Chapter 2
Light, Time, Mass, and Length

2.1 Light and Time

Let us think a little bit more about a light sender and a speedometer resting *relative* to each other, as in Fig. 2.1. Here we omit the speedometer. We put the whole setup into some transparent elevator as you find sometimes in department stores. We depicted it as a light-gray background box.

The man inside the elevator rests relative to the light-sender. He sees the light beam traveling horizontally. That's because the speed of the elevator has no absolute meaning to him, as we saw. He measures the time needed for the light beam to arrive at the tip of the arrow with the clock resting beside him.

Standing at the right outside the elevator, we will see the light starting at the sender. While the light moves to the right, we see that the elevator moves up together with the light beam. For example: at half the time it will be half the way up, so it just travels along the diagonal upwards.

This diagonal is *longer* than the horizontal distance. But remember that light travels always at the same speed c along the line whether measured inside or outside the elevator. We see that the **direction** of the light beam changes but the *magnitude* of the **speed of light** is for both of us the same. Hence we see the light traveling a *longer* time than the observer resting with the light source:

$$\begin{pmatrix} \text{time of clock} \\ \text{moving relative to us} \end{pmatrix} > \begin{pmatrix} \text{time of clock} \\ \text{resting relative to us} \end{pmatrix} \tag{2.1}$$

That means that light and time are connected. For us outside, time inside the elevator evolves *more slowly* than ours. When the observer sitting inside the elevator says: "1 second has passed", we outside say: "No, *more* than 1 second has passed". In other words:

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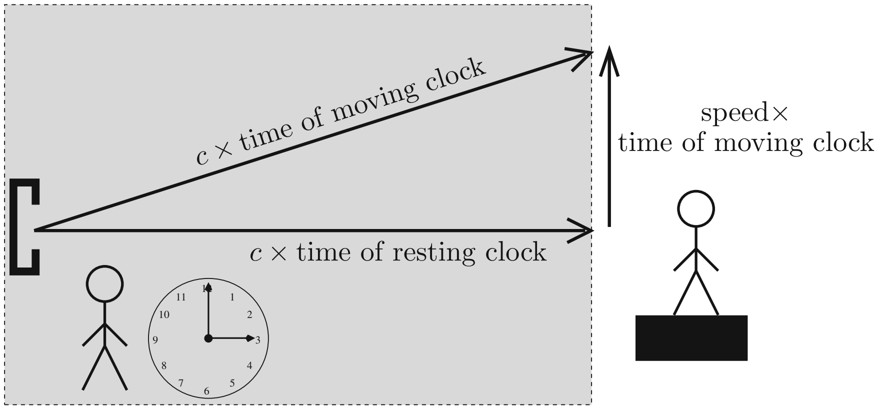


Fig. 2.1 We see the elevator moving upwards. Because speed is distance per time, we can write the distances as speed times time

Because the magnitude of the **speed of light is absolute**, the pace of time is *relative* to the speed of the observer.

We use the Greek letter **gamma** "γ" to describe this: it denotes a number between zero and one and connects the larger time of the moving clock with the smaller time of the resting clock:

$$\begin{pmatrix}\text{time of clock}\\ \text{moving relative to us}\end{pmatrix} \times \gamma = \begin{pmatrix}\text{time of clock}\\ \text{resting relative to us}\end{pmatrix}$$

We see: for speed zero, γ is one because then we agree with the observer in the elevator on our times. The larger the speed of the elevator, the longer the diagonal becomes, so the smaller the factor γ.

Please have again a look at Fig. 2.1. During the time of the moving clock in which the light travels along the longer diagonal at a speed c, the elevator travels the *shorter vertical* line at its speed. Hence its speed is always *less* than the speed of light. We see again:

Matter cannot travel **faster than the speed of light**, relative to us.

2.2 The Gamma Factor

We can actually compute the γ factor. We need only the **Pythagoras theorem** to get the answer. Please have a look at Fig. 2.2. We use the same triangle from Fig. 2.1. The theorem of Pythagoras tells us that

$$\begin{aligned}&(c \times \text{time of moving clock})^2\\ &= (c \times \text{time of resting clock})^2 + (\text{speed} \times \text{time of moving clock})^2\end{aligned}$$

We want the time of the resting clock, looking from the outside, so

$$\begin{aligned}&c^2 \times (\text{time of moving clock})^2 - \text{speed}^2 \times (\text{time of moving clock})^2\\ &= c^2 \times (\text{time of resting clock})^2\end{aligned}$$

Factor out the "time of the moving clock" on the left,

$$(\text{time of moving clock})^2 \times \left(c^2 - \text{speed}^2\right) = c^2 \times (\text{time of resting clock})^2$$

Then divide by c^2:

$$(\text{time of moving clock})^2 \left(1 - \frac{\text{speed}^2}{c^2}\right) = (\text{time of resting clock})^2$$

Because the square of the moving time and the resting time are never negative, the factor $\left(1 - \frac{\text{speed}^2}{c^2}\right)$ must be at least zero as well. Hence we see here again that the speed cannot be larger than the speed of light. Further, we can extract the square root to get the relation between our moving time and the resting time inside the elevator:

$$(\text{time of moving clock}) \times \sqrt{1 - \frac{\text{speed}^2}{c^2}} = \text{time of resting clock} \tag{2.2}$$

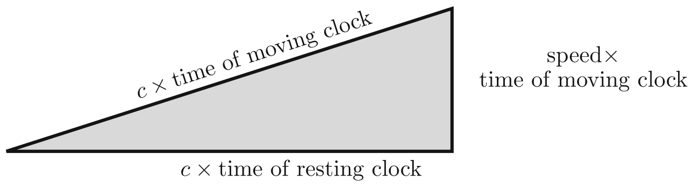


Fig. 2.2 The Pythagoras theorem tells us how the times of the resting and the moving clock depend on each other

The square root term is the γ **factor**,

$$\gamma = \sqrt{1 - \left(\frac{\text{speed}}{c}\right)^2} \tag{2.3}$$

We can also get the γ factor in a graphic way. We use again the triangle of Fig. 2.2 and express the lengths of all three sites as multiples of the longest side, that is we divide all three sides by (c $\times$ speed of moving clock). Then the longest side has length one, by construction. The lower side becomes

$$\frac{\cancel{c} \times (\text{time of resting clock})}{\cancel{c} \times (\text{time of moving clock})} = \gamma$$

and the right side becomes

$$\frac{\text{speed} \times \cancel{(\text{time of moving clock})}}{c \times \cancel{(\text{time of moving clock})}} = \frac{\text{speed}}{c}$$

In other words, γ and $\frac{\text{speed}}{c}$ are the coordinates of a point on a circle of radius one, as you can see in Fig. 2.3. Hence we can see the size of the γ factor also from this figure. We see again that as the speed decreases to zero, γ increases to one.

This factor γ appears nearly everywhere in the theory of relativity, once you start to calculate things. Sometimes it is easier to use γ than to use the speed.

How much differs the γ factor from one for ordinary speeds on Earth? An airplane travels at roughly 1000 kilometers per hour, which is one million meters per 3600 seconds, or roughly 300 meters per second, that is 3×10^2. The speed of light is 3×10^8, so the airplane travels roughly at 10^{-6}, that is one part of a million of the speed of light

$$\frac{\text{speed}}{c} \approx 10^{-6}$$

Now square this. It becomes practically zero, namely one part in a million-million, that is 10^{-12}. Hence the gamma factor is practically one for ordinary speeds on Earth. Can we conclude from this fact that the theory of relativity does not play any role for nature on Earth, as sometimes people state? No! We will see in Chap. 3 that even at speeds of less than one millimeter per second we can easily observe the time slip!

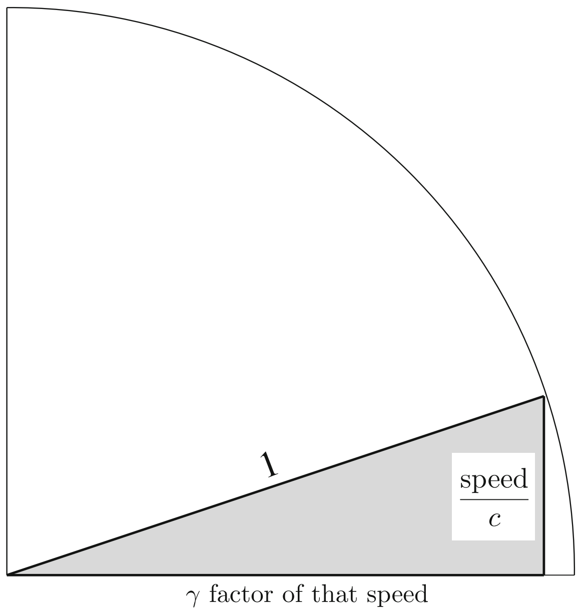


Fig. 2.3 The Pythagoras theorem tells us how the γ factor depends on the speed

For small speeds, we can estimate the γ factor, as you will find in Appendix A.3. The result is

$$\gamma = \sqrt{1 - \left(\frac{\text{speed}}{c}\right)^2} \approx 1 - \frac{1}{2}\left(\frac{\text{speed}}{c}\right)^2 \qquad (2.4)$$

We list the most important **properties of the gamma factor**:

1. For zero speed, γ is one.
2. The larger the speed, the smaller γ.
3. For nearly the speed of light, γ is nearly zero.
4. For very small speed, γ is smaller than one by a factor which is in proportion to the square of the speed.

2.3 Whose Clock Is Running More Slowly?

Let us now place the light-sender *outside on the left of* the elevator and let us be inside, as in Fig. 2.4. Let the elevator move upwards. Then the outside observer will see the light beam passing horizontally and we see it passing downwards. As for us,

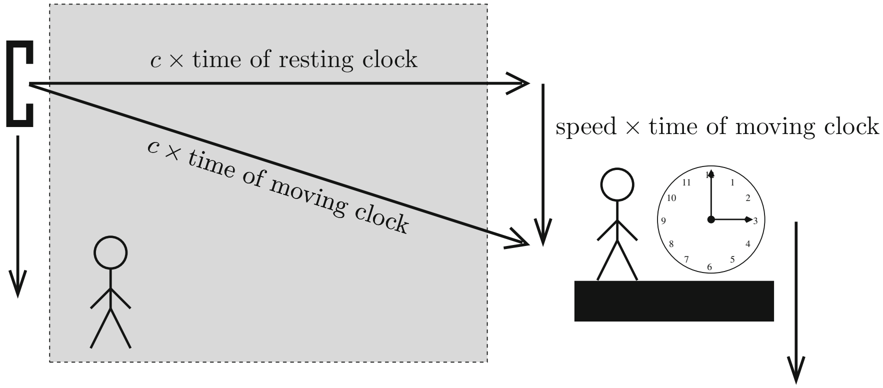


Fig. 2.4 We inside the elevator see the observer outside moving down, together with the light-sender

the light travels the longer, diagonal path and we conclude that for us the *time of the observer standing outside of the elevator* is running more slowly. However, in the last section we saw that for the outside observer the time *inside* the elevator runs more slowly than his own!

Compare this with Fig. 2.1. Then whose time is *really* running more slowly?

Answer: the question is wrong! It is the kind of question like "When is the air feeling colder: at night or outside?" We cannot decide because we cannot *compare* the two situations. The same goes for this situation: if the inside and outside observers want to compare their clocks, one of them or both have to *change* their speed to stop near each other. *Then* they can compare their clocks. However, by changing speed our clocks will change their pace! We will see in Chap. 4 what will happen then. For the time being, the observers pass each other and *continue* to travel at a steady speed. Hence *both* are correct in their statements that the other one's clock is going more slowly.

2.4 Light, Time, and Length

2.4.1 Length in the Direction of Movement

How do the observers measure the speed of the setup in Fig. 2.1? At first, we as the outside observer put down some rod in front of us. We sketch the rod as the solid black arrow pointing upwards in Fig. 2.5. Then we measure the length of the rod.

Because the rod rests relative to us, we call this the "length of resting rod". Then we measure the time needed for the setup to pass the rod. We choose the time of the *moving* clock. We get the speed

$$\text{speed} = \frac{\text{length of } \textit{resting} \text{ rod}}{\text{time of } \textit{moving} \text{ clock}}$$

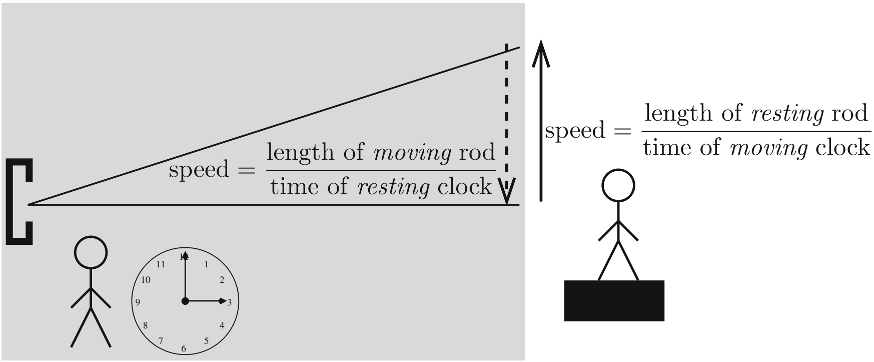


Fig. 2.5 Measuring the relative speed in terms of own length and time differences

What does the observer inside the setup measure? Because constant speed is relative, the observer can insist that *he* rests while the rod outside and we travel with at the same speed *downwards*. This is indicated by the dashed arrow pointing downwards. As a result, the observer measures the same speed in terms of the moving rod and clock,

$$\text{speed} = \frac{\text{length of } \textit{resting} \text{ rod}}{\text{time of } \textit{moving} \text{ clock}}$$

While for us outside the moving clock measures one second, the observer inside the elevator sees this clock resting and therefore showing only less than one second, namely $\gamma < 1$ seconds. Hence for him, the length of the *moving* rod must be *shorter* than for us, by the same γ factor, to get the same speed:

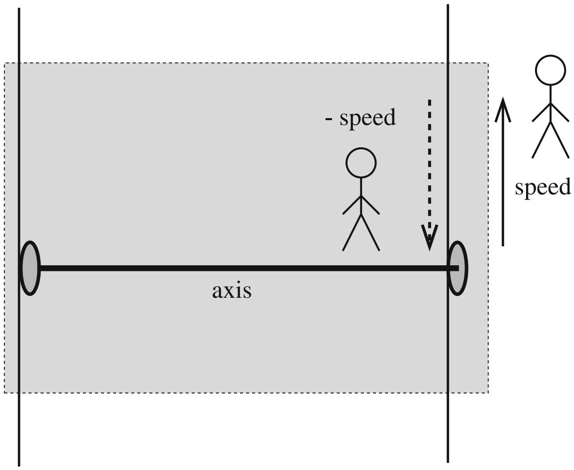
$$\begin{pmatrix}\textbf{length of rod moving}\\ \textbf{in direction of speed}\end{pmatrix} = \gamma \times \text{ (length of resting rod)} \tag{2.5}$$

2.4.2 Length at Right Angles to Movement

What happens at right angles to the movement? In Fig. 2.6, we drew also the rails on which the elevator moves up and down. We sketched two wheels and one axis. Both observers measure the *same* length for the axis.

Why? For the outside observer, the rails always rest. Their distance is just the length of the *resting* axis. If the elevator moves up, then for the observer inside the axis has still the same length because speed is only relative, and he can insist that he rests. If the outside observer measures another length of the axis than the resting length, this would then mean for him that the axis would be longer or shorter than

Fig. 2.6 The length of a rod at right angles to the movement does not change



the distance of the still resting two rails: the elevator would derail! This is absurd. Hence:

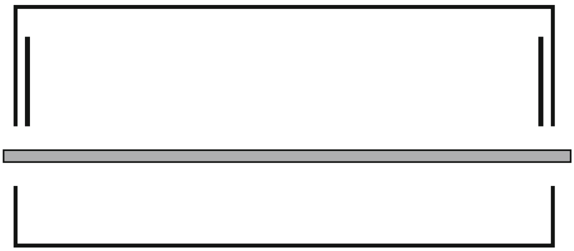
Lengths at right angles to movement do not change.

2.5 At the Same Time?

Let us see how the relativity of time and length go hand in hand with the help of a famous thought experiment called "The pole and the barn". Here light does not enter, only a pole enters a barn and apparently contracts.

A friend has a gray pole slightly longer than the barn into which he wants to place it. The barn has an electrical front door at the left and an electrical back door at the right. At first, we check that the resting pole is really longer than the barn, as in Fig. 2.7.

Fig. 2.7 The resting pole is longer than the barn



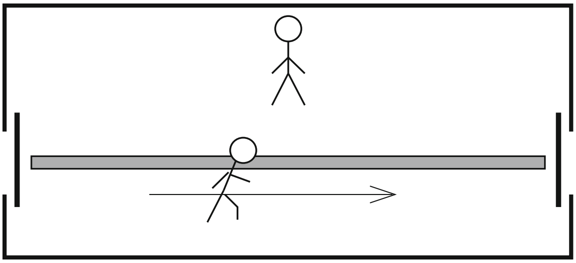


Fig. 2.8 The moving pole is shorter than the barn, for us standing inside the barn

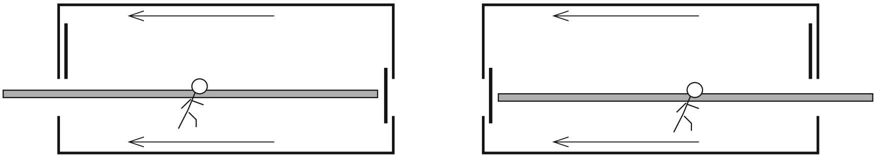


Fig. 2.9 For our running friend, the moving barn is shorter than the resting pole, but for him the right door closes and opens *before* the left door does

Then the friend takes the pole out of the barn and runs very fast through it from the left to the right. Because of the contraction described in Sect. 2.4.1 and for us standing in the barn, the pole looks *shorter* than the barn if the pole is only fast enough. We can arrange in advance that both doors close at the *same instant of time* and open shortly afterwards. If we choose the right moment for closing and opening the doors, the barn is *completely inside the barn* for a short moment as we see in Fig. 2.8.

However, what does our running friend see? The *barn* is moving towards him, so the *barn* is shorter for him. Hence the pole *does not fit* into the barn at all! By which miracle can the pole be completely *inside* the barn for some time, with both doors closed at least for a moment?

The point is that for the running friend the right door closes and opens again first and the left door closes and opens again afterwards, as you see in Fig. 2.9.

Why is that? Enter light. Suppose that some light flashes at the doors when pressing the switch to close both doors. As for us standing in the center of the barn, the flashes occur at the same time. Hence for us, the doors are closing *at the same time*.

What does our running friend see? He also sees the light of both flashes traveling at the speed of light c, as we know. However, the flash from the right door moves to the runner at a shorter distance because he runs *towards* it, and because he moves away from the flash coming from the left door. Hence he sees the right flash *before* the left flash: he sees that the right door is closing *before* the left door. The reason is twofold: first, the speed of light is absolute, and second, the two events of "the backdoor is closing" and "the front door is closing" happen at *different* places. We conclude:

The statement "**At the same time**" is not an absolute true statement for things happening at different places. Time depends on the relative speed of the clock and the observer.

2.6 Are There Any Time Machines?

Within the thought experiment of the pole and the barn, it seems to be possible to mix up the future and the past.

We can arrange to close the left door a little bit *before* the right door. If this "little bit" is short enough, we then have a strange situation which sounds as if a time machine will be possible.

For us standing in the center of the barn, the light from the left door reaches us *before* the light from the *right* door while our friend is running so fast towards the right door that *its* light reaches him *before* the light from the *left* door. In other words: as for us the left door closes *before* the right door while for our friend the left door closes *after* the right door.

Is it therefore possible to mix up the future and the past as in a time machine?

However, closing the left door does not *cause* the right door to close. Let us think about what will happen *if* the left door's closing *causes* the right door to close. Please have a look at Fig. 2.10.

When the left door begins to close, it sends light to us via the dashed arrow. At the same time, the left door sends a signal to the right door via the solid arrow. Via this signal it *causes* the right door to close, that is *when* the signal reaches the door. When the right door begins to close, this door itself sends some light to us via the dotted arrow.

We see the left door closing after the time it took light to travel *directly* via the dashed line to us. The signal causing the right door to close can travel no faster than the speed of light. Therefore we see the right door closing *at the earliest* after the time it took light to travel from the left door via the solid *and* dotted line to us. As

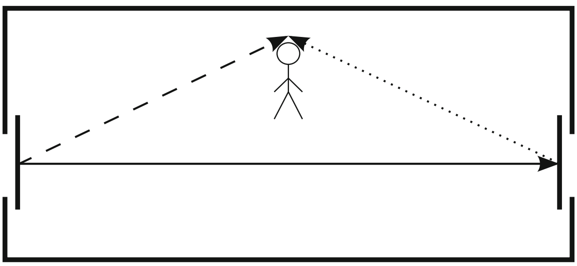


Fig. 2.10 If the left door's closing *causes* the right door to close we will see the left door's closing before the right door's closing, even if we are running in some direction relative to the doors

you see in the figure, the length of the dashed line is shorter than the length of the solid *plus* the length of the dotted line.

To sum up, we see the *cause before* the *effect*, that is the left door closing before the right door.

What does our running friend see? The runner sees the light traveling at the *same* speed of light, and also for him, the shortest path for the signal from the left door is the direct path. Hence he also sees that the left door closes before the right door!

In general: physical phenomena can not influence effects in the past. In other words: there is no such thing as a **time machine** "sending" us into the past, that is, influencing the past. Physicists call this the principle of **causality**.

2.7 Time and Mass

While we were standing outside the moving elevator in Sect. 2.1, we saw that time itself slows down for bodies inside that elevator. What are the consequences as to the bodies moving inside? It means that all movements slow down by exactly the same amount. So some property shared by all the bodies inside must change, at least from our standpoint. The obvious candidate is the inertia of the body. If they all become more inert, they move more sluggishly as if in slow motion. That fits with our observation before. For the same reason, when time slows down, the bodies will never be able to move faster than light moves from us away. To coin a catch-phrase:

> For bodies moving relative to us, time evolves more slowly than they experience themselves. Therefore they look more inert from our point of view. Their **inertia increases as their time** slows down, relative to us.

Let us see how much the inertia grows with speed. Suppose a ball with some resting-mass bounces at right angles, *slowly* against a wall and bounces back elastically as in Fig. 2.11.

When the ball bounces back, it changes from its original downwards speed to the same speed, but upwards. It makes no difference if the ball has say three times as much mass or if the same mass is moving three times as fast. So the "push" which the wall receives depends only on the product

$$\text{push} = \text{mass} \times \text{speed}$$

The wall receives *twice* this push: once when the wall absorbs the push from the ball and a second time when the wall pushes the ball elastically *back* at the same speed.

By the way, physicists call this push **momentum**.

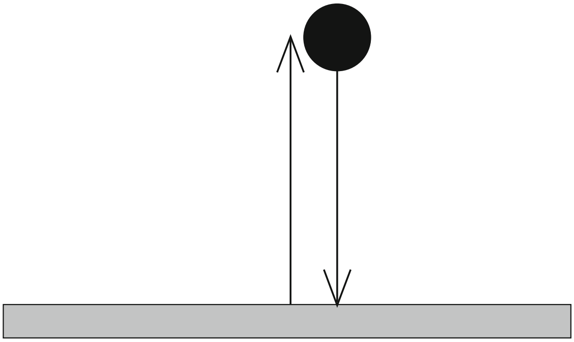


Fig. 2.11 A ball bounces against a resting wall

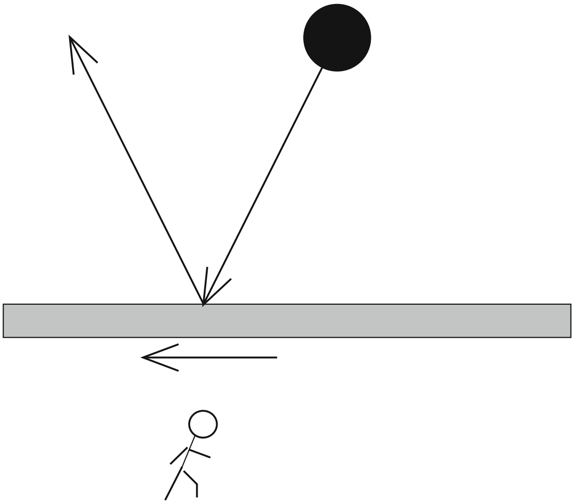


Fig. 2.12 The same ball from Fig. 2.11 bouncing at right angles from the same wall, seen by us, moving relative to the wall, along the wall

Further, the ball was slow enough so that its mass is by our experience nearly unchanged, that is nearly the resting-mass. Hence the wall receives twice the amount of

$$\text{push} = (\text{resting mass}) \times \text{speed}$$

Next, we imagine ourselves traveling fast *along* the wall, that is parallel to it, as in Fig. 2.12. That is relative to us, the wall travels from the right to the left.

All the same, the wall will receive the same push at right angles. Also the distance between the wall and the ball does not change for us because we learned in Sect. 2.4.2 that lengths at right angles to the motion do not change. The only difference is now that we see the time of the ball proceeding more slowly by the factor γ. Hence we see that the speed of the ball is *slower* by this factor. Therefore, to maintain the *same* push, the mass of the ball must be *larger* by the same amount that its speed is *slower*, that is by the same amount that its time is evolving more slowly:

$$\textbf{total mass} \text{ moving at speed } = \frac{\textbf{resting mass}}{\gamma} \tag{2.6}$$

We learned in Sect. 1.13 that this growing mass is due to the motion energy which the mass has gained. Therefore this **mass has the total energy**

$$\begin{aligned} &\text{total energy of mass moving at some speed} \\ &= \text{mass moving at speed} \times c^2 = \frac{\text{resting mass}}{\gamma} \times c^2 \end{aligned} \tag{2.7}$$

2.8 Speed Addition

We already know that we cannot travel faster than light. In Sect. 1.6 we demonstrated that with a peppercorn driven by a rocket. However, maybe there is another method *without* needing to accelerate things? Here comes a thought experiment as shown in Fig. 2.13. In a very light, yet strong box there are two balls with equal resting-mass moving to the left and the right so that we can ignore the mass of the box and concentrate on the mass of the two balls.

This box moves at say 70 percent of the speed of light to the right, relative to the ground. Let the upper ball move at the speed of say 70 percent of the speed of light to the left in the box, that is relative to the box. Let the lower ball move at the speed of say 70 percent of the speed of light to the right relative to the box. Then it seems that the lower ball is moving relative to the ground at 140 percent of the speed of light?

Let us estimate the masses of the balls. If we stand inside the box, the masses of both balls are equal and because of Eq. (2.6) they are larger than their resting-masses by the factor $1/\gamma$ belonging to the speed of 70 percent of c. Looking at it from outside, these two masses add up to the **resting-mass** of the box because we assumed the box itself to have nearly no mass.

Now look at these masses while standing still on the ground. The box moves at the same speed to the right as the lower ball inside the box, so its moving mass is again larger by the inverse gamma factor $1/\gamma$. Hence the total mass of the box is twice the resting-mass of one ball, divided by γ^2.

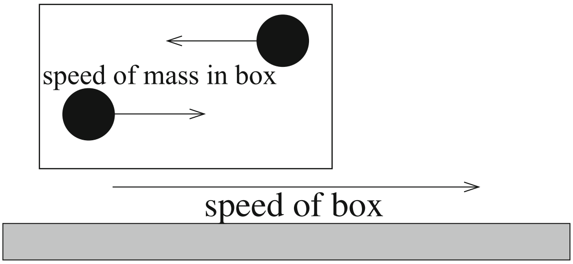


Fig. 2.13 Masses move inside a light moving box

Next look from the ground at each ball separately. The right ball moves at the same speed as the box, but to the left, so it rests relative to the ground. Hence its mass is now just its resting-mass. What is the mass of the left ball? The left ball has just such more mass as the right ball has less because their masses should add up to the same total mass of the box as before. However, we know that the higher the speed is the more mass increases. Hence the resultant speed of the left-hand ball, relative to the ground, must be less than twice the speed with which the balls are moving relative to the box. In fact, it is so much slower that again it is not faster than light! In other words:

Relativistic speed addition
If some box moves relative to the ground at some speed and some mass moves inside that box at some speed in the same direction, then that mass moves relative to the ground at a speed which is *so much less* than the sum of the speeds so that it never can be faster than light.

For the actual calculation, see Appendix A.4. The result is in terms of fractions of the speed of light

$$\frac{\text{total speed}}{c} = \frac{\dfrac{\text{speed of box}}{c} + \dfrac{\text{speed of mass in box}}{c}}{1 + \dfrac{\text{speed of box}}{c} \times \dfrac{\text{speed of mass in box}}{c}} \tag{2.8}$$

For our example we get speeds

$$\frac{\text{total speed}}{c} = \frac{0.7 + 0.7}{1 + 0.7 \times 0.7} \approx 0.94$$

which is of course less than the speed of light because it *must come out* that way!

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