

Chapter 2

Geometry of Quasi-Metric Spaces

The main goal of this chapter is to set the stage for the rest of this monograph by presenting a brief survey of some of the many facets of the theory of quasi-metric spaces. Quasi-metric spaces constitute generalizations of not only the classical Euclidean setting, but of quasi-Banach spaces and ultrametric spaces. In this work, quasi-metric spaces will constitute the natural geometric context in which our main results are going to be developed.

This chapter is organized as follows. In Sect. 2.1 we record an assortment of preliminary material, centered around the concept of quasi-metric spaces, and discuss the sharp metrization theory developed in [MiMiMiMo13]. For the sake of completeness, we will then survey various important tools used in this work such as the existence of a partition of unity subordinate to a Whitney decomposition for an open set in a geometrically doubling quasi-metric space. This is done in Sect. 2.2. In this vein, we also present a Vitali-type covering lemma in Sect. 2.3.

Regarding measure theoretic aspects pertinent to present work, Sect. 2.4 is devoted to developing and exploring a general notion of d -dimensional Ahlfors-regular quasi-metric spaces where we consider the possibility of a set, consisting just of a singleton, having strictly positive measure.

Section 2.5 is the final section of this chapter wherein we review basic definitions and results from [MiMiMiMo13] pertaining to the concept of the index of a quasi-metric space. This index will play an important role in the formulation of many of our subsequent key results.

2.1 Quasi-Metric Spaces

There are two main goals of this section. First, we review the notion of a quasi-metric space (along with related metric and topological matters) and lay out several necessary conventions with regards to the notation used in this monograph.

Second, we record a sharp metrization theorem recently obtained [MiMiMiMo13, Theorem 3.46, p. 144]. This theorem will prove to be a superior tool in establishing many of the results we have in mind.

To get started, given a nonempty set X , call a function $\rho : X \times X \rightarrow [0, \infty)$ a *quasi-distance* (or a *quasi-metric*) provided there exist two finite constants $C_0, C_1 > 0$ with the property that for every $x, y, z \in X$, one has

$$\begin{aligned} \rho(x, y) = 0 &\iff x = y, \quad \rho(y, x) \leq C_0 \rho(x, y) \\ \text{and} \quad \rho(x, y) &\leq C_1 \max\{\rho(x, z), \rho(z, y)\}. \end{aligned} \quad (2.1)$$

If X has cardinality at least 2 then necessarily the constants C_0 and C_1 appearing in (2.1) are ≥ 1 . In this context, we define C_ρ to be the smallest constant which can play the role of C_1 in the last inequality in (2.1), i.e.,

$$C_\rho := \sup_{\substack{x, y, z \in X \\ \text{not all equal}}} \frac{\rho(x, y)}{\max\{\rho(x, z), \rho(z, y)\}} \in [1, \infty), \quad (2.2)$$

and define \tilde{C}_ρ to be the smallest constant which can play the role of C_0 in the first inequality in (2.1), i.e.,

$$\tilde{C}_\rho := \sup_{\substack{x, y \in X \\ x \neq y}} \frac{\rho(y, x)}{\rho(x, y)} \in [1, \infty). \quad (2.3)$$

A quasi-metric ρ , as in (2.1), shall be referred to as *symmetric* whenever $\tilde{C}_\rho = 1$, i.e., whenever $\rho(x, y) = \rho(y, x)$ for every $x, y \in X$. Recall that a distance¹ d on the set X is called an *ultrametric* provided that in place of the triangle-inequality, d satisfies the stronger condition $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$, holds. Hence,

$$\rho \text{ ultrametric on } X \iff \rho \text{ is a quasi-distance on } X \text{ and } C_\rho = \tilde{C}_\rho = 1. \quad (2.4)$$

In light of this observation, it is natural to refer to the last condition in (2.1) as the *quasi-ultrametric condition* for ρ . Given the elementary inequality $\frac{1}{2}(a + b) \leq \max\{a, b\} \leq a + b$, $a, b \in [0, \infty)$, it is easy to see that this condition is equivalent to the more commonly used quasi-triangle inequality. Namely, the condition that there exists a constant $C \in (0, \infty)$ such that

$$\rho(x, y) \leq C(\rho(x, z) + \rho(z, y)) \quad \text{for every } x, y, z \in X. \quad (2.5)$$

¹A function $d : X \rightarrow [0, \infty)$ shall be referred to as a *distance* provided for every $x, y, z \in X$, the function d satisfies: $d(x, y) = 0 \iff x = y$, $d(x, y) = d(y, x)$, and $d(x, y) \leq d(x, z) + d(z, y)$.

However, as we will demonstrate below, it is the nature of the best constant C_ρ (appearing in the quasi-ultrametric condition) rather than C as in (2.5) which will prove to be of utmost importance.

In the sequel, we shall denote by $\Omega(X)$ the collection of all quasi-distances on X . It is clear that

$$\rho \in \Omega(X) \implies \rho^\beta \in \Omega(X) \quad \text{for every number } \beta \in (0, \infty), \quad (2.6)$$

where, in general, given any nonempty set \mathcal{X} , a function $f : \mathcal{X} \rightarrow [0, \infty]$, and an exponent $\beta \in (0, \infty)$ we define

$$f^\beta : \mathcal{X} \longrightarrow [0, \infty] \quad \text{by setting} \quad f^\beta(x) := (f(x))^\beta, \quad \forall x \in \mathcal{X}. \quad (2.7)$$

Also, with \mathcal{X} keeping its significance, call two functions $f, g : \mathcal{X} \rightarrow [0, \infty]$ equivalent, and write $f \approx g$, if there exists a constant $C \in [1, \infty)$ with the property that

$$C^{-1}f \leq g \leq Cf \quad \text{pointwise on } \mathcal{X}. \quad (2.8)$$

It follows that if $\rho \in \Omega(X)$ and $\varrho : X \times X \rightarrow [0, \infty)$ is a function such that $\varrho \approx \rho$, then $\varrho \in \Omega(X)$ as well. Thus (2.8) defines an equivalence relation \approx on $\Omega(X)$ and we will call each equivalence class $\mathbf{q} \in \Omega(X)/\approx$ a quasi-metric space structure on X . Finally, for each $\rho \in \Omega(X)$, denote $[\rho] \in \Omega(X)/\approx$ the equivalence class of ρ .

By a quasi-metric space we shall understand a pair (X, \mathbf{q}) where X is a set of cardinality at least 2, and $\mathbf{q} \in \Omega(X)/\approx$. If X is a set of cardinality at least 2 and $\rho \in \Omega(X)$ we will sometimes write (X, ρ) in place of $(X, [\rho])$. Given a quasi-metric space (X, \mathbf{q}) and $\rho \in \mathbf{q}$, the ρ -ball centered at $x \in X$ with radius $r \in (0, \infty)$ is naturally defined as

$$B_\rho(x, r) := \{y \in X : \rho(x, y) < r\}. \quad (2.9)$$

Given that the quasi-distance ρ is not assumed to be symmetric, care must be taken when discussing the membership of a point to any ρ -ball. We also remark here that it follows from (2.6) and (2.9) that whenever $\beta \in (0, \infty)$ there holds

$$B_{\rho^\beta}(x, r) = B_\rho(x, r^{1/\beta}) \quad \forall x \in X \text{ and } \forall r \in (0, \infty). \quad (2.10)$$

Given a quasi-metric space (X, \mathbf{q}) , call $E \subseteq X$ bounded if E is contained in a ρ -ball for some (hence all) $\rho \in \mathbf{q}$. In other words, a set $E \subseteq X$ is bounded, relative to the quasi-metric space structure \mathbf{q} on X , if and only if for some (hence all) $\rho \in \mathbf{q}$ we have $\text{diam}_\rho(E) < \infty$, where

$$\text{diam}_\rho(E) := \sup \{\rho(x, y) : x, y \in E\}. \quad (2.11)$$

Given a bounded set $E \subseteq X$, if we wish to emphasize the particular choice of quasi-distance $\rho \in \mathbf{q}$, then we will refer to E as being ρ -bounded. In this context, if $\rho \in \mathbf{q}$, we define the ρ -distance between two arbitrary, nonempty sets $E, F \subseteq X$ to be

$$\text{dist}_\rho(E, F) := \inf \{ \rho(x, y) : x \in E, y \in F \}, \quad (2.12)$$

and if $E = \{x\}$ for some $x \in X$ we shall abbreviate $\text{dist}_\rho(x, F) := \text{dist}_\rho(\{x\}, F)$.

Turning to topological considerations, we note that any quasi-metric space (X, \mathbf{q}) has a canonical topology, denoted $\tau_{\mathbf{q}}$, which is (unequivocally) defined as the topology τ_ρ naturally induced by a choice of quasi-distance $\rho \in \mathbf{q}$, the latter being characterized by

$$\mathcal{O} \in \tau_\rho \stackrel{\text{def}}{\iff} \mathcal{O} \subseteq X \text{ and } \forall x \in \mathcal{O}, \exists r \in (0, \infty) \text{ such that } B_\rho(x, r) \subseteq \mathcal{O}. \quad (2.13)$$

For a given quasi-distance $\rho \in \mathbf{q}$, we will refer to the elements of τ_ρ as ρ -open sets. It follows from the observation in (2.10) that the topology τ_ρ is invariant under power-rescalings of the quasi-distance ρ , i.e.,

$$\rho \in \mathbf{q}, \beta \in (0, \infty) \implies \tau_{\mathbf{q}} = \tau_\rho = \tau_{\rho^\beta}. \quad (2.14)$$

This is remarkable since, in general, it is not to be expected $\rho^\beta \approx \rho$ if $\beta \in (0, \infty)$ is a fixed number. For example, such an occurrence of this fact can be seen when ρ is the Euclidean distance and the underlying set is \mathbb{R}^d .

Additionally, it is important to note that in contrast to what would be the case in a genuine metric space, the relaxation of the triangle inequality precludes the guarantee that all ρ -balls belong to τ_ρ . In spite of this disparity, as is well-known, the topology induced by the given quasi-distance on a quasi-metric space is metrizable and we shall take a moment review a main result in [MiMiMiMo13] which is a sharp quantitative version of this fact.

To facilitate the subsequent discussion in this chapter we first make a couple of definitions. Assume that X is an arbitrary, nonempty set. Given an arbitrary function $\rho : X \times X \rightarrow [0, \infty]$ and an arbitrary exponent $\alpha \in (0, \infty]$ define the function

$$\rho_\alpha : X \times X \longrightarrow [0, \infty] \quad (2.15)$$

by setting for each $x, y \in X$

$$\begin{aligned} \rho_\alpha(x, y) := \inf \left\{ \left(\sum_{i=1}^N \rho(\xi_i, \xi_{i+1})^\alpha \right)^{\frac{1}{\alpha}} : \text{there exists } N \in \mathbb{N} \text{ and } \xi_1, \dots, \xi_{N+1} \in X, \right. \\ \left. (\text{not necessarily distinct}) \text{ such that } \xi_1 = x \text{ and } \xi_{N+1} = y \right\}, \end{aligned} \quad (2.16)$$

whenever $\alpha \neq \infty$, and its natural counterpart corresponding to the case when one has $\alpha = \infty$, i.e.,

$$\rho_\infty(x, y) := \inf \left\{ \max_{1 \leq i \leq N} \rho(\xi_i, \xi_{i+1}) : \text{there exists } N \in \mathbb{N}, \xi_1, \dots, \xi_{N+1} \in X, \right. \\ \left. (\text{not necessarily distinct}) \text{ such that } \xi_1 = x \text{ and } \xi_{N+1} = y \right\}. \quad (2.17)$$

It is then clear from definitions that

$$\forall \rho \in \mathfrak{Q}(X), \forall \alpha \in (0, \infty] \implies \begin{cases} \rho_\alpha \in \mathfrak{Q}(X), \\ C_{\rho_\alpha} \leq C_\rho, \text{ and} \\ \rho_\alpha \leq \rho \text{ pointwise on } X \times X. \end{cases} \quad (2.18)$$

Going further, if $\rho : X \times X \rightarrow [0, \infty]$ is an arbitrary function, consider its symmetrization $\rho_{\text{sym}} : X \times X \rightarrow [0, \infty]$ which is defined by

$$\rho_{\text{sym}}(x, y) := \max \{ \rho(x, y), \rho(y, x) \}, \quad \forall x, y \in X. \quad (2.19)$$

Then ρ_{sym} is symmetric, i.e., $\rho_{\text{sym}}(x, y) = \rho_{\text{sym}}(y, x)$ for every $x, y \in X$, and $\rho_{\text{sym}} \geq \rho$ on $X \times X$. In fact, ρ_{sym} is the smallest $[0, \infty]$ -valued function defined on $X \times X$ which is symmetric and pointwise $\geq \rho$. Furthermore, if $\rho \in \mathfrak{Q}(X)$ then

$$\rho_{\text{sym}} \in \mathfrak{Q}(X), \quad C_{\rho_{\text{sym}}} \leq C_\rho, \quad \tilde{C}_{\rho_{\text{sym}}} = 1, \quad \text{and} \quad \rho \leq \rho_{\text{sym}} \leq \tilde{C}_\rho \rho. \quad (2.20)$$

The reader is referred to [MiMiMiMo13, Theorem 3.26, p. 91] for a more systematic exposition regarding the properties of ρ_α and ρ_{sym} . Here is the quantitative metrization theorem from [MiMiMiMo13] alluded to above.

Theorem 2.1 *Let (X, \mathbf{q}) be a quasi-metric space, fix $\rho \in \mathbf{q}$, and assume that $C_\rho, \tilde{C}_\rho \in [1, \infty)$ are as in (2.2)–(2.3). In this context, define (cf. (2.16)–(2.17))*

$$\rho_\# := (\rho_{\text{sym}})_{\alpha_\rho} \text{ for } \alpha_\rho := [\log_2 C_\rho]^{-1} \in (0, \infty]. \quad (2.21)$$

Then $\rho_\# \in \mathbf{q}$ with $C_{\rho_\#} \leq C_\rho$ and $\tilde{C}_{\rho_\#} = 1$. Also, $(\rho^\gamma)_\# \approx (\rho_\#)^\gamma$ for every $\gamma \in (0, \infty)$. Moreover, for any finite number $\beta \in (0, \alpha_\rho]$, the function

$$d_{\rho, \beta} : X \times X \rightarrow [0, \infty), \quad d_{\rho, \beta}(x, y) := [\rho_\#(x, y)]^\beta, \quad \forall x, y \in X, \quad (2.22)$$

is a distance on X , i.e., for every $x, y, z \in X$, $d_{\rho, \beta}$ satisfies

$$d_{\rho, \beta}(x, y) = 0 \iff x = y \quad (2.23)$$

$$d_{\rho, \beta}(x, y) = d_{\rho, \beta}(y, x) \quad (2.24)$$

$$d_{\rho, \beta}(x, y) \leq d_{\rho, \beta}(x, z) + d_{\rho, \beta}(z, y) \quad (2.25)$$

and which has the property $(d_{\rho,\beta})^{1/\beta} \approx \rho$. More specifically,

$$(C_\rho)^{-2} \rho(x, y) \leq [d_{\rho,\beta}(x, y)]^{1/\beta} = \rho_\#(x, y) \leq \tilde{C}_\rho \rho(x, y), \quad \forall x, y \in X. \quad (2.26)$$

In particular, the topology induced by the distance $d_{\rho,\beta}$ on X is precisely $\tau_{\mathbf{q}}$.

Additionally, $\rho_\#$ satisfies the following local Hölder-type regularity condition of order β :

$$|\rho_\#(x, y) - \rho_\#(x, z)| \leq \frac{1}{\beta} \max \{ \rho_\#(x, y)^{1-\beta}, \rho_\#(x, z)^{1-\beta} \} [\rho_\#(y, z)]^\beta \quad (2.27)$$

whenever $x, y, z \in X$ (with the understanding that when $\beta \geq 1$ one also imposes the condition that $x \notin \{y, z\}$). In particular, it is straightforward to show, based on (2.27), that the function

$$\rho_\# : X \times X \longrightarrow [0, \infty) \quad \text{is continuous,} \quad (2.28)$$

when $X \times X$ is equipped with the natural product topology $\tau_{\mathbf{q}} \times \tau_{\mathbf{q}}$. Ergo, all $\rho_\#$ -balls are open in the topology $\tau_{\mathbf{q}}$.

The striking feature of the result discussed in Theorem 2.1 is the fact that if (X, \mathbf{q}) is any quasi-metric space and $\rho \in \mathbf{q}$ then ρ^β is equivalent to a distance on X for any finite number $\beta \in (0, (\log_2 C_\rho)^{-1}]$. This result improves upon an earlier version due to R.A. Macías and C. Segovia [MaSe79i, Theorem 2, p. 259], in which these authors have identified a non-optimal upper-bound for the exponent β . The non-optimality of the metrization theory in [MaSe79i] has presented widespread limitations to many subsequent publications. For example, as we will illustrate in this monograph, this exponent directly influences the range of p 's for which there exists a “rich” theory of Hardy spaces (H^p spaces). In addition, the ability to construct an approximation to the identity is an indispensable tool in analysis and this exponent governs the amount of smoothness such an approximate identity can possess. This alone has many overreaching consequences which others have taken note (see, e.g., [HuYaZh09, Remark 5.3, p.133]).

In this regard, it is instructive to note that it was shown in [MiMiMiMo13, p. 150] that the upper bound of $\alpha_\rho = [\log_2 C_\rho]^{-1}$ is sharp in the following sense. Given any finite number $C_1 > 1$, there exist a nonempty set X and a symmetric quasi-distance $\rho : X \times X \rightarrow [0, \infty)$ satisfying the quasi-ultrametric condition for the given C_1 and which has the property that if $\varrho : X \times X \rightarrow [0, \infty)$ is such that $\varrho \approx \rho$ and there exist $\beta \in (0, \infty)$ and $C \in [0, \infty)$ for which

$$|\varrho(x, y) - \varrho(x, z)| \leq C \max \{ \varrho(x, y)^{1-\beta}, \varrho(x, z)^{1-\beta} \} [\varrho(y, z)]^\beta \quad (2.29)$$

whenever $x, y, z \in X$ (and also $x \notin \{y, z\}$ if $\beta \geq 1$) then necessarily

$$\beta \leq \frac{1}{\log_2 C_1}. \quad (2.30)$$

We conclude this section by proving a result pertaining to the nature of the topology induced by a quasi-metric, which is going to be relevant in the context of the Lebesgue Differentiation Theorem discussed later, in Sect. 3.3.

Lemma 2.2 *Assume that (X, \mathbf{q}) is a quasi-metric space. Then any open set in the topology $\tau_{\mathbf{q}}$ can be written as a countable union of closed sets in the topology $\tau_{\mathbf{q}}$.*

Proof Let O be an open set in the topology $\tau_{\mathbf{q}}$. Fix a quasi-metric $\rho \in \mathbf{q}$ and let $\rho_{\#}$ be its regularization, as discussed in Theorem 2.1. For each $j \in \mathbb{N}$ then consider

$$C_j := \{x \in O : \rho_{\#}(x, y) \geq 1/j \text{ for every } y \in X \setminus O\}. \quad (2.31)$$

Clearly, $C_j \subseteq O$ for every j , hence $\bigcup_{j \in \mathbb{N}} C_j \subseteq O$. To prove the opposite inclusion, pick an arbitrary $x_0 \in O$. Since O is open in $\tau_{\mathbf{q}}$, it follows that there exists $r > 0$ with the property that $B_{\rho_{\#}}(x_0, r) \subseteq O$. Then for any $y \in X \setminus O$ we necessarily have $\rho_{\#}(x_0, y) \geq r$ which, in turn, goes to show that $x_0 \in C_j$ whenever $j \in \mathbb{N}$ satisfies $j \geq 1/r$. This establishes $O = \bigcup_{j \in \mathbb{N}} C_j$. There remains to show that, for each $j \in \mathbb{N}$, the set C_j is closed in $\tau_{\mathbf{q}}$. To this end, fix $x_1 \in X \setminus C_j$ and note that this entails the existence of some $y_1 \in X \setminus O$ such that $\rho_{\#}(x_1, y_1) < 1/j$. Select $\beta \in (0, [\log_2 C_{\rho}]^{-1})$ and pick a number r satisfying

$$0 < r < ((1/j)^{\beta} - \rho_{\#}(x_1, y_1)^{\beta})^{1/\beta}. \quad (2.32)$$

In light of (2.25), this choice ensures that for every $z \in B_{\rho_{\#}}(x_1, r)$ we have

$$\rho_{\#}(z, y_1)^{\beta} \leq \rho_{\#}(z, x_1)^{\beta} + \rho_{\#}(x_1, y_1)^{\beta} \leq r^{\beta} + \rho_{\#}(x_1, y_1)^{\beta} < (1/j)^{\beta}. \quad (2.33)$$

Hence, ultimately, $\rho_{\#}(z, y_1) < 1/j$ which places z in $X \setminus C_j$. Given that $z \in B_{\rho_{\#}}(x_1, r)$ has been arbitrarily chosen, it follows that $B_{\rho_{\#}}(x_1, r) \subseteq X \setminus C_j$ from which we conclude that $X \setminus C_j$ is open in $\tau_{\mathbf{q}}$. Thus, C_j is closed in $\tau_{\mathbf{q}}$, as wanted. \square

2.2 A Whitney-Type Decomposition and Partition of Unity

In the first part of this section, we present a version of the classical Whitney decomposition in the setting of geometrically doubling quasi-metric spaces recently obtained in [AlMiMi13]. A variation of this result in the Euclidean setting (as presented in, e.g., [St70, Theorem 1.1, p. 167]) has been worked out in [CoWe71, Theorem 3.1, p. 71] and [CoWe77, Theorem 3.2, p. 623] for bounded open sets and in [MaSe79ii, Lemma 2.9, p. 277] for proper open subsets of finite measure in the context of spaces of homogeneous type. Regarding a version absent of any measure theoretic structure, we wish to mention that in [MiMiMiMo13], the scope of this work has been further generalized as to apply to arbitrary open sets in a geometrically doubling quasi-metric space, equipped with a symmetric quasi-

distance. This result has further been refined in [AlMiMi13] to incorporate the scenario when the quasi-distances are not necessarily symmetric.

In the second part of this section we present a result obtained in [MiMiMiMo13] guaranteeing the existence of a partition of unity subordinate to the aforementioned Whitney-type decomposition, which is quantitative in the sense that the size of the functions involved is controlled in terms of the size of their respective supports. A formulation in the standard setting of \mathbb{R}^n may be found in [St70, p. 170]. More recently, such quantitative Whitney partitions of unity have been constructed on general metric spaces (see [KoShTu00, GoKoSh10, Lemma 2.4, p. 339]), and on quasi-metric spaces, as in [MaSe79ii, Lemma 2.16, p. 278]. Here we wish to improve upon the latter result both by allowing a more general set-theoretic framework and by providing a transparent description of the order of smoothness of the functions involved in such a Whitney-like partition of unity for an arbitrary quasi-metric space. Before formulating these results, in Theorems 2.4 and 2.5 below, we first define the class of geometrically doubling quasi-metric spaces.

Definition 2.3 A quasi-metric space (X, \mathbf{q}) is called *geometric doubling* if there exists $\rho \in \mathbf{q}$ for which one can find a number $N \in \mathbb{N}$, called the *geometric doubling constant* of (X, \mathbf{q}) , with the property that any ρ -ball of radius r in X may be covered by at most N ρ -balls in X of radii $r/2$. Finally, if X is an arbitrary, nonempty set and $\rho \in \mathfrak{Q}(X)$, call (X, ρ) *geometric doubling* if $(X, [\rho])$ is geometric doubling.

Note that a quasi-metric space (X, \mathbf{q}) is geometrically doubling if and only if

$$\forall \rho \in \mathbf{q} \quad \forall \theta \in (0, 1) \quad \exists N \in \mathbb{N} \text{ such that any } \rho\text{-ball of radius } r \quad (2.34)$$

in X may be covered by at most N ρ -balls in X of radii θr .

In particular, this ensures that the last part in Definition 2.3 is meaningful. Another useful consequence of the geometrically doubling property for a quasi-metric space (X, \mathbf{q}) is as follows.

$$\begin{aligned} &\text{If } (X, \mathbf{q}) \text{ is a geometric doubling quasi-metric space} \\ &\text{then the topological space } (X, \tau_{\mathbf{q}}) \text{ is separable.} \end{aligned} \quad (2.35)$$

Throughout the remainder of the work, given a set X , we denote by $\mathbf{1}_E$ the characteristic function of a set $E \subseteq X$. With this in mind we present the first main result of this section.

Theorem 2.4 (Whitney-Type Decomposition) *Suppose (X, \mathbf{q}) is a geometrically doubling quasi-metric space and fix $\rho \in \mathbf{q}$. Then for each number $\lambda \in (1, \infty)$ there exist constants $\Lambda \in (\lambda, \infty)$ and $M \in \mathbb{N}$, both depending only on C_ρ as in (2.2), λ and the geometric doubling constant of (X, \mathbf{q}) , and which have the following significance.*

For each proper, nonempty, open subset Ω of the topological space $(X, \tau_{\mathbf{q}})$ there exist a sequence of points $\{x_j\}_{j \in \mathbb{N}}$ in Ω along with a family of real numbers $r_j > 0$, $j \in \mathbb{N}$, for which the following properties are valid:

- (1) $\Omega = \bigcup_{j \in \mathbb{N}} B_\rho(x_j, r_j)$;
- (2) $\sum_{j \in \mathbb{N}} \mathbf{1}_{B_\rho(x_j, \lambda r_j)} \leq M$ on Ω . In fact, there exists $\varepsilon \in (0, 1)$, which depends only on C_ρ, λ and the geometric doubling constant of (X, \mathbf{q}) , with the property that for any $x_0 \in \Omega$

$$\# \left\{ j \in \mathbb{N} : B_\rho(x_0, \varepsilon \operatorname{dist}_\rho(x_0, X \setminus \Omega)) \cap B_\rho(x_j, \lambda r_j) \neq \emptyset \right\} \leq M, \quad (2.36)$$

where in general we define $\#E$ to be the cardinality of a set E .

- (3) $B_\rho(x_j, \lambda r_j) \subseteq \Omega$ and $B_\rho(x_j, \Lambda r_j) \cap [X \setminus \Omega] \neq \emptyset$ for every $j \in \mathbb{N}$.
- (4) $r_i \approx r_j$ uniformly for $i, j \in \mathbb{N}$ such that $B_\rho(x_i, \lambda r_i) \cap B_\rho(x_j, \lambda r_j) \neq \emptyset$.

Proof For the proof of Theorem 2.4, the reader is referred to [AlMiMi13]. See also [MiMiMiMo13, Theorem 4.21, p. 184] wherein the authors present a constructive proof in the case when the quasi-distance is assumed to be symmetric. \square

We will refer to the constant M appearing in (2) in the conclusion of Theorem 2.4 as the bounded overlap constant (for the given decomposition).

In Theorem 2.5 below, we present the existence of a partition of unity subordinate to such a decomposition produced in Theorem 2.4. A version of this result originally appeared in [MiMiMiMo13, Theorem 4.18, p. 178] in the class of Hölder-continuous functions and was subsequently generalized to a class of functions having a modulus of continuity in [AlMiMi13]. Theorem 2.5 below is a slight extension of the work in [MiMiMiMo13]. Before proceeding, we take a moment to recall the smoothness class of Hölder functions \mathcal{C}^β in the context of quasi-metric spaces.

Let (X, \mathbf{q}) be a quasi-metric space. Also, fix a number $\beta \in (0, \infty)$ and a quasi-distance $\rho \in \mathbf{q}$. Given a complex-valued function f on X , define Hölder semi-norm² (of order β , relative to the quasi-distance ρ) of the function f by setting

$$\|f\|_{\mathcal{C}^\beta(X, \rho)} := \sup_{x, y \in X, x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^\beta}. \quad (2.37)$$

We introduce the homogeneous Hölder space $\mathcal{C}^\beta(X, \mathbf{q})$ as

$$\begin{aligned} \mathcal{C}^\beta(X, \mathbf{q}) &:= \{f : X \rightarrow \mathbb{C} : \|f\|_{\mathcal{C}^\beta(X, \rho)} < \infty \text{ for some } \rho \in \mathbf{q}\} \\ &= \{f : X \rightarrow \mathbb{C} : \|f\|_{\mathcal{C}^\beta(X, \rho)} < \infty \text{ for every } \rho \in \mathbf{q}\}. \end{aligned} \quad (2.38)$$

²Given a vector space \mathcal{X} over \mathbb{C} , recall that a function $\|\cdot\| : \mathcal{X} \rightarrow [0, \infty)$ is called a semi-norm provided that for each $x, y \in \mathcal{X}$ the following three conditions hold (i) $x = 0$ implies $\|x\| = 0$, (ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$, $\forall \lambda \in \mathbb{C}$, and (iii) $\|x + y\| \leq \|x\| + \|y\|$.

Given any $\beta \in (0, \infty)$, it follows that $\{\|\cdot\|_{\mathcal{C}^\beta(X, \rho)} : \rho \in \mathbf{q}\}$ is a family of equivalent semi-norms on $\mathcal{C}^\beta(X, \mathbf{q})$. If $\rho \in \mathfrak{Q}(X)$ is given then we shall some times slightly simplify notation and write $\mathcal{C}^\beta(X, \rho)$ in place of $\mathcal{C}^\beta(X, [\rho])$. If we introduce an equivalence relation, \sim , on $\mathcal{C}^\beta(X, \rho)$ defined by $f \sim g$ if and only if $f - g$ is a constant function on X , then $\mathcal{C}^\beta(X, \rho)/\sim$ is a Banach space when equipped with the norm $\|\cdot\|_{\mathcal{C}^\beta(X, \rho)}$. Let us also note here that if $\rho \in \mathbf{q}$ and if $\beta > 0$ is a finite number then for any pair of real-valued functions $f, g \in \mathcal{C}^\beta(X, \mathbf{q})$ it follows that

$$\max\{f, g\} \in \mathcal{C}^\beta(X, \mathbf{q}), \quad \min\{f, g\} \in \mathcal{C}^\beta(X, \mathbf{q}), \quad (2.39)$$

with

$$\begin{aligned} & \max\left\{\|\max\{f, g\}\|_{\mathcal{C}^\beta(X, \rho)}, \|\min\{f, g\}\|_{\mathcal{C}^\beta(X, \rho)}\right\} \\ & \leq \max\left\{\|f\|_{\mathcal{C}^\beta(X, \rho)}, \|g\|_{\mathcal{C}^\beta(X, \rho)}\right\}. \end{aligned} \quad (2.40)$$

As a notational convention, given a quasi-metric space (X, \mathbf{q}) , we will write

$$\text{Lip}(X, \mathbf{q}) := \mathcal{C}^1(X, \mathbf{q}). \quad (2.41)$$

Maintaining the above assumptions on the ambient, given a complex-valued function f on X set

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}. \quad (2.42)$$

and define the inhomogeneous Hölder space $\mathcal{C}^\beta(X, \mathbf{q})$ as

$$\begin{aligned} \mathcal{C}^\beta(X, \mathbf{q}) &:= \{f : X \rightarrow \mathbb{C} : \|f\|_\infty + \|f\|_{\mathcal{C}^\beta(X, \rho)} < \infty \text{ for some } \rho \in \mathbf{q}\} \\ &= \{f : X \rightarrow \mathbb{C} : \|f\|_\infty + \|f\|_{\mathcal{C}^\beta(X, \rho)} < \infty \text{ for every } \rho \in \mathbf{q}\}. \end{aligned} \quad (2.43)$$

Note that for each fixed $\rho \in \mathbf{q}$, the space $\mathcal{C}^\beta(X, \mathbf{q})$, when equipped with the norm

$$\|\cdot\|_{\mathcal{C}^\beta(X, \rho)} := \|\cdot\|_\infty + \|\cdot\|_{\mathcal{C}^\beta(X, \rho)} \quad (2.44)$$

is a Banach space for every $\beta \in (0, \infty)$. In fact, similar to as above, given any $\beta \in (0, \infty)$, it follows that $\{\|\cdot\|_{\mathcal{C}^\beta(X, \rho)} : \rho \in \mathbf{q}\}$ is a family of equivalent norms on $\mathcal{C}^\beta(X, \mathbf{q})$.

It is instructive to note that the following general fact holds. Given a quasi-metric space (X, ρ) , one has

$$\text{Bdd}(X) \cap \mathcal{C}^\alpha(X, \rho) \subseteq \bigcap_{\beta \in (0, \alpha]} \mathcal{C}^\beta(X, \rho), \quad \forall \alpha \in (0, \infty), \quad (2.45)$$

where

$$\text{Bdd}(X) := \{ f : X \rightarrow \mathbb{C} : \|f\|_\infty < \infty \}. \quad (2.46)$$

Moreover, the inclusion in (2.45) is quantitative in the sense that for each $\alpha \in (0, \infty)$ and each $\beta \in (0, \alpha)$ there holds

$$\|f\|_{\dot{\mathcal{C}}^\beta(X, \rho)} \leq \max \{ 2\|f\|_\infty, \|f\|_{\dot{\mathcal{C}}^\alpha(X, \rho)} \}, \forall f \in \text{Bdd}(X) \cap \dot{\mathcal{C}}^\alpha(X, \rho). \quad (2.47)$$

Going further, we wish to note that the function spaces defined in (2.38) and (2.43) exhibit a certain type of homogeneity with respect to power-rescalings of the quasi-distance. Specifically, if (X, ρ) is a quasi-metric space and $\alpha \in (0, \infty)$ is fixed, then (X, ρ^α) is a quasi-metric space and

$$\dot{\mathcal{C}}^\beta(X, \rho^\alpha) = \dot{\mathcal{C}}^{\alpha\beta}(X, \rho) \text{ and } \mathcal{C}^\beta(X, \rho^\alpha) = \mathcal{C}^{\alpha\beta}(X, \rho), \quad \forall \beta \in (0, \infty). \quad (2.48)$$

We now present the result pertaining to the existence of a partition of unity.

Theorem 2.5 (Partition of Unity) *Let (X, \mathbf{q}) be a geometrically doubling quasi-metric space and suppose Ω is a proper nonempty subset of X . Fix $\rho \in \mathbf{q}$ along with a number $\lambda > C_\rho^2$, where C_ρ is as in (2.2), and consider the decomposition of Ω into the family $\{B_\rho(x_j, r_j)\}_{j \in \mathbb{N}}$ as given by Theorem 2.4 for this choice of λ . Finally, consider a number $\lambda' \in (C_\rho, \lambda/C_\rho)$. Then for every $\alpha \in \mathbb{R}$ satisfying*

$$0 < \alpha \leq [\log_2 C_\rho]^{-1}, \quad (2.49)$$

there exist a finite constant $C \geq 1$, depending only on ρ, α, M , and the proportionality constants in (4) of Theorem 2.4, along with a family of real-valued functions $\{\varphi_j\}_{j \in \mathbb{N}}$ defined on X such that the following conditions are valid:

(1) *for each $j \in \mathbb{N}$ one has*

$$\varphi_j \in \dot{\mathcal{C}}^\beta(X, \mathbf{q}) \quad \text{and} \quad \|\varphi_j\|_{\dot{\mathcal{C}}^\beta(X, \rho)} \leq C r_j^{-\beta}, \quad (2.50)$$

for every $\beta \in (0, \alpha]$;

(2) *for every $j \in \mathbb{N}$ one has*

$$\begin{aligned} 0 \leq \varphi_j \leq 1 \text{ on } X, \quad \varphi_j \equiv 0 \text{ on } X \setminus B_\rho(x_j, \lambda' r_j), \\ \text{and } \varphi_j \geq 1/C \text{ on } B_\rho(x_j, r_j); \end{aligned} \quad (2.51)$$

(3) *one has $\sum_{j \in \mathbb{N}} \varphi_j = \mathbf{1}_{\bigcup_{j \in \mathbb{N}} B_\rho(x_j, r_j)} = \mathbf{1}_{\bigcup_{j \in \mathbb{N}} B_\rho(x_j, \lambda' r_j)} = \mathbf{1}_{\bigcup_{j \in \mathbb{N}} B_\rho(x_j, \lambda r_j)}$.*

Proof The conclusion of this theorem is a direct result of Theorem 5.1 in [AlMiMi13] with the exception of (2.50), where it was only shown to be valid

for $\beta = \alpha$. However, if (2.50) is valid for $\beta = \alpha$ then the conditions in (2.51) ensure (2.50) also holds for every $\beta \in (0, \alpha]$. \square

The following result is quantitative version of the classical Urysohn's lemma which was originally proved in [MiMiMiMo13, Theorem 4.12, p. 165] and subsequently generalized in [AlMiMi13].

Theorem 2.6 *Let (X, \mathbf{q}) be a quasi-metric space and fix $\rho \in \mathbf{q}$. Let $C_\rho \in [1, \infty)$ be as in (2.2) and consider a finite number $\beta \in (0, (\log_2 C_\rho)^{-1}]$. Suppose $F_0, F_1 \subseteq X$ are two nonempty sets with the property that $\text{dist}_\rho(F_0, F_1) > 0$. Then, there exists a finite constant $C = C(\rho) > 0$ and a function $\psi \in \mathcal{C}^\beta(X, \mathbf{q})$ such that*

$$0 \leq \psi \leq 1 \text{ on } X, \quad \psi \equiv 0 \text{ on } F_0, \quad \psi \equiv 1 \text{ on } F_1, \quad (2.52)$$

and for which

$$\|\psi\|_{\mathcal{C}^\beta(X, \rho)} \leq C(\text{dist}_\rho(F_0, F_1))^{-\beta}. \quad (2.53)$$

As a corollary, the space $\mathcal{C}^\beta(X, \mathbf{q})$ separates the points in X . In particular, the space $\mathcal{C}^\beta(X, \mathbf{q})$ contains non-constant functions.

2.3 Vitali-Type Covering Lemma on Quasi-Metric Spaces

Proposition 2.8 below is the main result in the section where we further elaborate on the nature of the topological structure induced by a quasi-metric. As a preamble, we first record the following basic covering result, in the spirit of Vitali's covering lemma, proved in [MiMiMi13].

Lemma 2.7 *Let (X, ρ) be a quasi-metric space and fix a finite constant $C_o > C_\rho^2 \tilde{C}_\rho$. Consider a family of ρ -balls*

$$\mathcal{A} = \{B_\rho(x_\alpha, r_\alpha)\}_{\alpha \in I}, \quad x_\alpha \in X, \quad r_\alpha \in (0, \infty) \text{ for every } \alpha \in I, \quad (2.54)$$

such that

$$\sup_{\alpha \in I} r_\alpha < \infty. \quad (2.55)$$

In addition, suppose that either

$$(X, \tau_\rho) \text{ is separable}, \quad (2.56)$$

or

for every sequence $\{B_\rho(x_j, r_j)\}_{j \in \mathbb{N}} \subseteq \mathcal{A}$ consisting
of mutually disjoint ρ -balls one has $\lim_{j \rightarrow \infty} r_j = 0$. (2.57)

Then there exists an at most countable set $J \subseteq I$ with the property that

$$B_\rho(x_j, r_j) \cap B_\rho(x_k, r_k) = \emptyset \quad \forall j, k \in J \text{ with } j \neq k, \quad (2.58)$$

and each ρ -ball from \mathcal{A} is contained in a dilated ρ -ball of the form $B_\rho(x_j, C_\rho r_j)$ for some $j \in J$. In particular,

$$\bigcup_{\alpha \in I} B_\rho(x_\alpha, r_\alpha) \subseteq \bigcup_{j \in J} B_\rho(x_j, C_\rho r_j). \quad (2.59)$$

In turn, the above Vitali-type covering lemma is the main ingredient in establishing the following result pertaining to the nature of the open sets in the topology induced by a quasi-metric. To introduce some notation, suppose (X, ρ) is a quasi-metric space and, as usual, denote by τ_ρ the topology canonically induced by ρ on X . In this context, given any $A \subseteq X$ let \bar{A} and A° stand, respectively, for the closure and interior of A in the topology τ_ρ . In this regard, it is useful to recall from [MiMiMiMo13, p. 149, (3.544)–(3.545)] that

$$\theta \in (0, C_\rho^{-1}) \implies \overline{B_\rho(x, \theta r)} \subseteq B_\rho(x, r) \subseteq (B_\rho(x, \theta^{-1} r))^\circ, \quad \forall x \in X, \forall r \in (0, \infty). \quad (2.60)$$

Proposition 2.8 *Let (X, ρ) be a quasi-metric space such that (X, τ_ρ) is separable. Consider an arbitrary, nonempty open set \mathcal{O} (in the topology τ_ρ) and fix some $\varepsilon \in (0, \infty)$.*

Then there exist a sequence of points $\{x_j\}_{j \in \mathbb{N}}$ in X and a sequence of positive numbers $\{r_j\}_{j \in \mathbb{N}}$ such that the following properties hold:

- (i) $0 < r_j < \varepsilon$ for each $j \in \mathbb{N}$;
- (ii) $\mathcal{O} = \bigcup_{j \in \mathbb{N}} B_\rho(x_j, r_j) = \bigcup_{j \in \mathbb{N}} \overline{B_\rho(x_j, r_j)} = \bigcup_{j \in \mathbb{N}} (B_\rho(x_j, r_j))^\circ$;
- (iii) there exists $\theta \in (0, 1)$ with the property that the ρ -balls $B_\rho(x_j, \theta r_j)$, $j \in \mathbb{N}$, are mutually disjoint.

Proof Assume that $\varepsilon \in (0, \infty)$ is given and fix a finite number $M > 4C_\rho^4 \tilde{C}_\rho$. Since \mathcal{O} is open, it follows that for every $x \in \mathcal{O}$ there exists $r(x) \in (0, \infty)$ such that $B_\rho(x, r(x)) \subseteq \mathcal{O}$. Introduce $\bar{r}(x) := \min\{\varepsilon, r(x)\}$ and apply Lemma 2.7 to the family of ρ -balls with bounded radii

$$\left\{ B_\rho\left(x, \frac{\bar{r}(x)}{M}\right) \right\}_{x \in \mathcal{O}}. \quad (2.61)$$

Hence, since the topological space (X, τ_ρ) is separable, Lemma 2.7 applies and gives the existence of a sequence $\{x_j\}_{j \in \mathbb{N}}$ of points in \mathcal{O} with the property that the ρ -balls

$$B_\rho\left(x_j, \frac{\bar{r}(x_j)}{M}\right), \quad j \in \mathbb{N}, \quad \text{are mutually disjoint,} \quad (2.62)$$

and

$$\forall x \in \mathcal{O} \quad \exists j = j(x) \in \mathbb{N} \text{ such that } B_\rho\left(x, \frac{\bar{r}(x)}{M}\right) \subseteq B_\rho\left(x_j, \frac{2C_\rho^2 \tilde{C}_\rho \bar{r}(x_j)}{M}\right). \quad (2.63)$$

Define

$$r_j := \frac{4C_\rho^3 \tilde{C}_\rho \bar{r}(x_j)}{M} \quad \text{for each } j \in \mathbb{N}. \quad (2.64)$$

We claim that the x_j 's and r_j 's just constructed are such that properties (i)–(ii) are satisfied. To see this, note that since $M > L$ and $\bar{r}(x_j) < \varepsilon$, it is immediate that $r_j < \varepsilon$ for every $j \in \mathbb{N}$. Moreover, the above choices ensure that

$$B_\rho\left(x_j, \frac{2C_\rho^2 \tilde{C}_\rho \bar{r}(x_j)}{M}\right) = B_\rho\left(x_j, \frac{r_j}{2C_\rho}\right) \subseteq (B_\rho(x_j, r_j))^\circ, \quad \text{for every } j \in \mathbb{N}, \quad (2.65)$$

thanks to (2.60). Based on (2.65) and (2.63), we may therefore conclude that

$$\mathcal{O} \subseteq \bigcup_{j \in \mathbb{N}} (B_\rho(x_j, r_j))^\circ. \quad (2.66)$$

Moving on, whenever $\lambda \in (C_\rho, \frac{M}{4C_\rho^3 \tilde{C}_\rho})$, which is a non-degenerate interval given that $M > 4C_\rho^4 \tilde{C}_\rho$, then $\lambda r_j \leq \bar{r}(x_j) \leq r(x_j)$ for every $j \in \mathbb{N}$ so that, by (2.60),

$$\overline{B_\rho(x_j, r_j)} \subseteq B_\rho(x_j, \lambda r_j) \subseteq B_\rho(x_j, r(x_j)) \subseteq \mathcal{O}, \quad \forall j \in \mathbb{N}. \quad (2.67)$$

Hence,

$$\bigcup_{j \in \mathbb{N}} \overline{B_\rho(x_j, r_j)} \subseteq \mathcal{O}. \quad (2.68)$$

By combining (2.66) and (2.68), we may therefore conclude that (ii) holds. Finally, choose $\theta \in (0, 1)$ so that $0 < \theta < \frac{1}{4C_\rho^3 \tilde{C}_\rho}$. Then $\theta r_j \leq \frac{\bar{r}(x_j)}{M}$ which, in view of (2.62), shows that (iii) holds for this choice of θ , completing the proof of the proposition. \square

2.4 Ahlfors-Regular Quasi-Metric Spaces

The bulk of this section is devoted to developing an important subclass of spaces of homogeneous type in which we will choose to establish a theory of Hardy spaces which generalizes well-known results in the d -dimensional Euclidean setting (where $d \in \mathbb{N}$). \mathbb{R}^d is a very resourceful environment which, among other things, has a vector space structure as well as the notion of differentiability. In contrast, we wish to work in a setting which has minimal assumptions on the geometric and measure theoretic aspects since, from the perspective of applications, it is not often that we get to work in such a resourceful environment.

One such general context which has provided an environment rich enough to do a good deal of analysis on is a space of homogeneous type introduced by R.R. Coifman and G. Weiss in [CoWe71, p. 66] (see also [CoWe77, p. 587] where the measure is assumed to be doubling (see (2.80) below). In this setting, although a theory of Hardy spaces exists, the assumptions are so general that it is even difficult to identify when these named spaces are trivial (i.e., reduce to just constants). It is this qualitative nature of the Hardy space theory which is undesirable for application purposes.

In this work, we will ask more of our measure (in a fashion which would not compromise our desire for minimal assumptions on the ambient) and in turn we will be able to produce a theory which generalizes results in the Euclidean setting to a more general geometric measure theoretic context. More importantly, this is done without compromising the quantitative aspects of such a theory.

Given the generality of the framework of a space of homogeneous type, it may be the case that the measure of a singleton is positive.³ However, as it was shown in [MaSe79i], there can only be at most countably many such points. For the completeness of the theory developed in the subsequent sections of this work, we wish to consider a space which still allows for the existence of atoms. The specifics of this space are described in Definition 2.11 below. However, a few preliminary notions must first be discussed.

Moving on, we make the following convention, an arbitrary set X and a topology τ on X , we denote by $Borel_\tau(X)$, the smallest sigma-algebra of X containing τ . With this in mind we now record a few measure theoretic notions in Definition 2.9 below.

Definition 2.9 Suppose X is a set and τ is any topology on X . Assume \mathfrak{M} is a sigma-algebra of subsets of X and consider a measure $\mu : \mathfrak{M} \rightarrow [0, \infty]$.

1. Call μ a Borel measure on (X, τ) (or simply on X if the topology is understood) provided $Borel_\tau(X) \subseteq \mathfrak{M}$.
2. The measure μ is said to be a Borel-regular measure (again, on (X, τ) or simply on X if the topology is understood) provided μ is a Borel measure on X satisfying

³Such points have been historically referred to as “atoms”.

$$\begin{aligned} &\text{for every } A \in \mathfrak{M}, \text{ there exists } B \in \text{Borel}_\tau(X) \\ &\text{with the property that } A \subseteq B \text{ and } \mu(A) = \mu(B). \end{aligned} \quad (2.69)$$

3. Given a quasi-metric structure \mathbf{q} on X , call the measure μ locally finite provided the μ -measure of every bounded subset of X is finite.

Comment 2.10 In regards to parts 1 and 2 of Definition 2.9, the reader is alerted to the fact that for a measure $\mu : \mathfrak{M} \rightarrow [0, \infty]$ to be Borel measure we merely demand that \mathfrak{M} contains $\text{Borel}_\tau(X)$ and not necessarily that $\mathfrak{M} = \text{Borel}_\tau(X)$. In fact, in the latter case the measure μ would automatically be Borel-regular. In particular, if $\mu : \mathfrak{M} \rightarrow [0, \infty]$ is a Borel measure then $\mu|_{\text{Borel}_\tau(X)}$ is Borel-regular measure. ■

We next record some definitions regarding certain aspects of the geometry of a quasi-metric space. Suppose (X, \mathbf{q}) is a quasi-metric space, $\rho \in \mathbf{q}$, and μ is a nonnegative measure X . In this setting we define for each $x \in X$

$$R_\rho(x) := \begin{cases} \sup\{r \in (0, \infty) : B_\rho(x, r) \neq X\} & \text{if } \mu(X) < \infty, \\ \infty & \text{if } \mu(X) = \infty, \end{cases} \quad (2.70)$$

and

$$r_\rho(x) := \inf\{r \in (0, \infty) : B_\rho(x, r) \neq \{x\}\}. \quad (2.71)$$

In the definition of a spaces of homogeneous type one typically demands that the measure of every ball is finite (see (3.1) below for more details). This assumption implies that the underlying set is bounded whenever the space has infinite measure. In this regard, at least roughly speaking, the additional assumption in (2.70) that $R_\rho(x) = \infty$ whenever $\mu(X) = \infty$ can be thought of as an analogous condition in this setting.

It is readily seen from the definitions in (2.70)–(2.71) that

$$r_\rho(x) \in [0, \infty) \text{ and } R_\rho(x) \in (0, \infty] \text{ are well-defined for every } x \in X, \quad (2.72)$$

$$r_\rho(x) \leq R_\rho(x) \text{ for every } x \in X, \quad (2.73)$$

$$\begin{aligned} \forall \beta \in (0, \infty) \implies r_{\rho^\beta}(x) &= [r_\rho(x)]^\beta \text{ and } R_{\rho^\beta}(x) = [R_\rho(x)]^\beta \\ &\text{for every } x \in X, \end{aligned} \quad (2.74)$$

$$\text{for every } x \in X, r_\rho(x) > 0 \implies B_\rho(x, r_\rho(x)) = \{x\}, \quad (2.75)$$

$$\begin{aligned} \text{for every } x \in X, R_\rho(x) < \infty \implies X \setminus B_\rho(x, R_\rho(x)) \\ = \{y \in X : \rho(x, y) = R_\rho(x)\}, \end{aligned} \quad (2.76)$$

and also

if $\varrho \in \mathbf{q}$, that is, if $C_1, C_2 \in (0, \infty)$ are such that $C_1\varrho \leq \rho \leq C_2\varrho$ pointwise on $X \times X$ then $C_1R_\varrho \leq R_\rho \leq C_2R_\varrho$ and $C_1r_\varrho \leq r_\rho \leq C_2r_\varrho$ pointwise on X . (2.77)

Observe that if (X, \mathbf{q}) is a quasi-metric space, $\rho \in \mathbf{q}$, and μ is a nonnegative measure on X with the property that all ρ -balls are μ -measurable then every singleton in X is μ -measurable. With this in mind, we make the following definition.

Definition 2.11 Call a triplet (X, \mathbf{q}, μ) a d -Ahlfors-regular (quasi-metric) space (or simply, a d -AR space) if the pair (X, \mathbf{q}) is a quasi-metric space, μ is a nonnegative measure on X and if for some number $d \in (0, \infty)$ there exist $\rho \in \mathbf{q}$ and four constants $C_1, C_2, c_1, c_2 \in (0, \infty)$ with $c_1 \leq 1 \leq c_2$ having the following property: all ρ -balls are μ -measurable and

$$C_1r^d \leq \mu(B_\rho(x, r)) \leq C_2r^d, \quad \text{for all } x \in X \quad (2.78)$$

and $r \in (0, \infty)$ with $c_1r_\rho(x) \leq r \leq c_2R_\rho(x)$,

where r_ρ and R_ρ are as in (2.70)–(2.71).

Additionally, call a d -Ahlfors-regular quasi-metric space, (X, \mathbf{q}, μ) , a standard d -Ahlfors-regular (quasi-metric) space if $r_\rho(x) = 0$ for every $x \in X$.

Note that by possibly decreasing and increasing, respectively, the constants C_1 and C_2 in (2.78), we can assume without consequence that $C_1 \in (0, 1]$ and $C_2 \in [1, \infty)$. The constants c_1, c_2, C_1 , and C_2 will be referred to as constants depending on μ . Going further, given a set X with cardinality at least 2 along with a quasi-distance $\rho \in \mathfrak{Q}(X)$ and a nonnegative measure μ on X satisfying the Ahlfors-regularity condition described in (2.78) with ρ , we let (X, ρ, μ) denote the d -AR space $(X, [\rho], \mu)$.

We now collect some basic properties of d -AR spaces.

Proposition 2.12 Suppose (X, \mathbf{q}, μ) is a d -AR space for some $d \in (0, \infty)$. Specifically, suppose $\rho \in \mathbf{q}$ satisfies (2.78). Then there exists $C \in (0, \infty)$ such that the following hold.

1. If $\mu(X) < \infty$ then $\text{diam}_\rho(X) < \infty$ and

$$0 < (C_\rho \tilde{C}_\rho)^{-1} \text{diam}_\rho(X) \leq \inf_{x \in X} R_\rho(x) \leq \sup_{x \in X} R_\rho(x) \leq \text{diam}_\rho(X); \quad (2.79)$$

where $C_\rho, \tilde{C}_\rho \in [1, \infty)$ are as in (2.2)–(2.3);

2. $\mu(B_\rho(x, r)) \leq Cr^d$, for every $x \in X$ and positive $r \in [c_1r_\rho(x), \infty)$, where $c_1 \in (0, 1]$ is as Definition 2.11; this property will be referred to as the upper-Ahlfors-regularity condition for μ ;

3. $C^{-1}r^d \leq \mu(B_\rho(x, r))$, for every $x \in X$ and finite $r \in (0, c_2 R_\rho(x)]$, where the constant $c_2 \in [1, \infty)$ is as Definition 2.11; this property will be referred to as the lower-Ahlfors-regularity condition for μ ;
4. $\sup_{x \in X} r_\rho(x) \leq \text{diam}_\rho(X)$;
5. $C^{-1}[r_\rho(x)]^d \leq \mu(\{x\}) \leq C[r_\rho(x)]^d$ for every $x \in X$;
6. $C^{-1}[R_\rho(x)]^d \leq \mu(X) \leq C[R_\rho(x)]^d$ for every $x \in X$;
7. $\text{diam}_\rho(X) < \infty$ if and only if $\mu(X) < \infty$;
8. for every parameter $\lambda \in [1, \infty)$, there exists some finite constant $c > 0$ such that $cr^d \leq \mu(B_\rho(x, r))$, for every $x \in X$ and finite $r \in (0, \lambda R_\rho(x)]$; in particular, for every parameter $\lambda \in [1, \infty)$, there exists some finite constant $c > 0$ such that $cr^d \leq \mu(B_\rho(x, r))$, for every $x \in X$ and finite $r \in (0, \lambda \text{diam}_\rho(X)]$;
9. $\mu(B_\rho(x, r)) \in (0, \infty)$ for every $x \in X$ and $r \in (0, \infty)$;
10. μ satisfies (2.78) (with the same constants c_1, c_2) for any other $\varrho \in \mathbf{q}$ having the property that all ϱ -balls are μ measurable;
11. for every point $x \in X$ and every radius $r \in (0, \infty)$, $B_\rho(x, r) = \{x\}$ if and only if there holds $r \in (0, r_\rho(x)]$;
12. for every point $x \in X$ and every radius $r \in (0, \infty)$, $B_\rho(x, r) = X$ if and only if there holds $r \in (R_\rho(x), \infty)$;
13. μ satisfies the following doubling property: there exists a finite constant $\kappa > 0$ such that

$$0 < \mu(B_\rho(x, 2r)) \leq \kappa \mu(B_\rho(x, r)) < \infty, \quad \forall x \in X, \quad \forall r \in (0, \infty); \quad (2.80)$$

14. one has

$$\mu \text{ is a Borel measure on } (X, \tau_{\mathbf{q}}), \quad (2.81)$$

where $\tau_{\mathbf{q}}$ is the topology induced by the quasi-metric space structure \mathbf{q} on X ;

15. there holds

$$(X, \rho^\beta, \mu) \text{ is a } \frac{d}{\beta}\text{-AR space for each fixed } \beta \in (0, \infty); \quad (2.82)$$

more specifically, if $\beta \in (0, \infty)$ is fixed then μ satisfies the regularity condition listed in (2.78) in Definition 2.11 with ρ^β and with constants C_1, C_2, c_1^β , and c_2^β .

Proof We begin proving 1. First observe that if $\mu(X) < \infty$ then the condition in (2.78) implies $\sup_{x \in X} R_\rho(x) < \infty$. Combining this fact with (2.76) gives $B_\rho(x, R_\rho(x) + 1) = X$ for every $x \in X$. Hence, $\text{diam}_\rho(X) < \infty$. Turning our attention to proving the inequalities in (2.79), fix $x \in X$. Observe that by the definition of a ρ -ball and the nondegeneracy of the quasi-distance ρ , we have for every $y \in X$ with $y \neq x$ that $\rho(x, y) > 0$ and $y \in X \setminus B_\rho(x, \rho(x, y))$. In particular, $B_\rho(x, \rho(x, y)) \neq X$. Therefore, by (2.70) we have $\rho(x, y) \leq R_\rho(x)$. As such, if $y, z \in X$ then

$$\rho(z, y) \leq C_\rho \max\{\rho(z, x), \rho(x, y)\} \leq C_\rho \tilde{C}_\rho R_\rho(x), \quad (2.83)$$

which further implies

$$\text{diam}_\rho(X) \leq C_\rho \tilde{C}_\rho R_\rho(x), \quad (2.84)$$

given that $y, z \in X$ were arbitrary. Moving on, if $r \in (0, \infty)$ is such that $B_\rho(x, r) \neq X$ then we may choose $y \in X \setminus B_\rho(x, r)$ and write

$$r \leq \rho(x, y) \leq \text{diam}_\rho(X). \quad (2.85)$$

Taking the supremum over all such r (recalling that in this case we are assuming $\mu(X) < \infty$) gives $R_\rho(x) \leq \text{diam}_\rho(X)$. Given that $x \in X$ was arbitrary, the inequalities in *I* follow from this and the estimate in (2.84).

Moving on, we next prove 2. Pick $x \in X$ and $r \in (0, \infty)$ such that $r \geq c_1 r_\rho(x)$. From (2.78) we know that $\mu(B_\rho(x, r)) \leq C_2 r^d$ whenever $r \leq c_2 R_\rho(x)$. Thus suppose $r > c_2 R_\rho(x)$. In this case, we necessarily have that $R_\rho(x)$, and therefore $\mu(X)$, is finite (cf. (2.70)). Also, from (2.76) and the fact that $c_2 \geq 1$ we have $B_\rho(x, r) = X$. Thus, 3 will follow once we show the existence of a constant $C \in (0, \infty)$, which is independent of x and r , such that

$$\mu(X) \leq C r^d. \quad (2.86)$$

Given that $\mu(X) < \infty$, it is possible to choose a number $C \in (0, \infty)$ satisfying

$$C > (C_\rho \tilde{C}_\rho)^d \mu(X) / \text{diam}_\rho(X)^d. \quad (2.87)$$

Note that it follows from (2.79) in *I* that such a choice of C implies (2.86) holds granted that

$$(C_\rho \tilde{C}_\rho)^d \mu(X) / \text{diam}_\rho(X)^d \geq \mu(X) / R_\rho(x)^d, \quad (2.88)$$

and $r > R_\rho(x)$. This completes the proof of 2.

Disposing next of the claim in 3 pick $x \in X$ and $r \in (0, \infty)$ such that $r \leq c_2 R_\rho(x)$. From (2.78) we know that $C_1 r^d \leq \mu(B_\rho(x, r))$ whenever $r \geq c_1 r_\rho(x)$. Thus suppose $r < c_1 r_\rho(x)$. Then necessarily we have that $r_\rho(x) > 0$. Moreover, collectively (2.75) and the fact that $c_1 \leq 1$ imply $B_\rho(x, r) = \{x\}$, for $r < c_1 r_\rho(x)$. Therefore, in order to finish the proof of 3, we want a constant $C \in (0, \infty)$, independent of x and r , such that

$$C r^d \leq \mu(\{x\}). \quad (2.89)$$

Observe that given $0 < r < c_1 r_\rho(x)$, the condition in (2.78) (with $r_\rho(x)$ in place of r) implies

$$\mu(\{x\}) \geq C_1 [r_\rho(x)]^d \geq C_1 c_1^{-d} r^d. \quad (2.90)$$

Note that the usage of (2.78) is valid in this scenario granted (2.73) along with the fact that $c_1 \leq 1$ give $c_1 r_\rho(x) \leq r_\rho(x) \leq c_2 R_\rho(x)$. Hence, (2.89) holds whenever $C \in (0, C_1 c_1^{-d})$.

Moving on, we next address the claim in 4. Fix $x \in X$ and note that since the cardinality of X is at least 2, we may choose a point $y \in X$ with $y \neq x$. Then for every $\varepsilon \in (1, \infty)$ we have

$$r_\rho(x) < \varepsilon \rho(x, y) \leq \varepsilon \text{diam}_\rho(X), \quad (2.91)$$

where the first inequality above is a consequence of (2.71), (2.75), and the fact $y \in B_\rho(x, \varepsilon \rho(x, y))$ with $x \neq y$. Hence,

$$\sup_{x \in X} r_\rho(x) \leq \varepsilon \text{diam}_\rho(X), \quad (2.92)$$

from which the desired conclusion follows granted $\varepsilon \in (1, \infty)$ was arbitrary.

Disposing next of the claim in 5, fix $x \in X$ and note that if $r_\rho(x) > 0$, then the desired conclusion follows immediately from combining (2.75) and (2.78). Note that the use of (2.78) is valid since $c_1 r_\rho(x) \leq r_\rho(x) \leq c_2 R_\rho(x)$ given (2.73) and the fact that $c_1 \leq 1 \leq c_2$. On the other hand, if $r_\rho(x) = 0$, then it follows from what has been established in 2 that $\mu(\{x\}) = 0$. Hence, the estimates in 5 hold in this case as well.

We move forward to the proof of 6. Fix $x \in X$ and note that in light of (2.70), the desired conclusion follows if $\mu(X) = \infty$. If on the other hand, $\mu(X) < \infty$ then necessarily we have $R_\rho(x) \in (0, \infty)$ by (2.72) and 1. Consequently, the first inequality in 6 follows from (2.78) and the fact that $c_1 r_\rho(x) \leq R_\rho(x) \leq c_2 R_\rho(x)$. Regarding the second inequality, observe that (2.76) implies $B_\rho(x, 2R_\rho(x)) = X$ which in conjunction with 2 gives

$$\mu(X) = \mu(B_\rho(x, 2R_\rho(x))) \leq C[2R_\rho(x)]^d \quad (2.93)$$

as desired.

Regarding the claim in 7, the fact that $\text{diam}_\rho(X) < \infty$ whenever $\mu(X) < \infty$ follows from 1. Conversely, if $\text{diam}_\rho(X) < \infty$ then fix $x \in X$ and choose the radius $r \in (r_\rho(x), \infty)$ large enough so that $B_\rho(x, r) = X$. Note that such a choice of r is possible granted 4. From 2 we have

$$\mu(X) = \mu(B_\rho(x, r)) \leq Cr^d < \infty \quad (2.94)$$

completing the proof of 7.

We prove 8 in a similar fashion as 2 except that if the radius $r \in (0, \infty)$ is such that $c_2 R_\rho(x) < r \leq \lambda R_\rho(x)$ then we demand $C \in (0, \infty)$ satisfies

$$C < \mu(X)/\text{diam}_\rho(X)^d \leq \mu(X)/\lambda R_\rho(x)^d. \quad (2.95)$$

Again, such a choice of C is guaranteed in the current scenario by I .

Moving on, note that 9 now follows from 2 and 3 and that 10 is an immediate consequence of parts 2–3 as well as (2.77) and (2.78).

As for the claim in 11 , it is clear that if $x \in X$ and $r \in (0, r_\rho(x)]$ then $r_\rho(x) > 0$. It therefore follows from (2.75) that $B_\rho(x, r) = \{x\}$. Conversely, if $B_\rho(x, r) = \{x\}$, then combining parts 9 and 5 we have that

$$C[r_\rho(x)]^d \leq \mu(\{x\}) = \mu(B_\rho(x, r)) > 0. \quad (2.96)$$

Hence, $r_\rho(x) > 0$ and the fact that $r \in (0, r_\rho(x)]$ follows from (2.71) and (2.75). This completes the proof of 11 . The justification for 12 follows along a similar line of reasoning used in the proof of 11 .

Observing that (2.80) follows from using 2–3 we address next the claim in (2.81). It is well known, doubling condition in (2.80) implies the ambient space is geometrically doubling in the sense of Definition 2.3 (cf. [CoWe71]). Consequently, (2.81) follows from part (I) in Theorem 2.4 and (2.78).

There remains the matter of justifying 15 . In this regard, fix $\beta \in (0, \infty)$ and recall from (2.10) that

$$B_{\rho^\beta}(x, r) = B_\rho(x, r^{1/\beta}) \quad \text{for every } x \in X \text{ and every } r \in (0, \infty). \quad (2.97)$$

From this observation, we can see immediately that all balls with respect to the quasi-distance ρ^β are μ -measurable given the measurability of the ρ -balls. Moreover, the equality in (2.97) when used in conjunction with the fact that μ satisfies the Ahlfors-regularity condition in (2.78) (with ρ) gives

$$\begin{aligned} \mu(B_{\rho^\beta}(x, r)) &= \mu(B_\rho(x, r^{1/\beta})) \approx r^{d/\beta} \quad \text{uniformly for every } x \in X \\ &\text{and } r \in (0, \infty) \text{ satisfying } c_1 r_\rho(x) \leq r^{1/\beta} \leq c_2 R_\rho(x). \end{aligned} \quad (2.98)$$

On the other hand, by (2.74) we have

$$\left. \begin{aligned} x \in X \text{ and } r \in (0, \infty) \text{ with } \\ c_1 r_\rho(x) \leq r^{1/\beta} \leq c_2 R_\rho(x) \end{aligned} \right\} \implies c_1^\beta r_{\rho^\beta}(x) \leq r \leq c_2^\beta R_{\rho^\beta}(x), \quad (2.99)$$

which in concert with (2.98) ultimately yields (2.82). This completes the proof of the proposition. \square

Comment 2.13 As a consequence of Proposition 2.12, the following fact holds. Suppose (X, \mathbf{q}, μ) is a d -AR space for some $d \in (0, \infty)$. Then, one has

$$\rho \in \mathbf{q} \implies \begin{cases} \mu \text{ satisfies the } d\text{-dimensional Ahlfors-regularity} \\ \text{condition stated in (2.78) with } \rho_\# \in \mathbf{q} \end{cases} \quad (2.100)$$

where the quasi-distance $\rho_{\#}$ denotes the regularized version of ρ defined in (2.21). Indeed, this is an immediate consequence of Theorem 2.1 and (2.81) in Proposition 2.12. \blacksquare

Let us further augment the list of properties in Proposition 2.12 with the following result pertaining to the nature of a Cartesian product of Ahlfors-regular quasi-metric spaces.

Proposition 2.14 *Let $N \in \mathbb{N}$ be fixed and assume that (X_i, ρ_i) , $1 \leq i \leq N$, are quasi-metric spaces. Define $X := \prod_{i=1}^N X_i$ and consider $\rho := \bigvee_{i=1}^N \rho_i : X \times X \rightarrow [0, \infty)$ concretely given by*

$$\begin{aligned} \rho(x, y) &:= \max_{1 \leq i \leq N} \rho_i(x_i, y_i) \text{ for all } x = (x_1, \dots, x_N), \\ y &= (y_1, \dots, y_N) \in X. \end{aligned} \quad (2.101)$$

Then $\rho \in \mathfrak{Q}(X)$. Moreover, assume that each (X_i, ρ_i) is equipped with a measure μ_i which renders the triplet (X_i, ρ_i, μ_i) a d_i -AR space for some $d_i \in (0, \infty)$, and consider the product measure defined by $\mu := \mu_1 \otimes \mu_2 \otimes \dots \otimes \mu_N$ on X . Then

$$(X, \rho, \mu) \text{ is } \left(\sum_{1 \leq i \leq N} d_i \right)\text{-AR}. \quad (2.102)$$

Proof All claims are straightforward consequences of definitions. \square

We conclude this section by making some remarks. First, in the context of a d -Ahlfors-regular space, we do not need to assume initially that the measure μ is Borel, but rather (as Proposition 2.12 outlines) this is a quality that μ inherits as a consequence (2.78). It is remarkable that this phenomenon still remains valid in the more general setting of spaces of homogeneous type where the measure is only assumed to be doubling in the sense that μ satisfies the condition described in (2.80). Secondly, the doubling condition in (2.80) along with (2.81) implies that every Ahlfors-regular quasi-metric space of dimension $d \in (0, \infty)$ is a space of homogeneous type in the sense of [CoWe71] and [CoWe77].

Lastly, granted Proposition 2.12, if we consider symmetric quasi-distances, then it is straightforward to check that when $d = 1$, the definition of a 1-AR space is equivalent to the notion of a normal space in [MaSe79i, p.258] and [MaSe79ii, p.272] due to R.A. Macías and C. Segovia. Moreover, regarding the notion of normal spaces of a given order (cf. [MaSe79ii, 1.9 on p.272]), recall a normal space (X, ρ, μ) shall be referred to as a normal space of order $\alpha \in (0, \infty)$ if ρ is symmetric and there exists a finite constant $K_0 > 0$ with the property that

$$|\rho(x, z) - \rho(y, z)| \leq K_0 r^{1-\alpha} \rho(x, y)^\alpha, \quad (2.103)$$

for every $x, y, z \in X$ satisfying $\max\{\rho(x, z), \rho(y, z)\} < r$. Although, in principle, the notion of a normal space is valid for all $\alpha \in (0, \infty)$, the authors proved in [MaSe79i, Theorem 2, p. 259], that given an arbitrary space of homogeneous type, there exists a normal space only of order $\alpha \in (0, 1)$. In comparison, we wish mention that in light of Theorem 2.1, any given 1-AR space is a normal space of order $\min\{1, \beta\}$ for every finite $\beta \in (0, \alpha]$ where α is defined as in (2.21). Hence, d -AR spaces constitute a generalization of the spaces considered in [MaSe79ii].

We now conclude this section by giving a few interesting examples of d -AR spaces, the first of which may be regarded as the prototypical example.

Example 1 Given $d \in \mathbb{N}$, and a number $\beta \in (0, \infty)$, then

$$(\mathbb{R}^d, |\cdot - \cdot|^\beta, \mathcal{L}^d), \quad (2.104)$$

where \mathcal{L}^d is the d -dimensional Lebesgue measure on \mathbb{R}^d , is an Ahlfors-regular space of dimension d/β . ■

The next example often arises in several areas of analysis.

Example 2 Given $d \in \mathbb{N}$, $d \geq 2$, suppose that $\Sigma \subseteq \mathbb{R}^d$ is the graph of a real-valued Lipschitz function defined in \mathbb{R}^{d-1} . Fix $\beta \in (0, \infty)$ and consider

$$(\Sigma, |\cdot - \cdot|^\beta, \mathcal{H}^{d-1}|_\Sigma), \quad (2.105)$$

where \mathcal{H}^{d-1} is the $(d-1)$ -dimensional Hausdorff measure on \mathbb{R}^d restricted to the set Σ is an Ahlfors-regular space of dimension $(d-1)/\beta$. ■

In the previous example, the set Σ possessed a fair amount of regularity. In contrast the following example highlights the fact the underlying set can be rather rough and yet still be equipped with an Ahlfors-regular measure.

Example 3 (The Four-Corner Planar Cantor Set) Consider $E_0 := [0, 1]^2$, the unit square in \mathbb{R}^2 , and let \mathcal{C}_1 be the set consisting of the four (closed) squares $\{Q_1^j\}_{j=1, \dots, 4}$, of side-length 4^{-1} which are located in the corners of E_0 and set $E_1 := \bigcup_{j=1}^4 Q_1^j$. Iteratively, for each $n \in \mathbb{N}$ we let \mathcal{C}_n denote the n -th generation of squares defined as the collection of 4^n squares $\{Q_n^j\}_{j=1, \dots, 4^n}$, of side-length $\ell(Q_n^j) = 4^{-n}$, which are located in the corners of E_{n-1} (i.e., each Q_n^j , $j = 1, \dots, 4^n$, is located in one of the corners of the square Q_{n-1}^k , for some $k \in \{1, \dots, 4^{n-1}\}$) and set $E_n := \bigcup_{j=1}^{4^n} Q_n^j$. Having introduced this notation, the four-corner Cantor set in \mathbb{R}^2 , is then given by (Fig. 2.1)

$$E := \bigcap_{n=0}^{\infty} E_n. \quad (2.106)$$

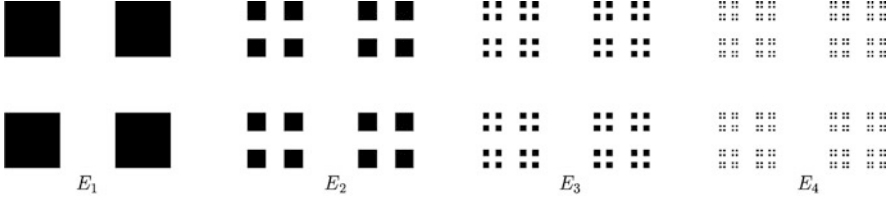


Fig. 2.1 The first four iterations in the construction of the four-corner Cantor set

It has been shown in [MiMiMiMo13, Proposition 4.79, p.238] (see also [MiMiMiMo13, Corollary 4.80,p.245]) that for each fixed $\beta \in (0, \infty)$, the space

$$(E, |\cdot - \cdot|^\beta|_E, \mathcal{H}^1 \llcorner E) \quad (2.107)$$

is a $1/\beta$ -Ahlfors-regular quasi-metric space. ■

As is apparent from the above examples, the Hausdorff outer-measure plays a conspicuous role, at least in the Euclidean setting. Recently, in [MiMiMi13] it has been shown that the Hausdorff outer-measure defined on quasi-metric spaces continues to enjoy most of the properties of its counterpart from the setting of Euclidean spaces (see, e.g., [EvGa92] for a good reference of these properties). For example, it is a basic result in the Euclidean setting that the Hausdorff outer-measure is a Borel-regular outer-measure. This phenomenon, to some degree, continues to transpire in the more general context of quasi-metric spaces. We present this result, from [MiMiMi13], in Proposition 2.16 below. First, a definition is in order.

Definition 2.15 Let (X, ρ) be a quasi-metric space, and fix $d \in [0, \infty)$. Given a set $E \subseteq X$, for every $\varepsilon \in (0, \infty)$ define

$$\mathcal{H}_{X,\rho,\varepsilon}^d(E) := \inf \left\{ \sum_{j=1}^{\infty} r_j^d : E \subseteq \bigcup_{j=1}^{\infty} B_\rho(x_j, r_j) \text{ and } r_j \leq \varepsilon \text{ for every } j \right\} \quad (2.108)$$

(with the convention that $\inf \emptyset := \infty$), then define the Hausdorff outer-measure⁴ of dimension d in (X, ρ) of the set E as

$$\mathcal{H}_{X,\rho}^d(E) := \lim_{\varepsilon \rightarrow 0^+} \mathcal{H}_{X,\rho,\varepsilon}^d(E) = \sup_{\varepsilon > 0} \mathcal{H}_{X,\rho,\varepsilon}^d(E) \in [0, \infty]. \quad (2.109)$$

Also, define the Hausdorff dimension in (X, ρ) of the set E by the formula

$$\dim_{X,\rho}^{\mathcal{H}}(E) := \inf \{d \in [0, \infty) : \mathcal{H}_{X,\rho}^d(E) = 0\} \quad (2.110)$$

again, with the convention that $\inf \emptyset := \infty$.

⁴In general, given a nonempty set X , call a function $\mu : 2^X \rightarrow [0, \infty]$ an outer-measure if $\mu(\emptyset) = 0$ and $\mu(E) \leq \sum_{j \in \mathbb{N}} \mu(E_j)$ whenever $E, \{E_j\}_{j \in \mathbb{N}} \subseteq 2^X$ satisfy $E \subseteq \bigcup_{j \in \mathbb{N}} E_j$.

We now make a few notational conventions. Given a quasi-metric space (X, ρ) , and nonempty subset $E \subseteq X$, we will denote by $\rho|_E$, the function defined on $E \times E$ obtained by restricting the function ρ to the set $E \times E$. It is clear that the function $\rho|_E$ is a quasi-distance on E . As such, we can consider the canonical topology induced by the quasi-distance $\rho|_E$ on E , which we will denote by $\tau_{\rho|_E}$. We are now in a position to state the aforementioned proposition (see [EvGa92, p. 5,61] for a version of this result specialized to the Euclidean setting, and [MiMiMi13] for the more general setting considered here).

Proposition 2.16 *Let (X, ρ) be a quasi-metric space and fix a number $d \in (0, \infty)$. Also, consider the regularized quasi-distance $\rho_{\#}$ (constructed in relation to ρ) defined as in (2.21). Then for any $E \subseteq X$, the restriction of the Hausdorff outer-measure $\mathcal{H}_{X, \rho_{\#}}^d$ to E , i.e., $\mathcal{H}_{X, \rho_{\#}}^d|_E$, is a Borel-regular outer-measure on $(E, \tau_{\rho|_E})$, and the measure associated with it (via restriction to the sigma-algebra of $\mathcal{H}_{X, \rho_{\#}}^d$ -measurable subsets of E , in the sense of Carathéodory) is a Borel-regular measure on $(E, \tau_{\rho|_E})$.*

Furthermore, if E is $\mathcal{H}_{X, \rho_{\#}}^d$ -measurable (in the sense of Carathéodory; hence, in particular, if E is a Borel subset of (E, τ_{ρ})), then the restriction to E of the measure associated with the outer-measure $\mathcal{H}_{X, \rho_{\#}}^d$ (as above) is a Borel-regular measure on $(E, \tau_{\rho|_E})$.

At this stage we are prepared to shed light on the following issue. Given a quasi-metric space (X, ρ) , characterize all Borel measures on X which satisfy an Ahlfors-regularity condition with a given exponent $d \in (0, \infty)$. In Proposition 2.17 below we shall show that if there is such a measure μ on X , then the d -dimensional Hausdorff measure $\mathcal{H}_{X, \rho_{\#}}^d$ on X also satisfies the aforementioned Ahlfors-regularity condition. Moreover, if μ is Borel-regular then necessarily μ is comparable with $\mathcal{H}_{X, \rho_{\#}}^d$. In particular, this explains the ubiquitous role played by the Hausdorff measure in the examples of Ahlfors-regular spaces presented earlier in (2.104)–(2.107).

Proposition 2.17 *Assume that (X, \mathbf{q}, μ) is a standard d -Ahlfors-regular quasi-metric space for some $d \in (0, \infty)$, i.e., assume (X, \mathbf{q}) is a quasi-metric space and suppose μ is a measure on X with the property that there exists $\rho \in \mathbf{q}$ and $\kappa_1, \kappa_2 \in (0, \infty)$ such that all ρ -balls are μ -measurable and*

$$\kappa_1 r^d \leq \mu(B_{\rho}(x, r)) \leq \kappa_2 r^d, \text{ for all } x \in X \text{ and all finite } r \in (0, \text{diam}_{\rho}(X)]. \quad (2.111)$$

Then, with $\rho_{\#}$ denoting the regularized version of ρ as in (2.21),

$$\mathcal{H}_{X, \rho_{\#}}^d(B_{\rho}(x, r)) \approx r^d, \text{ uniformly for all } x \in X \text{ and all finite } r \in (0, \text{diam}_{\rho}(X)]. \quad (2.112)$$

Also, μ is a Borel measure and there exist two finite constants $C_1, C_2 > 0$ such that, if τ_ρ denotes the topology canonically induced by ρ on X , one has

$$\mu(E) \leq C_2 \mathcal{H}_{X, \rho\#}^d(E) \quad \text{for every } \mu\text{-measurable set } E \subseteq X, \quad \text{and} \quad (2.113)$$

$$C_1 \mathcal{H}_{X, \rho\#}^d(E) \leq \inf_{E \subseteq \mathcal{O} \in \tau_\rho} \mu(\mathcal{O}) \quad \text{for every set } E \subseteq X. \quad (2.114)$$

Moreover, there exists a unique function f satisfying the following properties:

- (i) f is $\text{Borel}_{\tau_\rho}(X)$ -measurable,
- (ii) $\exists C_3, C_4 \in (0, \infty)$ and $\exists A \in \text{Borel}_{\tau_\rho}(X)$ with $\mathcal{H}_{X, \rho\#}^d(A) = 0$
such that $C_3 \leq f(x) \leq C_4$ for every point $x \in X \setminus A$,
- (iii) $\mu|_{\text{Borel}_{\tau_\rho}(X)} = f \mathcal{H}_{X, \rho\#}^d|_{\text{Borel}_{\tau_\rho}(X)}$.

Hence, in particular,

$$\mu|_{\text{Borel}_{\tau_\rho}(X)} \approx \mathcal{H}_{X, \rho\#}^d|_{\text{Borel}_{\tau_\rho}(X)}. \quad (2.116)$$

In addition, if the measure μ is actually Borel-regular, then for the same constants C_1, C_2 as above

$$C_1 \mathcal{H}_{X, \rho\#}^d(E) \leq \mu(E) \leq C_2 \mathcal{H}_{X, \rho\#}^d(E) \quad \text{for all } \mu\text{-measurable sets } E \subseteq X. \quad (2.117)$$

Proof We begin by observing that μ is a Borel measure, as noted in part 14 of Proposition 2.12. Moving on, from assumption (2.111) it follows that μ is a doubling measure (in the sense that μ satisfies the condition described in (2.80)). In turn, this implies that (X, ρ) is geometrically doubling (cf. [CoWe71, p. 67]), hence

$$(X, \tau_\rho) \quad \text{is separable} \quad (2.118)$$

by (2.35). Our first goal is to show that the upper bound in (2.112) holds. For this purpose, let $x \in X$ and some finite $r \in (0, \text{diam}_\rho(X)]$ be fixed. Also, consider some $\varepsilon \in (0, r)$. From Lemma 2.7 it follows that it is possible to cover $B_\rho(x, r)$ with an at most countable family of ρ -balls of radii equal to ε , i.e., one can choose a family of points $x_j \in X, j \in I$ with I at most countable, such that

$$B_\rho(x, r) \subseteq \bigcup_{j \in I} B_\rho(x_j, \varepsilon) \quad \text{and} \quad B_\rho(x_j, \varepsilon) \cap B_\rho(x, r) \neq \emptyset \quad \text{for all } j \in I. \quad (2.119)$$

By once more applying Vitali's lemma (cf. Lemma 2.7), there exists a set $J \subseteq I$ (which makes J at most countable) such that $\{B_\rho(x_j, \varepsilon)\}_{j \in J}$ are mutually disjoint and

$$B_\rho(x, r) \subseteq \bigcup_{j \in J} B_\rho(x_j, 3C_\rho^2 \varepsilon). \quad (2.120)$$

Since by the second part of (2.119) we have $B_\rho(x_j, \varepsilon) \subseteq B_\rho(x, C_\rho(r + 2C_\rho\varepsilon))$ for each $j \in J$, we obtain

$$\begin{aligned} \mathcal{H}_{X, \rho_\#, \varepsilon}^d(B_\rho(x, r)) &\leq c \sum_{j \in J} \varepsilon^d \leq c' \sum_{j \in J} \mu(B_\rho(x_j, \varepsilon)) = c' \mu\left(\bigcup_{j \in J} B_\rho(x_j, \varepsilon)\right) \\ &\leq c' \mu(B_\rho(x, C_\rho(r + 2C_\rho\varepsilon))) \leq c' \mu(B_\rho(x, C_\rho(1 + 2C_\rho)r)) \\ &\leq c' c_2 (C_\rho(1 + 2C_\rho)r)^d, \end{aligned} \quad (2.121)$$

where we have used (2.111) and the fact that the ρ -balls are μ -measurable. After passing to the limit as $\varepsilon \rightarrow 0^+$, we therefore arrive at

$$\mathcal{H}_{X, \rho_\#}^d(B_\rho(x, r)) = \lim_{\varepsilon \rightarrow 0^+} \mathcal{H}_{X, \rho_\#, \varepsilon}^d(B_\rho(x, r)) \leq Cr^d, \quad (2.122)$$

which is the upper bound in (2.112).

Regarding the lower bound in (2.112), let x, r retain their earlier significance and fix an arbitrary $\varepsilon \in (0, \infty)$. If we now cover $B_\rho(x, r) \subseteq \bigcup_{j=1}^\infty B_\rho(x_j, r_j)$ for some $x_j \in X$, $0 < r_j < \varepsilon$, $j \in \mathbb{N}$, (as before, such a cover always exists) then by the upper bound in (2.111),

$$\mu(B_\rho(x, r)) \leq \sum_{j=1}^\infty \mu(B_\rho(x_j, r_j)) \leq \kappa_2 \sum_{j=1}^\infty r_j^d. \quad (2.123)$$

Taking the infimum of the two most extreme sides of (2.123) over all such covers with $0 < r_j < \varepsilon$ gives $\mu(B_\rho(x, r)) \leq c \mathcal{H}_{X, \rho_\#, \varepsilon}^d(B_\rho(x, r))$ hence, using the lower bound in (2.111),

$$cr^d \leq \lim_{\varepsilon \rightarrow 0^+} \mathcal{H}_{X, \rho_\#, \varepsilon}^d(B_\rho(x, r)) = \mathcal{H}_{X, \rho_\#}^d(B_\rho(x, r)), \quad (2.124)$$

as wanted. In summary, the above reasoning shows that

$$\begin{aligned} r^d &\approx \mu(B_\rho(x, r)) \approx \mathcal{H}_{X, \rho_\#}^d(B_\rho(x, r)) \quad \text{uniformly} \\ &\text{for all } x \in X \text{ and all finite } r \in (0, \text{diam}_\rho(X)], \end{aligned} \quad (2.125)$$

proving (2.112).

Consider next (2.113). To proceed, fix an arbitrary μ -measurable set $E \subseteq X$ and assume that $\mathcal{H}_{X, \rho_\#}^d(E) < \infty$ (since otherwise there is nothing to prove). Also, fix some finite $\varepsilon > 0$. Then for any cover $\{B_\rho(x_j, r_j)\}_{j \in \mathbb{N}}$ of E with $x_j \in X$ and $0 < r_j < \varepsilon$

for all $j \in \mathbb{N}$ (that such a cover exists is implicit in the fact that $\mathcal{H}_{X,\rho\#}^d(E) < \infty$) we can write, based on the monotonicity and subadditivity of the measure μ ,

$$\mu(E) \leq \mu\left(\bigcup_{j=1}^{\infty} B_{\rho}(x_j, r_j)\right) \leq \sum_{j=1}^{\infty} \mu(B_{\rho}(x_j, r_j)) \leq C \sum_{j=1}^{\infty} r_j^d, \quad (2.126)$$

where for the last inequality we have used the upper-bound in (2.111). Hence, taking the infimum over all such covers we obtain

$$\mu(E) \leq C \mathcal{H}_{X,\rho\#, \varepsilon}^d(E) \leq C \mathcal{H}_{X,\rho\#}^d(E), \quad (2.127)$$

proving (2.113).

To prove (2.114), suppose next that $E \subseteq X$ is arbitrary. Let $\mathcal{O} \subseteq X$ be an open set in τ_{ρ} such that $E \subseteq \mathcal{O}$ and assume that $\{B_{\rho}(x_j, r_j)\}_{j \in \mathbb{N}}$ and $\theta \in (0, 1)$ are as in Proposition 2.8. Then making use of (2.125) we have (again, recall that ρ -balls are μ -measurable in the current case):

$$\begin{aligned} \mathcal{H}_{X,\rho\#}^d(E) &\leq \mathcal{H}_{X,\rho\#}^d(\mathcal{O}) \leq \sum_{j \in \mathbb{N}} \mathcal{H}_{X,\rho\#}^d(B_{\rho}(x_j, r_j)) \approx \sum_{j \in \mathbb{N}} \mu(B_{\rho}(x_j, r_j)) \\ &\leq C \sum_{j \in \mathbb{N}} \mu(B_{\rho}(x_j, \theta r_j)) = C \mu\left(\bigcup_{j \in \mathbb{N}} B_{\rho}(x_j, \theta r_j)\right) \leq C \mu(\mathcal{O}). \end{aligned} \quad (2.128)$$

Taking the infimum over all open sets \mathcal{O} containing E now yields (2.114).

Consider next the issue of existence of a function f as in (2.115). First observe that by (2.112) and Proposition 2.16 we have that

$$(X, \text{Borel}_{\tau_{\rho}}(X), \mathcal{H}_{X,\rho\#}^d|_{\text{Borel}_{\tau_{\rho}}(X)}) \quad \text{is a sigma-finite measure space.} \quad (2.129)$$

On the other hand, $\mu|_{\text{Borel}_{\tau_{\rho}}(X)}$ is a Borel measure on X and estimate (2.113) entails

$$\mu|_{\text{Borel}_{\tau_{\rho}}(X)} \ll \mathcal{H}_{X,\rho\#}^d|_{\text{Borel}_{\tau_{\rho}}(X)}. \quad (2.130)$$

Having established (2.129)–(2.130), the Radon-Nikodym Theorem gives the existence of a nonnegative function f satisfying (i) and (iii) in (2.115). Moreover, (see [Ru76i, Theorem 1.40, p. 30]) there exists $A \in \text{Borel}_{\tau_{\rho}}(X)$ with the property that $\mathcal{H}_{X,\rho\#}^d(A) = 0$ and for $x \in X \setminus A$,

$$f(x) \in \overline{\left\{ \frac{1}{\mathcal{H}_{X,\rho\#}^d(E)} \int_E f \, d\mathcal{H}_{X,\rho\#}^d : E \in \text{Borel}_{\tau_{\rho}}(X), \mathcal{H}_{X,\rho\#}^d(E) > 0 \right\}} \quad (2.131)$$

with the closure taken in the canonical topology of \mathbb{R} . On the other hand, if the set $E \in \text{Borel}_{\tau_\rho}(X)$ is such that $\mathcal{H}_{X,\rho\#}^d(E) > 0$, (2.113) gives

$$\frac{1}{\mathcal{H}_{X,\rho\#}^d(E)} \int_E f d\mathcal{H}_{X,\rho\#}^d = \frac{\mu(E)}{\mathcal{H}_{X,\rho\#}^d(E)} \leq C_2. \quad (2.132)$$

With this in hand, we deduce from (2.131) that f also satisfies $0 \leq f(x) \leq C_2$ for each $x \in X \setminus A$, for some $A \in \text{Borel}_{\tau_\rho}(X)$ with $\mathcal{H}_{X,\rho\#}^d(A) = 0$. Thus, in order to complete the proof of (ii) in (2.115), there remains to establish a bound from below (away from the zero) for f . To this end, based on (2.114), the fact that ρ -balls are open and (iii) in (2.115), we may write

$$C_1 \mathcal{H}_{X,\rho}^d(B_\rho(x, r)) \leq \mu(B_\rho(x, r)) = \int_{B_\rho(x, r)} f d\mathcal{H}_{X,\rho}^d. \quad (2.133)$$

Employing Lebesgue's Differentiation Theorem (see the implication (1) \Rightarrow (3) in Theorem 3.14 below for details) there exists $A \in \text{Borel}_{\tau_\rho}(X)$ such that $\mathcal{H}_{X,\rho\#}^d(A) = 0$ and

$$\lim_{r \rightarrow 0^+} \int_{B_\rho(x, r)} f d\mathcal{H}_{X,\rho\#}^d = f(x) \quad \forall x \in X \setminus A. \quad (2.134)$$

Thus, based on (2.133) and (2.134) the lower bound from (ii) in (2.115) follows, as desired.

As far as (2.117) is concerned, observe that Proposition 2.8 and (2.111) show that the measure μ has the property that

$$\exists \{O_j\}_{j \in \mathbb{N}} \subseteq \tau_\rho \text{ so that } X = \bigcup_{j \in \mathbb{N}} O_j \text{ and } \mu(O_j) < \infty \quad \forall j \in \mathbb{N}. \quad (2.135)$$

The relevance of this property stems from the implication (cf. [MiMiMi13] for details)

$$\begin{aligned} & \mu \text{ Borel-regular measure on } X \text{ satisfying (2.135)} \\ \implies & \mu(E) = \inf_{E \subseteq \mathcal{O} \in \tau_\rho} \mu(\mathcal{O}), \text{ for all } \mu\text{-measurable sets } E \subseteq X. \end{aligned} \quad (2.136)$$

As such, (2.117) follows from this, (2.113) and (2.114), finishing the proof of the proposition. \square

Comment 2.18 A careful inspection of the proof of Proposition 2.17 reveals that the arguments made in justifying the upper-bound in (2.112), and the estimate in (2.113) yields the following more nuanced conclusions. Assume that (X, ρ) is a quasi-metric space and let μ be an upper d -Ahlfors-regular measure on X , i.e.,

suppose there exists a quasi-distance $\rho \in \mathbf{q}$ with the property that all ρ -balls are μ -measurable and assume for some $d \in (0, \infty)$ and some $c \in (0, \infty)$ there holds

$$\mu(B_\rho(x, r)) \leq cr^d, \quad \text{for all } x \in X \text{ and all finite } r \in (0, \text{diam}_\rho(X)]. \quad (2.137)$$

Then, with $\rho_\#$ denoting the regularized version of ρ as in (2.21), there exists a finite constant $C > 0$ such that

$$\begin{aligned} \mathcal{H}_{X, \rho_\#}^d(B_\rho(x, r)) &\leq Cr^d, \quad \text{uniformly for all } x \in X \\ &\text{and all finite } r \in (0, \text{diam}_\rho(X)], \end{aligned} \quad (2.138)$$

and

$$\mu(E) \leq C_2 \mathcal{H}_{X, \rho_\#}^d(E) \quad \text{for every } \mu\text{-measurable set } E \subseteq X. \quad (2.139)$$

■

2.5 The Smoothness Indices of a Quasi-Metric Space

The goal of this section is to briefly survey some of the new concepts presented in [MiMiMiMo13] regarding to what the authors refer to as the lower smoothness and Hölder indices. One issue that arises in working with Hardy spaces, $H^p(X)$, in the setting of spaces of homogeneous type is that unless p is “near” to 1, then the spaces become trivial. This is a consequence of the fact that Hölder spaces may reduce to just constant functions if the order is too large. (cf., e.g., the comment on the footnote on p. 591 in [CoWe77] where the authors qualitatively mention an unspecified range of p ’s for which this occurs). This phenomenon is well-known in the Euclidean setting where the space of Hölder functions $\mathcal{C}^\beta(\mathbb{R}^d)$ is trivial (i.e., reduces to just constant functions) whenever $\beta \in (1, \infty)$. However, given an arbitrary quasi-metric space, this upper bound, in principle, may not be 1. Therefore, the natural questions are, how should one interpret this upper bound for β , and is it possible to identify such a bound in the context of a more general setting?

In an effort to answer these questions in quantifiable manner, the authors in [MiMiMiMo13, pp. 196–246] have provided a new angle on this question by introducing the notion of “index” (see Definition 2.19 below). In this work, this notion of index is going to play a fundamental role in the formulation and proofs of many of our main results. For example, the index will help identify the optimal range of p ’s for which there exists a rich theory of Hardy spaces in spaces of homogeneous type. More specifically, the index permits us to determine just how far p can be below 1 while still having a maximal characterization of the atomic Hardy spaces introduced in [CoWe77].

For the purposes we have in mind for this work, we only wish to touch briefly upon this notion of index. The reader is referred to [MiMiMiMo13, pp. 196–246], wherein the authors provide a systematic treatment in exploring this relatively new concept.

Definition 2.19 Suppose (X, \mathbf{q}) is a given a quasi-metric space.

(I) The lower smoothness index of (X, \mathbf{q}) is defined as

$$\text{ind}(X, \mathbf{q}) := \sup \{ [\log_2 C_\rho]^{-1} : \rho \in \mathbf{q} \} \in (0, \infty] \quad (2.140)$$

where, for every $\rho \in \Omega(X)$, the constant C_ρ has been introduced in (2.2).

(II) The Hölder index of (X, \mathbf{q}) is defined as

$$\text{ind}_H(X, \mathbf{q}) := \inf \{ \alpha \in (0, \infty) : \forall x, y \in X \text{ and } \forall \varepsilon > 0 \quad (2.141)$$

$$\exists \xi_1, \dots, \xi_{N+1} \in X \text{ such that } \xi_1 = x, \xi_{N+1} = y \text{ and } \sum_{i=1}^N \rho(\xi_i, \xi_{i+1})^\alpha < \varepsilon \},$$

with the agreement that $\inf \emptyset := \infty$.

Whenever X is an arbitrary set of cardinality at least 2 and $\rho \in \Omega(X)$, abbreviate $\text{ind}(X, \rho) := \text{ind}(X, [\rho])$ and $\text{ind}_H(X, \rho) := \text{ind}_H(X, [\rho])$.

The terminology of “Hölder index” used for (2.141) is justified by the fact that

$$\text{ind}_H(X, \mathbf{q}) = \sup \{ \alpha \in (0, \infty) : \mathcal{C}^\alpha(X, \mathbf{q}) \neq \mathbb{C} \} \in (0, \infty], \quad (2.142)$$

which follows from [MiMiMiMo13, Theorem 4.59, p. 215].

In the context of Definition 2.19, one could ask if the supremum listed in (2.140) is ever attained. In other words, does there exists a quasi-distance $\rho \in \mathbf{q}$ for which the corresponding value of C_ρ is the smallest among all other quasi-distances belonging to \mathbf{q} ? As the next proposition will highlight, the answer is yes in the Euclidean setting, $(\mathbb{R}^d, |\cdot - \cdot|)$. Howbeit, this anomaly is not to be expected in arbitrary quasi-metric spaces. In [BriMi13] the authors successfully managed to construct a quasi-metric space for which the lower smoothness index is not attained (which has been recorded in Chap. 1 as Theorem 1.1). Hence, the issue of whether or not the lower smoothness index of a given ambient is attained is a delicate matter. In fact, as we will see later in this work, this directly affects the range of p ’s for which we are guaranteed nontrivial Hardy spaces. What is becoming apparent is that, in developing a Hardy space theory in this degree of generality, the nature of the geometry of the ambient and the amount of analysis which can be performed on it are intimately connected.

In order to obtain a better understanding of $\text{ind}(X, \mathbf{q})$ and $\text{ind}_H(X, \mathbf{q})$, the following proposition collects just a few of their properties. Again, the reader is referred to [MiMiMiMo13, pp. 196–246] for further results as well as complete

proofs of the statements provided below. In this regard, recall that a quasi-metric space (X, \mathbf{q}) is said to be *imperfect* provided there exist a quasi-distance $\rho \in \mathbf{q}$, a point $x_0 \in X$, and a number $r \in (0, \infty)$, with the property that

$$X \setminus B_\rho(x_0, r) \neq \emptyset \quad \text{and} \quad \text{dist}_\rho(X \setminus B_\rho(x_0, r), B_\rho(x_0, r)) > 0. \quad (2.143)$$

To the point, this condition amounts to the ambient space having an “island”. With this definition in mind we now present the following proposition.

Proposition 2.20 *Suppose (X, \mathbf{q}) is a quasi-metric space and $\rho \in \mathbf{q}$. Then*

1. $[\log_2 C_\rho]^{-1} \leq \text{ind}(X, \mathbf{q}) \leq \text{ind}_H(X, \mathbf{q})$;
2. $\text{ind}(X, \rho^\alpha) = \frac{1}{\alpha} \text{ind}(X, \rho)$ and also $\text{ind}_H(X, \rho^\alpha) = \frac{1}{\alpha} \text{ind}_H(X, \rho)$, for every number $\alpha \in (0, \infty)$.
3. *There holds*

$$\text{ind}(X, \rho) = \sup \{ \alpha \in (0, \infty) : \rho_\alpha \approx \rho \text{ pointwise on } X \times X \}, \quad (2.144)$$

$$\text{ind}_H(X, \rho) = \inf \{ \alpha \in (0, \infty) : \rho_\alpha = 0 \text{ pointwise on } X \times X \}, \quad (2.145)$$

where ρ_α is defined as in (2.16).

4. *There holds*

- (a) ρ ultrametric on $X \implies \text{ind}(X, \rho) = \infty$; in particular, if X is a set of finite cardinality then $\text{ind}(X, \rho) = \infty$;
- (b) ρ distance on $X \implies \text{ind}(X, \rho) \geq 1$;
- (c) (X, \mathbf{q}) imperfect $\implies \text{ind}_H(X, \mathbf{q}) = \infty$;
- (d) $\text{ind}(Y, \mathbf{q}) \geq \text{ind}(X, \mathbf{q})$ for any subset Y of X ;
- (e) if $(X, \|\cdot\|)$ is a nontrivial normed vector space and if \mathbf{q} stands for the quasi-metric space structure induced by the norm $\|\cdot\|$, then

$$\begin{aligned} \text{ind}(Y, \mathbf{q}) &= \text{ind}_H(Y, \mathbf{q}) = 1, \quad \text{for any} \\ &\text{convex subset } Y \text{ of } X \text{ of cardinality } \geq 2; \end{aligned} \quad (2.146)$$

- (f) $\text{ind}(X, \mathbf{q}) \leq 1$ whenever the interval $[0, 1]$ may be bi-Lipschitzly⁵ embedded into (X, \mathbf{q}) ; and
- (g) if $\text{ind}(X, \mathbf{q}) < 1$, then (X, \mathbf{q}) cannot be bi-Lipschitzly embedded into some \mathbb{R}^d , $d \in \mathbb{N}$.

Comment 2.21 Given quasi-metric space (X, \mathbf{q}) , part 1 in Proposition 2.20 gives that the Hölder index of (X, \mathbf{q}) always dominates the lower smoothness index however we cannot expect that these two quantities should coincide given such an

⁵Recall that given two arbitrary quasi-metric spaces (X_j, \mathbf{q}_j) , $j = 0, 1$, a mapping $\Phi : (X_0, \mathbf{q}_0) \rightarrow (X_1, \mathbf{q}_1)$ is called bi-Lipschitz provided for some (hence, any) $\rho_j \in \mathbf{q}_j$, $j = 0, 1$, one has $\rho_1(\Phi(x), \Phi(y)) \approx \rho_0(x, y)$, uniformly for $x, y \in X_0$.

abstract setting. In particular, although there exist nonconstant Hölder functions of order $\alpha \in [\text{ind}(X, \mathbf{q}), \text{ind}_H(X, \mathbf{q})]$ whenever α is finite, it is not clear if these Hölder spaces have any good properties. Going further, if it was known that $\text{ind}(X, \mathbf{q})$ was attained and was finite, then the corresponding class of Hölder functions of order $\text{ind}(X, \mathbf{q})$ would consist of plenty of nonconstant functions. We will see in Example 1 below that this occurs in the Euclidean setting but should not be expected to happen in general. ■

We continue by recording a result from [MiMiMiMo13] (cf. Proposition 4.28, p. 198) detailing on the nature of the index of a Cartesian product of quasi-metric spaces.

Proposition 2.22 *Let $N \in \mathbb{N}$ be fixed and assume that (X_i, ρ_i) , $1 \leq i \leq N$, are quasi-metric spaces. Define $X := \prod_{i=1}^N X_i$ and consider $\rho := \bigvee_{i=1}^N \rho_i : X \times X \rightarrow [0, \infty)$ as in (2.101). Then*

$$\text{ind}(X, \rho) = \min_{1 \leq i \leq N} \text{ind}(X_i, \rho_i) \quad (2.147)$$

$$\text{ind}_H(X, \rho) = \max_{1 \leq i \leq N} \text{ind}_H(X_i, \rho_i) \quad (2.148)$$

We now take a moment to provide a few examples of ambient spaces and their corresponding indices.

Example 1 As a consequence of (2.146), for any $d \in \mathbb{N}$ and $\alpha \in (0, \infty)$ one has

$$\begin{aligned} \text{ind}(\mathbb{R}^d, |\cdot - \cdot|^\alpha) &= \text{ind}_H(\mathbb{R}^d, |\cdot - \cdot|^\alpha) = \frac{1}{\alpha}, \\ \text{ind}([0, 1]^d, |\cdot - \cdot|^\alpha) &= \text{ind}_H([0, 1]^d, |\cdot - \cdot|^\alpha) = \frac{1}{\alpha}, \end{aligned} \quad (2.149)$$

where $|\cdot|$ denotes the standard Euclidean norm in \mathbb{R}^d . Additionally, for any exponent $p \in (0, \infty]$ one also has⁶

$$\text{ind}\left(L^p(\mathbb{R}), \|\cdot - \cdot\|_{L^p(\mathbb{R})}\right) = \text{ind}\left(\ell^p(\mathbb{N}), \|\cdot - \cdot\|_{\ell^p(\mathbb{N})}\right) = \min\{1, p\}, \quad (2.150)$$

and

$$\text{ind}_H\left(L^p(\mathbb{R}), \|\cdot - \cdot\|_{L^p(\mathbb{R})}\right) = \text{ind}_H\left(\ell^p(\mathbb{N}), \|\cdot - \cdot\|_{\ell^p(\mathbb{N})}\right) = \min\{1, p\}. \quad (2.151)$$

■

⁶Here $L^p(\mathbb{R})$ and $\ell^p(\mathbb{N})$ are defined in a natural fashion. See Sects. 3.2 and 5.1 below for details.

Although the notion of index is of a purely geometric nature, it is remarkable, as the following example describes, that there is still an interaction between the index and measure theoretic aspects of a given ambient.

Example 2 Let (X, ρ) be a pathwise connected quasi-metric space.⁷ With $\dim_{X,\rho}^{\mathcal{H}}$ as in (2.110), suppose that there exists $d \in (0, \infty)$ satisfying

$$\forall x, y \in X \ \exists \Gamma \text{ continuous path joining } x \text{ and } y \text{ with } \dim_{X,\rho}^{\mathcal{H}}(\Gamma) \leq d; \quad (2.152)$$

Then

$$\text{ind}_H(X, \rho) \leq d. \quad (2.153)$$

Therefore, one has

$$\text{ind}(X, \rho) \leq d. \quad (2.154)$$

As a consequence of this result and the observation made in Comment 2.18, given any pathwise connected quasi-metric space (X, ρ) having the property that there exists a nonnegative measure μ on X satisfying the following upper-Ahlfors-regular for some $d \in (0, \infty)$,

$$\begin{aligned} &\text{all } \rho\text{-balls are } \mu\text{-measurable and } \exists c \in (0, \infty) \text{ such that} \\ &\mu(B_\rho(x, r)) \leq cr^d, \quad \text{for all } x \in X \text{ and all finite } r \in (0, \text{diam}_\rho(X)], \end{aligned} \quad (2.155)$$

one necessarily has $\text{ind}_H(X, \rho) \leq d$. Hence, (2.154) holds in this case as well.

A particular case of the above setting which is worth mentioning is (2.105) where the ambient considered, Σ , was the graph of a real-valued Lipschitz function defined in \mathbb{R}^{d-1} . In this case, for any fixed $\beta \in (0, \infty)$, one has that

$$\left(\Sigma, |\cdot - \cdot|^\beta, \mathcal{H}^{d-1}|_\Sigma \right) \quad (2.156)$$

is an Ahlfors-regular space of dimension $(d-1)/\beta$ which is pathwise connected. Hence, in this context

$$\text{ind}(X, \rho) \leq \text{ind}_H(X, \rho) \leq (d-1)/\beta. \quad (2.157)$$

■

⁷Call a quasi-metric space (X, ρ) pathwise connected provided for every pair of points $x, y \in X$, there exists a continuous path $f : [0, 1] \rightarrow (X, \tau_\rho)$ with $f(0) = x$ and $f(1) = y$, where τ_ρ represents the canonical topology induced by the quasi-distance ρ on X . We shall refer to the set $\Gamma := f([0, 1]) \subseteq X$ as a continuous path joining x and y .

The previous example highlighted the fact that if the underlying set of a quasi-metric space exhibits enough regularity (here measured by the connectivity of the set), then the indices listed in Definition 2.19 can not be too large relative to the Hausdorff dimension of the space itself or the Hausdorff dimension of the continuous paths joining various points in the space in question. In contrast, the next two examples illustrate the fact in the absences of any sort of connectivity on the underlying set, both the Hölder and lower smoothness indices can very large.

Example 3 Let

$$X := \{a = (a^{(i)})_{i \in \mathbb{N}} : a^{(i)} \in \{0, 1\} \text{ for each } i \in \mathbb{N}\} \quad (2.158)$$

and define $d : X \times X \rightarrow [0, \infty)$ by setting

$$\begin{aligned} d(a, b) &:= 2^{-D(a, b)}, \quad \forall a = (a^{(i)})_{i \in \mathbb{N}} \in X, \quad \forall b = (b^{(i)})_{i \in \mathbb{N}} \in X, \\ \text{where } D(a, b) &:= \inf \{i \in \mathbb{N} : a^{(i)} \neq b^{(i)}\}, \end{aligned} \quad (2.159)$$

with the convention that $\inf \emptyset = \infty$.

Then, for each $\beta \in (0, \infty)$ it follows that $(X, d^\beta, \mathcal{H}_{X, d^\beta}^{1/\beta})$ is a $1/\beta$ -Ahlfors-regular ultrametric space.⁸ Thus, in particular,

$$\text{ind}_H(X, d^\beta) = \text{ind}(X, d^\beta) = \infty. \quad (2.160)$$

It follows that (X, τ_d) is totally disconnected and, as such, any continuous path in (X, τ_d^β) reduces to just a point. ■

We shall describe next a similar phenomenon to the one presented in Example 3, this time in the context the four-corner planar Cantor set described in Example 4 of Sect. 2.4.

Example 4 If E is the four-corner planar Cantor set from (2.106) and define the function $d_\star : E \times E \rightarrow [0, \infty)$ by setting

$$\begin{aligned} d_\star(x, y) &:= \inf \left\{ r > 0 : \exists \xi_1, \dots, \xi_{N+1} \in E, N \in \mathbb{N}, \text{ such that} \right. \\ &\quad \left. x = \xi_1, y = \xi_{N+1} \text{ and } |\xi_i - \xi_{i+1}| < r, \quad \forall i \in \{1, \dots, N\} \right\}, \end{aligned} \quad (2.161)$$

for each $x, y \in E$. Then, for each fixed $\beta \in (0, \infty)$ it follows that d_\star^β is a well-defined ultrametric on E and $(E, d_\star^\beta, \mathcal{H}_{X, d_\star^\beta}^{1/\beta})$ is a $1/\beta$ -Ahlfors-regular ultrametric space. That is,

$$\text{ind}_H(X, d_\star^\beta) = \text{ind}(X, d_\star^\beta) = \infty. \quad (2.162)$$

⁸In general, call (X, \mathbf{q}, μ) a d -Ahlfors-regular ultrametric space for some $d \in (0, \infty)$ if (X, \mathbf{q}, μ) is a d -AR space and \mathbf{q} contains an ultrametric.

Moreover, while the Euclidean distance restricted to E is not an ultrametric, it is equivalent to d_\star . That is, one has $(E, |\cdot - \cdot|^\beta, \mathcal{H}_{X, d_\star}^{1/\beta})$ a $1/\beta$ -Ahlfors-regular ultrametric space. ■

Additionally, the authors in [MiMiMiMo13] provided another example of an ultrametric on the four-corner Cantor set which is equivalent with the restriction of the Euclidean distance to this set. We include this example in the following comment and refer the reader to [MiMiMiMo13, Comment 4.81, p. 245] for further details.

Comment 2.23 Given a dyadic square Q in \mathbb{R}^2 (always considered to be closed), denote by \tilde{Q} the set consisting of Q with the upper horizontal and right vertical sides removed. In particular, for every $n \in \mathbb{Z}$ the plane \mathbb{R}^2 decomposes into the disjoint union of all \tilde{Q} 's where Q runs through the collection of all dyadic cubes with side-length 2^{-n} . Then the function $\tilde{d} : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow [0, \infty)$ given by

$$\tilde{d}(x, y) := \inf\{\ell(Q) : Q \text{ dyadic cube such that } x, y \in \tilde{Q}\}, \quad \forall x, y \in \mathbb{R}^2, \quad (2.163)$$

is a well-defined ultrametric on \mathbb{R}^2 . In particular, with E denoting the four-corner planar Cantor set in (2.106), it follows that $\tilde{d}|_E$ is an ultrametric on E . Additionally, with d_\star as in (2.161),

$$\tilde{d}|_E \approx d_\star. \quad (2.164)$$

■

The claims made in Comment 2.23 have natural formulations in all space dimensions. In particular, a result related to the one-dimensional version reads as follows.

Example 5 Let $X := [0, 1)$ and for each $x, y \in X$ set

$$d(x, y) := \begin{cases} \ell(x, y), & \text{if } x \neq y, \\ 0, & \text{if } x = y, \end{cases} \quad (2.165)$$

where, for $x, y \in X$ such that $x \neq y$,

$$\begin{aligned} \ell(x, y) &\text{ is the length of the smallest dyadic interval } \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right), \\ &\text{containing both } x \text{ and } y, \text{ where } k \in \mathbb{N} \text{ is such that } 1 \leq k \leq 2^n - 1. \end{aligned} \quad (2.166)$$

Then d is a well-defined ultrametric on X . Hence $\text{ind}_H(X, d) = \text{ind}(X, d) = \infty$. ■

The last example we wish to discuss here illustrates that the inequality $\text{ind}(X, \mathbf{q}) \leq \text{ind}_H(X, \mathbf{q})$ appearing in Proposition 2.20 for any quasi-metric space (X, \mathbf{q}) can be strict. See Comment 4.38 on p.206 and Remark 4.49 on p.211 in [MiMiMiMo13].

Example 6 Let a, b, c, d be four real numbers with the property that $a < b < c < d$. Then,

$$\text{ind}([a, b] \cup [c, d], |\cdot - \cdot|) = 1 \quad (2.167)$$

whereas

$$\text{ind}_H([a, b] \cup [c, d], |\cdot - \cdot|) = \infty. \quad (2.168)$$



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