Preface

This research monograph is an analytical treatment of a geometric problem that recently arose in an applied community [6, 7, 10] focused on developing numerical methods for understanding the pathways of rare transition events in stochastic dynamical systems with small noise. For years, it had been a reoccurring problem that the underlying mathematical framework, Wentzell-Freidlin theory [8], is typically formulated in terms of time-parameterized paths, and that in that formulation no “maximum likelihood transition path” exists. This was leading to numerical problems since algorithms had no well-defined object to converge to.

In a collaboration of Eric Vanden-Eijnden (NYU) and myself [9, 10], it was then found that a geometric reformulation of the theory, i.e., one based on *un*parameterized rectifiable curves[1] γ, promised to resolve this issue because the main reason for this non-existence (the time parameterization) had been eliminated. Indeed, an algorithm based on this approach, the geometric minimum action method (gMAM), turned out to converge reliably in our applications.

This in turn seemed to suggest that in this geometric formulation an (unparameterized) maximum likelihood transition curve $\gamma^\star$ does indeed exist, defined as the minimizer of a certain non-negative geometric functional $S(\gamma)$. Motivated by the prospects of finally having a well-defined object to work with, I then took up the exciting task of developing criteria for rigorously proving this existence in the most general framework possible. The results of this effort are the content of this monograph.

The key problem in dealing with our functionals of interest is a degeneracy[2] they share that allows for curves γ with positive Euclidean length but with vanishing

[1]These are the same curves that the reader will know from the Cauchy integral theorem in complex analysis, which also treats its curves as geometric objects that are not tied to any specific parameterization.

[2]To prevent confusion for those familiar with Wentzell-Freidlin theory, it should be pointed out that this property is *not* related to degeneracies in the diffusion matrix of the given SDE. In fact, in our applications we can only consider *non*-degenerate diffusions.

action, $S(\gamma) = 0$. Many of the techniques and concepts that we develop here in order to address this difficulty are fundamentally new and have value in their own right, as they may be of use in other problems related to such actions.

The effort that this investigation required is justified by more than just academic curiosity: No algorithm for finding a minimizer $\gamma^\star$ of S can work without the interaction with a human who tweaks its parameters and who verifies whether its output looks reasonable. Now if no minimizer exists, then naturally the algorithm will fail to find one, but without any analytical insight the user may falsely blame himself/herself instead and keep trying to tweak the algorithm parameters. Furthermore, any analytically obtained knowledge about *properties* of $\gamma^\star$ can be used either to gain confidence in the numerically obtained curve (by checking whether it indeed has these properties) or to speed up the algorithm (by restricting its search for $\gamma^\star$ to only those curves that fulfill these properties).

In short: *Solid analytical knowledge about the existence and properties of $\gamma^\star$ are invaluable to the person who uses an algorithm for finding it.*

I hope that this monograph will not only impact how people within the large deviation community view and work with transition curves, but that the generality of its results will also spark some interest outside of this field and lead to applications that go beyond my original motivation for this work.

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