Chapter 2
Possibilistic Models of Risk Management

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Abstract In the traditional treatment, risk situations are modeled by random variables. This chapter focuses on risk situations described by fuzzy numbers. The first goal of the chapter is to define and characterize possibilistic risk aversion and study some of its indicators. The second goal is the study of two possibilistic models of risk management: a coinsurance problem and an investment portfolio problem.

Keywords Risk management · Fuzzy sets · Possibility · Static portfolio · Coinsurance

2.1 Introduction

Risk aversion is an important topic in decision making under uncertainty. The first crucial contributions on this topic were brought by Arrow (1965, 1970) and Pratt (1964). They defined the risk aversion of an agent, they showed how it could be

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evaluated and how two agents' risk aversions could be compared. Then the literature dedicated to risk aversion considerably increased (see e.g. Eeckhoudt et al. (2005), Laffont (1993), Mas-Colell et al. (1995), Ross (1981)), which led to several applications in risk management. The monograph (Eeckhoudt et al. 2005) presents such applications in insurance decision, static and dynamic portfolio choices, consumption and saving, optimal prevention, etc.

The whole risk theory was based on probability theory. The notions and theorems on risk are formulated in terms of probabilistic indicators (expected value, variance, covariance, etc.). However the probability theory cannot model all risk situations related to economic and social phenomena. Zadeh's possibility theory (Zadeh 1978) offers another way of treating mathematically uncertainty (see also Carlsson and Fullér (2011), Dubois and Prade (1988), (1987)). In Georgescu (2009), (2010), (2012), Georgescu and Kinnunen (2012) a few possibilistic models of risk aversion based on a notion of possibilistic expected utility are studied. In the possibilistic approach, the risk is modeled by fuzzy numbers and possibilistic indicators of fuzzy numbers (expected value, variance, covariance, etc.) are used in order to formulate the definitions and the theorems of possibilistic theory.

This chapter continues the investigations of Georgescu (2009), (2010). It has two main goals:

- to develop some new aspects of possibilistic risk aversion
- to apply this theory to two models of risk management: the coinsurance problem and an investment portfolio problem.

The chapter is organized as follows.

In Sect. 2.2 fuzzy numbers and their indicators are presented. Due to their remarkable properties, the fuzzy numbers constitute the most important class of possibilistic distributions (Carlsson and Fullér 2011; Dubois and Prade 1988; Georgescu 2012). They allow us to define some possibilistic indicators analogues with the well-known indicators of random variables. The possibilistic expected value $E_f(A)$ from Carlsson and Fullér (2001), (2011), two notions of possibilistic variance $Var_f(A)$ from Carlsson and Fullér (2001), Fullér and Majlender (2003), (2004) and $Var_f^*(A)$ from Georgescu (2009) are recalled. $Var_f^*(A)$ is more useful than $Var_f(A)$ in the evaluation of possibilistic risk aversion (see Georgescu (2009), (2010), (2012), Georgescu and Kinnunen (2012)). For example, in Georgescu (2009), (2012) the possibilistic risk premium is expressed in terms of $E_f(A)$ and $Var_f(A)$.

Section 2.3 is dedicated to a notion of possibilistic expected utility (associated with a fuzzy number, a utility function and a weighting function) and some of their properties (Georgescu 2009). Among the results of this section we mention an approximation formula of possibilistic expected utility.

In Georgescu (2009), (2012), Georgescu (2010), Georgescu and Kinnunen (2012) we studied the risk aversion of an agent faced to a risk situation described by a fuzzy number. We defined the possibilistic risk premium as a measure for risk aversion and we proved some basic properties of this indicator. However in these papers there exists no definition of what we mean that an agent is risk–averse. In Sect. 2.4 we define a possibilistic risk–averse, a possibilistic risk–lover and a possibilistic risk–neutral agent (represented by a utility function u). These three concepts are characterized by the concavity, convexity and linearity of the utility function u. Some surprising conclusions are reached: an agent is possibilistic risk–averse iff it is probabilistic risk–averse, etc. (in the sense of Eeckhoudt et al. (2005), p. 8).

In Sect. 2.5 two new notions of possibilistic risk premium are defined and they are connected with the one from Georgescu (2009). The section also contains an approximate calculation formula for these indicators of risk aversion and a more complete form of possibilistic Pratt theorem of Georgescu (2010). Finally a characterization theorem of those utility functions for which the possibilistic risk premium is decreasing in wealth is proved.

Section 2.6 tackles the coinsurance problem in the context of possibilistic risk. Insurance contracts for which the loss is modeled by a fuzzy number are studied. Then the mean sum retrieved by the policyholder is a possibilistic expected utility and on its basis the possibilistic premium for insurance indemnity is defined. The optimal coinsurance rate is determined as a solution of a decision problem for which the objective function is expressed as a possibilistic expected utility. Properties of optimal coinsurance rate, its calculation and the way it changes with the variation of the initial wealth are studied.

Section 2.7 deals with a possibilistic model of an investment portfolio problem. The case of a risk–averse agent who invests in a risk–free asset and a risky asset is studied. Our model is based on the hypothesis that the return of the risky asset is described by a fuzzy number. To determine an investment with a maximum payoff a decision problem should be solved whose objective function is a possibilistic expected utility. Several properties of the optimal solution are studied and an approximate calculation formula is proved.

2.2 Fuzzy Numbers and Their Indicators

In this section we recall the definition of fuzzy numbers, their operations and two of their indicators (expected value and variance) (see Carlsson and Fullér (2011), Dubois and Prade (1980), (1988), Majlender (2004)).

Let X be a non–empty set. Following Zadeh (1965), a *fuzzy subset* of X is a function $A : X \to [0, 1]$. A fuzzy subset A of X is *normal* if there exists $x \in X$ such that $A(x) = 1$. The *support* of a fuzzy set A is $\sup p(A) = \{x \in X | A(x) > 0\}$.

Throughout this chapter,we shall consider that $X = R$. For $\gamma \in [0, 1]$, let the γ-level set $[A]^\gamma$ of a fuzzy subset A of R (see Carlsson and Fullér (2011), Dubois and Prade (1980)). The fuzzy set A is called *fuzzy convex* if $[A]^\gamma$ is a convex subset in R for any $\gamma \in [0, 1]$.

Definition 2.1 A fuzzy subset A of R is called fuzzy number if A is normal, fuzzy convex, upper semicontinuous and with bounded support.

If A, B are two fuzzy numbers and $\lambda \in R$, the fuzzy numbers $A + B$ and λA are defined by

$$(A+B)(x) = \sup_{y+z=x} \min(A(y), B(z));$$
$$(\lambda A)(x) = \sup_{\lambda y=x} A(y).$$

If $A_1, \ldots, A_n$ are fuzzy numbers and $\lambda_1, \ldots, \lambda_n \in R$, then one can consider the fuzzy number $\sum_{i=1}^n \lambda_i A_i$.

A non–negative and monotone increasing function $f : [0, 1] \to R$ is a *weighting function* if it satisfies the normality condition $\int_0^1 f(\gamma)d\gamma = 1$.

We fix a fuzzy number A and a weighting function f such that $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for all $\gamma \in [0, 1]$.

Definition 2.2 (Fullér and Majlender 2003) The f-weighted possibilistic expected value of A is defined by $E_f(A) = 1/2 \int_0^1 (a_1(\gamma) + a_2(\gamma))f(\gamma)d\gamma$.

Remark 2.3 (Carlsson and Fullér 2011) If $A_1, \ldots, A_n$ are fuzzy numbers and $\lambda_1, \ldots, \lambda_n \in R$ then $E_f(\sum_{i=1}^n \lambda_i A_i) = \sum_{i=1}^n \lambda_i E_f(A_i)$.

Definition 2.4 (Fullér and Majlender 2003) The f-weighted possibilistic variance of A is defined by $Var_f(A) = 1/2 \int_0^1 (a_1(\gamma) - a_2(\gamma))^2 f(\gamma)d\gamma$.

These two possibilistic indicators have important mathematical properties and they have been used in the construction of models with applications to strategic investment planning, fuzzy real options for strategic decisions, portfolio selection with imprecise data, risk assessment in grid computing, etc. (see Carlsson and Fullér (2011), Majlender (2004), Mezei (2011)).

In Georgescu (2009) another notion of possibilistic variance $Var_f^*(A)$ was defined, necessary to the possibilistic risk aversion model from that paper.

Definition 2.5 (Georgescu 2009)

$Var_f^*(A) = \int_0^1 [\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} (x - E_f(A))^2 dx] f(\gamma)d\gamma.$

The next proposition contains a computation formula for $Var_f^*(A)$.

Proposition 2.6 (Georgescu 2009)

$Var_f^*(A) = 1/3 \int_0^1 [a_1^2(\gamma) + a_2^2(\gamma) + a_1(\gamma)a_2(\gamma)]f(\gamma)d\gamma - E_f^2(A).$

2.3 Possibilistic Expected Utility

As far as we know from Eeckhoudt et al. (2005), Georgescu and Kinnunen (in press), Mas-Colell et al. (1995), Quiggin (1993), the probabilistic risk theory is developed in the framework of expected utility theory. A notion of possibilistic expected utility was introduced in Fullér and Majlender (2003), Georgescu (2009), then it was used in the construction of some possibilistic models (Carlsson and Fullér 2011; Georgescu 2009, 2012; Georgescu 2010; Georgescu and Kinnunen 2012; Majlender 2004; Mezei 2011).

In this section we recall this notion of possibilistic expected utility and some of its main properties.

We fix a weighting function $f : [0, 1] \to R$ and a fuzzy number A such that $[A]^{\gamma} = [a_1(\gamma), a_2(\gamma)]$ for any $\gamma \in [0, 1]$.

We consider a utility function u of class C^2. Sometimes the domain of the utility function will be $[0, \infty)$ or a real interval $[m, M]$.

Definition 3.1 (Fullér and Majlender 2003) The possibilistic expected utility $E_f(u(A))$ associated with *f*, A and u is $E_f(u(A)) = \int_0^1 [\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u(x)dx]f(\gamma)d\gamma$.

Each time we use the right hand side integral above we will assume that this integral is finite.

Remark 3.2 (i) If u is the identity function then $E_f(u(A)) = E_f(A)$; (ii) If $u(x) = (x - E_f(A))^2$ for all $x \in R$, then $E_f(u(A)) = Var_f^*(A)$; (iii) If $\lambda \in R$ and $A(x) = \lambda$ for all $x \in R$, then $E_f(u(A)) = \lambda$.

Proposition 3.3 (Georgescu 2009) *Let* $g : R \to R$ *and* $h : R \to R$ *be two utility functions and* $a, b \in R$. *We consider the utility function* $u = ag + bh$. *Then* $E_f(u(A)) = aE_f(g(A)) + bE_f(h(A))$.

Proposition 3.4 (Georgescu 2009) *Let* $g : R \to R$ *and* $h : R \to R$ *be two utility functions such that* $g(x) \le h(x)$ *for all* $x \in R$. *Then* $E_f(g(A)) \le E_f(h(A))$.

Corollary 3.5 *Let* $g : R \to R$ *be a utility function.*
(i) If $g(x) \ge 0$ *for all* $x \in R$, *then* $E_f(u(A)) \ge 0$.
(ii) If $g(x) \le 0$ *for all* $x \in R$, *then* $E_f(u(A)) \le 0$.

Corollary 3.6 *Let* $g : R \to R$ *be a utility function and* $a \le b$ *be two real numbers. If* $a \le g(x) \le b$ *for any* $x \in R$, *then* $a \le E_f(g(A)) \le b$.

Proposition 3.7 *Let* A *be a fuzzy number and* $\lambda \in R$. *Then* $Var_f^*(\lambda + A) = Var_f^*(A)$.

The following result establishes an approximation formula of the possibilistic expected utility $E_f(u(A))$.

Proposition 3.8 *If the utility function u is of class C_2 then*

$$E_f(u(A)) \approx u(E_f(A)) + \frac{1}{2}u''(E_f(A))Var_f(A)$$

Proof According to the Taylor approximation formula of order II:

$$u(x) \approx u(E_f(A)) + u'(E_f(A))(x - E_f(A)) + \frac{1}{2}u''(E_f(A))(x - E_f(A))^2$$

Let us consider the following functions:

$$\begin{aligned} g(x) &= x - E_f(A), \quad x \in R \\ h(x) &= (x - E_f(A))^2, \quad x \in R \end{aligned}$$

We remark then g = 1_A − $E_f(A)$. Let us denote $a = u(E_f(A))$, $b = u'(E_f(A))$, $c = \frac{1}{2}u''(E_f(A))$. It follows that $u \approx a + bg + ch$.

By Proposition 3.3 one gets

$$E_f(u(A)) \approx E_f((a + bg + ch)(A)) = a + bE_f(g(A)) + cE_f(h(A))$$

Since g = 1_A − $E_f(A)$) it follows that

$$E_f(g(A)) = E_f(x - E_f(A))(A)) = E_f(A) - E_f(A) = 0.$$

According to Remark 3.2(ii), $E_f(h(A)) = Var_f(A)$ therefore

$$E_f(u(A)) \approx u(E_f(A)) + \frac{1}{2}u''(E_f(A))Var_f(A).$$

□

Remark 3.9 If the integral of Definition 3.1 is not finite then one can define the value of possibilistic utility $E_f(u(A))$ by the right member of the equality of Proposition 3.8.

Example 3.10 Let us consider the triangular fuzzy number A = (a, α, β) defined by

$$A(t) = \begin{cases} 1 - \frac{a-t}{\alpha}, & if\ a - \alpha \le t \le a \\ 1 - \frac{t-a}{\beta}, & if\ a \le t \le a + \beta \\ 0, & otherwise \end{cases}$$

(a ∈ **R** and α, β > 0)

We assume that the weighting function f has the form $f(\gamma) = 2\gamma$, for any $\gamma \in [0, 1]$.

According to Georgescu (2012, p. 11), the level sets of the triangular fuzzy number A = (a, α, β) have the form $[A]^{\gamma} = [a_1(\gamma), a_2(\gamma)]$, where $a_1(\gamma) = a - (1 - \gamma)\alpha$ and $a_2(\gamma) = a + (1 - \gamma)\beta$.

By Georgescu (2012, p. 25, 29), the possibilistic expected value $E_f(A)$ and the possibilistic variance $Var_f^*(A)$ have the following expressions:

$$E_f(A) = a + \frac{\beta - \alpha}{6}, Var_f^*(A) = \frac{\alpha^2 + \beta^2}{36}$$

Applying Proposition 3.8, we find the following approximate value of $E_f(u(A))$:

$$E_f(u(A)) \approx u(a + \frac{\beta - \alpha}{6}) + \frac{\alpha^2 + \beta^2}{72} u''(a + \frac{\beta - \alpha}{6})$$

As regards to the form of the utility function u and the numerical values of a, α, β, we will be able to compute the approximate value of $E_f(u(A))$.

For example, if $u(x) = \ln x$ and A = (4, 2, 1) then $E_f(u(A)) \approx 1.331$.

2.4 Possibilistic Risk Aversion

In this section we will consider an agent faced with a risk situation. The agent is represented by a utility function and the risk is described by a fuzzy number. Using the possibilistic expected utility we will define a risk–averse, a risk-lover and a risk–neutral agent. We will prove that these notions are characterized by the concavity, convexity or linearity of the utility function. We identify an agent with its utility function.

We fix a weighting function and a utility function u of class C^2.

Definition 4.1 The agent u is possibilistic risk–averse if for any wealth level w and for any fuzzy number A the following inequality holds:
(1) $E_f(u(w + A)) \leq u(w + E_f(A))$.

When the opposite inequality holds, the agent u is possibilistic risk–lover, and if (1) becomes equality the agent u is possibilistic risk–neutral.

Lemma 4.2 *The following assertions are equivalent:*
(a) The agent u is risk–averse.
(b) For any wealth level w and any fuzzy number B with $E_f(B) = 0$, *the following inequality holds:*
(2) $E_f(u(w + B)) \leq u(w)$.

Proof $(a) \Rightarrow (b)$ is obvious; $(b) \Rightarrow (a)$: Denoting $B = A - E_f(A)$, we have $E_f(B) = 0$. Applying (2) for this B and for $w + E_f(A)$ instead of w it follows that $E_f(u(w + A)) = E_f(u(w + E_f(A) + B)) \leq u(w + E_f(A))$. □

Proposition 4.3 *The following assertions are equivalent:*
(i) The function u is concave.
(ii) The agent u is risk–averse.

Proof $(i) \Rightarrow (ii)$: Let A be an arbitrary fuzzy number and $m = E_f(A)$. The second order Taylor approximation of $u(w+x)$ around $w+m$ gives us
(3) $u(w+x) = u(w+m) + u'(w+m)(x-m) + {}^1\!/_2 u''(\xi(x))(x-m)^2$, where $\xi(x)$ is a real number between x and m.

By Proposition 3.3,

$$E_f(u(w+A)) = u(w+m) + u'(w+m)E_f(A-m) + {}^1\!/_2 E_f(u''(\xi(A))(A-m)^2)$$

Since $E_f(A-m) = E_f(A) - m = 0$, we obtain
(4) $E_f(u(w+A)) = u(w+m) + {}^1\!/_2 E_f(u''(\xi(A))(A-m)^2)$.

Let $g(x) = u''(\xi(x))(x-m)^2$ for $x \in R$. Since u is concave, we have $u''(\xi(x)) \leq 0$, therefore $g(x) \leq 0$ for any $x \in R$. Applying Corollary 3.5(ii) it follows that $E_f(u''(\xi(A))(A-m)^2) = E_f(g(A)) \leq 0$.

Then, by (4), $E_f(u(w+A)) \leq u(w+A)$ for any w. Thus the agent u is possibilistic risk–averse.

$(ii) \Rightarrow (i)$: Assume that the function u is not concave. Then there exists $w \in R$ and an interval $I = [w-\delta, w+\delta]$ such that $u'(x) > 0$ for any $x \in I$.

We choose a fuzzy number A such that $supp(A) \subseteq I$. If $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for $\gamma \in [0,1]$, then $[a_1(0), a_2(0)] = supp(A) \subseteq I$. For any $\gamma \in [0,1]$, $[a_1(\gamma), a_2(\gamma)] \subseteq [a_1(0), a_2(0)] \subseteq I$.

We consider the function $g(x) = u''(\xi(x))(x-m)^2$ for any $x \in R$ (associated with the Taylor expansion (3)). Then
(5) $E_f(g(A)) = \int_0^1 [\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u''(\xi(x))(x-m)^2 dx] f(\gamma) d\gamma$.

One notices that $a_1(0) \leq E_f(A) \leq a_2(0)$. Thus $m = E_f(A) \in I$. Accordingly, $u''(\xi(x)) < 0$. Thus $u''(\xi(x))(x-m)^2 < 0$ for any $x \in [a_1(\gamma), a_2(\gamma)] - \{m\}$. It follows that $\int_{a_1(\gamma)}^{a_2(\gamma)} u''(\xi(x))(x-m)^2 dx > 0$ for any $\gamma \in [0,1]$. Since $a_2(\gamma) - a_1(\gamma) > 0$ for any $\gamma \in [0,1]$, it follows that $\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u''(\xi(x))(x-m)^2 dx > 0$.

Using this inequality and the properties of f from (5) it follows that $E_f(g(A)) > 0$. Now (4) can be written

$E_f(u(w+A)) = u(w+m) + \frac{1}{2} E_f(g(A))$,

thus $E_f(u(w+A)) > u(w+m) + {}^1\!/_2 E_f(g(A))$. Thus $E_f(u(w+A)) > u(w+m)$. Then the agent u is not risk–averse. □

Corollary 4.4 *The following assertions are equivalent:*
(i) The function u is convex.
(ii) The agent u is risk–lover.

Proof It follows from Proposition 4.3 and from the fact that u is convex iff $-u$ is concave. □

Corollary 4.5 *The following assertions are equivalent:*
(i) The function u is linear.
(ii) The agent u is risk–neutral.

Proof From real analysis it is known that u is linear iff u is simultaneously convex and concave. Proposition 4.3 and Corollary 4.4 are applied then. □

By Proposition 1.2 from Eeckhoudt et al. (2005), an agent u is possibilistic risk–averse iff u is concave. Combining this result with Proposition 4.3 it follows:

Proposition 4.6 *Given a utility function u the following assertions are equivalent:*
(i) The agent u is probabilistic risk–averse.
(ii) The agent u is possibilistic risk–averse.

Due to Proposition 4.6, we will use the term risk–averse agent instead of probabilistic or possibilistic risk–averse agent.

2.5 Possibilistic Risk Aversion Indicators

In papers Georgescu (2009) and Georgescu (2010) the study of possibilistic risk aversion started. In Georgescu (2009) a notion of possibilistic risk premium was defined, and in Georgescu (2010) a possibilistic version of Pratt theorem was proved (Pratt 1964).

In this section two more notions of possibilistic risk premium are introduced and they are compared with the one from Georgescu (2009). Approximation formulas are obtained and the Pratt-type theorem from Georgescu (2010) is strengthened. A necessary and sufficient condition for the possibilistic risk aversion to be decreasing in wealth is found.

We fix a weighting function f and an injective utility function u.

Definition 5.1 (Georgescu 2009) Let A be a fuzzy number. We define the possibilistic risk premium $\rho(A, u)$ associated with A and u as the unique solution of the equation

$$E_f(u(A)) = u(E_f(A) - \rho(A, u)) \tag{2.1}$$

Proposition 5.2 (Georgescu 2009) *Assume that u has the class* C^2 *and* $u' > 0$. *Then an approximate solution of* Eq. (2.1) *is*

$$\rho(A, u) \approx -{}^1\!/_2 \frac{u''(E_f(A))}{u'(E_f(A))} Var_f^*(A) \tag{2.2}$$

We recall from Arrow (1965) and Pratt (1964) the definition of the Arrow–Pratt index of the utility function u:

$$r_u(x) = -\frac{u''(x)}{u'(x)} \text{ for } x \in R \tag{2.3}$$

Then (2.2) can be written as:

$$\rho(A,u) \approx {}^1\!/_2 Var_f^*(A) r_u(E_f(A)) \tag{2.4}$$

We define now two more notions of possibilistic risk premium.

Definition 5.3 Let A be a fuzzy number and $x \in R$. We define $\pi(x,A,u)$ as the unique solution of the equation

$$E_f(u(x+A)) = u(x + E_f(A) - \pi(x,A,u)) \tag{2.5}$$

Definition 5.4 Let $y \in R$ and B a fuzzy number such that $E_f(B) = 0$. We define $\pi_1(y,B,u)$ as the unique solution of the equation

$$E_f(u(y+B)) = u(y - \pi_1(y,B,u)) \tag{2.6}$$

$\pi(x,A,u)$ is the possibilistic analogue of the probabilistic risk premium from Pratt (1964), and $\pi_1(y,B,u)$ is the possibilistic analogue of the probabilistic risk premium from Ross (1981).

Next we study the relationship between the three notions of possibilistic risk premium $\pi(x,A,u)$, $\pi_1(y,B,u)$ and $\rho(A,u)$.

Lemma 5.5 *For any $\mu \in R$ we have*

$$\pi(x,A,u) = \pi(x+\mu, A-\mu, u) \tag{2.7}$$

Proof One notices that $E_f(A-\mu) = E_f(A) - \mu$. Therefore applying twice (2.5) one obtains
$u(x + E_f(A) - \pi(x+\mu, A-\mu, u))= u(x+\mu+E_f(A-\mu) - \pi(x+\mu, A-\mu, u)$-
$= E_f(u(x+A)) = u(x+E_f(A) - \pi(x,A,u))$.
Then (2.7) results from u's injectivity. □

Proposition 5.6 *(i) If $y \in R$ and B is a fuzzy number with $E_f(B) = 0$, then $\pi(y,B,u) = \pi_1(y,B,u)$; (ii) If $x \in R$ and A is an arbitrary fuzzy number, then $\pi(x,A,u) = \pi_1(x + E_f(A), A - E_f(A), u)$.*

Proof (i) Since $E_f(B) = 0$ from (2.5) and (2.6) it follows that $u(y - \pi(y,B,u)) = E_f(u(y+B)) = u(y - \pi_1(y,B,u))$ from where, due to u's injectivity, one obtains

$\pi(y, B, u) = \pi_1(y, B, u)$; (ii) One notices that $E_f(A - E_f(A)) = 0$. Thus, by Lemma 5.5 and (i), it follows that

$\pi(x, A, u) = \pi(x + E_f(A), A - E_f(A), u) + \pi_1(x + E_f(A), A - E_f(A), u)$. □

By Proposition 5.6 we will always write $\pi(y, B, u)$ instead of $\pi_1(y, B, u)$.

Proposition 5.7 *Let $x \in R$ and A an arbitrary fuzzy number. Then*
(i) $\rho(A, u) = \pi(0, A, u)$;
(ii) $\pi(x, A, u) = \rho(x + A, u)$.

Proof (i) By applying (2.5) for $x = 0$ and then (2.1) it follows that $u(E_f(A) - \pi(0, A, u)) = E_f(u(A)) = u(E_f(A) - \rho(A, u))$ from where $\rho(A, u) = \pi(0, A, u)$.

(ii) We apply (2.1) and (2.5):

$u(E_f(x + A) - \rho(x + A, u)) = E_f(u(x + A))$
$= u(x + E_f(A) - \pi(x, A, u)) = u(E_f(x + A) - \pi(x, A, u))$ from where $\pi(x, A, u) = \rho(x + A, u)$ follows. □

The relationship between the indicators π, π_1 and ρ established by Propositions 5.6 and 5.7 allows a result obtained for one of them to be able to be transferred to the others. We will exemplify next this idea.

Proposition 5.8 *Let $x \in R$, A a fuzzy number and u a utility function of class C^2 such that $u' > 0$. Then*

$$\pi(x, A, u) \approx {}^1\!/_2\, r_u(x + E_f(A)) Var_f^*(A) \tag{2.8}$$

Proof By Proposition 5.7, $Var_f^*(x + A) = Var_f^*(A)$. Then, applying Propositions 5.2 and 5.7 it follows that $\pi(x, A, u) = \rho(x + A, u) \approx {}^1\!/_2\, Var_f^*(x + A) r_u (E_f(x + A)) = {}^1\!/_2\, Var_f^*(A) r_u(x + E_f(A))$. □

Remark 5.9 If $E_f(A) = 0$, then (2.8) becomes:

$$\pi(x, A, u) \approx {}^1\!/_2\, Var_f^*(A) r_u(x) \tag{2.9}$$

Applying in this case Proposition 2.6, $Var_f^*(A) = {}^1\!/_3 \int_0^1 [a_1^2(\gamma) + a_2^2(\gamma) + a_1(\gamma) a_2(\gamma)] f(\gamma) d\gamma$, thus (2.9) can be written as:

$$\pi(x, A, u) \approx {}^1\!/_6\, r_u(x) \int_0^1 [a_1^2(\gamma) + a_2^2(\gamma) + a_1(\gamma) a_2(\gamma)] f(\gamma) d\gamma \tag{2.10}$$

Example 5.10 Let A be the triangular fuzzy number (a, α, β). According to Example 3.10 *and* (2.4), (2.10) *the following approximation formulas are obtained:*

$$\rho(A,u) \approx \frac{\alpha^2+\beta^2}{72} r_u\left(a+\frac{\beta-\alpha}{6}\right)$$
$$\pi(x,A,u) \approx \frac{\alpha^2+\beta^2}{72} r_u\left(x+a+\frac{\beta-\alpha}{6}\right).$$

Let u_1 and u_2 be two utility functions of class C^2 such that $u_1' > 0$, $u_2' > 0$, $u_1'' < 0$, $u_2'' < 0$. We denote by $r_1(x) = r_{u1}(x)$ and $r_2(x) = r_{u2}(x)$ the Arrow–Pratt indexes of u_1 and u_2.

The following result is a possibilistic version of Pratt theorem (Pratt 1964).

Proposition 5.11 *The following assertions are equivalent:*
(a) For any $x \in R$ and for any fuzzy number A, we have $\pi(x,A,u_1) \geq \pi(x,A,u_2)$.
(b) For any $y \in R$ and for any fuzzy number B with $E_f(B) = 0$, we have $\pi(x,B,u_1) \geq \pi(x,B,u_2)$.
(c) For any fuzzy number A, we have $\rho(A,u_1) \geq \rho(A,u_2)$.
(d) $r_1(x) \geq r_2(x)$ for any $x \in R$.
(e) u_1 is more concave than u_2: there exists a function $\phi : R \to R$ with $\phi' > 0$ and $\phi'' \leq 0$ such that $u_2(x) = \phi(u_1(x))$ for any $x \in R$.

Proof The equivalences $(c) \Leftrightarrow (d) \Leftrightarrow (e)$ were proved in Georgescu (2009) (see also Georgescu (2012), Proposition 4.3.7); $(a) \Leftrightarrow (b)$ follows from Proposition 5.6; and $(a) \Leftrightarrow (c)$ follows from Proposition 5.7. □

Definition 5.12 Consider two agents with the utility functions u_1 and u_2. If the equivalent conditions of Proposition 5.11 are fulfilled, then we say that the agent u_1 is possibilistic more risk–averse than u_2.

Remark 5.13 One notices that the equivalent conditions (d) and (e) from Proposition 5.11 also appear in Pratt theorem from probabilistic risk aversion (see Pratt (1964) or Eeckhoudt et al. (2005), Proposition 1.5, p. 14). Then, by combining Pratt theorem with Proposition 5.11, it follows that u_1 is probabilistic more risk–averse than u_2 iff u_1 is possibilistic more risk–averse than u_2. In this case we say that u_1 is more risk averse than u_2.

Let u be a utility function of class C^2 with $u' > 0$, $u'' < 0$, $u''' > 0$. Then the function $v = -u'$ has the class C^2 and the properties $v' > 0$, $v'' < 0$. Thus u and v are utility functions verifying the hypotheses in which Proposition 5.11 can be applied.

The following result establishes a necessary and sufficient condition for the possibilistic risk premium $\pi(x,A,u)$ to be decreasing in wealth.

Proposition 5.14 *The following assertions are equivalent:*
(i) For any fuzzy number A, the possibilistic risk premium $\pi(x,A,u)$ is decreasing in wealth; $x_1 \leq x_2$ implies that $\pi(x_2,A,u) \leq \pi(x_1,A,u)$.

(ii) For any fuzzy number A with $E_f(A) = 0$, the possibilistic risk premium $\pi(x, A, u)$ is decreasing in wealth.
(iii) v is more concave than u.

Proof $(i) \Leftrightarrow (ii)$: by Proposition 5.6; $(ii) \Leftrightarrow (iii)$: Let A be a fuzzy number with $E_f(A) = 0$. Assume that $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for all $\gamma \in [0, 1]$. By (2.6) we have

$$\begin{aligned} u(x - \pi(x, A, u)) =& E_f(u(x + A)) \\ &= \int_0^1 \left[\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u(x + t)dt \right] f(\gamma)d\gamma. \end{aligned}$$

Deriving with respect to x and taking into account (2.6) applied to v it follows that

$$\begin{aligned} (1 - \pi'(x, A, u))u'(x - \pi(x, A, u)) &= \int_0^1 \left[\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u'(x + t)dt \right] f(\gamma)d\gamma \\ &= -\int_0^1 \left[\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} v(x + t)dt \right] f(\gamma)d\gamma \\ &= -E_f(v(x + A)) = -v(x - \pi(x, A, v)) \end{aligned}$$

From the above equalities it follows that $\pi'(x, A, u)$
$= \dfrac{u'(x - \pi(x, A, u)) + v(x - \pi(x, A, v)}{u'(x - \pi(x, A, u))} = \frac{v(x - \pi(x, A, v)) - v(x - \pi(x, A, u)}{u'(x - \pi(x, A, u))}$.
But $u'(x - \pi(x, A, u)) > 0$ and v and is strictly increasing. Thus the following assertions are equivalent:

- $\pi(x, A, u)$ is decreasing in x;
- For all x, $\pi'(x, A, u) \leq 0$;
- For all x, $v(x - \pi(x, A, v)) \leq v(x - \pi(x, A, u))$;
- For all x, $\pi(x, A, v) \geq \pi(x, A, u)$.

Then (ii) is equivalent with condition (b) of Proposition 5.11 stated for the utility functions u and v. According to the equivalence $(b) \Leftrightarrow (c)$ from Proposition 5.11, it follows that $(ii) \Leftrightarrow (iii)$. □

Definition 5.15 (Eeckhoudt et al. 2005) The Arrow–Pratt index of the utility function $v = -u'$:

$$P_u(x) = r_v(x) = -\frac{u'''(x)}{u''(x)} \tag{2.11}$$

is called the degree of absolute prudence of the agent u.

Remark 5.16 According to the equivalence $(d) \Leftrightarrow (c)$ of Proposition 5.11, the three conditions of Proposition 5.14 are equivalent with the following property:

$$\text{For all} \quad x \in R, P_u(x) \geq r_u(x) \tag{2.12}$$

(i.e., prudence is larger than absolute risk aversion).

2.6 Possibilistic Coinsurance Problem

We consider a risk–averse agent with a utility function u and an initial wealth w_0. The agent faces a risk situation where it can lose a part of w_0. We will assume that the loss is described mathematically by the fuzzy number A.

To retrieve a part of the loss the agent will close an insurance contract. By Eeckhoudt et al. (2005, p. 46), an insurance contract consists of a premium P to be paid by the policyholder and an indemnity schedule $I(x)$ representing the amount to be paid by the insurer for a loss of size x.

$I(x)$ will be considered a utility function. The form of the premium P depends on the mathematical modeling of the loss: a random variable in the probabilistic approach (Eeckhoudt et al. 2005) and a fuzzy number in the possibilistic approach. If the loss is a random variable $X \geq 0$, then the mean sum retrieved by the agent will be the probabilistic expected utility $M(I(X))$ and P will be defined with respect to this indicator (Eeckhoudt et al. 2005, p. 49).

In this section we will assume that the loss is a fuzzy number A whose level sets are $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for $\gamma \in [0, 1]$. We will assume that A is not a fuzzy point and $supp(A) \subseteq R_+$, thus $[A]^\gamma \subseteq R_+$ for any $\gamma \in [0, 1]$. We will fix a weighting function $f : [0, 1] \to R$.

We will consider the possibilistic expected utility associated with f, A and I:

$$E_f(I(A)) = \int_0^1 \left[\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} I(x)dx \right] f(\gamma)d\gamma \tag{2.13}$$

In our possibilistic model, $E_f(I(A))$ is the mean sum retrieved by the agent through the insurance contract.

The *possibilistic premium for insurance indemnity* will be

$$P = (1 + \lambda)E_f(I(A)) \tag{2.14}$$

where λ is a loading factor.

Equation (2.14) is inspired from the form of probabilistic premium for insurance indemnity from Eeckhoudt et al. (2005, pp. 49–50), $E_f(I(A))$ replaces the probabilistic actuarial value from Eeckhoudt et al. (2005).

The function I may have various forms. We will assume that $I(x) = \beta x$ for all x. The constant β is called *coinsurance rate* and $1 - \beta$ is called *retention rate* (see Eeckhoudt et al. (2005), p. 49),

For the function $I(x) = \beta x$ one gets the form:

$$E_f(I(A)) = \beta E_f(A) \tag{2.15}$$

Then by (2.13) and (2.15) the possibilistic premium for insurance indemnity will depend on the parameter β and will have the form

$$P(\beta) = (1 - \beta)E_f(I(A)) = \beta(1 - \beta)E_f(A) \tag{2.16}$$

By denoting $P_0 = (1 + \lambda)E_f(A)$ (= the full possibilistic insurance premium), (2.16) becomes

$$P(\beta) = \beta P_0 \tag{2.17}$$

The coinsurance rate β represents the fraction of loss which returns to the policyholder and is a priori fixed by it. Next we study an optimization problem to choose β.

We consider the function g defined as

$$g(x, \beta) = w_0 - \beta P_0 - (1 - \beta)x \tag{2.18}$$

$g(x, \beta)$ represents the sum from w_0 that remains to the agent if the loss is x and if it closed an insurance contract with the premium P and the coinsurance rate β. Since x is one of the values which the fuzzy number A can take, representing the loss, the *final wealth* of the policyholder with loss A and the coinsurance rate β will be the fuzzy number

$$g(A, \beta) = w_0 - \beta P_0 - (1 - \beta)A \tag{2.19}$$

We recall that the utility function u has the class C^2. The agent being risk–averse, by Proposition 4.3 the function u will be concave.

We consider the function

$$h(x, \beta) = u(g(x, \beta)) = u(w_0 - \beta P_0 - (1 - \beta)x) \tag{2.20}$$

and the possibilistic expected utility associated with f, A and h:

$$H(\beta) = E_f(h(A, \beta)) = \int_0^1 \left[\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} h(x, \beta)dx\right] f(\gamma)d\gamma \tag{2.21}$$

Remark 6.1 By Proposition 3.8, we have the following approximation formula of $H(\beta)$:

$$H(\beta) \approx h(E_f(A), \beta) + \frac{1}{2} h''(E_f(a), \beta) Var_f^*(A)$$

In our model, $H(\beta)$ is the *possibilistic expected final wealth* of the policyholder with loss A and coinsurance rate β. The choice of β by the policyholder will maximize $H(\beta)$. This way we reach the following optimization problem:

$$H(\beta^*) = \max_{\beta} H(\beta) \tag{2.22}$$

whose solution β^* is called *optimal coinsurance rate*.

Next we deal with the calculation and the properties of β^*. By (2.20), $H(\beta)$ is written as

$$H(\beta) = \int_0^1 \left[\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u(g(x, \beta)) dx \right] f(\gamma) d\gamma \tag{2.23}$$

Proposition 6.2 *(i) The function H is concave.*
(ii) The necessary and sufficient condition for the real number β^* to be the optimal solution of problem (2.22) *is $H'(\beta^*) = 0$.*

Proof (i) We compute the first derivative of H:
$H'(\beta) = \int_0^1 [\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u'(g(x, \beta)) \frac{\partial g(x,\beta)}{\partial \beta} dx] f(\gamma) d\gamma.$
By (2.19), $\frac{\partial g(x,\beta)}{\partial \beta} = x - P_0$. Thus,

$$H'(\beta) = \int_0^1 \left[\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u'(g(x, \beta))(x - P_0) dx \right] f(\gamma) d\gamma \tag{2.24}$$

Similarly we obtain the second derivative of H:

$$H''(\beta) = \int_0^1 \left[\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u''(g(x, \beta))(x - P_0)^2 dx \right] f(\gamma) d\gamma \tag{2.25}$$

Since u is concave, $u''(g(x, \beta)) \le 0$ for all x and β. Thus for any $\gamma \in [0, 1]$ we have $H''(\beta) = \int_{a_1(\gamma)}^{a_2(\gamma)} u''(g(x, \beta))(x - P_0)^2 dx \le 0$. Since $a_2(\gamma) - a_1(\gamma) \ge 0$ for any $\lambda_1, \ldots, \lambda_n \in R$, from (2.5) it follows $H''(\beta) \le 0$ for all β. Thus, H is concave.
(ii) follows from (i). □

The following result is the possibilistic version of a theorem of Mossin (see Mossin (1968) or Proposition 3.1 of Eeckhoudt et al. (2005, p. 51)).

Proposition 6.3 *Assume that $u' > 0$ and $u'' \le 0$.*

(i) If $\lambda = 0$*, then* $\beta^* = 1$*;*
(ii) If $\lambda > 0$*, then* $\beta^* < 1$.

Proof (i) If $\beta = 1$, then by (2.6) $g(x, 1) = w_0 - P_0$ for all x. By applying (2.25) to this particular case:

$$\begin{aligned} H'(1) &= \int_0^1 \left[\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u'(g(x,\beta))(x - P_0)dx\right] f(\gamma)d\gamma \\ &= u'(w_0 - P_0) \int_0^1 \left[\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} (x - P_0)dx\right] f(\gamma)d\gamma \\ &= u'(w_0 - P_0) \int_0^1 \left[\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} xdx - P_0\right] f(\gamma)d\gamma \\ &= u'(w_0 - P_0)(E_f(A) - P_0). \end{aligned}$$

But $E_f(A) - P_0 = E_f(A) - (1 + \lambda)E_f(A) = -\lambda E_f(A)$. Thus:

$$H'(1) = -\lambda u'(w_0 - P_0)E_f(A) \tag{2.26}$$

For $\lambda = 0$, we obtain $H'(1) = 0$. Thus, by Proposition 6.2(ii), $\beta^* = 1$ is the optimal solution of the problem (2.22).
(ii) One knows that $a_1(0) \leq E_f(A) \leq a_2(0)$. By the hypothesis that A is not a fuzzy point and $supp(A) = [a_1(0), a_2(0)] \in R_+$, thus, $E_f(A) > 0$ follows. Since $u'(w_0 - P_0) > 0$ and $\lambda > 0$, by (2.26) we have $H'(1) = -\lambda u'(w_0 - P_0)E_f(A) < 0$. Assume that the optimal solution β^* of the problem (2.10) verifies that $\beta^* \geq 1$. By Proposition 6.2(i), H is concave. Thus, its derivative H' is decreasing. It follows that $H'(\beta^*) \leq H'(1) < 0$, which contradicts $H'(\beta^*) = 0$. Accordingly, $\beta^* < 1$. □

Proposition 6.4 *If* $\lambda = 0$*, then the possibilistic expected final wealth* $E_f(g(A, \beta))$ *is constant.*

Proof If $\lambda = 0$, then $P_0 = E_f(A)$. Thus, $g(x, \beta) = w_0 - \beta E_f(A) - (1 - \beta)x$ for all x and β. We compute $E_f(g(A, \beta))$ by the formula

$E_f(g(A,\beta)) = \int_0^1 [\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} g(x, \beta))dx] f(\gamma)d\gamma.$

A simple calculation shows that

$\int_{a_1(\gamma)}^{a_2(\gamma)} g(x,\beta))dx = \int_{a_1(\gamma)}^{a_2(\gamma)} (w_0 - \beta E_f(A) - (1 - \beta)x)dx = (w_0 - \beta E_f(A))(a_2(\gamma) - a_1(\gamma)) - \frac{1-\beta}{2}(a_2(\gamma) - a_1(\gamma)).$

Replacing in the expression of $E_f(g(A, \beta))$ one obtains:

$$\begin{aligned}E_f(g(A,\beta)) &= \int_0^1 \left[w_0 - \beta E_f(A) - \frac{1-\beta}{2}(a_1(\gamma)+a_2(\gamma))\right] f(\gamma)d\gamma \\ &= (w_0 - \beta E_f(A)) \int_0^1 f(\gamma)d\gamma - (1-\beta)\int_0^1 \frac{a_1(\gamma)+a_2(\gamma)}{2} f(\gamma)d\gamma \\ &= w_0 - \beta E_f(A) - (1-\beta)E_f(A) = w_0 - E_f(A).\end{aligned}$$

□

For the rest of the section we assume that $f(\gamma) = 2\gamma$ for any $\gamma \in [0,1]$.

Proposition 6.5 *If $u''(x) < 0$ for all x, then $H''(\beta) < 0$ for all β.*

Proof By (2.25) we have

$$H''(\beta) = 2\int_0^1 \left[\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u''(g(x,\beta))(x-P_0)^2 dx\right] f(\gamma)d\gamma \qquad (2.27)$$

A is not a fuzzy point. Thus, $a_2(\gamma) - a_1(\gamma) > 0$ for any $\gamma \in [0,1]$. Also $u''(g(x,\beta)) < 0$ and $(x-P_0)^2 > 0$ for any $x \in [a_1(\gamma), a_2(\gamma)] - \{P_0\}$. Therefore, $\int_{a_1(\gamma)}^{a_2(\gamma)} u''(g(x,\beta))(x-P_0)^2 dx < 0$ for any $\gamma \in [0,1]$. Thus, for any $\gamma \in [0,1]$, $\frac{\gamma}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u''(g(x,\beta))(x-P_0)^2 dx < 0$.

It follows that $H''(\beta) < 0$ for any β. □

We consider two agents with the utility functions u_1 and u_2 such that $u_1' > 0$, $u_2' > 0$, $u_1'' < 0$, $u_2'' < 0$. Let β_1^* and β_2^* be the optimal coinsurance rates associated with u_1 and u_2 (in the sense of problem (2.22)).

Proposition 6.6 *If u_1 is more risk–averse than u_2 then $\beta_1^* \geq \beta_2^*$.*

Proof We consider the possibilistic expected final wealths $H_1(\beta)$ and $H_2(\beta)$ associated with u_1 and u_2:

$$H_1(\beta) = 2\int_0^1 \left[\frac{\gamma}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u_1(g(x,\beta))dx\right] d\gamma;$$

$$H_2(\beta) = 2\int_0^1 \left[\frac{\gamma}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u_2(g(x,\beta))dx\right] d\gamma.$$

Then $H_1'(\beta_1^*) = 0$ and $H_2'(\beta_2^*) = 0$.

If $\lambda = 0$, then, by Proposition 6.3(i), $\beta_1^* = \beta_2^* = 1$ and the assertion is verified.

Assume $\lambda > 0$. Using condition (c) of Proposition 5.11 and judging the same way as in the proof of Proposition 3.2 of Eeckhoudt et al. (2005) one proves that for any x:

$(x-P_0)u_2'(w_0 - (1-\beta_1^*)x - \beta_1^* P_0) \leq (x-P_0)u_1'(w_0 - (1-\beta_1^*)x - \beta_1^* P_0)$, which by (2.18) can be written as $(x-P_0)u_2'(g(x,\beta_1^*)) \leq (x-P_0)u_1'(g(x,\beta_1^*))$.

Then for any $\gamma \in [0, 1]$:
$\int_{a_1(\gamma)}^{a_2(\gamma)} (x - P_0)u_2'(g(x, \beta_1^*))dx \leq \int_{a_1(\gamma)}^{a_2(\gamma)} (x - P_0)u_1'(g(x, \beta_1^*))dx$.

Taking into account that $a_2(\gamma) - a_1(\gamma) > 0$ for any $\gamma \in [0, 1]$, it follows, by (2.24), that
$H_2'(\beta_1^*) = 2\int_0^1 [\frac{\gamma}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} (x - P_0)u_2'(g(x, \beta_1^*))dx]d\gamma \leq 2\int_0^1 [\frac{\gamma}{a_2(\gamma)-a_1(\gamma)}$ $\int_{a_1(\gamma)}^{a_2(\gamma)} (x - P_0)u_1'(g(x, \beta_1^*))dx]d\gamma = H_1'(\beta_1^*) = 0$.

By hypothesis $u_2'' < 0$, thus by Proposition 6.5, $H_2''(\beta) < 0$ for all β. Therefore, H_2' is strictly decreasing. Thus, from $H_2'(\beta_1^*) \leq 0 = H_2'(\beta_2^*)$, $\beta_2^* \leq \beta_1^*$ follows. □

2.7 Static Portfolio Choices: A Possibilistic Model

By Eeckhoudt et al. (2005, p. 65), we consider an agent with a wealth w_0, which it invests in a risk–free asset and in a risky asset. In the probabilistic approach of Eeckhoudt et al. (2005), the return of the risky asset is a random variable. In this section we will study a model in which the return of the risky asset is a fuzzy number.

Let r be the risk–free return of the first asset and x the value of the return of the risky asset. The agent invests the sum α in the risky asset and $w_0 - \alpha$ in the risk–free asset. Then the value of the portfolio $(w_0 - \alpha,\ \alpha)$ at the end of the considered period is by Eeckhoudt et al. (2005, p. 66): $(w_0 - \alpha)(1 + r) + \alpha(1 + x) = w_0(1 + r) + \alpha(x - r) = w + \alpha(x - r)$, where $w = w_0(1 + r)$ is the future wealth obtained with risk–free strategy.

In the probabilistic model of Eeckhoudt et al. (2005), x is the value of a random variable. In the possibilistic model, which we will develop, x will be the value of a fuzzy number A.

We consider the function

$$g(\alpha, w, x) = w + \alpha(x - r) \tag{2.28}$$

If the fuzzy number A is the return of the risky asset, then the fuzzy number $g(\alpha, w, A) = w + \alpha(A - r)$ is the value of the portfolio at the end of the period.

We fix a weighting function $f : [0, 1] \to R$. Assume that the agent has an increasing and concave utility function u of class C^2. Also, we assume that $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for $\gamma \in [0, 1]$. We consider the function

$$h(\alpha, w, x) = u(g(\alpha, w, x)) \tag{2.29}$$

and the possibilistic expected utility associated with f, A and h: $K(\alpha, w) = E_f(h(\alpha, w, A)) = \int_0^1 \frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} h(\alpha, w, x)dx]f(\gamma)d\gamma$.

By (2.2) we have

$$K(\alpha, w) = \int_0^1 \left[\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u(g(\alpha, w, x))dx\right] f(\gamma)d\gamma \tag{2.30}$$

Remark 7.1 By Proposition 3.9, we have the following approximation formula for $K(\alpha, w)$:

$$K(\alpha, w) \approx h(\alpha, w, E_f(A)) + \frac{1}{2}\frac{\partial^2 h(\alpha, w, E_f(A))}{\partial^2 x} \cdot Var_f^*(A)$$

The investor's problem is to choose a value α^* such that

$$K(\alpha^*, w) = \max_{\alpha} K(\alpha, w) \tag{2.31}$$

We prove next some properties of K and the optimal solution α^*.

Proposition 7.2 *(i) The function $K(\alpha, w)$ is concave in α;*

(ii) The necessary and sufficient condition for the real number α^* to be the optimal solution of (2.31) *is* $\frac{\partial K(\alpha^*, w)}{\partial \alpha} = 0$.

Proof (i) From (2.30) it follows that
$\frac{\partial K(\alpha., w)}{\partial \alpha} = \int_0^1 [\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u'(g(\alpha, w, x)) \frac{\partial g(\alpha, w, x)}{\partial \alpha} dx] f(\gamma)d\gamma.$
But $\frac{\partial g(\alpha, w, x)}{\partial \alpha} = x - r$. Thus,

$$\frac{\partial K(\alpha, w)}{\partial \alpha} = \int_0^1 \left[\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u'(g(\alpha, w, x))(x - r)dx\right] f(\gamma)d\gamma \tag{2.32}$$

From (2.5) one obtains

$$\frac{\partial^2 K(\alpha, w)}{\partial \alpha^2} \tag{2.33}$$

$$= \int_0^1 \left[\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u''(g(\alpha, w, x))(x - r)^2 dx\right] f(\gamma)d\gamma$$

u is concave and $u''(g(\alpha, w, x)) \leq 0$. Thus, for any $\gamma \in [0, 1]$, we have $\int_{a_1(\gamma)}^{a_2(\gamma)} u''(g(\alpha, w, x))(x - r)^2 dx \leq 0$.

Then from (2.33) it follows that $\frac{\partial^2 K(\alpha, w)}{\partial \alpha^2} \leq 0$ for any α. Thus, $K(\alpha, w)$ is concave in α.
(ii) follows from (i). □

Proposition 7.3 *Assume that $u' > 0$ and $u'' < 0$.*

(i) If $E_f(A) = r$, *then* $\alpha^* = 0$;
(ii) If $E_f(A) > r$, *then* $\alpha^* > 0$.

Proof We notice that $g(0, w, x) = w$, thus making $\alpha = 0$ in (2.32) it follows that

$$\frac{\partial K(\alpha, w)}{\partial \alpha} \approx u'(w) \int_0^1 \left[\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} (x - r)dx \right] f(\gamma)d\gamma$$
$$= u'(w) \left[\int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma)d\gamma - r \right] = u'(w)[E_f(A) - r].$$

If $E_f(A) = r$, then $\frac{\partial K(0,w)}{\partial \alpha} = 0$. Thus, by Proposition 7.2(ii), $\alpha^* = 0$ is the optimal solution of (2.31).

(ii) Assume by absurdum that $\alpha^* \leq 0$. If $r < E_f(A)$, then $\frac{\partial K(0,w)}{\partial \alpha} = u'(w)(E_f(A) - r) > 0$, since $u'(w) > 0$. Then from $\alpha^* \leq 0$ it follows that $0 = \frac{\partial K(\alpha^*,w)}{\partial \alpha} > \frac{\partial K(0,w)}{\partial \alpha} > 0$.

The obtained contradiction shows that $\alpha^* = 0$. □

Proposition 7.4 *An approximate value of the optimal solution of* (2.31) *is*

$$\alpha^* \approx -\frac{u'(w)}{u''(w)} \frac{E_f(A) - r}{Var_f^*(A) + (E_f(A) - r)^2} \tag{2.34}$$

Proof We use the first–order Taylor approximation of $u'(w + \alpha(x - r))$ around w:

$$u'(g(\alpha, w, x)) = u'(w + \alpha(x - r)) \approx u'(w) + \alpha(x - r)u''(w) \tag{2.35}$$

Replacing $u'(g(\alpha, w, x))$ in (2.32) with the approximate value from (2.35) it follows that

$\frac{\partial K(\alpha,w)}{\partial \alpha} \approx \int_0^1 [\frac{1}{a_2(\gamma)-a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} (u'(w) + \alpha(x - r)u''(w)(x - r))(x - r)dx] f(\gamma)d\gamma.$

We write this relation under the form:

$$\frac{\partial K(\alpha, w)}{\partial \alpha} \approx u'(w)I_1 + \alpha u''(w)I_2, \tag{2.36}$$

where

$$I_1 = \int_0^1 \left[\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} (x - r)dx \right] f(\gamma)d\gamma; \quad \textit{and}$$
$$I_2 = \int_0^1 \left[\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} (x - r)^2 dx \right] f(\gamma)d\gamma$$

We compute I_1:

$I_1 = {}^1\!/_2 \int_0^1 [a_1(\gamma) + a_2(\gamma) - 2r]f(\gamma)d\gamma = \int_0^1 \frac{a_1(\gamma)+a_2(\gamma)}{2} f(\gamma)d\gamma - r = E_f(A) - r.$

We compute I_2:

$I_2 = {}^1\!/_3 \int_0^1 [(a_1(\gamma) - r)^2 + (a_2(\gamma) - r)^2 + (a_1(\gamma) - r)(a_2(\gamma) - r)]f(\gamma)d\gamma.$

We notice that $(a_1(\gamma) - r)^2 + (a_2(\gamma) - r)^2 + (a_1(\gamma) - r)(a_2(\gamma) - r) = a_1^2(\gamma) + a_2^2(\gamma) + a_1(\gamma)a_2(\gamma) - 3r(a_1(\gamma) + a_2(\gamma)) + 3r^2$. Thus,

$I_2 = {}^1\!/_3 \int_0^1 [a_1^2(\gamma) + a_2^2(\gamma) + a_1(\gamma)a_2(\gamma)]f(\gamma)dv - 2r \int_0^1 \frac{a_1(\gamma)+a_2(\gamma)}{2} f(\gamma)d\gamma + r^2$ and by Proposition 2.6:

$I_2 = Var_f^*(A) + E_f^2(A) - 2rE_f(A) + r^2 = Var_f^*(A) + (E_f(A) - r)^2.$

Introducing the above values of I_1 and I_2 in (2.36), we find that $\frac{\partial K(\alpha,w)}{\partial\alpha} \approx u'(w)(E_f(A) - r) + \alpha u''(w)[Var_f^*(A) + (E_f(A) - r)^2]$.

An approximate value of the equation $\frac{\partial K(\alpha^*,w)}{\partial\alpha} = 0$ will be obtained by equaling with 0 the right hand side member of the previous equation. It follows that $\alpha^* \approx -\frac{u'(w)}{u''(w)} \frac{E_f(A)-r}{Var_f^*(A)+(E_f(A)-r)^2}$. □

Example 7.5 Let us assume that the return of the risky asset is the triangular fuzzy number A = (a, α, β) and the weighting function f has the form $f(\gamma) = 2\gamma$, for $\gamma \in [0, 1]$.

Therefore $E_f(A) = a + \frac{\beta-\alpha}{6}$ and $Var_f^*(A) = \frac{\alpha^2+\beta^2}{36}$, hence by Proposition 7.4, the agent will invest in the risky asset the amount

$$\alpha^* \approx -\frac{u'(w)}{u''(w)} \cdot \frac{a + \frac{\beta-\alpha}{6} - r}{\frac{\alpha^2+\beta^2}{36} + (a + \frac{\beta-\alpha}{6} - r)^2}$$

If a is exactly the risk-free return r (a = r) we obtain :

$\alpha^* \approx -3\frac{u'(w)}{u''(w)} \cdot \frac{\beta-\alpha}{\alpha^2+\beta^2+\alpha\beta}.$

2.8 Conclusions

This work fits in a recent research direction in which risk is treated by the possibility theory (Carlsson and Fullér 2011; Georgescu 2009, 2012; Georgescu 2010; Georgescu and Kinnunen 2012).

First the theme of possibilistic risk aversion, whose study began in Georgescu (2009), (2010), (2012), Georgescu and Kinnunen (2012) is deepened. This will provide a framework for the two risk management models of the chapter. Both models are about a risk–averse agent in front of a situation of uncertainty.

In the first case we deal with an insurance contract the agent closes to recover a part of the loss due to the risk situation. In the second case we develop a two-agent investment model in which the risk has a possibilistic representation.

Both models lead to optimization problems: in the first one the optimal coinsurance rate (in the possibilistic sense) should be achieved, and in the second one the optimal investment problem should be found. The main results of the chapter focus on the optimal solutions of the problems: existence, properties, calculation, behaviour towards risk aversion.

We present now some directions for further research.

1. A rich literature was dedicated to probabilistic models of multidimensional risk (see the paper Jouini et al. (2013) for a survey of such models). In Georgescu (2012) a possibilistic model for the situations with many risk parameters was proposed. This model studies the risk aversion of an agent in the face of a multidimensional possibilistic risk where the components are fuzzy numbers. An open problem is to extend the models of Sects. 2.6 and 2.7 (coinsurance problem and investment portfolio problem) to multidimensional possibilistic risk.

2. Credibility theory invented by Liu and Liu in (2002) is another way to describe phenomena with incomplete information. A complete expose of this field can be found in the monograph (Liu 2007). The paper Georgescu and Kinnunen (in press) proposes a risk aversion approach by credibility theory. In particular, a risk-prudent agent in credibilistic sense is defined and a credibilistic Pratt-type theorem is proved. The treatment of coinsurance problem and investment portfolio problem in the framework of such credibilistic models may be a topic for further research.

3. In paper Wu et al. (2014) various Principal—Agent Problems are studied by credibility theory. Among others, necessary and sufficient conditions are established for the optimal solution when the principal is risk-averse or risk-neutral (in a credibilistic sense). To our knowledge, an approach of the Principal—Agent Problem by possibility theory has not been discussed yet. In case of such approach the results of the chapter would be certainly useful.

4. In Georgescu and Kinnunen (2011) and Georgescu (2012) a risk aversion model for mixed parameter situations was considered: some parameters are modeled by random variables and others by fuzzy numbers. An open problem is to define a risk-averse, risk-lover or risk-neutral agent in the context of such mixed models. It would be interesting coinsurance problem and investment portfolio problem to be tackled for mixed parameter problems.

5. The mixed parameter risk models from Georgescu and Kinnunen (2011), Georgescu (2012) combine probabilistic and possibilistic risk modeling. We can figure out situations with three types of risk parameters: some probabilistically modeled by random variables, some by fuzzy numbers and others by credibilistic distributions. Can the concepts and results of this chapter be generalized to this tridimensional context? Can concrete risk situations be found for which such hybrid models offer an appropriate modeling?

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