

Abstract In this chapter we begin by teaching the reader all the necessary basics of rectifiable curves and absolutely continuous functions. We then introduce the class of geometric action functionals to which our theory can be applied (and in particular the subclass of Hamiltonian geometric actions), give several examples of geometric actions, and prove a lower semi-continuity property for them. Finally, we define the notion of a "drift" of an action, as a generalization of the drift vector field entering the Wentzell-Freidlin action.

2.1 Curves

Let us begin by reviewing some basic definitions and facts related to curves, and let us then introduce the various classes of curves that we will use.

2.1.1 Rectifiable Curves and Absolutely Continuous Functions

An unparameterized oriented curve γ is an equivalence class of functions $\varphi \in C([0,T],D)$, $T > 0$, that are identical up to continuous non-decreasing changes of their parameterizations, or more formally, whose Fréchet distance to each other vanishes. *In this monograph we will tacitly assume that all our curves are unparameterized and oriented.*

A curve γ is called ***rectifiable*** [18, p. 115] if for some (and thus for every) parameterization $\varphi \in C([0,T],D)$ of γ we have

$$\text{length}(\gamma) := \text{length}(\varphi) := \sup_{\substack{N\in\mathbb{N} \\ 0=t_0<\cdots<t_N=T}} \sum_{i=1}^{N} \left|\varphi(t_i) - \varphi(t_{i-1})\right| < \infty.$$

It is easy to see that length(φ) is in fact the same for any parameterization φ of γ, and that it is finite if and only if all the component functions of φ are of bounded variation [18, Theorem 3.1]. We will denote the set of rectifiable curves in D by Γ.

A function $\varphi\colon [0,T] \to D$ is said to be ***absolutely continuous*** [18, p. 127] if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any finite collection of disjoint

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intervals $[t_{i-1}, t_i) \subset [0, T]$, $i = 1, \ldots, N$, we have

$$\sum_{i=1}^{N}(t_i - t_{i-1}) < \delta \qquad \Rightarrow \qquad \sum_{i=1}^{N}\big|\varphi(t_i) - \varphi(t_{i-1})\big| < \varepsilon.$$

We will denote the space of absolutely continuous functions with values in D by $\bar{C}(0, T)$. One can show [18, Proposition 1.12 (ii) and Theorem 3.11] that a function φ is in $\bar{C}(0, T)$ if and only if there exists an L^1-function which we denote by φ' such that $\varphi(t) = \varphi(0) + \int_0^t \varphi'(\tau)\,\mathrm{d}\tau$ for $\forall t \in [0, T]$. In that case, φ is differentiable in the classical sense at almost every $t \in [0, T]$, with derivative $\varphi'(t)$.

Clearly, every function $\varphi \in \bar{C}(0, T)$ is the parameterization of a rectifiable curve γ since for every partition $0 = t_0 < \cdots < t_N = T$ we have

$$\sum_{i=1}^{N}\big|\varphi(t_i) - \varphi(t_{i-1})\big| = \sum_{i=1}^{N}\left|\int_{t_i}^{t_{i-1}} \varphi'\,\mathrm{d}t\right| \le \int_0^T |\varphi'|\,\mathrm{d}t < \infty,$$

and it is not hard to show [18, Theorem 4.1] that $\mathrm{length}(\gamma) = \int_0^T |\varphi'|\,\mathrm{d}t$. The reverse is not true: Not every function φ that parameterizes a rectifiable curve γ is necessarily absolutely continuous (a counterexample can be constructed using the Cantor function [18, p. 125]). However, we have the following:

Lemma 2.1 (Parameterization by Arclength)

(i) Any curve $\gamma \in \Gamma$ can be parameterized by a unique function $\varphi_\gamma \in \bar{C}(0, 1)$ with $|\varphi_\gamma'| \equiv \mathrm{length}(\gamma)$ a.e.

(ii) If $\varphi \in \bar{C}(0, T)$ is any absolutely continuous parameterization of γ then $\varphi = \varphi_\gamma \circ \beta$ for some absolutely continuous function $\beta\colon [0, T] \to [0, 1]$, and we have $\varphi' = (\varphi_\gamma' \circ \beta)\beta'$ and $\beta' \ge 0$ a.e. on $[0, 1]$.

Proof (i) This is a trivial modification of [18, p. 136].

(ii) In the proof in [18, p. 136] it is shown that for any parameterization $\varphi \in C([0, T], D)$ of γ the function φ_γ fulfills $\varphi(t) = \varphi_\gamma(\beta(t))$ for $\forall t \in [0, T]$, where $\beta\colon [0, T] \to [0, 1]$ is defined by $\beta(t) := \mathrm{length}\big(\varphi|_{[0,t]}\big)/\,\mathrm{length}(\gamma)$. For any collection of disjoint intervals $[t_{i-1}, t_i) \subset [0, T]$, $i = 1, \ldots, N$, we have

$$\begin{aligned}
\sum_{i=1}^{N}\big(\beta(t_i) - \beta(t_{i-1})\big) &= \frac{1}{\mathrm{length}(\gamma)}\sum_{i=1}^{N}\mathrm{length}\big(\varphi|_{[t_{i-1},t_i]}\big) \\
&= \frac{1}{\mathrm{length}(\gamma)}\sum_{i=1}^{N}\sup_{\substack{M_i\in\mathbb{N}\\ t_{i-1}=s_0^i<\cdots<s_{M_i}^i=t_i}}\sum_{k=1}^{M_i}\big|\varphi(s_k^i) - \varphi(s_{k-1}^i)\big| \\
&= \frac{1}{\mathrm{length}(\gamma)}\sup_{\substack{M_1\in\mathbb{N}\\ t_0=s_0^1<\cdots<s_{M_1}^1=t_1}}\cdots\sup_{\substack{M_N\in\mathbb{N}\\ t_{N-1}=s_0^N<\cdots<s_{M_N}^N=t_N}}\sum_{i=1}^{N}\sum_{k=1}^{M_i}\big|\varphi(s_k^i) - \varphi(s_{k-1}^i)\big|,
\end{aligned}$$

and since for $\varphi \in \bar{C}(0, T)$ the last double sum can be made arbitrarily small by ensuring that $\sum_{i=1}^{N} \sum_{k=1}^{M_i} (s_k^i - s_{k-1}^i) = \sum_{i=1}^{N} (t_i - t_{i-1})$ is sufficiently small, this shows that β is absolutely continuous. Clearly, $\beta' \geq 0$ a.e. since β is non-decreasing, and for $\forall t \in [0, T]$ we have

$$\begin{aligned} \int_0^t \varphi' \, \mathrm{d}\tau &= \varphi(t) - \varphi(0) = \varphi_\gamma(\beta(t)) - \varphi_\gamma(\beta(0)) \\ &= \int_{\beta(0)}^{\beta(t)} \varphi_\gamma' \, \mathrm{d}\alpha = \int_0^t \varphi_\gamma'(\beta(\tau))\beta'(\tau) \, \mathrm{d}\tau \end{aligned}$$

(for the last step, see [18, p. 149, Exercise 21]), which implies that $\varphi' = (\varphi_\gamma' \circ \beta)\beta'$ a.e. on $[0, T]$. □

The following lemma is a result about the uniform convergence of absolutely continuous functions. We will use the notation $\varphi \subset G$ (for a function $\varphi \in \bar{C}(0, 1)$ and a set $G \subset \mathbb{R}^n$) to indicate that $\varphi(\alpha) \in G$ for $\forall \alpha \in [0, 1]$. Similarly, for a curve $\gamma \in \Gamma$ we will write $\gamma \subset G$ to indicate that $\varphi_\gamma \subset G$.

Lemma 2.2 *(i) If a sequence $(\varphi_n)_{n \in \mathbb{N}} \subset \bar{C}(0, 1)$ fulfills $\varphi_n \subset K$ for $\forall n \in \mathbb{N}$ and some compact set $K \subset D$, and if*

$$M := \sup_{n \in \mathbb{N}} \operatorname*{ess\,sup}_{\alpha \in [0,1]} |\varphi_n'(\alpha)| < \infty, \tag{2.1}$$

then there exists a uniformly converging subsequence.

(ii) If a sequence $(\varphi_n)_{n \in \mathbb{N}} \subset \bar{C}(0, 1)$ fulfilling the conditions of part (i) converges uniformly then its limit φ is in $\bar{C}(0, 1)$ and fulfills $|\varphi'| \leq M$ a.e.

Proof (i) The sequence $(\varphi_n)_{n \in \mathbb{N}}$ is equicontinuous since by (2.1) we have

$$|\varphi_n(\alpha_1) - \varphi_n(\alpha_0)| = \left| \int_{\alpha_0}^{\alpha_1} \varphi_n' \, \mathrm{d}\alpha \right| \leq \int_{\alpha_0}^{\alpha_1} |\varphi_n'| \, \mathrm{d}\alpha \leq M(\alpha_1 - \alpha_0)$$

for $\alpha_0 < \alpha_1$ and $\forall n \in \mathbb{N}$, and so we can apply the Arzelà-Ascoli theorem.

(ii) By the same estimate, for any collection of disjoint intervals $[\alpha_{i-1}, \alpha_i) \subset [0, 1]$, $i = 1, \ldots, N$, we have

$$\sum_{i=1}^{N} \left|\varphi(\alpha_i) - \varphi(\alpha_{i-1})\right| = \lim_{n \to \infty} \sum_{i=1}^{N} \left|\varphi_n(\alpha_i) - \varphi_n(\alpha_{i-1})\right| \leq M \sum_{i=1}^{N} (\alpha_i - \alpha_{i-1}).$$

This shows that φ is absolutely continuous, and (taking $N = 1$ and recalling that φ' is the classical derivative a.e.) that $|\varphi'| \leq M$ a.e. Since K is compact and $\varphi_n \subset K$ for $\forall n \in \mathbb{N}$, we have $\varphi \subset K \subset D$ and thus $\varphi \in \bar{C}(0, 1)$. □

2.1.2 Curves that Pass Points in Infinite Length

Sometimes we will have to work with curves that do not have finite length (i.e., that are not rectifiable). We denote by $\tilde{C}(0,1) \supset \bar{C}(0,1)$ the space of all functions in $C([0,1], D)$ that are absolutely continuous in neighborhoods of all but at most finitely many $\alpha_i \in [0,1]$, and we denote by $\tilde{\Gamma} \supset \Gamma$ the set of all curves that can be parameterized by a function $\varphi \in \tilde{C}(0,1)$.

Note that for $\forall \varphi \in \tilde{C}(0,1)$, φ' is still defined a.e., but one can see that for these exceptional values α_i we have $\int_{[0,1]\cap[\alpha_i-\varepsilon,\alpha_i+\varepsilon]} |\varphi'|\, d\alpha = \infty$ for $\forall \varepsilon > 0$.[1] We therefore say that the curve $\gamma \in \tilde{\Gamma}$ given by φ ***passes the points*** $\varphi(\alpha_i)$ ***in infinite length***.

Of particular use in our work is, for fixed $x \in D$, the set $\tilde{\Gamma}(x)$ of all curves that are either of finite length (i.e., rectifiable) or that pass x once in infinite length (note that $\Gamma \subset \tilde{\Gamma}(x) \subset \tilde{\Gamma}$). More precisely, these are the curves that can be parameterized by functions in the set $\tilde{C}(x)$, which we define to be the set of functions $\varphi \in C([0,1], D)$ such that

either $\varphi \in \bar{C}(0,1)$,

or $\varphi(\frac{1}{2}) = x$,
and $\varphi|_{[0,1/2-a]}$ and $\varphi|_{[1/2+a,1]}$ are absolutely continuous for $\forall a \in (0, \frac{1}{2})$.

See Sect. 2.1.3 and Fig. 2.1 for an illustration of these classes of curves.

In preparation for Lemma 2.3, which is the equivalent of Lemma 2.2 for sequences of functions in $\tilde{C}(x)$, we introduce the following notation: For a curve γ and a point x we say that γ ***passes*** x ***at most once*** if for any parameterization $\varphi \in C([0,1])$ of γ we have

$$\big(\exists 0 \le \alpha_1 < \alpha_2 \le 1\colon \varphi(\alpha_1) = \varphi(\alpha_2) = x\big) \quad \Rightarrow \quad \forall \alpha \in [\alpha_1, \alpha_2]\colon \varphi(\alpha) = x. \tag{2.2}$$

For a Borel set $E \subset D$ and a curve $\gamma \in \tilde{\Gamma}$ we define

$$\text{length}(\gamma|_E) := \int_\gamma \mathbb{1}_{z\in E}\, |dz| = \int_0^1 |\varphi'| \mathbb{1}_{\varphi\in E}\, d\alpha \ \in [0,\infty]$$

for any parameterization $\varphi \in \tilde{C}(0,1)$ of γ.

Lemma 2.3 *Let $x \in D$, let the sequence $(\gamma_n)_{n\in\mathbb{N}} \subset \tilde{\Gamma}(x)$ fulfill $\gamma_n \subset K$ for $\forall n \in \mathbb{N}$ and some compact set $K \subset D$, suppose that every curve γ_n passes x at most once,*

[1]The key argument for this can be found at the end of the proof of Proposition 3.25.

and suppose that there exists a function $\eta\colon (0,\infty) \to [0,\infty)$ *such that*

$$\forall n \in \mathbb{N}\ \ \forall u > 0\colon\ \operatorname{length}\bigl(\gamma_n|_{\bar{B}_u(x)^c}\bigr) \le \eta(u). \tag{2.3}$$

Then there exist parameterizations $\varphi_n \in \tilde{C}(x)$ *of the curves* γ_n *such that a subsequence* $(\varphi_{n_k})_{k\in\mathbb{N}}$ *converges pointwise on* $[0,1]$ *and uniformly on the sets* $[0,\frac{1}{2}-a] \cup [\frac{1}{2}+a,1]$, $a \in (0,\frac{1}{2})$. *The limit* φ *is in* $\tilde{C}(x)$, *and the corresponding curve* $\gamma \in \tilde{\Gamma}(x)$ *fulfills*

$$\forall u > 0\colon\ \operatorname{length}\bigl(\gamma|_{\bar{B}_u(x)^c}\bigr) \le \eta(u). \tag{2.4}$$

Proof See Appendix A.1. This proof uses Lemma 2.6 (i). □

Introducing some final notation, for two sets $A_1, A_2 \subset \tilde{D}$ we write

$$\Gamma_{A_1}^{A_2} := \bigl\{\gamma \in \Gamma \bigm| \gamma \subset \tilde{D},\, \gamma \text{ starts in } A_1 \text{ and ends in } A_2\bigr\},$$
$$\bar{C}_{A_1}^{A_2}(0,1) := \bigl\{\varphi \in \bar{C}(0,1) \bigm| \varphi \subset \tilde{D},\ \varphi(0) \in A_1,\ \varphi(1) \in A_2\bigr\},$$

and for two points $x_1, x_2 \in \tilde{D}$ we similarly define $\Gamma_{x_1}^{x_2}$ and $\bar{C}_{x_1}^{x_2}(0,1)$. The sets $\tilde{\Gamma}_{A_1}^{A_2}$, $\tilde{C}_{A_1}^{A_2}(0,1)$, $\tilde{\Gamma}_{x_1}^{x_2}$, $\tilde{C}_{x_1}^{x_2}(0,1)$, $\tilde{\Gamma}_{x_1}^{x_2}(x)$ and $\tilde{C}_{x_1}^{x_2}(x)$ are defined analogously.

2.1.3 Summary of the Various Classes of Curves

(See Fig. 2.1 for illustrations.) All curves are unparameterized and oriented, and they may have loops and cusps. The class Γ contains only curves with finite length, while curves in $\tilde{\Gamma} \supset \Gamma$ may reach and/or leave finitely many points in infinite length, also repeatedly. For some fixed $x \in D$ (marked by the cross), $\tilde{\Gamma}(x)$ contains all of Γ, plus all the curves that pass x once in infinite length; they cannot pass any other point in infinite length, and they cannot pass x twice in infinite length. The sub- and

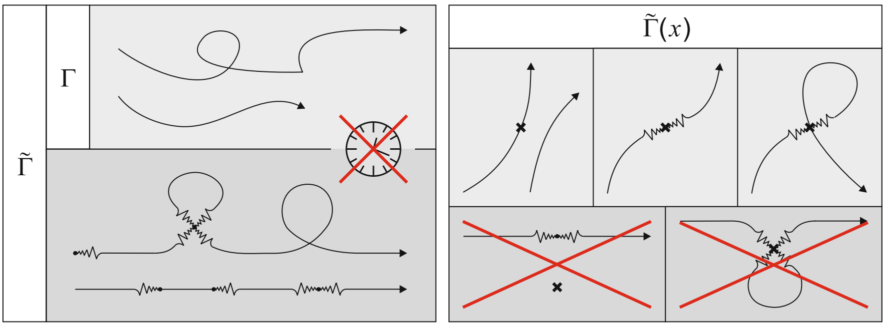

Fig. 2.1 Illustration of the various classes of curves

superscripts x_1 and x_2 or A_1 and A_2 add constraints to the start and end points of these functions and curves and in addition require them to take their values in $\tilde{D}$.

2.2 Geometric Actions, Drift Vector Fields

In this section we will define the class $\mathscr{G}$ of ***geometric action functionals***, and we will generalize the concept of a "drift vector field" $b(x)$ from the large deviation geometric action of the SDE (1.3), given by (1.7), to general geometric actions $S \in \mathscr{G}$.

Definition 2.4 We denote by $\mathscr{G}$ the set of all functionals $S\colon \tilde{\Gamma} \to [0, \infty]$ of the form

$$S(\gamma) := \int_\gamma \ell(z, \mathrm{d}z) := \int_0^1 \ell(\varphi, \varphi')\, \mathrm{d}\alpha, \tag{2.5}$$

where $\varphi \in \tilde{C}(0,1)$ is an arbitrary parameterization of γ, and where the ***local action*** $\ell \in C(D \times \mathbb{R}^n, [0, \infty))$ has the following properties:

(i) $\forall x \in D \;\; \forall y \in \mathbb{R}^n \;\; \forall c \geq 0\colon \;\; \ell(x, cy) = c\ell(x, y)$,
(ii) for every fixed $x \in D$ the function $\ell(x, \cdot\,)$ is convex.

For $\varphi \in \tilde{C}(0,1)$ we will sometimes use the notation $S(\varphi) := \int_0^1 \ell(\varphi, \varphi')\, \mathrm{d}\alpha$, and for any interval $[\alpha_1, \alpha_2] \subset [0,1]$ we will denote by $S(\varphi|_{[\alpha_1,\alpha_2]}) := \int_{\alpha_1}^{\alpha_2} \ell(\varphi, \varphi')\, \mathrm{d}\alpha$ the action of the curve segment parameterized by $\varphi|_{[\alpha_1,\alpha_2]}$.

As we will see next, (i) is needed to show that (2.5) is independent of the specific choice of φ, while (ii) is essential to show that S is lower semi-continuous in a certain sense (Lemma 2.6). Observe also that (i) implies that $\ell(x, 0) = 0$ for $\forall x \in D$.

Lemma 2.5 *Functionals $S \in \mathscr{G}$ and their local actions $\ell(x, y)$ have the following properties:*

(i) $S(\gamma)$ is well-defined, i.e., (2.5) is independent of the specific choice of φ.
(ii) For $\forall$ compact $K \subset D$ $\exists c_1 = c_1(K) > 0$ $\forall x \in K$ $\forall y \in \mathbb{R}^n$: $\ell(x, y) \leq c_1|y|$. In particular, we have for $\forall \gamma \in \tilde{\Gamma}$ with $\gamma \subset K$: $S(\gamma) \leq c_1 \operatorname{length}(\gamma)$.

Proof (i) Given a curve $\gamma \in \Gamma$ and any parameterization $\varphi \in \bar{C}(0,1)$ of γ, we use the representation $\varphi = \varphi_\gamma \circ \beta$ of Lemma 2.1 (ii) and Definition 2.4 (i) to find that

$$\begin{aligned}
\int_0^1 \ell(\varphi, \varphi')\, \mathrm{d}\alpha &= \int_0^1 \ell\big(\varphi_\gamma \circ \beta, (\varphi_\gamma' \circ \beta)\beta'\big)\, \mathrm{d}\alpha \\
&= \int_0^1 \ell(\varphi_\gamma \circ \beta, \varphi_\gamma' \circ \beta)\beta'\, \mathrm{d}\alpha \\
&= \int_0^1 \ell(\varphi_\gamma, \varphi_\gamma')\, \mathrm{d}\beta,
\end{aligned}$$

where the last step follows again from [18, p. 149, Exercise 21]. By the uniqueness of φ_γ, the right-hand side only depends on γ. The proof for general curves $\gamma \in \tilde{\Gamma}$ is based on the same calculation.

(ii) Given any K, set $c_1 := 1 + \max_{x\in K,|y|=1} \ell(x, y) > 0$, use Definition 2.4 (i) to show that $\ell(x, y) = |y|\ell\big(x, \frac{y}{|y|}\big) \le c_1|y|$ for $\forall y \neq 0$, and recall that $\ell(x, 0) = 0$. In particular, if $\varphi \in \tilde{C}(0, 1)$ is a parameterization of some $\gamma \in \tilde{\Gamma}$ with $\gamma \subset K$ then $S(\gamma) = \int_0^1 \ell(\varphi, \varphi')\, d\alpha \le c_1 \int_0^1 |\varphi'|\, d\alpha = c_1\, \text{length}(\gamma)$. □

Lemma 2.6 (Lower Semi-Continuity) *For $\forall S \in \mathscr{G}$ we have the following:*

(i) If a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \bar{C}(0, 1)$ fulfilling (2.1) has a uniform limit $\varphi \in \bar{C}(0, 1)$ then $\liminf_{n\to\infty} S(\varphi_n) \ge S(\varphi)$.

(ii) The limit γ constructed in Lemma 2.3 fulfills $\liminf_{n\to\infty} S(\gamma_n) \ge S(\gamma)$.

Proof See Appendix A.2. □

Definition 2.7 Let $S \in \mathscr{G}$. A vector field $b \in C^1(D, \mathbb{R}^n)$ is called a ***drift*** of S if for $\forall$ compact $K \subset D$ $\exists c_2 = c_2(K) > 0$ $\forall x \in K$ $\forall y \in \mathbb{R}^n$:

$$\ell(x, y) \ge c_2\big(|b(x)||y| - \langle b(x), y\rangle\big). \tag{2.6}$$

The right-hand side of (2.6) is a constant multiple of the local large deviation geometric action (1.7) of the SDE (1.3) with drift $b(x)$ and homogeneous noise, and thus we see that for the geometric action associated to (1.3), the vector field $b(x)$ in (1.3) is clearly a drift also in this generalized sense (take $c_2 = 1$). The inequality (2.6), which will only be used to obtain the key estimate Lemma 6.13 (and a weaker version thereof in the proof of Lemma 4.2), effectively reduces our proofs for an arbitrary action $S \in \mathscr{G}$ to the case of the specific action given by (1.7), and it is ultimately the reason why the conditions of our main criteria, Propositions 3.23 and 3.25, solely depend on the drift and not on any other aspect of the action S.

The drift vector field $b(x)$ in Definition 2.7 is not a uniquely defined object: If b is a drift of some action $S \in \mathscr{G}$ and if $\beta \in C^1(D, [0,\infty))$ then βb is a drift of S as well (with modified constants c_2), and in particular the vector field $b(x) \equiv 0$ is a drift of *any* action $S \in \mathscr{G}$. Note however that (i) if $\beta(x) > 0$ for $\forall x \in D$ then the vector fields b and βb have the same flowline diagrams, and we will find that our criteria will not distinguish between these two choices; (ii) if on the other hand $\beta(x) = 0$ and $b(x) \neq 0$ for some $x \in D$ then the flowline diagrams of b and βb are different, and our criteria may only apply to b but not to βb. In general, a good choice for the drift (i.e., one that lets us get the most out of our criteria) will be one with only as many roots as necessary.

Definition 2.8 For a given vector field $b \in C^1(D, \mathbb{R}^n)$ we define the flow $\psi \in C^1(D \times \mathbb{R}, D)$ as the unique solution of the ODE

$$\begin{cases} \partial_t \psi(x,t) = b(\psi(x,t)) & \text{for} \quad x \in D,\ t \in \mathbb{R}, \\ \psi(x,0) = x & \text{for} \quad x \in D. \end{cases} \tag{2.7}$$

By a standard result from the theory of ODEs [1, Sect. 7.3, Corollary 4], our regularity assumption on b implies that the solution $\psi(x,t)$ is well-defined *locally* (i.e., for small t), unique, and C^1 in (x,t). However, since b will always play the role of a drift, we may assume that $\psi(x,t)$ is in fact defined *globally*, i.e., for $\forall t \in \mathbb{R}$: Indeed, if this is not the case then we can instead consider the modified drift βb, for some function $\beta \in C^1(D, (0,\infty))$ that vanishes so fast near the boundary ∂D that the associated flow $\tilde{\psi}$ only reaches ∂D in infinite time (i.e., $\tilde{\psi}(x,t)$ is defined for $\forall (x,t) \in D \times \mathbb{R}$), and the only aspect of the flow that will be relevant to us (the flowline diagram) remains invariant under this change.

Finally, recall that under this additional assumption we have $\psi(\psi(x,t),s) = \psi(x,t+s)$ and $\partial_t \nabla\psi(x,t) = \nabla b(\psi(x,t))$ for $\forall x \in D$ and $\forall t,s \in \mathbb{R}$.

We conclude this section by classifying the points in state space according to the type of difficulty that they will pose for our existence theory.

Definition 2.9 Let $S \in \mathscr{G}$ be given by the local action $\ell(x,y)$, and let $x \in D$.

(i) x is called a ***degenerate point*** of S if $\exists y \in \mathbb{R}^n \setminus \{0\}$: $\ell(x,y) = 0$.
(ii) x is called a ***critical point*** of S if $\forall y \in \mathbb{R}^n$: $\ell(x,y) = 0$.

We denote by $D_{S+} := \{x \in D \mid \forall y \in \mathbb{R}^n \setminus \{0\}\colon \ell(x,y) > 0\}$ the set of non-degenerate points of S.

In other words, degenerate points are those at which there is *at least one* direction into which one can locally move at no cost, while at critical points one can move into *any* direction at no cost. At non-degenerate points of S, every direction comes at a positive cost. Note that every critical point is degenerate.

Since directions with zero cost make it hard for us to control the length of curves that pass the point in question, critical points will be the hardest to deal with in our existence theory, while non-degenerate points will be the easiest.

Example 2.10 (i) For the geometric action S given by (1.7), i.e., by $\ell(x,y) = |b(x)||y| - \langle b(x), y\rangle$, every point in D is degenerate (i.e., $D_{S+} = \varnothing$), and the critical points are those points x for which $b(x) = 0$. Indeed, if $x \in D$ is such that $b(x) = 0$ then clearly we have $\ell(x,y) = 0$ for $\forall y \in \mathbb{R}^n$, and for all other points only the direction given by $y = b(x) \neq 0$ fulfills $\ell(x,y) = 0$.

(ii) For the Euclidean length, i.e., the geometric action S given by $\ell(x,y) = |y|$, there are no degenerate or even critical points, and so we have $D_{S+} = D$. □

2.3 The Subclass of Hamiltonian Geometric Actions

We will now consider a particular way of constructing a geometric action from a Hamiltonian $H(x, \theta)$, which was introduced in [9, 10] in the context of large deviation theory.[2]

Lemma 2.11 *Let the function $H \in C(D \times \mathbb{R}^n, \mathbb{R})$ fulfill the assumptions*

(H1) $\forall x \in D$: $H(x, 0) \leq 0$,
(H2) *the derivatives H_θ and $H_{\theta\theta}$ exist and are continuous in (x, θ),*
(H3) $\forall$ *compact* $K \subset D\ \exists m_K > 0\ \forall x \in K\ \forall \theta, \xi \in \mathbb{R}^n$: $\langle \xi, H_{\theta\theta}(x, \theta)\xi \rangle \geq m_K |\xi|^2$.

Then the function $\ell : D \times \mathbb{R}^n \to [0, \infty)$ defined by

$$\ell(x, y) := \max\big\{\langle y, \theta\rangle \,\big|\, \theta \in \mathbb{R}^n,\ H(x, \theta) \leq 0\big\} \tag{2.8a}$$

$$= \max\big\{\langle y, \theta\rangle \,\big|\, \theta \in \mathbb{R}^n,\ H(x, \theta) = 0\big\} \tag{2.8b}$$

has the properties of Definition 2.4, and so it defines a geometric action $S \in \mathscr{G}$.

Proof The sets $L_x := \{\theta \in \mathbb{R}^n \,|\, H(x, \theta) \leq 0\}$ are bounded, in fact uniformly for all x in any compact set $K \subset D$, since for $\forall x \in K\ \forall \theta \in L_x\ \exists \tilde{\theta} \in \mathbb{R}^n$:

$$\begin{aligned} 0 \geq H(x, \theta) &= H(x, 0) + \langle H_\theta(x, 0), \theta\rangle + \tfrac{1}{2}\big\langle \theta, H_{\theta\theta}(x, \tilde{\theta})\theta \big\rangle \\ &\geq -\max_{x \in K} |H(x, 0)| - \max_{x \in K} |H_\theta(x, 0)||\theta| + \tfrac{1}{2} m_K |\theta|^2. \end{aligned} \tag{2.9}$$

This shows that ℓ is finite-valued, and since $0 \in L_x$ by (H1) we have $\ell(x, y) \geq \langle y, 0\rangle = 0$ for $\forall y \in \mathbb{R}^n$. The fact that the representations (2.8a) and (2.8b) are equivalent is obvious for $y = 0$; for $y \neq 0$ observe that for $\forall \theta \in \mathbb{R}^n$ with $H(x, \theta) < 0$ the boundedness of L_x implies that there $\exists c > 0$ such that $H(x, \theta + cy) = 0$, and $\langle y, \theta + cy\rangle \geq \langle y, \theta\rangle$. The relation $\ell(x, cy) = c\ell(x, y)$ for $\forall c \geq 0$ is clear, and $\ell(x, \cdot)$ is convex as the supremum of linear functions. The continuity at any point $(x_0, y_0 = 0)$ follows from the estimate $\ell(x, y) \leq M|y|$ for $\forall y \in \mathbb{R}^n$ and all x in some ball $\bar{B}_\varepsilon(x_0) \subset D$, where $M := \sup\big\{|\theta| \,\big|\, \theta \in \bigcup_{x \in \bar{B}_\varepsilon(x_0)} L_x\big\}$. The continuity everywhere else will follow from Lemma 2.14 (i). □

Definition 2.12 (i) We call a function H fulfilling the properties (H1)–(H3) a ***Hamiltonian***, and we say that H ***induces the geometric action*** S defined in Lemma 2.11.

(ii) We denote the class of all ***Hamiltonian geometric actions***, i.e., of all actions S constructed as in Lemma 2.11, by $\mathscr{H} \subset \mathscr{G}$.

[2]This work also proposed an algorithm, called the geometric minimum action method (gMAM), for numerically computing minimizing curves of such geometric actions.

(iii) We denote by $\mathscr{H}_0 \subset \mathscr{H}$ the class of all geometric actions $S \in \mathscr{H}$ that are constructed from a Hamiltonian H that fulfills the stronger assumption

(H1') $\forall x \in D$: $H(x, 0) = 0$.

Note that since ℓ depends on H only through its 0-level sets, different Hamiltonians H can define the same local action ℓ via (2.8), i.e., they can induce the same geometric action $S \in \mathscr{H}$. In particular, for $\forall \beta \in C(D, (0, \infty))$ the Hamiltonians $H(x, \theta)$ and $\beta(x)H(x, \theta)$ induce the same action S. The next lemma shows how Definition 2.9 can be expressed in terms of H, and that Assumption (H1') does not depend on the specific choice of H.

Lemma 2.13 *Let $S \in \mathscr{H}$, and let H be a Hamiltonian that induces S.*

(i) A point $x \in D$ is critical if and only if

$$H_\theta(x, 0) = 0 \qquad \text{and} \qquad H(x, 0) = 0, \tag{2.10}$$

and in that case (2.10) holds in fact for every Hamiltonian that induces S.

(ii) A point $x \in D$ is degenerate if and only if $H(x, 0) = 0$.

(iii) If some H inducing S fulfills (H1') then all of them do.

Proof See Appendix A.3. For part (ii) see also Fig. 2.2b. □

In particular, Lemma 2.13 (ii) and (iii) imply that $\mathscr{H}_0$ is the class of all Hamiltonian actions S such that D only consists of degenerate points, i.e., such that $D_{S+} = \varnothing$. Furthermore, we learn that for $\forall S \in \mathscr{H}$ we have $D_{S+} = \{x \in D \mid H(x, 0) < 0\}$.

To actually compute $\ell(x, y)$ from a given Hamiltonian H, and for many proofs, the following alternative representation of ℓ is oftentimes useful. It can be derived by carrying out the constraint maximization in (2.8b) with the method of Lagrange multipliers.

Lemma 2.14 *(i) For every fixed $x \in D$ and $y \in \mathbb{R}^n \setminus \{0\}$ the system*

$$H_\theta(x, \vartheta) = \lambda y, \qquad H(x, \vartheta) = 0, \qquad \lambda \geq 0 \tag{2.11}$$

has a unique solution $(\vartheta(x, y), \lambda(x, y))$, the functions $\vartheta : D \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ and $\lambda : D \times (\mathbb{R}^n \setminus \{0\}) \to [0, \infty)$ are continuous, and the function ℓ defined in (2.8a) can be written as

$$\ell(x, y) = \begin{cases} \langle y, \vartheta(x, y) \rangle & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases} \tag{2.12}$$

(ii) If $S \in \mathscr{H}$ is induced by H then a point $x \in D$ is critical if and only if $\exists y \neq 0$: $\lambda(x, y) = 0$. In that case, we have in fact $\lambda(x, y) = 0$ for $\forall y \neq 0$.

Proof See Appendix A.4. □

See Fig. 2.2a for a geometric interpretation of (2.8a)–(2.8b) and (2.11)–(2.12): By Assumption (H3) the function $H(x,\cdot)$ and thus also its 0-sublevel set $\{\theta \in \mathbb{R}^n \,|\, H(x,\theta) \le 0\}$ is strictly convex, and by Assumption (H1) it contains the origin. The maximizer in (2.8a), $\theta = \vartheta(x,y)$, is the unique point on its boundary where the outer normal aligns with y, and the local action $\ell(x,y)$ is $|y|$ times the component of $\vartheta(x,y)$ in the direction y.

The following lemma provides a quick way to obtain a drift for any Hamiltonian geometric action. The examples at the end of this section will illustrate its use.

Lemma 2.15 *If $S \in \mathscr{H}$ is induced by H then $b(x) := H_\theta(x,0)$ fulfills the estimate in Definition 2.7, and thus if b is C^1 then it is a drift of S. We call a drift obtained in this way a* ***natural drift*** *of S.*

Proof Let $b(x) := H_\theta(x,0)$, and let $K \subset D$ be compact. Define $a := \sup_{x\in K}|b(x)|$ and $c_2 := \big[2 + \sup\big\{|H_{\theta\theta}(x,\theta)| \,\big|\, x \in K,\ |\theta| \le a\big\}\big]^{-1} \in (0, \frac{1}{2}]$, and let $x \in K$ and $y \in \mathbb{R}^n$.

If $y = 0$ then (2.6) is trivial since both sides vanish. Also, if $y \ne 0$ and $\lambda(x,y) = 0$ then by Lemmas 2.14 (ii) and 2.13 (i) we have $b(x) = 0$, so (2.6) is trivial again. Therefore let us now assume that $y \ne 0$ and that $\lambda(x,y) > 0$.

Setting $\theta_0 := c_2\big(\frac{|b(x)|}{|y|}y - b(x)\big)$, a Taylor expansion of $H(x,\theta_0)$ around $\theta = 0$ gives us a θ' on the straight line between 0 and θ_0 (thus fulfilling $|\theta'| \le |\theta_0| \le 2c_2|b(x)| \le 2ac_2 \le a$) such that

$$
\begin{aligned}
H(x,\theta_0) &= H(x,0) + \big\langle H_\theta(x,0), \theta_0\big\rangle + \tfrac{1}{2}\big\langle \theta_0, H_{\theta\theta}(x,\theta')\theta_0\big\rangle \\
&\le 0 + \big\langle b(x), \theta_0\big\rangle + \tfrac{1}{2}c_2^{-1}|\theta_0|^2 \\
&= \big\langle b(x) + \tfrac{1}{2}c_2^{-1}\theta_0, \theta_0\big\rangle \\
&= \big\langle \tfrac{1}{2}\big(\tfrac{|b(x)|}{|y|}y + b(x)\big),\ c_2\big(\tfrac{|b(x)|}{|y|}y - b(x)\big)\big\rangle \\
&= \tfrac{1}{2}c_2\big(\big|\tfrac{|b(x)|}{|y|}y\big|^2 - |b(x)|^2\big) = 0.
\end{aligned}
$$

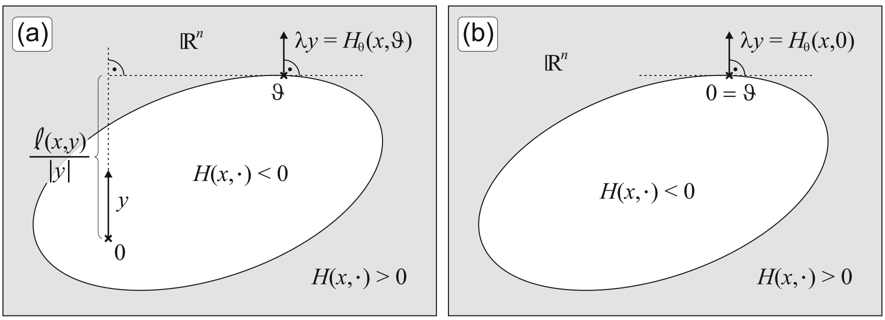

Fig. 2.2 (**a**) Illustration of (2.8a)–(2.8b) and (2.11)–(2.12), for fixed $x \in D$ and $y \in \mathbb{R}^n \setminus \{0\}$, in the case $H(x,0) < 0$. (**b**) If $H(x,0) = 0$ and if y aligns with $H_\theta(x,0)$ then $\vartheta = 0$ and thus $\ell(x,y) = 0$

Another Taylor expansion, this time around $\theta = \vartheta := \vartheta(x, y)$, now gives us a θ'' such that

$$\begin{aligned}0 &\geq H(x, \theta_0)\\ &= H(x, \vartheta) + \big\langle H_\theta(x, \vartheta), \theta_0 - \vartheta\big\rangle + \tfrac{1}{2}\big\langle \theta_0 - \vartheta, H_{\theta\theta}(x, \theta'')(\theta_0 - \vartheta)\big\rangle\\ &\geq 0 + \lambda(x, y)\langle y, \theta_0 - \vartheta\rangle + 0,\end{aligned}$$

where we used both equations in (2.11), and Assumption (H3). Since $\lambda(x, y) > 0$, this implies that

$$\ell(x, y) = \langle \vartheta, y\rangle \geq \langle \theta_0, y\rangle = c_2\big\langle \tfrac{|b(x)|}{|y|} y - b(x), y\big\rangle = c_2\big(|b(x)||y| - \langle b(x), y\rangle\big).$$

□

Note that since there is not a unique Hamiltonian associated to S, there is not a unique natural drift either; in particular, the remark following Definition 2.12 implies that with b also βb is a natural drift for $\forall \beta \in C^1(D, (0, \infty))$, with the same flowline diagram. The next remark shows that for actions $S \in \mathscr{H}_0$ in fact *every* natural drift has the same flowline diagram.

Remark 2.16 For $S \in \mathscr{H}_0$ we have the following:

(i) All natural drifts b share the same roots since by Lemma 2.13 (i) and (H1') we have $b(x) = 0$ if and only if x is a critical point. In particular, this means that natural drifts are optimal in the sense that by (2.6) they only vanish where necessary.
(ii) At non-critical points x, the direction $y := \frac{b(x)}{|b(x)|}$ is the same for every natural drift b, since Lemma 4.3 (i)–(ii) will characterize it as the unique unit vector y such that $\ell(x, y) = 0$.

Thus, for any fixed $S \in \mathscr{H}_0$ all natural drifts have the same flowline diagram.

In contrast, for actions $S \in \mathscr{H} \setminus \mathscr{H}_0$ (i.e., if S has any non-degenerate points) the natural drift is not always the optimal choice: In Examples 2.20 and 2.21 below the natural drift will even turn out to be the trivial (and thus useless) drift $b \equiv 0$. (See Example 3.32 in Sect. 3.4.3 for how to find a better one.) Furthermore, Example 3.33 illustrates two cases in which the natural drift is non-trivial but contains a limit cycle, which would usually prevent us from using it in our existence criteria.

However, since in that example we assume that there is a non-degenerate point on the limit cycle, the following lemma turns out to resolve the problem in this case: It says that we are allowed to modify the obtained natural drift in a closed subset of the region D_{S+} in any way we want.

Lemma 2.17 *Suppose that b is a drift of $S \in \mathscr{G}$, and that $\tilde{b} \in C^1(D, \mathbb{R}^n)$ is another vector field that coincides with b outside of some closed subset of D_{S+}. Then $\tilde{b}$ is a drift of S, too.*

Proof See Appendix A.5. □

Finally, the next lemma states the key property of Hamiltonian geometric actions in particular in the context of large deviation theory: It shows how a double minimization problem such as (1.4)–(1.5) can be reduced to a simple minimization problem over a Hamiltonian geometric action.

Lemma 2.18 *Let H be a Hamiltonian fulfilling (H1)–(H3), and define for $\forall T > 0$ the functional $S_T\colon \bar{C}(0,T) \to [0,\infty]$ as*

$$S_T(\chi) := \int_0^T L(\chi,\dot{\chi})\,\mathrm{d}t, \qquad \textit{where} \tag{2.13}$$

$$L(x,y) := \sup_{\theta\in\mathbb{R}^n} \big(\langle y,\theta\rangle - H(x,\theta)\big) \qquad \textit{for } \forall x\in D\ \forall y\in\mathbb{R}^n \tag{2.14}$$

is the Legendre transform of $H(x,\cdot)$. Then for $\forall A_1, A_2 \subset D$ we have

$$\inf_{\substack{T>0\\ \chi\in\bar{C}_{A_1}^{A_2}(0,T)}} S_T(\chi) = \inf_{\gamma\in\Gamma_{A_1}^{A_2}} S(\gamma), \tag{2.15}$$

where $S\in\mathscr{H}$ is the geometric action induced by H.

Proof Using the bijection $(T,\chi) \leftrightarrow (\gamma,T,\beta)$ given in Lemma 2.1 (ii) that assigns to every $\chi\in\bar{C}(0,T)$ its curve $\gamma\in\Gamma$ and its parameterization $\beta\in\bar{C}([0,T],[0,1])$ via the relation $\chi = \varphi_\gamma\circ\beta$, we have

$$\inf_{\substack{T>0\\ \chi\in\bar{C}_{A_1}^{A_2}(0,T)}} S_T(\chi) = \inf_{\gamma\in\Gamma_{A_1}^{A_2}} \inf_{\substack{T>0\\ \beta\in\bar{C}([0,T],[0,1])\\ \beta\text{ non-decr., surjective}}} S_T(\varphi_\gamma\circ\beta) = \inf_{\gamma\in\Gamma_{A_1}^{A_2}} S(\gamma),$$

where the functional

$$S(\gamma) := \inf_{\substack{T>0\\ \beta\in\bar{C}([0,T],[0,1])\\ \beta\text{ non-decr., surjective}}} S_T(\varphi_\gamma\circ\beta)$$

was found in [10] to have the integral representation (2.5) with the local action given by (2.8a)–(2.8b) (or equivalently, by (2.11)–(2.12)).[3] □

We conclude this section with three examples of Hamiltonian geometric actions.

Example 2.19 (Large Deviation Theory, Part I) Stochastic dynamical systems with a small noise parameter $\varepsilon > 0$ often satisfy a large deviation principle whose

[3] At the beginning of [10], additional smoothness assumptions on H were made, but they do not enter the proofs of these representations.

action functional S_T is of the form (2.13)–(2.14). Examples include (i) stochastic differential equations (SDEs) in $\mathbb{R}^n$ [8]

$$\mathrm{d}X_t^\varepsilon = b(X_t^\varepsilon)\,\mathrm{d}t + \sqrt{\varepsilon}\,\sigma(X_t^\varepsilon)\,\mathrm{d}W_t, \qquad X_{t=0}^\varepsilon = x_1, \tag{2.16}$$

where $b(x)$ is the drift vector field and $\sigma(x)$ is the diffusion matrix of the SDE, and (ii) continuous-time Markov jump processes in $\mathbb{R}^n$ [16] with jump vectors $\varepsilon e_i \in \mathbb{R}^n$, $i = 1, \dots, N$, and corresponding jump rates $\varepsilon^{-1}\nu_i(\varepsilon x) > 0$. Here we assume that b, $A := \sigma\sigma^T$ and ν_i are C^1 functions, and that for each fixed $x \in D$, $A(x)$ is a positive definite matrix.

Using the notation $\langle w_1, w_2\rangle_M := \langle w_1, M w_2\rangle$ and soon also $|w|_M := \langle w, w\rangle_M^{1/2}$ for $\forall w_1, w_2, w \in \mathbb{R}^n$ and for any positive definite symmetric matrix M, the Hamiltonians used in (2.13)–(2.14) to define S_T are

$$H(x,\theta) = \langle b(x), \theta\rangle + \tfrac{1}{2}|\theta|_{A(x)}^2, \qquad \text{(SDE)} \tag{2.17a}$$

$$H(x,\theta) = \sum_{i=1}^N \nu_i(x)\big(\mathrm{e}^{\langle e_i,\theta\rangle} - 1\big). \qquad \text{(Markov jump process)} \tag{2.17b}$$

In the SDE case, the function $L(x, y)$ defined in (2.14) can easily be found to be

$$L(x,y) = \tfrac{1}{2}|b(x) - y|_{A^{-1}(x)}^2, \qquad \text{(SDE)} \tag{2.18}$$

whereas for Markov jump processes no closed form of $L(x, y)$ is available.

The central object of large deviation theory for answering various questions about rare events in the zero-noise-limit $\varepsilon \to 0$, such as the transition from one stable equilibrium point of b to another, is the ***quasipotential*** $V(x_1, x_2)$. Originally defined by (1.4) using the action S_T given by (2.13)–(2.14), Lemma 2.18 allows us to rewrite it as

$$V(x_1, x_2) = \inf_{\gamma \in \Gamma_{x_1}^{x_2}} S(\gamma), \tag{2.19}$$

where $S \in \mathscr{H}$ is the Hamiltonian geometric action defined via (2.8a)–(2.8b), or equivalently, via (2.11)–(2.12). The minimizing curve $\gamma^\star$ in (2.19) (if it exists) can be interpreted as the maximum likelihood transition curve.

In the SDE case, (2.11) can in fact be solved explicitly: Its solution is given by $\lambda = |b(x)|_{A(x)^{-1}}/|y|_{A(x)^{-1}}$ and $\vartheta = A(x)^{-1}(\lambda y - b(x))$, and so we obtain the local geometric action

$$\ell(x,y) = |b(x)|_{A^{-1}(x)}|y|_{A^{-1}(x)} - \langle b(x), y\rangle_{A^{-1}(x)}. \qquad \text{(SDE)} \tag{2.20}$$

For Markov jump processes no explicit expression for $\ell(x, y)$ exists.

Finally, we observe that in the SDE case (2.17a) the expression $H_\theta(x, 0)$ for the natural drift given in Lemma 2.15 indeed recovers the given vector field $b(x)$, while in the case (2.17b) of a Markov jump process we obtain

$$b(x) = \sum_{i=1}^{N} \nu_i(x) e_i\,, \qquad \text{(Markov jump process)}$$

which is the vector field that defines the zero-noise-limit of Kurtz's Theorem (see [12] or [16, Theorem 5.3]). □

Example 2.20 (Riemannian Metric) Suppose that $A \in C(D, \mathbb{R}^{n\times n})$ is a function whose values are positive definite symmetric matrices $A(x)$. Then the action $S \in \mathscr{G}$ given by

$$\ell(x, y) = |y|_{A(x)} \tag{2.21}$$

is a Hamiltonian action, $S \in \mathscr{H} \setminus \mathscr{H}_0$, with associated Hamiltonian

$$H(x, \theta) = |\theta|^2_{A(x)^{-1}} - 1. \qquad \text{(Riemannian metric)}$$

Indeed, as one can easily check, for this choice of H the Eq. (2.11) are fulfilled by $\lambda = 2/|y|_{A(x)}$ and $\vartheta := A(x)y/|y|_{A(x)}$, and thus the local geometric action defined by (2.12) yields (2.21).

Note that the natural drift for this Hamiltonian is $b(x) \equiv 0$. As we shall see, however, this will be made up for by the fact that $H(x, 0) < 0$ for $\forall x \in D$, see Proposition 3.16 and Example 3.32 in Sect. 3.4.3. □

Example 2.21 (Quantum Tunneling) The instanton by which quantum tunneling arises is the minimizer $\gamma^\star$ of the Agmon distance [17, Eq. (1.4)], i.e., of (2.19), where $S \in \mathscr{G}$ is given by the local action

$$\ell(x, y) = \sqrt{2U(x)}|y|. \tag{2.22}$$

Here, x_1 and x_2 are the minima of the potential $U \in C(D, [0, \infty))$, and it is assumed that $U(x_1) = U(x_2) = 0$.

If U did not have any roots then this would be a special case of Example 2.20, with $A(x) := 2U(x)I$, which leads us to the Hamiltonian $H(x, \theta) = |\theta|^2/(2U(x)) - 1$. According to the remark following (2.11), we could multiply H by the function $U(x)$ without changing the associated action, and so we would find that (2.22) is given by

$$H(x, \theta) = \tfrac{1}{2}|\theta|^2 - U(x). \qquad \text{(quantum tunneling)}$$

We can now check that this choice in fact leads to (2.22) even if U does have roots (with $\lambda = \sqrt{2U(x)}/|y|$ and $\vartheta = \lambda y$), and so we have $S \in \mathscr{H} \setminus \mathscr{H}_0$. Again, the natural drift is $b(x) \equiv 0$. □

Example 2.22 (Large Deviation Theory, Part II) Now consider again the SDE (2.16), but equipped with the additional feature that the process jumps to some "dead" state at the rate $\varepsilon^{-1}r(X_t)$, for some given bounded absorption rate function $r \in C(D, [0,\infty))$. Then this ***killed diffusion process*** is fulfilling a large deviation principle as well,[4] and assuming for simplicity that $A(x) \equiv I$, the large deviation action S_T is given by

$$H(x,\theta) = \langle b(x), \theta\rangle + \tfrac{1}{2}|\theta|^2 - r(x), \tag{2.23}$$

$$L(x,y) = \tfrac{1}{2}|b(x) - y|^2 + r(x), \tag{2.24}$$

thus penalizing curves for spending time in regions where $r(x) > 0$. Solving the system (2.11), we find that $\lambda = |y|^{-1}\sqrt{|b(x)|^2 + 2r(x)}$ and $\vartheta = \lambda y - b(x)$, which leads us to the corresponding geometric local action

$$\ell(x,y) = |y|\sqrt{|b(x)|^2 + 2r(x)} - \langle y, b(x)\rangle. \tag{2.25}$$

For general $A(x)$ all scalar products and norms only have to be replaced as in Example 2.19, which then makes (2.25) a generalization of (2.20). Observe also how our expression $H_\theta(x,0)$ for the natural drift defined in Lemma 2.15 still recovers the given vector field b.

In summary, adding the continuous and bounded absorption rate $\varepsilon^{-1}r(x)$ to the SDE (2.16) had the effect of subtracting $r(x)$ from $H(x,\theta)$ and adding it to $L(x,y)$, which leaves the natural drift unchanged but results in $H(x,0) = -r(x)$ being negative wherever $r(x) > 0$. As a result, by Lemma 2.13 (ii) the set of non-degenerate points in Definition 2.9 is given by $D_{S+} = \{r \in D \mid r(x) > 0\}$.

In fact, the comments in Appendix A.6 show that adding a properly scaled absorption rate to *any* other process will have the same effect on its large deviation action. □

[4] Probabilists will find some comments in Appendix A.6.

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