2
Second-Order Linear Equations

In this chapter we study second-order linear differential equations of the form

$$ax'' + bx' + cx = f(t)$$

and their applications to classical mechanics and electrical circuits. These applications are standard fare and a centerpiece in both elementary physics and engineering courses, and they serve as prototypes for oscillating systems, oscillating systems with dissipation, or damping, and forced vibrations that occur in all areas of pure and applied science. In the final sections of the chapter we extend the coverage to linear equations with variable coefficients.

2.1 Classical Mechanics

Newton's second law is familiar from high school physics and beginning calculus courses. It is the fundamental law of classical particle dynamics and is perhaps the most well known law in elementary physics. Its simple statement is: *force equals mass times acceleration*, or $F = ma$. When an external force acts on a particle of mass m, it changes the momentum, or inertia, in the system. If $x = x(t)$ denotes the position of the particle, then the particle undergoes an acceleration given by the second derivative, $x''(t)$. Thus Newton's law takes the form $mx'' = F$, which is a second-order differential equation for the position x. In the general case, the external force could depend on time t, the position

J. D. Logan, *A First Course in Differential Equations*, Undergraduate Texts in Mathematics, DOI 10.1007/978-3-319-17852-3_2

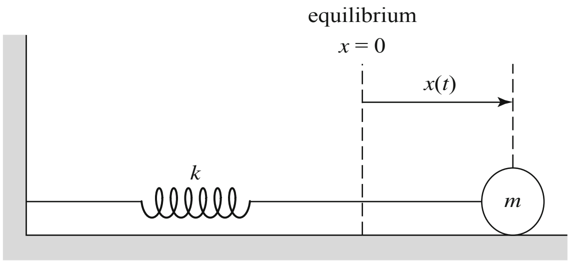

Figure 2.1 Spring–mass oscillator. $x(t)$ is the displacement of the mass at time t, where $x = 0$ is the equilibrium positon; $x(t) > 0$ when the mass is displaced to the right.

x, and the velocity x'. Therefore Newton's law of motion can be expressed generally as

$$mx'' = F(t, x, x'), \qquad (\textbf{Newton's Second Law})$$

where the form of the force $F(t, x, x')$ is prescribed.

In dynamics we expect to impose *two* initial conditions,

$$x(0) = x_0, \quad x'(0) = x_1,$$

where x_0 is the initial position and x_1 is the initial velocity. In general terms, the program of classical mechanics is *deterministic*; that is, if the initial state (position and velocity) of a system is known, as well as the forces acting on the system, then the state of the system is determined for all times $t > 0$. Practically, this means we solve the initial value problem above associated with Newton's law to determine the evolution of the system $x = x(t)$.

2.1.1 Oscillations and Dissipation

Oscillatory behavior is a common phenomenon in mechanical, biological, electrical, atomic, and other physical systems. We begin with a prototype of a simple oscillatory system, a mass connected to a spring.

Example 2.1

(**Oscillator**) Imagine a mass m lying on a table and connected to a spring, which is in turn attached to a rigid wall (Figure 2.1). At time $t = 0$ we displace the mass a positive distance x_0 to the right of equilibrium and then release it. If we ignore friction on the table then the mass executes *simple harmonic motion*; that is, it slides back and forth at a fixed frequency and amplitude. Following the doctrine of mechanics we write Newton's second law of motion, $mx'' = F_s$, where the state function $x = x(t)$ is the position of the mass at time t; we take $x = 0$ to be the equilibrium position and $x > 0$ to the right; the spring exerts external force F_s, which must be prescribed. Experiments confirm that if the displacement is not too large (which we assume), then the force exerted by the spring on the mass is proportional to its displacement from equilibrium, or

$$F_s = -kx. \qquad \text{(Hooke's Law)} \tag{2.1}$$

The minus sign appears because the force opposes positive motion, which is to the right. The force is negative when $x > 0$, and positive if $x < 0$. The proportionality constant $k > 0$ (having dimensions of force per unit distance) is called the **spring constant**, or **stiffness** of the spring, and Equation (2.1) is called **Hooke's law**. Not every spring behaves in this manner, but Hooke's law is used as a model for some springs. In engineering such a law is called a **constitutive relation**; it is an empirical result rather than a law of nature. This is an example of a linear force that depends only on the position x.

To give more justification for Hooke's law, suppose the force F_s depends on the displacement x through $F = F_s(x)$, with $F_s(0) = 0$. By Taylor's theorem,

$$\begin{aligned} F_s(x) &= F_s(0) + F_s'(0)x + \frac{1}{2}F_s''(0)x^2 + \cdots \\ &= -kx + \frac{1}{2}F_s''(0)x^2 + \cdots, \end{aligned}$$

where we *defined* $F_s'(0) = -k$. So Hooke's law is a good approximation if the displacement is small, allowing the higher-order terms in the series to be neglected.

We can measure the stiffness k of a spring by attaching it to the ceiling without the mass. Then we attach the mass m and measure the elongation L after it comes to rest. The force of gravity mg (downward) must balance the restoring force kx (upward) of the spring, so $mg = kL$. Therefore,

$$k = \frac{mg}{L}.$$

Newton's law, or the equation of motion of the system, is therefore

$$mx'' = -kx. \tag{2.2}$$

This is the **spring-mass equation**. The initial conditions are $x(0) = x_0$, the position where the mass is released, and the velocity $x'(0) = x_1$ given to it at time $t = 0$.

As a special case, suppose the initial velocity is zero. That is, we just displace the mass to x_0 and release it. The initial conditions are

$$x(0) = x_0, \quad x'(0) = 0.$$

We expect oscillatory motion. Assuming a solution of (2.2) of the form $x(t) = A\cos\omega t$ for some unknown frequency ω and amplitude A, we find upon substitution into (2.2) that $\omega = \sqrt{k/m}$ and $A = x_0$. (Verify this!) Therefore, the position of the mass at time t is given by

$$x(t) = x_0 \cos\sqrt{k/m}\, t.$$

This solution is an oscillation of amplitude x_0, natural frequency $\sqrt{k/m}$, and period 2π divided by the frequency, or $2\pi\sqrt{m/k}$. □

Now we add an additional force, that of the frictional force of the table exerted on the mass. Friction is a force that opposes positive motion and it **dissipates**, or decreases, the energy in the system. Dissipation in mechanical systems include friction, air resistance, and so on, which are also called damping forces. In circuits, electrical resistance dissipates the energy in the circuit. These energies often are transformed to heat energy.

Example 2.2

(**Damped Oscillator**) Assume there is friction as the mass slides on the table. The simplest constitutive relation is to take the frictional force to be proportional to how fast the mass is moving, or its velocity x'. Thus

$$F_d = -\gamma x', \qquad \text{(damping force)}$$

where $\gamma > 0$ is the **damping coefficient** (mass per time). If the mass is moving to the right, or $x' > 0$, the damping retards the motion and $F_d < 0$. Therefore, the total external force is the sum

$$F = F_s + F_d = -kx - \gamma x'.$$

The equation of motion is

$$mx'' = -\gamma x' - kx. \qquad \text{(damped oscillator)}$$

This equation is called the **damped oscillator equation**. Both forces have negative signs because each opposes positive (to the right) motion. For this case we expect an oscillatory solution with a decreasing amplitude during each oscillation because of the presence of friction. One such a solution, a damped oscillation, takes the form of

$$x(t) = Ae^{-\lambda t} \cos \omega t,$$

where A is some amplitude, λ is a decay rate, and ω is the frequency. $\square$

The Mechanical-Electrical Analogy

There is great similarity between the damped spring-mass system and an RCL circuit. We can write the damped oscillator equation with unknown displacement $x(t)$ as

$$mx'' + \gamma x' + kx = 0. \qquad (\textbf{damped oscillator})$$

Interestingly enough, from Section 1.5, the current $I(t)$ in an RCL circuit with no electromotive force (emf) satisfies

$$LI'' + RI' + \frac{1}{C}I = 0, \qquad (\textbf{RCL circuit})$$

which has *exactly* the same form. This similarity is a classical example of the unifying nature of mathematics in science. The similarity between these two models is called the **mechanical–electrical analogy**:

- The spring constant k is analogous to the inverse capacitance $1/C$; both a spring and a capacitor *store energy* in the system.
- The damping constant γ is analogous to the resistance R; both friction in a mechanical system and a resistance in an electrical system *dissipate energy*, often in the form of heat.
- The mass m is analogous to the inductance L; both represent *inertia* in the system; a large mass or inductance causes the velocity or current, respectively, to resist change. In the case of a circuit, an inductor (or coil) stores energy in its magnetic field which resists changes in current.

In every equation we encounter, we want to understand the meaning of each term. In the analogy above the first term is the inertia term, the second term is the dissipation or energy loss term, and the third is an energy storage term. Many of the equations we examine in this chapter can be regarded as either circuit equations or mechanical equations. Common among them are the physical properties of inertia, damping, and oscillation. In Section 2.3 we consider additional forces on the system due to external forcing or an electromotive force.

Energy Considerations

To get a better idea about the role of dissipation in a system, let us think a little deeper about energy. For a damped spring-mass system, the governing equation is

$$mx'' + \gamma x' + kx = 0.$$

To get energy expressions, multiply this equation by the velocity x' to obtain

$$mx'x'' + \gamma x'x' + kxx' = 0.$$

Each term has units of energy per time. By the chain rule

$$\frac{d}{dt}x'(t)^2 = 2x'(t)x''(t) \quad \text{and} \quad \frac{d}{dt}x(t)^2 = 2x(t)x'(t).$$

Therefore, the energy equation can be written

$$\frac{d}{dt}\left[\frac{1}{2}m(x')^2 + \frac{1}{2}kx^2\right] = -\gamma x'x'. \tag{2.3}$$

The two terms inside the left bracket are the kinetic energy and potential energy in the system:

$$T = \frac{1}{2}m(x')^2 \quad (\textbf{kinetic energy}); \quad V = \frac{1}{2}kx^2 \quad (\textbf{potential energy})$$

To understand potential energy, recall that the *force is the negative derivative of the potential.* The force is $F(x) = -kx$, so the potential energy due to that force is the negative integral of the force, or

$$V(x) = -\int(-kx)dx = \frac{1}{2}kx^2.$$

Therefore (2.3) is an energy dissipation law

$$\frac{dE}{dt} = \frac{d}{dt}[T+V] = -\gamma x'x',$$

stating that the *total* energy $E = T + V$ in the system is dissipated at the rate $-\gamma x'x'$. Energy per time is *power*, so (2.3) is the power lost in the system.

If there is no damping in the system then $\gamma = 0$ and we have

$$\frac{1}{2}m(x')^2 + \frac{1}{2}kx^2 = E, \quad E \text{ constant}.$$

This is the **conservation of energy** law.

EXERCISES

1. When a mass of 0.3 kg is placed on a spring hanging at rest from the ceiling, it elongates the spring 5 cm. What is the stiffness k of the spring?

2. **(a)** Beginning with the RCL circuit equation expressed in terms of charge Q on the capacitor, or

$$LQ'' + RQ' + \frac{1}{C}Q = 0,$$

derive the energy dissipation law corresponding to a spring-mass oscillator (2.3). **(b)** Identify the energy in the inductor and on the capacitor. **(c)** Show that the power lost by the resistor is $-RI^2$. **(d)** If there is no resistor, what is the conservation of energy law for the circuit?

3. We derived the spring-mass equation for a mass moving horizontally on a table top. This exercise shows that the same equation holds for a mass oscillating on a vertical, perfectly elastic string (or a spring) under the influence of the force of gravity mg. We assume Hooke's law holds for the string, that is, the force exerted by the string is proportional to its displacement. See the set up in Figure 2.2, where we introduce two coordinate systems, x and y, to measure displacement. First, only the string of natural length L is attached with no mass. This is position $y = 0$. Then we attach the mass, and at equilibrium it reaches a natural length $L + \Delta L$; this is position $x = 0$. Next we pull the mass down a positive distance and release it, and it undergoes oscillatory motion.

   a) Use Newton's second law and Hooke's law to justify the equation of motion
   $$my'' = -ky + mg.$$

   b) In the x coordinate system show that the equation of motion is
   $$mx'' = -kx,$$
   where the gravitational force does not appear. Hint: Note that $y = x + \Delta L$ and use the definition of the stiffness, $k\Delta L = mg$.

   c) Show that if damping occurs, then the governing equation is $mx'' = -kx - \gamma x$.

2.2 Equations with Constant Coefficients

Both the damped spring-mass equation and the RCL circuit equation have the form, namely,

$$ax'' + bx' + cx = 0, \tag{2.4}$$

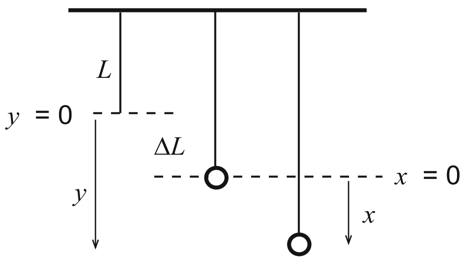

Figure 2.2 Two coordinate systems x and y for the motion of a mass m attached to a perfectly elastic string, or a spring, subject to both a gravitational force mg downward and a restoring force given by Hooke's law.

where a, b, and c are constants. An equation of the form (2.4) is called a **homogeneous linear equation with constant coefficients**. The word **homogeneous** refers to the fact that the right side is zero, meaning there is no external forces acting on the mass or no emf in the circuit. (In Section 2.3 we include these types of forces.) Equation (2.4) can be accompanied by initial conditions of the form

$$x(0) = x_0, \quad x'(0) = x_1. \tag{2.5}$$

Here, we are using $x = x(t)$ as the generic state function.

The problem of solving (2.4) subject to (2.5) is called the **initial value problem** (IVP). In (2.5) the initial conditions are given at $t = 0$, which is the common case, but they could be given at any time $t = t_0$. Finally, in the spring-mass problem and RCL circuit problem the constants a, b, and c are nonnegative, but our analysis is valid for any values of the constants.

2.2.1 The General Solution

We develop a simple technique to solve the homogeneous linear equation

$$ax'' + bx' + cx = 0.$$

Fundamental to the discussion is the following existence–uniqueness theorem, which is proved in advanced texts; it also includes a definitive statement about the interval where solutions are valid.

Theorem 2.3

(**Existence-Uniqueness**) The initial value problem (2.4)–(2.5) has a unique solution that exists on $-\infty < t < \infty$. □

The issue is how to find the solution. For the constant coefficient equation (2.4), with no initial conditions, we demonstrate that there are always exactly two **independent solutions**, say $x_1(t)$ and $x_2(t)$, meaning one is not a constant multiple of the other; they are not proportional. Such a set of solutions, $x_1(t), x_2(t)$, is called a basic, or **fundamental set**. Further, if we multiply each by an arbitrary constant and form the linear combination

$$x(t) = c_1 x_1(t) + c_2 x_2(t), \tag{2.6}$$

where c_1 and c_2 are arbitrary constants, then we can easily check that $x(t)$ is also a solution to the differential equation (2.4). (This is the superposition principle—see Exercise 3.) The linear combination (2.6) is called the **general solution** to (2.4), which means that *all* solutions to (2.4) are contained in this linear combination for different choices of the constants c_1 and c_2. In solving the initial value problem we use the initial conditions (2.5) to *uniquely* determine the constants c_1 and c_2 in (2.6).

Theorem 2.4

(**General Solution**) Let $x_1(t)$, $x_2(t)$ be a fundamental set of solutions of (2.4), and let $\phi(t)$ be any other solution. Then there exists unique values of the constants c_1 and c_2 such that

$$\phi(t) = c_1 x_1(t) + c_2 x_2(t). \quad \square$$

To prove this result, let $x_1(t)$ and $x_2(t)$ be the unique solutions that satisfy the initial conditions

$$x_1(0) = 1, \;\; x_1'(0) = 0, \qquad x_2(0) = 0, \;\; x_2'(0) = 1,$$

respectively, and let $\phi(t)$ be *any* solution of (2.4). $\phi(t)$ will satisfy some conditions at $t = 0$, say, $\phi(0) = A$ and $\phi'(0) = B$. But the function

$$x(t) = Ax_1(t) + Bx_2(t)$$

satisfies those same initial conditions, $x(0) = A$ and $x'(0) = B$. By the uniqueness theorem, Theorem 2.3, $\phi(t) = x(t)$ and so $c_1 = A$, $c_2 = B$. So the solution $\phi(t)$ is contained in the general solution. $\square$

Construction of Solutions

Our strategy now is to find two independent solutions $x_1(t)$ and $x_2(t)$ of (2.4). We suspect something of the form

$$x(t) = e^{\lambda t},$$

where λ is a constant to be determined, might work because every term in (2.4) has to be the same type of function for cancelation to occur; thus, x, x', and x'' must be the same form, which suggests an exponential function for x. Substitution of $x = e^{\lambda t}$ into (2.4) instantly leads to

$$a\lambda^2 + b\lambda + c = 0, \qquad \text{(characteristic equation)} \tag{2.7}$$

which is a quadratic equation for the unknown λ. Equation (2.7) is called the **characteristic equation**. Using the quadratic formula, we obtain its roots

$$\lambda = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right). \qquad \text{(eigenvalues)}$$

These roots of the characteristic equation are called the **eigenvalues** corresponding to the differential equation (2.4). Each value of λ gives a solution $x(t) = e^{\lambda t}$ to the equation

$$ax'' + bx' + cx = 0.$$

Clearly, the values of λ could be real numbers or complex numbers. Thus, there are three cases, depending upon whether the discriminant $b^2 - 4ac$ is positive, zero, or negative.

Case 1. If $b^2 - 4ac > 0$, then there are two real unequal eigenvalues λ_1 and λ_2. Hence, there are two independent, exponential-type solutions

$$x_1(t) = e^{\lambda_1 t}, \quad x_2(t) = e^{\lambda_2 t}, \qquad \lambda_1 \neq \lambda_2.$$

Therefore the general solution to (2.4) is

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}. \tag{2.8}$$

Case 2. If $b^2 - 4ac = 0$ then there is a double root $\lambda_1 = -b/2a$, $\lambda_2 = -b/2a$. Then one solution is $x_1(t) = e^{\lambda t}$, where $\lambda = -b/2a$. A second independent solution in this case is $x_2(t) = te^{\lambda t}$. (Later we show why this solution occurs.) Therefore the general solution to (2.4) in this case is

$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}. \tag{2.9}$$

Case 3. If $b^2 - 4ac < 0$ then the roots, or eigenvalues, of the characteristic equation are complex conjugates having the form[4]

$$\lambda_1 = \alpha + i\beta, \qquad \lambda_2 = \alpha - i\beta,$$

[4] Notation: Any complex number z can be written $z = u + iv$, where u and v are real numbers; u is called *the real part of* z and v is called *the imaginary part of* z. Similarly, if $z(t) = u(t) + iv(t)$ is a complex function, then $u(t)$ and $v(t)$ are its real and imaginary parts, respectively. The numbers $u + iv$ and $u - iv$ are called *complex conjugates*.

http://www.springer.com/978-3-319-17851-6

A First Course in Differential Equations
Logan, J.D.
2015, XIII, 369 p. 101 illus., 13 illus. in color., Hardcover
ISBN: 978-3-319-17851-6