Chapter 2
Propositional Annotated Logics $P\tau$

Abstract This chapter introduces the propositional annotated logics $P\tau$. We present a Hilbert style axiomatization of $P\tau$ and their semantics. We show some formal results including completeness.

2.1 Language

Here, we start by presenting the *language* of the propositional annotated logics $P\tau$ (also denoted $P\mathscr{T}$) following da Costa et al. [66]; also see Abe [1]. We denote by L the language of $P\tau$.

Annotated logics are based on some arbitrary fixed finite lattice called a *lattice of truth-values* denoted by $\tau = \langle | \tau |, \leq, \sim \rangle$, which is the complete lattice with the ordering $\leq$ and the operator $\sim: | \tau | \rightarrow | \tau |$.

Here, $\sim$ gives the "meaning" of atomic-level negation of $P\tau$. We also assume that $\top$ is the top element and $\perp$ is the bottom element, respectively. In addition, we use two lattice-theoretic operations: $\vee$ for the least upper bound and $\wedge$ for the greatest lower bound.[1]

Definition 2.1 (*Symbols*) The symbols of $P\tau$ are defined as follows:

1. Propositional symbols: $p, q, \ldots$ (possibly with subscript)
2. Annotated constants: $\mu, \lambda, \ldots \in | \tau |$
3. Logical connectives: $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), and $\neg$ (negation)
4. Parentheses: (and).

Definition 2.2 (*Formulas*) Formulas are defined as follows:

1. If p is a propositional symbol and $\mu \in | \tau |$ is an annotated constant, then p_μ is a formula called an *annotated atom*.
2. If F is a formula, then $\neg F$ is a formula.

[1] We employ the same symbols for lattice-theoretical operations as the corresponding logical connectives.

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3. If F and G are formulas, then $F \wedge G$, $F \vee G$, $F \to G$ are formulas.
4. If p is a propositional symbol and $\mu \in |\tau|$ is an annotated constant, then a formula of the form $\neg^k p_\mu$ $(k \geq 0)$ is called a *hyper-literal*. A formula which is not a hyper-literal is called a *complex formula*.

Here, some remarks are in order. The annotation is attached only at the atomic level. An annotated atom of the form p_μ can be read "it is believed that p's truth-value is at least μ". In this sense, annotated logics incorporate the feature of many-valued logics.

A hyper-literal is a special kind of formula in annotated logics. In the hyper-literal of the form $\neg^k p_\mu$, $\neg^k$ denotes the k's repetition of $\neg$. More formally, if A is an annotated atom, then $\neg^0 A$ is A, $\neg^1 A$ is $\neg A$, and $\neg^k A$ is $\neg(\neg^{k-1} A)$. The convention is also used for $\sim$.

Next, we define some abbreviations.

Definition 2.3 Let A and B be formulas. Then, we put:

$$A \leftrightarrow B =_{def} (A \to B) \wedge (B \to A)$$
$$\neg_* A =_{def} A \to (A \to A) \wedge \neg(A \to A)$$

Here, $\leftrightarrow$ is called the *equivalence* and $\neg_*$ *strong negation*, respectively.

Observe that strong negation in annotated logics behaves classically in that it has all the properties of classical negation.

2.2 Semantics

A *semantics* specifies the meaning of formulas in a logical system. The semantics for $P\tau$ can be given in various ways. Now, we describe a *model-theoretic semantics* for $P\tau$.

Let $\mathbf{P}$ be the set of propositional variables. An *interpretation* I is a function $I : \mathbf{P} \to \tau$. To each interpretation I, we associate a *valuation* $v_I : \mathbf{F} \to \mathbf{2}$, where $\mathbf{F}$ is a set of all formulas and $\mathbf{2} = \{0, 1\}$ is the set of truth-values. Henceforth, the subscript is suppressed when the context is clear.

Definition 2.4 (*Valuation*) A valuation v is defined as follows:
If p_λ is an annotated atom, then

$$v(p_\lambda) = 1 \text{ iff } I(p) \geq \lambda,$$
$$v(p_\lambda) = 0 \text{ otherwise},$$
$$v(\neg^k p_\lambda) = v(\neg^{k-1} p_{\sim\lambda}), \text{ where } k \geq 1.$$

If A and B are formulas, then

$$v(A \wedge B) = 1 \text{ iff } v(A) = v(B)) = 1,$$
$$v(A \vee B) = 0 \text{ iff } v(A) = v(B) = 0,$$
$$v(A \to B) = 0 \text{ iff } v(A) = 1 \text{ and } v(B) = 0.$$

If A is a complex formula, then

$$v(\neg A) = 1 - v(A).$$

We say that the valuation v *satisfies* the formula A if $v(A) = 1$ and that v *falsifies* A if $v(A) = 0$. For the valuation v, we can obtain the following lemmas.

Lemma 2.1 *Let p be a propositional variable and $\mu \in |\tau|$ $(k \geq 0)$, then we have:*

$$v(\neg^k p_\mu) = v(p_{\sim^k \mu}).$$

Proof Immediate from the definition of the valuation of hyper-literal.

Lemma 2.2 *Let p be a propositional variable, then we have:*

$$v(p_\perp) = 1$$

Proof Since $\perp$ is the bottom element, it can be derived by the definition of $v(p_\lambda)$.

Lemma 2.3 *For any complex formula A and B and any formula F, the valuation v satisfies the following:*

1. $v(A \leftrightarrow B) = 1$ iff $v(A) = v(B)$
2. $v((A \rightarrow A) \wedge \neg(A \rightarrow A)) = 0$
3. $v(\neg_* A) = 1 - v(A)$
4. $v(\neg F \leftrightarrow \neg_* F) = 1$

Proof (1): For ($\Rightarrow$), suppose $v(A \leftrightarrow B) = 1$. Then, we have:

$$v((A \rightarrow B) \wedge (B \rightarrow A)) = 1,$$

namely, both (a) $v(A \rightarrow B) = 1$ and (b) $v(B \rightarrow A) = 1$ hold. Now, assume that $v(A) \neq v(B)$. This means that either the case that $v(A) = 1$ and $v(B) = 0$ or the case that $v(A) = 0$ and $v(B) = 1$. In other words, $v(A \rightarrow B) = 0$ or $v(B \rightarrow A) = 0$. This contradicts the hypothesis that both (a) and (b). Thus, we have $v(A) = v(B)$.

For ($\Leftarrow$), suppose $v(A) = v(B)$. Then, there are two cases (a) $v(A) = v(B) = 1$ and (b) $v(A) = v(B) = 0$. For (a), by definition, we have $v(A \rightarrow B) = v(B \rightarrow A) = 1$. Then, we have $v((A \rightarrow B) \wedge (B \rightarrow A)) = 1$, i.e., $v(A \leftrightarrow B) = 1$. For (b), by definition, we have $v(A \rightarrow B) = v(B \rightarrow A) = 1$. Thus, $v((A \rightarrow B) \wedge (B \rightarrow A)) = v(A \leftrightarrow B) = 1$ follows.

(2): We have that $v(A \rightarrow A) \neq v(\neg(A \rightarrow A))$, because $v(A \rightarrow A) = 1 - v(\neg(A \rightarrow A))$. This leads to the conclusion that $v((A \rightarrow A) \wedge \neg(A \rightarrow A)) = 0$.

(3): We have two cases (a) $v(A) = 0$ and (b) $v(A) = 1$. For (a), $v(A \rightarrow (A \rightarrow A) \wedge \neg(A \rightarrow A)) = 1$. By the definition of strong negation, $v(\neg_* A) = 1$. This implies that $v(\neg_* A) = 1 - v(A)$. For (b), by (2), $v((A \rightarrow A) \wedge \neg(A \rightarrow A)) = 0$ holds. Then, from the valuation of implication, we have that $v(A \rightarrow (A \rightarrow A) \wedge \neg(A \rightarrow A)) = 0$, i.e., $v(\neg_* A) = 0$. We can thus conclude that $v(\neg_* A) = 1 - v(A)$.

(4): Suppose that $v(\neg F) \neq v(\neg_* F)$. Here, $v(F) = 1 - v(\neg_* F)$ from (3). The valuation of $\neg$ gives rise to the fact that $v(F) = 1 - v(\neg F)$. From these facts, $v(\neg F) = v(\neg_* F)$ follows. But, it contradicts the assumption. Therefore, $v(\neg F \leftrightarrow \neg_* F) = 1$ by (1).

We here define the notion of semantic consequence relation denoted by $\models$. Let Γ be a set of formulas and F be a formula. Then, F is a *semantic consequence* of Γ, written $\Gamma \models F$, iff for every v such that $v(A) = 1$ for each $A \in \Gamma$, it is the case that $v(F) = 1$. If $v(A) = 1$ for each $A \in \Gamma$, then v is called a *model* of Γ. If Γ is empty, then $\Gamma \models F$ is simply written as $\models F$ to mean that F is *valid*.

Lemma 2.4 *Let p be a propositional variable and $\mu, \lambda \in \mid \tau \mid$. Then, we have:*

1. $\models p_\perp$
2. $\models p_\mu \to p_\lambda,\ \ \mu \geq \lambda$
3. $\models \neg^k p_\mu \leftrightarrow p_{\sim^k \mu},\ \ k \geq 0$

Proof (1): By the definition of v, we have $I(p) \geq \perp$ for any interpretation I. Therefore, $\models p_\perp$ holds.
(2): Suppose that there exists an interpretation such that $\not\models p_\mu \to p_\lambda$. This implies that $\models p_\mu$ and $\not\models p_\lambda$. So, $I(p) \geq \mu$ and $I(p) \not\geq \lambda$. This contradicts the assumption. Thus, we have that $\models p_\mu \to p_\lambda$, if $\mu \geq \lambda$.
(3): Immediate from Lemma 2.1.

The consequence relation $\models$ satisfies the next property.

Lemma 2.5 *Let A, B be formulas. Then, if $\models A$ and $\models A \to B$ then $\models B$.*

Proof Suppose $v(A) = 1$ and $v(A \to B) = 1$ but $v(B) = 0$. By the definition of $v(A \to B)$, we have $v(A \to B) = 0$ contradicting the assumption.

Lemma 2.6 *Let F be a formula, p a propositional variable, and $(\mu_i)_{i \in J}$ be an annotated constant, where J is an indexed set. Then, if $\models F \to p_\mu$, then $F \to p_{\mu_i}$, where $\mu = \bigvee \mu_i$.*

Proof Suppose that we have the valuation such that $v(F \to p_{\mu_i}) = 1$ and $v(F \to p_\mu) = 0$. Then, $v(F) = 1, v(p_{\mu_i}) = 1$ and $v(\mu) = 0$ hold. The third condition implies that $I(p) \not\geq \mu$. In other words, we have that $I(p) \not\geq \mu_j$ with $j \in J$. But, by the second condition, $I(p) \geq \mu_j$ holds. Since $\mu = \bigvee \mu_j$ holds, we have that $v(F \to p_\mu) = 1$. A contradiction.

As a corollary to Lemma 2.6, we can obtain the following lemma.

Lemma 2.7 $\models p_{\lambda_1} \wedge p_{\lambda_2} \wedge \cdots \wedge p_{\lambda_m} \to p_\lambda$, *where* $\lambda = \bigvee_{i=1}^{m} \lambda_i$.

Next, we discuss some results related to paraconsistency and paracompleteness.

Definition 2.5 (*Complementary property*) A truth-value $\mu \in \tau$ has the *complementary property* if there is a λ such that $\lambda \leq \mu$ and $\sim \lambda \leq \mu$. A set $\tau' \subseteq \tau$ has the *complementary property* iff there is some $\mu \in \tau'$ such that μ has the complementary property.

Definition 2.6 (*Range*) Suppose I is an interpretation of the language L. The *range* of I, denoted $range(I)$, is defined to be $range(I) = \{\mu \mid (\exists A \in B_L) I(A) = \mu\}$, where B_L denotes the set of all ground atoms in L.

For $P\tau$, ground atoms correspond to propositional variables. If the range of the interpretation I satisfies the complementary property, then the following theorem can be established.

Theorem 2.1 *Let I be an interpretation such that $range(I)$ has the complementary property. Then, there is a propositional variable p and $\mu \in |\tau|$ such that*

$$v(p_\mu) = v(\neg p_\mu) = 1.$$

Proof Since $range(I)$ has the complementary property, there is a propositional variable p and a $\delta \in \tau$, satisfying (1) $I(p) = \delta$ and (2) there is a $\gamma \in \tau$ such that $\gamma \leq \delta$ and $\sim\gamma \leq \delta$. By (1), $I(p) \geq \delta$ holds. Thus, we have that $v(p_\gamma) = 1$. Similarly, we have that $v(\neg p_\gamma) = 1$ by (2). From both, we can reach the theorem by Definition 2.5.

Theorem 2.1 states that there is a case in which for some propositional variable it is both true and false, i.e., inconsistent. The fact is closely tied with the notion of paraconsistency.

Definition 2.7 (*¬-inconsistency*) We say that an interpretation I is *¬-inconsistent* iff there is a propositional variable p and an annotated constant $\mu \in |\tau|$ such that $v(p_\mu) = v(\neg p_\mu) = 1$.

Therefore, ¬-inconsistency means that both A and $\neg A$ are simultaneously true for some atomic A. Below, we formally define the concepts of non-triviality, paraconsistency and paracompleteness.

Definition 2.8 (*Non-triviality*) We say that an interpretation I is *non-trivial* iff there is a propositional variable p and an annotated constant $\mu \in |\tau|$ such that $v(p_\mu) = 0$.

By Definition 2.8, we mean that not every atom is valid if an interpretation is non-trivial.

Definition 2.9 (*Paraconsistency*) We say that an interpretation I is *paraconsistent* iff it is both ¬-inconsistent and non-trivial. $P\tau$ is called *paraconsistent* iff there is an interpretation of I of $P\tau$ such that I is paraconsistent.

Definition 2.9 allows the case in which both A and $\neg A$ are true, but some formula B is false in some paraconsistent interpretation I.

Definition 2.10 (*Paracompleteness*) We say that an interpretation I is *paracomplete* iff there is a propositional variable p and an annotated constant $\lambda \in |\tau|$ such that $v(p_\lambda) = v(\neg p_\lambda) = 0$. $P\tau$ is called *paracomplete* iff there is an interpretation I of $P\tau$ such that I is paracomplete.

From Definition 2.10, we can see that in the paracomplete interpretation I, both A and $\neg A$ are false. We say that $P\tau$ is *non-alethic* iff it is both paraconsistent and paracomplete. Intuitively speaking, paraconsistent logic can deal with inconsistent information and paracomplete logic can handle incomplete information.

This means that non-alethic logics like annotated logics can serve as logics for expressing both inconsistent and incomplete information. This is one of the starting points of our study of annotated logics.

As the following Theorems 2.2 and 2.3 indicate, paraconsistency and paracompleteness in $P\tau$ depend on the cardinality of τ.

Theorem 2.2 *$P\tau$ is paraconsistent iff $card(\tau) \geq 2$, where $card(\tau)$ denotes the cardinality (cardinal number) of the set τ.*

Proof For ($\Rightarrow$), let I be a paraconsistent interpretation. Then, there are propositional variables p, q and annotated constants μ, λ such that (1) $v(p_\mu) = v(\neg p_\mu) = 1$ and (2) $v(q_\lambda) = 0$. Since τ is a complete lattice, $card(\mu) \geq 1$. Now, assume that $card(\mu) = 1$, namely τ has one element μ, and $\mu = \bot = \top$. This means that $\bot$ and $\top$ agree in the lattice. Here, the only possible interpretation is to assign $\mu = \bot = \top$ to all propositional variables. It follows that for any propositional variable r, $v(r_\top) = v(r_\bot) = 1$. But, it contradicts the condition (2).

For ($\Leftarrow$), suppose $card(\tau) \geq 2$, with $\bot \neq \top$. Here, we can define the interpretation I such that $I(p) = \top$ and $I(q) = \bot$. Then, $v(p_\top) = 1$ follows. As $\sim\top \leq \top$, we have that $v(p_{\sim\top}) = v(\neg p_\top) = 1$. Since $\bot \leq \top$, we have that $v(q_\top) = 0$. Consequently, I is paraconsistent, and we can conclude that $P\tau$ is paraconsistent.

Theorem 2.3 *$P\tau$ is paracomplete iff $card(\tau) \geq 2$.*

Proof For ($\Rightarrow$), let I be a paracomplete interpretation. Then, there is a propositional variable p and an annotated constant μ such that $v(p_\mu) = v(\neg p_\mu) = 0$. Here, we assume that $card(\tau) = 1$ and set $\mu = \bot = \top$. Thus, $v(p_\bot) = v(p_\top) = 0$ holds. But, the only possible interpretation is $v(p_\bot) = v(p_\top) = 1$ for any propositional variable p. This is a contradiction.

For ($\Leftarrow$), suppose $card(\tau) \geq 2$, with $\bot \neq \top$. Now, assume that $I(p) = \bot$. Then, $v(p_\top) = 0$ since $I(p) = \bot \not\geq \top$. Now, we define the negation operator $\sim$ satisfying $\sim\top = \top$. Therefore, $v(\neg p_\top) = v(p_{\sim\top}) = v(p_\top) = 0$. As a consequence, I is shown to be a paracomplete interpretation. In other words, $P\tau$ is paracomplete.

The above two theorems imply that to formalize a non-alethic logic based on annotated logics we need at least both the top and bottom elements of truth-values. The simplest lattice of truth-values is *FOUR* in Belnap [49, 50], which is shown in Fig. 2.1.

Definition 2.11 (*Theory*) Given an interpretation I, we can define the theory $Th(I)$ associated with I to be a set:

$$Th(I) = Cn(\{p_\mu \mid p \in \mathbf{P} \text{ and } I(p) \geq \mu\}).$$

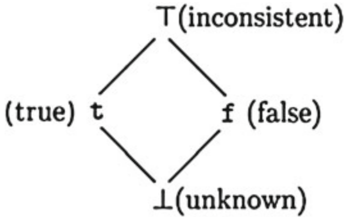

Fig. 2.1 The lattice *FOUR*

Here, Cn is the semantic consequence relation, i.e.,

$$Cn(\Gamma) = \{F \mid F \in \mathbf{F} \text{ and } \Gamma \models F\}.$$

Here, Γ is a set of formulas.

$Th(I)$ can be extended for any set of formulas.

Theorem 2.4 *An interpretation I is $\neg$-inconsistent iff $Th(\Gamma)$ is $\neg$-inconsistent.*

Proof For ($\Rightarrow$), suppose I is $\neg$-inconsistent. Then, there is an interpretation I and a hyper-literal p_μ such that $v(\neg p_\mu) = 1$. By the definition of $Th(I)$, we have that $p_\mu, \neg p_\mu \in Th(I)$. Next, consider a complex formula A for $\neg$-inconsistent interpretation. $v(A) = v(\neg A) = 1$ holds. Because $v(\neg A) = 1 - v(A)$, we have that $v(A) \neq v(\neg A)$. But it contradicts the assumption. Therefore, $Th(I)$ is $\neg$-inconsistent.

For ($\Leftarrow$), suppose $Th(I)$ is $\neg$-inconsistent. Then, $p_\mu, \neg p_\mu \in Th(I)$. It follows that $v(p_\mu) = v(\neg p_\mu) = 1$, concluding that I is $\neg$-inconsistent. Next, for complex formula A, $A, \neg A \in Th(I)$ holds. Then, we can $v(A) = 1$ and $v(\neg A)$ follows since $v(\neg A) = 1$. From these facts, I is $\neg$-inconsistent.

Theorem 2.5 *An interpretation I is paraconsistent iff $Th(I)$ is paraconsistent.*

Proof For ($\Rightarrow$), let I be a paraconsistent interpretation. Then, for some hyper-literals p_μ, q_μ, we have that $v(p_\mu = v(\neg p_\mu) = 1$ and $v(q_\mu) = 0$. Then, $p_\mu, \neg p_\mu \in Th(I)$, but $q_\mu \notin Th(I)$. For complex formulas A, B, we have that $v(A) = v(\neg A) = 1$ and $v(B) = 0$. From this, $A, \neg A \in Th(I)$, but $B \notin Th(I)$. This means that $Th(I)$ is paraconsistent.

For ($\Leftarrow$), $p_\mu, \neg p_\mu \in Th(I)$ but $q_\mu \notin Th(I)$ for some hyper-literals p_μ, q_μ. Then, $v(p_\mu) = v(\neg p_\mu) = 1$ and $v(q_\mu) = 0$ follows. This shows that I is paraconsistent.

The next lemma states that the replacement of equivalent formulas within the scope of $\neg$ does not hold in $P\tau$ as in other paraconsistent logics.

Lemma 2.8 *Let A be any hyper-literal. Then, we have:*

1. $\models A \leftrightarrow ((A \to A) \to A)$
2. $\not\models \neg A \leftrightarrow \neg(((A \to A) \to A))$
3. $\models A \leftrightarrow (A \wedge A)$
4. $\not\models \neg A \leftrightarrow \neg(A \wedge A)$

5. $\models A \leftrightarrow (A \vee A)$
6. $\not\models \neg A \leftrightarrow \neg(A \vee A)$

Proof Since A is a hyper-literal, then A is of the form p_μ, and $\mu \in |\tau|$. To prove these propositions, we consider the cases that $v(A) = 1$ and the case that $v(A) = 0$. Let $A = p_\top$ and define the negation operator $\sim$ satisfying $\sim\top = \top$.
(1): (a) If $v(A) = 1$, then $v((A \to A) \to A) = 1$ as required. (b) if $v(A) = 0$, then $v(A \to A) = 1$. Then, we have that $v((A \to A) \to A) = 0 = v(A)$.
(2) (a) If $v(p_\top) = 1$, then $v(\neg p_\top) = v(p_{\sim\top}) = v(p_\top) = 1$. Here, we have that $v(\neg((p_\top \to p_\top) \to p_\top)) = 1 - v((p_\top \to p_\top) \to p_\top) = 0 \neq v(p_\top)$. (b) If $v(p_\top) = 0$, then $v(\neg p_\top) = 0$. Here, we have that $v(\neg((p_\top \to p_\top) \to p_\top)) = 1 \neq v(p_\top)$.
(3) (a) If $v(A) = 1$, then $v(A \wedge A) = 1$. (b) If $v(A) = 0$, then $v(A \wedge A) = 0$.
(4) (a) If $v(p_\top) = 1$, then $v(\neg p_\top) = 1$. We can see that $v(\neg(p_\top \wedge p_\top)) = 1 - v(p_\top \wedge p_\top) = 0 \neq v(p_\top)$. (b) If $v(p_\top) = 0$, then $v(\neg p_\top) = 0$. Here, $v(\neg(p_\top \wedge p_\top)) = 1 - v(p_\top \wedge p_\top) = 1 \neq v(p_\top)$, as required.

As obvious from the above proofs, (1), (3) and (5) hold for any formula A. But, (2), (4) and (6) cannot be generalized for any A.

By the next theorem, we can find the connection of $P\tau$ and the positive fragment of classical propositional logic C.

Theorem 2.6 *If $F_1, \ldots, F_n$ are complex formulas and $K(A_1, \ldots, A_n)$ is a tautology of C, where $A_1, \ldots, A_n$ are the sole propositional variable occurring in the tautology, then $K(F_1, \ldots, F_n)$ is valid in $P\tau$. Here, $K(F_1, \ldots, F_n)$ is obtained by replacing each occurrence of A_i, $1 \leq i \leq n$, in K by F_i.*

Proof Proved by induction on n. For example, consider the formulas

$$K(p, q) = (p \wedge q) \to (q \wedge p)$$

which is a well-known tautology of C. Let $F_1 = F$, $F_2 = \neg G$ be complex formulas. By definition,

$$K(F, \neg G) = (F \wedge \neg G) \to (\neg G \wedge F)$$

is obtained. It suffices to show that $K(F, \neg G)$ is valid in $P\tau$. In other words, $v(K(F, \neg G)) = 1$ for any v. Suppose that $v(K(F, \neg G)) = 1$. This is equivalent to the following:

$v(F \wedge \neg G) = 0$ or $v(\neg G \wedge F) = 1$
iff ($v(F) = 0$ or $v(\neg G) = 0$) or ($v(\neg G) = 1$ and $v(F) = 1$)
iff ($v(F) = 0$ or $v(G) = 1$) or ($v(G) = 0$ and $v(F) = 1$)

It is easy to check that the last clause is satisfied by any v. Thus, $\models K(F, \neg G)$, that is, $K(F, \neg G)$ is valid in $P\tau$.

Next, we consider the properties of strong negation $\neg_*$.

Theorem 2.7 *Let A, B be any formulas. Then,*

1. $\models (A \to B) \to ((A \to \neg_* B) \to \neg_* A)$
2. $\models A \to (\neg_* A \to B)$
3. $\models A \vee \neg_* A$

Proof For (1), assume that there is a valuation v such that $v(A \to B) = 1$ and $v((A \to \neg_* B) \to \neg_* A) = 0$. From the latter, we have that $v(A \to \neg_* B) = 1$ and $v(\neg_* A) = 0$. Thus, $v(A) = 1$ because $v(A) = 1 - v(\neg_* A)$. From the former, we have that $v(A \to B) = 1$. So $v(A) = 0$ or $v(B) = 1$ holds. But, the claim that $v(A) = 1$ enables us to infer that $v(B) = 1$. As a consequence, we have:

$$v(A) = 1 \text{ and } v(B) = 1$$

Here, the latter also ensures that $v(A \to \neg_* B) = 1$. However, since $v(A) = 1$, we have to obtain $v(\neg_* B) = 1$, i.e. $v(B) = 0$. This induces a contradiction.

For (2), assume that we have a v satisfying that $v(A) = 1$ and $v(\neg_* A \to B) = 0$. The latter gives rise to the condition that $v(\neg_* A) = 1$ and $v(B) = 0$. Here, we have that $v(A) = 0$ from $v(\neg_* A) = 1$. However, this is impossible.

For (3), assume the existence of v such that $v(A \vee \neg_* A) = 0$. This implies that $v(A) = v(\neg_* A) = 0$. However, this is impossible.

Theorem 2.7 tells us that strong negation has all the basic properties of classical negation. Namely, (1) is a principle of *reductio ad abusurdum*, (2) is the related principle of the law of non-contradiction, and (3) is the law of excluded middle. Note that $\neg$ does not satisfy these properties. It is also noticed that for any complex formula $A \models \neg A \leftrightarrow \neg_* A$ but that for any hyper-literal $Q \not\models \neg Q \leftrightarrow \neg_* Q$.

From these observations, $P\tau$ is a paraconsistent and paracomplete logic, but adding strong negation enables us to perform classical reasoning.

2.3 Axiomatization

In this section, we provide an axiomatization of $P\tau$ in the Hilbert style. There are many ways to axiomatize a logical system, one of which is the *Hilbert system*. We discuss other proof systems in Chap. 4.

A Hilbert system can be defined by the set of *axioms* and *rules of inference*. Here, an axiom is a formula to be postulated as valid, and rules of inference specify how to prove a formula.

We are now ready to give a Hilbert style axiomatization of $P\tau$, called $\mathscr{A}\tau$. Let A, B, C be arbitrary formulas, F, G be complex formulas, p be a propositional variable, and λ, μ, λ_i be annotated constant. Then, the postulates are as follows (cf. Abe [1]):

Postulates for $\mathscr{A}\tau$

$(\to_1)$ $(A \to (B \to A)$
$(\to_2)$ $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

$(\rightarrow_3)$ $((A \rightarrow B) \rightarrow A) \rightarrow A$
$(\rightarrow_4)$ $A, A \rightarrow B/B$
$(\wedge_1)$ $(A \wedge B) \rightarrow A$
$(\wedge_2)$ $(A \wedge B) \rightarrow B$
$(\wedge_3)$ $A \rightarrow (B \rightarrow (A \wedge B))$
$(\vee_1)$ $A \rightarrow (A \vee B)$
$(\vee_2)$ $B \rightarrow (A \vee B)$
$(\vee_3)$ $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$
$(\neg_1)$ $(F \rightarrow G) \rightarrow ((F \rightarrow \neg G) \rightarrow \neg F)$
$(\neg_2)$ $F \rightarrow (\neg F \rightarrow A)$
$(\neg_3)$ $F \vee \neg F$
(τ_1) $p_\perp$
(τ_2) $\neg^k p_\lambda \leftrightarrow \neg^{k-1} p_{\sim\lambda}$
(τ_3) $p_\lambda \rightarrow p_\mu$, where $\lambda \geq \mu$
(τ_4) $p_{\lambda_1} \wedge p_{\lambda_2} \wedge \cdots \wedge p_{\lambda_m} \rightarrow p_\lambda$, where $\lambda = \bigvee_{i=1}^{m} \lambda_i$

Here, except $(\rightarrow_4)$, these postulates are axioms. $(\rightarrow_4)$ is a rule of inferences called *modus ponens* (MP).

In da Costa et al. [66], a different axiomatization is given, but it is essentially the same as ours. There, the postulates for implication are different. Namely, although $(\rightarrow_1)$ and $(\rightarrow_3)$ are the same (although the naming differs), the remaining axiom is:

$$(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$$

It is well known that there are many ways to axiomatize the implicational fragment of classical logic C. In the absence of negation, we need the so-called *Pierce's law* $(\rightarrow_3)$ for C.

In $(\neg_1)$, $(\neg_2)$, $(\neg_3)$, F and G are complex formulas. In general, without this restriction on F and G, these are not sound rules due to the fact that they are not admitted in annotated logics.

da Costa et al. [66] fuses (τ_1) and (τ_2) as the single axiom in conjunctive form. But, we separate it into two axioms for our purposes. Also there is a difference in the final axiom. They present it for infinite lattices as

$$A \rightarrow p_{\lambda_j} \text{ for every } j \in J\text{, then } A \rightarrow p_\lambda\text{, where } \lambda = \bigvee_{j \in J} \lambda_j.$$

If τ is a finite lattice, this is equivalent to the form of (τ_2).

As usual, we can define a *syntactic consequence relation* in $P\tau$. Let Γ be a set of formulas and G be a formula. Then, G is a syntactic consequence of Γ, written $\Gamma \vdash G$, iff there is a finite sequence of formulas $F_1, F_2, \ldots, F_n$, where F_i belongs to Γ, or F_i is an axiom $(1 \leq i \leq n)$, or F_j is an immediate consequence of the previous two formulas by $(\rightarrow_4)$. This definition can extend for the transfinite case in which n is an ordinal number. If $\Gamma = \emptyset$, i.e. $\vdash G$, G is a *theorem* of $P\tau$.

Let Γ, Δ be sets of formulas and A, B be formulas. Then, the consequence relation $\vdash$ satisfies the following conditions.

1. if $\Gamma \vdash A$ and $\Gamma \subset \Delta$ then $\Delta \vdash A$.
2. if $\Gamma \vdash A$ and $\Delta, A \vdash B$ then $\Gamma, \Delta \vdash B$.
3. if $\Gamma \vdash A$, then there is a finite subset $\Delta \subset \Gamma$ such that $\Delta \vdash A$.

In the Hilbert system above, the so-called *deduction theorem* holds.

Theorem 2.8 (Deduction theorem) *Let Γ be a set of formulas and A, B be formulas. Then, we have:*

$$\Gamma, A \vdash B \ \Rightarrow\ \Gamma \vdash A \rightarrow B.$$

Proof See Kleene [103].

The following theorem shows some theorems related to strong negation.

Theorem 2.9 *Let A and B be any formula. Then,*

1. $\vdash A \vee \neg_* A$
2. $\vdash A \rightarrow (\neg_* A \rightarrow B)$
3. $\vdash (A \rightarrow B) \rightarrow ((A \rightarrow \neg_* B) \rightarrow \neg_* A)$

Proof For (1), we use the theorem $A \vee (A \rightarrow B)$ of classical propositional logic. If we set $B = (A \rightarrow A) \wedge \neg(A \rightarrow A)$, then we obtain the following:

$$\vdash A \vee \neg_* A$$

by the definition of strong negation, as required.

For (2), the following two hold:

(a) $(A \rightarrow A) \wedge \neg(A \rightarrow A) \vdash B$
(b) $A, A \rightarrow (A \rightarrow A) \wedge \neg(A \rightarrow A) \vdash (A \rightarrow A) \wedge \neg(A \rightarrow A)$

By the property of $\vdash$, we have (c).

(c) $A, A \rightarrow (A \rightarrow A) \wedge \neg(A \rightarrow A) \vdash B$

By applying the deduction theorem to (c) twice, (d) is obtained.

(d) $\vdash A \rightarrow ((A \rightarrow (A \rightarrow A) \wedge \neg(A \rightarrow A)) \rightarrow B)$

By the definition of $\neg_*$, (e) follows.

(e) $\vdash A \rightarrow (\neg_* A \rightarrow B)$.

For (3), we use the theorem:

$$(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)).$$

Now, set $C = (A \rightarrow A) \wedge \neg(A \rightarrow A)$. Then, we have (a):

(a) $\vdash (A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow (A \rightarrow A) \wedge \neg(A \rightarrow A)) \rightarrow (A \rightarrow (A \rightarrow A) \wedge \neg(A \rightarrow A)))$

By the definition of $\neg_*$, we can reach the following:

(b) $\vdash (A \to B) \to ((A \to \neg_* B) \to \neg_* A)$

From Theorems 2.9 and 2.10 follows.

Theorem 2.10 *For arbitrary formulas A and B, the following hold:*

1. $\vdash \neg_*(A \wedge \neg_* A)$
2. $\vdash A \leftrightarrow \neg_*\neg_* A$
3. $\vdash (A \wedge B) \leftrightarrow \neg_*(\neg_* A \vee \neg_* B)$
4. $\vdash (A \to B) \leftrightarrow (\neg_* A \vee B)$
5. $\vdash (A \vee B) \leftrightarrow \neg_*(\neg_* A \wedge \neg_* B)$

Theorem 2.10 implies that by using strong negation and a logical connective other logical connectives can be defined as in classical logic. If $\tau = \{t, f\}$, with its operations appropriately defined, we can obtain classical propositional logic in which $\neg_*$ is classical negation.

2.4 Formal Results

In this section, we provide some formal results of $P\tau$ including completeness and decidability.

Lemma 2.9 *Let p be a propositional variable and $\mu, \lambda, \theta \in |\tau|$. Then, the following hold:*

1. $\vdash p_{\lambda\vee\mu} \to p_\lambda$
2. $\vdash p_{\lambda\vee\mu} \to p_\mu$
3. $\lambda \geq \mu$ *and* $\lambda \geq \theta \Rightarrow \vdash p_\lambda \to p_{\mu\vee\theta}$
4. $\vdash p_\mu \to p_{\mu\wedge\theta}$.
5. $\vdash p_\theta \to p_{\mu\wedge\theta}$.
6. $\lambda \leq \mu$ *and* $\lambda \leq \theta \Rightarrow \vdash p_{\mu\wedge\theta}$
7. $\vdash p_\mu \leftrightarrow p_{\mu\vee\mu}$, $\vdash p_\mu \leftrightarrow p_{\mu\wedge\mu}$
8. $\vdash p_{\mu\vee\lambda} \leftrightarrow p_{\lambda\vee\mu}$, $\vdash p_{\mu\wedge\lambda} \leftrightarrow p_{\lambda\wedge\mu}$
9. $\vdash p_{(\mu\vee\lambda)\vee\theta}\vee \to p_{\mu\vee(\lambda\vee\theta)}$, $\vdash p_{(\mu\wedge\lambda)\wedge\theta}\vee \to p_{\mu\wedge(\lambda\wedge\theta)}$
10. $p_{(\mu\vee\lambda)\wedge\mu} \to p_\mu$, $p_{(\mu\wedge\lambda)\vee\mu} \to p_\mu$
11. $\lambda \leq \mu \Rightarrow \vdash p_{\lambda\vee\mu} \to p_\mu$
12. $\lambda \vee \mu = \mu \Rightarrow \vdash p_\mu \to p_\lambda$
13. $\mu \geq \lambda \Rightarrow \forall\theta \in |\tau|$ $(\vdash p_{\mu\vee\theta} \to p_{\lambda\vee\theta}$ *and* $\vdash p_{\mu\wedge\theta} \to p_{\lambda\wedge\theta})$
14. $\mu \geq \lambda$ *and* $\theta \geq \varphi \Rightarrow \vdash p_{\mu\vee\theta} \to p_{\lambda\vee\varphi}$ *and* $p_{\mu\wedge\theta} \to p_{\lambda\wedge\varphi}$

15. $\vdash p_{\mu\wedge(\lambda\vee\theta)} \rightarrow p_{(\mu\wedge\lambda)\vee(\mu\wedge\theta)}$, $\vdash p_{\mu\vee(\lambda\wedge\theta)} \rightarrow p_{(\mu\vee\lambda)\wedge(\mu\vee\theta)}$
16. $\vdash p_\mu \wedge p_\lambda \leftrightarrow p_{\mu\wedge\lambda}$
17. $\vdash p_{\mu\vee\lambda} \rightarrow p_\mu \vee p_\lambda$

Proof Immediate from the properties of τ.

Example 2.1 Consider the complete lattice $\tau = N \cup \omega$, where N is the set of natural numbers. The ordering on τ is the usual ordering on ordinals, restricted to the set τ. Consider the set $\Gamma = \{p_0, p_1, p_2, \ldots\}$, where $p_\omega \notin \Gamma$. It is clear that $\Gamma \vdash p_\omega$, but an infinitary deduction is required to establish this.

Definition 2.12 $\overline{\Delta} = \{A \in \mathbf{F} \mid \Delta \vdash A\}$.

Definition 2.13 Δ is said to be *trivial* iff $\overline{\Delta} = \mathbf{F}$ (i.e., every formula in our language is a syntactic consequence of Δ); otherwise, Δ is said to be *non-trivial*. Δ is said to be *inconsistent* iff there is some formula A such that $\Delta \vdash A$ and $\Delta \vdash \neg A$; otherwise, Δ is *consistent*.

From the definition of triviality, the next theorem follows:

Theorem 2.11 *Δ is trivial iff $\Delta \vdash A \wedge \neg A$ (or $\Delta \vdash A$ and $\Delta \vdash \neg_* A$) for some formula A.*

Proof Obvious from the axiom $(\neg_2)$ and Theorem 2.9(2).

Theorem 2.12 *Let Γ be a set of formulas, A, B be any formulas, and F be any complex formula. Then, the following hold.*

1. $\Gamma \vdash A$ *and* $\Gamma \vdash A \rightarrow B \;\Rightarrow\; \Gamma \vdash B$
2. $A \wedge B \vdash A$
3. $A \wedge B \vdash B$
4. $A, B \vdash A \wedge B$
5. $A \vdash A \vee B$
6. $B \vdash A \vee B$
7. $\Gamma, A \vdash C$ *and* $\Gamma, B \vdash C \;\Rightarrow\; \Gamma, A \vee B \vdash C$
8. $\vdash F \leftrightarrow \neg_* F$
9. $\Gamma, A \vdash B$ *and* $\Gamma, A \vdash \neg_* B \;\Rightarrow\; \Gamma \vdash \neg_* A$
10. $\Gamma, A \vdash B$ *and* $\Gamma, \neg_* A \vdash B \Rightarrow\; \Gamma \vdash B$.

Proof We here only prove (8), (9) and (10). For (8), we first prove $\neg F \rightarrow \neg_* F$. By definition, we have $\neg_* F =_{def} F \rightarrow (F \rightarrow F) \wedge \neg(F \rightarrow F)$. Since $\neg F, F \vdash (F \rightarrow F) \wedge \neg(F \rightarrow F)$, by the deduction theorem,

(a) $\vdash \neg F \rightarrow (F \rightarrow (F \rightarrow F) \wedge \neg(F \rightarrow F))$

holds. By the definition of $(\neg_*)$, we have (b).

(b) $\vdash \neg F \to \neg_* F$

Next, we prove $\neg_* F \to \neg F$. By ($\neg_1$), we have (a):

(a) $\vdash (F \to (F \to F) \wedge \neg(F \to F)) \to ((F \to \neg((F \to F) \wedge \neg(F \to F))) \to \neg F)$

By hypothesis, (b) holds.

(b) $\vdash F \to (F \to F) \wedge \neg(F \to F) = \neg_* F$

By ($\to_4$) from (a) and (b), we obtain (c).

(c) $\vdash (F \to \neg((F \to F) \wedge \neg(F \to F))) \to \neg F$

By ($\to_1$), (d) can be derived.

(d) $\vdash \neg((F \to F) \wedge \neg(F \to F)) \to (F \to \neg((F \to F) \wedge \neg(F \to F)))$

To proceed, we prove $\neg(G \wedge \neg G)$ for any complex formula G. By ($\neg_1$), we have (e):

(e) $\vdash ((G \wedge \neg G) \to G) \to (((G \wedge \neg G) \to \neg G) \to \neg(G \wedge \neg G))$

By ($\wedge_1$), ($\wedge_2$), (f) and (g) hold.

(f) $\vdash (G \wedge \neg G) \to G$
(g) $\vdash (G \wedge \neg G) \to \neg G$

By ($\to_4$) from (e) and (f), we have (h).

(h) $\vdash ((G \wedge \neg G) \to \neg G) \to \neg(G \wedge \neg G)$

By ($\to_4$) from (g) and (h), we have (i).

(i) $\vdash \neg(G \wedge \neg G)$

Set $G = F \to F$. Then, (i) is (j).

(j) $\neg((F \to F) \wedge \neg(F \to F))$

By ($\to_4$) from (d) and (j), (k) is derived.

(k) $F \to \neg((F \to F) \wedge \neg(F \to F))$

Applying ($\to_4$) to (c) and (k), we obtain (l).

(l) $\neg F$

Therefore, (m) is proved.

(m) $\vdash \neg_* F \to F$

For (9), by the deduction theorem, we have (a) and (b) from the assumptions.

(a) $\Gamma \vdash A \to B$
(b) $\Gamma \vdash A \to \neg_* B$

By Theorem 2.9(3), (c) can be derived.

(c) $\vdash (A \to B) \to ((A \to \neg_* B) \to \neg_* A)$

By ($\to_4$) from (a) and (c), we obtain (d).

(d) $\Gamma \vdash (A \to \neg_* B) \to \neg_* A$

By ($\to_4$) from (b) and (d), we obtain (e).

(e) $\Gamma \vdash \neg_* A$

For (10), assume that $\Gamma, A \vdash B$ and $\Gamma, \neg_* A \vdash A$. By Theorem 2.12(7), (a) holds.

(a) $\Gamma, A \vee \neg_* A \vdash B$

By the deduction theorem, (b) is proved.

(b) $\Gamma \vdash (A \vee \neg_* A) \to B$

By Theorem 2.9, we have (c).

(c) $\vdash A \vee \neg_* A$

By Theorem 2.12(1) from (b) and (c), (d) can be derived.

(d) $\Gamma \vdash B$

Note here that the counterpart of Theorem 2.12(10) obtained by replacing the occurrence of $\neg_*$ by $\neg$ is not valid.

Now, we are in a position to prove the soundness and completeness of $P\tau$. Our proof method for completeness is based on maximal non-trivial set of formulas; see Abe [1] and Abe and Akama [8]. da Costa et al. [66] presented another proof using Zorn's Lemma.

Theorem 2.13 (Soundness) *Let Γ be a set of formulas and A be any formula. $\mathscr{A}\tau$ is a sound axiomatization of $P\tau$, i.e., if $\Gamma \vdash A$ then $\Gamma \models A$.*

Proof It is easy to prove that all the postulates of $\mathscr{A}\tau$ is valid. From the property of $\vdash$, soundness follows.

For proving the completeness theorem, we need some theorems.

Theorem 2.14 *Let Γ be a non-trivial set of formulas. Suppose that τ is finite. Then, Γ can be extended to a maximal (with respect to inclusion of sets) non-trivial set with respect to* **F**.

Proof Let Γ be a non-trivial subset of formulas of **F**. To show that Γ can be extended to a maximal non-trivial set, we construct a sequence $\Gamma_0, \Gamma_1, \ldots, \Gamma_n, \ldots$ as follows. As a vocabulary is composed by a denumerable set of symbols, the set of formulas of **F** is denumerable. Let $\Gamma_0 = \Gamma$ and inductively construct the rest of the sequence by taking $\Gamma_{i+1} = \Gamma \cup \{A_{i+1}\}$ if this set is non-trivial and otherwise by taking $\Gamma_{i+1} = \Gamma_i$.

It is easy to see that each set of the sequence $\Gamma_0, \Gamma_1, \ldots$ is non-trivial, and this is a non-decreasing sequence of sets such that $\Gamma \subseteq \Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \cdots \Gamma \subseteq \cdots$ Set Γ^* as follows:

$$\Gamma^* = \bigcup_{i=0}^{\infty} \Gamma_i$$

Then, Γ^* is a maximal non-trivial set containing Γ. Each finite subset of Γ^* must be contained in some Γ_i for some i, and thus must be non-trivia (since Γ_i is non-trivial).

For suppose that $A \in \mathbf{F}$ and $A \notin \Gamma^*$. As A is a formulas of $\mathbf{F}$, it must appear in our enumeration, say as A_k. If $\Gamma \cup \{A_k\}$ were non-trivial, then our construction would guarantee that $A_k \in \Gamma_{k+1}$, and hence $A_k \in \Gamma^*$. Because $A_k \notin \Gamma^*$, it follows that $\Gamma_k \cup \{A\}$ is also trivial. Hence $\Gamma^* \cup \{A\}$ is also trivial. It follows that Γ^* is a maximal non-trivial set.

As $\Gamma \subseteq \Gamma_i$ and $i \in \omega$, we have that $\Gamma \subseteq \Gamma^* = \bigcup_{i=0}^{\infty} \Gamma_i$. On the other hand, suppose that Γ^* is trivial. Thus, $\Gamma^* = \mathbf{F}$. Thus, $p_\lambda, \neg_* p_\lambda \in \Gamma^*$. As τ is finite, we have that any application of *modus ponens* has only a finite number of premises. Thus, there are $n, m < \omega$ such that $p_\lambda \in \Gamma_n$ and $\neg_* p_\lambda \in \Gamma_m$. Therefore, $p_\lambda, \neg_* p_\lambda \in \Gamma_{n_0}$, where $n_0 = max(n, m)$. Thus, Γ_{n_0} is trivial, but it is a contradiction.

Theorem 2.15 *Let Γ be a maximal non-trivial set of formulas. Then, we have the following:*

1. *if A is an axiom of $P\tau$, then $A \in \Gamma$*
2. *$A, B \in \Gamma$ iff $A \wedge B \in \Gamma$*
3. *$A \vee B \in \Gamma$ iff $A \in \Gamma$ or $B \in \Gamma$*
4. *if $p_\lambda, p_\mu \in \Gamma$, then $p_\theta \in \Gamma$, where $\theta = max(\lambda, \mu)$*
5. *$\neg^k p_\mu \in \Gamma$ iff $\neg^{k-1} p_{\sim\mu} \in \Gamma$, where $k \geq 1$*
6. *if $A, A \to B \in \Gamma$, then $B \in \Gamma$*
7. *$A \to B \in \Gamma$ iff $A \notin \Gamma$ or $B \in \Gamma$*

Proof We only prove (4). (5) is clear by definition. The remaining cases are proved as in the classical cases. The proof of (4) is as follows. From p_λ, p_μ, by (2), we have that (a).

(a) $p_\lambda \wedge p_\mu \in \Gamma$

From the axiom (τ_4), (b) holds.

(b) $p_\lambda \wedge p_\mu \to p_\theta$

where $\theta = max(\lambda, \mu)$. By (1), from (b), (c) follows.

(c) $p_\lambda \wedge p_\mu \to p_\theta \in \Gamma$

By (7) from (a) and (c), we can derive (d).

(d) $p_\theta \in \Gamma$

Theorem 2.16 *Let Γ be a maximal non-trivial set of formulas. Then, the characteristic function χ of Γ, that is, $\chi_\Gamma \to \mathbf{2}$ is the valuation function of some interpretation $I : \mathbf{P} \to |\tau|$.*

Proof Let us define the function $I : \mathbf{P} \to |\tau|$ putting $I(p) = \bigvee\{\mu \in |\tau| | \ p_\mu \in \Gamma\}$. Such a function is well defined, so $p_\perp \in \Gamma$.

Let $v_I : \mathbf{F} \to \mathbf{2}$ be the valuation associated to I. Hereafter, we omit the subscript. We need to show $v = \chi_\Gamma$. To show this, let $p_\mu \in \Gamma$. Thus, $\chi_\Gamma(p_\mu) = 1$. On the other hand, it is clear that $I(p) \geq \mu$. So, $v(p_\mu) = 1$. If $p_\mu \notin \Gamma$, $\chi_\Gamma(p_\mu) = 0$. Also, $I(p) \ngeq \mu$, because if so, that is $I(p) \geq \mu$, we have $p_{I(p)} \in \Gamma$, which is a contradiction. Therefore, $I(p) \ngeq \mu$, and thus $v_{p_\mu} = 0$.

By Theorem 2.15(5), $\neg^k p_\mu \in \Gamma$ iff $\neg^{k-1} p_{\sim\mu} \in \Gamma$, where $k \geq 1$. Thus, $\chi_\Gamma(\neg^k p_\mu) = \chi_\Gamma(\neg^{k-1} p_{\sim\mu})$, where $k \geq 1$. We show that $v(\neg^k p_\mu) = \chi_\Gamma(\neg^k p_\mu)$. We proceed by induction on k. If $k = 0$, it is just the previous case. Suppose that it holds for $k - 1$ ($k \geq 1$). Then, we have that $\chi_\Gamma(\neg^k p_\mu) = \chi_\Gamma(\neg^{k-1} p_{\sim\mu}) = v(\neg^{k-1} p_{\sim\mu}) = v(\neg^k p_\mu)$.

Now, let A be any formula. We proceed by induction on the number of occurrences of connectives in A. Thus, suppose that:
(1) A is of the form $\neg B$: Due to the previous discussion, we can suppose that B is a complex formula. So, $\chi_\Gamma(B) = v(B)$. If $A \in \Gamma$, then $B \notin \Gamma$, and $\chi_\Gamma(A) = 0$ and $\chi_\Gamma(B) = 1$. But, $v(A) = 1 - v(B)$, Therefore, $v(A) = 0$.
(2) A is of the form $B \wedge C$: $A \in \Gamma$ iff $B, C \in \Gamma$. By induction hypothesis, $\chi_\Gamma(B) = v(B)$ and $\chi_\Gamma(C) = v(C)$. Thus, $\chi_\Gamma = v(A)$. The other cases can also easily proved.

Here is the completeness theorem for $P\tau$.

Theorem 2.17 (Completeness) *Let Γ be a set of formulas and A be any formula. If τ is finite, then $\mathscr{A}\tau$ is a complete axiomatization for $P\tau$, i.e., if $\Gamma \models A$ then $\Gamma \vdash A$.*

Proof It can be proved by contraposition. Suppose that $\Gamma \notin A$. Thus, $\Gamma_0 = \Gamma \cup \{\neg_* A\}$ is non-trivial. By Theorem 2.14, Γ_0 is contained in a maximal non-trivial set Γ. Let $v : \mathbf{F} \to \mathbf{2}$ be the valuation obtained from Γ. We have that $v(A) = 1 - v(\neg_* A) = 0$. Thus, $\Gamma \not\models A$.

The decidability theorem also holds for finite lattice.

Theorem 2.18 (Decidability) *If τ is finite, then $P\tau$ is decidable.*

Proof Let A be a formula. We denote by $sf(A)$ the set of all subformulas of A and by $at(A)$ the set of all atomic subformulas composing A. We write $\sharp A$ for the cardinality of the set A. So, by using the valuation defined above, we can check in $\sharp sf(A) - \sharp at(A)$ steps as in the classical case up to analyze $\sharp at(A)$ atomic formulas. The validity of each atomic formula is checked in $\sharp\tau$ times. So, $at(A)$ can be checked at most $k\sharp\tau\sharp at(A)$ times, where k is a constant. Thus, it is possible to check whether A is valid or not in a finite number of steps. This means that $P\tau$ is decidable.

The completeness does not in general hold for an infinite lattice. But, it holds for a special case.

Definition 2.14 (*Finite annotation property*) Suppose that Γ be a set of formulas such that the set of annotated constants occurring in Γ is included in a finite substructure of τ (Γ itself may be infinite). In this case, Γ is said to have the *finite annotation property*.

Note that if τ' is a substructure of τ then τ' is closed under the operations $\sim$, $\vee$ and $\wedge$. One can easily prove the following from Theorem 2.17.

Theorem 2.19 (Finitary Completeness) *Suppose that Γ has the finite annotation property. If A is any formula such that $\Gamma \vdash A$, then there is a finite proof of A from Γ.*

Theorem 2.19 tells us that even if the set of the underlying truth-values of $P\tau$ is infinite (countably or uncountably), as long as theories have the finite annotation property, the completeness result applies to them, i.e., $\mathscr{A}\tau$ is complete with respect to such theories.

In general, when we consider theories that do not possess the finite annotation property, it may be necessary to guarantee completeness by adding a new infinitary inference rule (ω-rule), similar in spirit to the rule used by da Costa [60] in order to cope with certain models in a particular family of infinitary language. Observe that for such cases a desired axiomatization of $P\tau$ is not finitary.

From the classical result of compactness, we can state a version of the compactness theorem.

Theorem 2.20 (Weak Compactness) *Suppose that Γ has the finite annotation property. If A is any formula such that $\Gamma \vdash A$, then there is a finite subset Γ' of Γ such that $\Gamma' \vdash A$.*

Annotated logics $P\tau$ provide a general framework, and can be used to reasoning about many different logics. Below we present some examples.

The set of truth-values $FOUR = \{t, f, \perp, \top\}$, with $\neg$ defined as: $\neg t = f, \neg f = t, \neg\perp = \perp, \neg\top = \top$. Four-valued logic based on $FOUR$ was originally due to Belnap [49, 50] to model internal states in a computer. Subrahmanian [149] formalized an annotated logic with $FOUR$ as a foundation for paraconsistent logic programming; also see Blair and Subrahmanian [53]. In Chap. 6, we will give a detailed exposition of paraconsistent logic programming.

Their annotated logic may be used for reasoning about inconsistent knowledge bases. For example, we may allow logic programs to be finite collections of formulas of the form:

$$(A : \mu_0) \leftrightarrow (B_1 : \mu_1) \& \ldots \& (B_n : \mu_n)$$

where A and B_i ($1 \leq i \leq n$) are atoms and μ_j ($0 \leq j \leq n$) are truth-values in $FOUR$.

Intuitively, such programs may contain "intuitive" inconsistencies–for example, the pair

$((p : f), (p : t))$

is inconsistent. If we append this program to a consistent program P, then the resulting union of these two programs may be inconsistent, even though the predicate symbols p occurs nowhere in program P.

Such inconsistencies can easily occur in knowledge based systems, and should not be allowed to trivialize the meaning of a program. However, knowledge based systems based on classical logic cannot handle the situation since the program is trivial. In Blair and Subrahmanian [53], it is shown how the four-valued annotated logic may be used to describe this situation. Later, Blair and Subrahmanian's annotated logic was extended as *generalized annotated logics* by Kifer and Subrahmanian [100].

There are also other examples which can be dealt with by annotated logics. The set of truth-values *FOUR* with negation defined as boolean complementation forms an annotated logic.

The unit interval [0, 1] of truth-values with $\neg x = 1 - x$ is considered as the base of annotated logic for qualitative or fuzzy reasoning. In this sense, probabilistic and fuzzy logics could be generalized as annotated logics.

The interval $[0, 1] \times [0, 1]$ of truth-values can be used for annotated logics for evidential reasoning. Here, the assignment of the truth-value (μ_1, μ_2) to proposition p may be thought of as saying that the degree of belief in p is μ_1, while the degree of disbelief is μ_2. Negation can be defined as $\neg(\mu_1, \mu_2) = (\mu_2, \mu_1)$.

Note that the assignment of $[\mu_1, \mu_2]$ to a proposition p by an interpretation I does not necessarily satisfy the condition $\mu_1 + \mu_2 \leq 1$. This contrasts with probabilistic reasoning. Knowledge about a particular domain may be gathered from different experts (in that domain), and these experts may hold different views.

Some of these views may lead to a "strong" belief in a proposition; likewise, other experts may have a "strong" disbelief in the same proposition. In such a situation, it seems appropriate to report the existence of conflicting opinions, rather than use ad-hoc means to resolve this conflict.

The above examples can be described by annotated logics $P\tau$ or its suitable extensions. These issues will be taken up in Chap. 5.

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