Fractional Brownian Motion and an Application to Fluids

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Abstract In this chapter, we review the fractional Brownian motion (fBm) and some of its properties. It is a Gaussian process that is characterized by its covariance function and its hurst parameter $H \in (0, 1)$. When $H > 1/2$, we introduce the stochastic integral with respect to a fBm by using fractional integrals. This is a pathwise approach based on the Riemann Stieltjes construction using the Hölder continuity of the process. An application related to fluids is provided. This is an integrodifferential equation representing the dynamic of a vortex filament associated to an inviscid, incompressible, homogeneous fluid in $\mathbb{R}^3$. We prove existence and uniqueness of solutions in a functional space of Sobolev type.

1 Introduction

This chapter begins with a short introduction to fractional Brownian motion (fBm) and a description on the vortex filament associated to a 3D turbulent fluid flow. In Sect. 2, assumptions and the functional setting of our problem have been introduced. We recall the notions of fractional integrals and some related a priori estimates in Sect. 3. In Sect. 4, we introduce the notion of integration with respect to a fBm. Section 5 contains our main results about the existence and uniqueness of a global solution for the Eq. (5) with most of their proofs.

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1.1 Fractional Brownian Motion

Fractional Brownian motion (fBm) was first introduced by Kolmogorov in 1940 by the name "Wiener spiral" [26] within a Hilbert space setting. The name "fractional Brownian motion" is due to Mandelbrot and Van Ness as they gave a representation to the fractional Brownian motion through a fractional integral with respect to the standard Brownian motion in 1968 [31].

Definition 1 Let $B^H = \{B^H_\eta, \eta \geq 0\}$ be a stochastic process, and $H \in (0, 1)$. B^H is called a fractional Brownian motion (fBm) with Hurst parameter H, if it is a centered Gaussian process with the covariance function

$$R_H(\gamma, \eta) = E[B^H_\gamma B^H_\eta] = \frac{1}{2}(\eta^{2H} + \gamma^{2H} - |\eta - \gamma|^{2H}), \quad \gamma, \eta \geq 0. \tag{1}$$

Fractional Brownian motion is parameterized by a parameter H, known as the 'Hurst parameter', and it is also due to Mandelbrot. Mandelbrot named the parameter after the hydrologist Hurst who did a statistical study of the yearly water run-offs of the Nile river [23]. The parameter $H \in (0, 1)$ and when $H = \frac{1}{2}$, we recover the well known standard Brownian motion. Hence fractional Brownian motion is a generalization of the standard Brownian motion. Many authors treat the fBm in three different families corresponding respectively to $0 < H < \frac{1}{2}$, $H = \frac{1}{2}$ and $\frac{1}{2} < H < 1$.

Fractional Brownian motion finds applications in diverse fields such as finance, economics, biology, physics and engineering. Whenever we are interested in processes with self similarity, long memory, stationary increments fBm is a natural candidate. Applications of fBm have been studied by many authors. For instance in mathematical finance, Cheridito [10] constructed arbitrage strategies using fBm, Comte and Renault [12] presents continuous time models with long memory with fBm, and Rogers [38] discussed use of fBm in modeling long range dependence of share returns. In engineering applications for instance, in [19] a queue with an infinite buffer space and fBm as a long-range dependent input have been studied. In [34] Norros presented a model for connectionless traffic using fBm. In [28] the self similarity of fBm has been studied in capturing fractal behavior in Ethernet local area network traffic.

1.2 Properties of Fractional Brownian Motion

Many authors have studied fBm in different approaches. For instance, [20] describes fBm together with a detailed presentation on self similarity which is an interesting property of fBm. Theory and applications of the long range dependence, another widely used property of fBm can be found in [17]. For a comprehensive treatment of fBm, see for instance [8, 33, 35]. The Hurst parameter H relates with the sign of the correlation of the increments, regularity of the sample paths, and many other properties of fBm as described below.

- *Self-similarity*: Fractional Brownian motion with Hurst parameter H is H self similar. For any constant $a > 0$, the two processes $\{B^H(at),\ t \geq 0\}$ and $\{a^H B^H(t),\ t \geq 0\}$ are equal in their probability distributions. This is a fractal property of fBm and results from its covariance function (1).

- *Stationary increments*: Fractional Brownian motion has stationary increments. Using the covariance function (1), one could verify that $\{B^H(t+h) - B^H(h)\}$ and $\{B^H(t)\}$ have the same probability distribution for $h > 0$.

- *Independent increments*: Fractional Brownian motion has independent increments if and only if $H = \frac{1}{2}$. When $H = \frac{1}{2}$ it can be seen that $\mathbb{E}[B^H(t)B^H(s)] = \min\{s, t\}$ and we recover the ordinary Brownian motion. For $H \neq \frac{1}{2}$ increments of fBm are not independent. When $H > \frac{1}{2}$, the increments are positively correlated, and for $H < \frac{1}{2}$, they are negatively correlated.

- *Long range dependence*: Let $\{X(t), t \geq 0\}$ be an $H-$self similar process with stationary increments, and non-degenerate for all t, with $\mathbb{E}[X(1)^2] < \infty$. Suppose $\xi(n) = X(n+1) - X(n)$ and $r(n) = \mathbb{E}[\xi(0)\xi(n)]$, for $n = 0, 1, 2 \dots$. Then, for $\frac{1}{2} < H < 1$ we have $\sum_n |r(n)| = \infty$ and this property is known as long range dependence. Fractional Brownian motion exhibits long range dependence or long memory when $H > \frac{1}{2}$. The coupling between values at distinct points of time decreases quite slowly as the time difference increases. This is in contrast with the short range dependence of some processes where the coupling between values at distinct points of time decreases quite rapidly (exponentially) as the time difference increases.

- *Markovian property*: Recall that a Gaussian process with covariance R is Markovian if and only if $R(s,u) = \frac{R(s,t)R(s,u)}{R(t,t)}$ for all $s \leq t \leq u$. Applying the covariance function in (1) one can see that fBm is Markovian if and only if $H = \frac{1}{2}$.

- *β Hölder continuity*: For the sample path properties of fBm, it has continuous trajectories. In particular, fBm admits a modification which is Hölder continuous of order β if and only if $\beta \in (0, H)$. This can be seen ny applying Kolmogorov's continuity criterion, for instance see [20]. Thus, value of the Hurst parameter H decides the regularity of sample paths.

- *Differentiability*: Fractional Brownian motion is almost surely nowhere differentiable. As fBm has stationary increments, consider the point $t = 0$. If the derivative $(B^H)'(0)$ exists, then we have $B^H(s) \leq (\epsilon + (B^H)'(0))s$, for some positive $s \leq s_\epsilon$ and B^H is 1-Hölder continuous at $t = 0$, contradicting the Hölder continuity of fBm.

• *p-variation*: Let f be a real valued function on $[0, T]$, $\Pi := \{t_k \mid 0 = t_0 < t_1 < \cdots < t_n = T\}$, be a partition of $[0, T]$, and $p \in [1, \infty)$. Then the p-variation of f along the partition Π is $V_p(f : \Pi) := \sum_{t_k \in \Pi} |f(t_k) - f(t_{k-1})|^p$. If $V_p(f) := \sup_\Pi V_p(f : \Pi)$ is finite, then f has bounded p-variation. Fractional Brownian motion has zero bounded p-variation when $p > \frac{1}{H}$ a.s. and unbounded p-variation when $p < \frac{1}{H}$.

• *Semimartingale*: A semimartingale can be written as the sum of a bounded variation process and a local martingale which has finite quadratic variation. If $H < \frac{1}{2}$ the quadratic variation is infinite, and if $H > \frac{1}{2}$ the 1-variation is infinite. Thus, most importantly the fBm with $H \neq \frac{1}{2}$ is not a semimartingale. As a consequence, the well developed Itô calculus which is applicable to semi martingales can't be used to define the stochastic integral with respect to the fBm. As a result, different approaches have been introduced when working with fBm.

The definition of stochastic integrals with respect to the fractional Brownian motion $\int_0^t u_s dB_s^H$ where u is a stochastic process has been investigated intensively by several authors. One approach is deterministic pathwise approach that is based on the Riemann-Stieltjes construction using the Hölder continuity and is due to Young [40]. This is applicable only for Hurst parameters $H > \frac{1}{2}$. This idea has been extended, see [14, 15, 22, 29, 30] and a rough path theory which is valid for Hurst parameters $H > \frac{1}{4}$ has been introduced and studied. As fBm is a Gaussian process another approach is to use Malliavin calculus or stochastic calculus of variation with the use of divergence operator and for more details refer [27, 35, 36]. For a comprehensive treatment of stochastic integrals with respect to the fractional Brownian motion refer [1, 2, 9, 13, 16, 18, 25, 37].

1.3 An Application of fBm to Fluids

In the present work, we are using the pathwise argument to solve an integral equation which is an approximation of the line vortex equation. In particular, we will assume that the vorticity field associated to an ideal inviscid incompressible homogeneous fluid in $\mathbb{R}^3$ is described by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ and we will study its evolution through a pathline equation. Let us denote by ω this vorticity field then

$$\boldsymbol{\omega} := \nabla \times \mathbf{u},$$

where $\mathbf{u}$ is the velocity field in $\mathbb{R}^3$. If we denote by $\mathbf{X}_t(\mathbf{x})$ the position at time t of the fluid particle that at time 0 was at $\mathbf{x} \in \mathbb{R}^3$. We have the following path-lines equation

$$\frac{d\mathbf{X}_t(\mathbf{x})}{dt} = \mathbf{u}(t, \mathbf{X}_t(\mathbf{x})). \tag{2}$$

Now, let us assume that the vorticity field is concentrated on a fractional Brownian curve $\mathbf{B}^H$ as follows

$$\boldsymbol{\omega}(t, \mathbf{x}) = \Gamma \int_0^1 \delta(\mathbf{x} - \mathbf{B}_\xi^H(t))\, d\mathbf{B}_\xi^H(t), \tag{3}$$

where δ is the usual "Dirac delta function", $\Gamma > 0$ is the intensity of vorticity, $\xi \in [0, 1]$ is the arc-length, while the parameter t represents the time. Using the Biot-Savart formula, the Eq. (2) becomes

$$\frac{d\mathbf{X}_\xi(t)}{dt} = \int_0^1 Q(\mathbf{X}_\xi(t) - \mathbf{B}_\eta^H(t))\, d\mathbf{B}_\eta^H(t), \tag{4}$$

with

$$\mathbf{X}_\xi(0) = \boldsymbol{\phi}_\xi.$$

Here $\boldsymbol{\phi}$ is the initial condition and the matrix valued function Q is the singular matrix

$$\frac{-\Gamma}{4\pi |\mathbf{y}|^3} \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix}.$$

For an introduction to this topic we refer to [5, 11, 32, 39] and for a more probabilistic approach to [21]. When $\mathbf{B}^H$ is replaced by $\mathbf{X}$, the Eq. (4) has been studied by [3] for a smooth closed curve $\mathbf{X}$ in the Sobolev space $W^{1,2}$ and an existence and uniqueness theorem has been proved for local solutions in time. Later, local solutions in some spaces of Hölder continuous functions have been investigated in [6] that have been extended to global solutions in [4].

In this work, we will be dealing with an approximation of Eq. (4), the approximation will be on the matrix Q, while the study of Eq. (4) is left for a subsequent paper. Furthermore, in order to make the exposition of our results easier to understand, we will make the assumption that the function $\mathbf{X}$ is a real valued function, however, our results will still be true for a vector valued function $\mathbf{X}$. More precisely, we are interested in the following integrodifferential equation

$$Y_\xi(t) = \phi_\xi + \int_0^t \int_0^\xi A(Y_\eta(s))\, dB_\eta^H(s) ds, \tag{5}$$

where for all $t \in [0, T]$, $B^H(t) = \left\{B_\xi^H(t),\ \xi \in [0, 1]\right\}$ is a real valued fractional Brownian motion of Hurst parameter $H > \frac{1}{2}$, A is a bounded and differentiable real valued function with a Lipschitz continuity property, ξ is a parameter in [0,1].

2 Some Preliminaries

2.1 The Integrodifferential Equation

Here we study the following equation.

$$\frac{\partial Y_\xi(t)}{\partial t} = \int_0^\xi A(Y_\eta(t))\, dB_\eta^H(t),\ \ \xi \in [0, 1] \text{ and } t \in [0, T] \tag{6}$$

with

$$Y_\xi(0) = \phi_\xi$$

or alternatively, we can consider the integral form

$$Y_\xi(t) = \phi_\xi + \int_0^t \int_0^\xi A(Y_\eta(s))\, dB_\eta^H(s)ds, \tag{7}$$

where B^H is a fractional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Here the stochasticity is with respect to ξ and this is deterministic with respect to time. The initial condition is ϕ_ξ and $A : \mathbb{R} \to \mathbb{R}$ is a measurable function that satisfies the assumptions given below.

2.2 Assumptions

Let us assume that:
A1. A is differentiable.
A2. There exists $M_1 > 0$ such that $|A(x) - A(y)| \leq M_1|x - y|$ for all $x, y \in \mathbb{R}$.
A3. There exists $M_2 > 0$ such that $|A(x)| \leq M_2$ for all $x \in \mathbb{R}$.
A4. For every N there exists $M_N > 0$, such that
$|A'(x) - A'(y)| \leq M_N|x - y|$ for all $|x|, |y| \leq N$.

2.3 Functional Setting

Let $\frac{1}{2} < H < 1, 1 - H < \alpha < \frac{1}{2}$. We will introduce the following functional spaces.

Let $C([0, T], W^{\alpha,\infty}[0, 1])$ be the space of measurable functions $f : [0, T] \times [0, 1] \to \mathbb{R}$ such that

$$\| f \|_{\alpha,\infty} := \sup_{t\in[0,T]} \sup_{\xi\in[0,1]} \left(|f(t, \xi)| + \int_0^\xi \frac{|f(t, \xi) - f(t, \eta)|}{(\xi - \eta)^{\alpha+1}} d\eta \right) < \infty. \tag{8}$$

Let $C([0,T], W_0^{1-\alpha,\infty}[0,1])$ be the space of measurable functions $f : [0,T] \times [0,1] \to \mathbb{R}$ such that

$$\| f \|_{1-\alpha,\infty,0} := \sup_{t\in[0,T]} \sup_{0<\eta<\xi<1} \left(\frac{|f(t,\xi) - f(t,\eta)|}{(\xi-\eta)^{1-\alpha}} + \int_\eta^\xi \frac{|f(t,\gamma) - f(t,\eta)|}{(\gamma-\eta)^{2-\alpha}} d\gamma \right) < \infty. \tag{9}$$

Let $W^{\alpha,1}([0,1])$ be the space of measurable functions $f : [0,1] \to \mathbb{R}$ such that

$$\| f \|_{\alpha,1} := \int_0^1 \frac{|f(\eta)|}{\eta^\alpha} d\eta + \int_0^1 \int_0^\eta \frac{|f(\eta) - f(\delta)|}{(\eta-\delta)^{\alpha+1}} d\delta d\eta < \infty. \tag{10}$$

3 Some a Priori Estimates

Since fBm with Hurst parameters $H > \frac{1}{2}$ have sample paths that are λ-Hölder continuous for all $\lambda \in (0, H)$, the construction of the integral with respect to a fBm will be performed using a pathwise argument by means of fractional derivatives and integrals. We refer to [24, 37] for more details.

3.1 Fractional Integrals and Derivatives

As usual we denote by $L^p(a,b)$ the space of all Lebesgue measurable functions $f : (a,b) \to \mathbb{R}$ such that

$$\| f \|_{L^p(a,b)} := \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \tag{11}$$

for $a < b$ and $1 \le p < \infty$. Let us recall some definitions on Riemann-Liouville fractional integrals and Weyl derivative.

Definition 2 Let $f \in L^1(a,b)$ and $\alpha > 0$. The left-sided and right-sided Riemann-Liouville fractional integrals of f of order α are defined for almost all $x \in (a,b)$ by

$$I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(y)}{(x-y)^{1-\alpha}} dy \tag{12}$$

and

$$I_{b-}^\alpha f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(y)}{(y-x)^{1-\alpha}} dy \tag{13}$$

respectively, where $(-1)^{-\alpha} = e^{-i\pi\alpha}$ and $\Gamma(\alpha) = \int_0^\infty r^{(\alpha-1)}e^{-r}dr$ is the Gamma function or the Euler integral of the second kind.

Definition 3 Suppose $I^\alpha_{a+}(L^p)$ is the image of $L^p(a,b)$ under the operator I^α_{a+} and $I^\alpha_{b-}(L^p)$ is the image of $L^p(a,b)$ under the operator I^α_{b-}. Let $0 < \alpha < 1$, then we define the Weyl derivative for almost all $x \in (a,b)$ as

$$D^\alpha_{a+}f(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^\alpha} + \alpha\int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}}dy\right)1_{(a,b)}(x) \tag{14}$$

when $f \in I^\alpha_{a+}(L^p)$, and

$$D^\alpha_{b-}f(x) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(b-x)^\alpha} + \alpha\int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha+1}}dy\right)1_{(a,b)}(x) \tag{15}$$

when $f \in I^\alpha_{b-}(L^p)$.

The convergence of the integrals at the singularity $y = x$ holds pointwise for almost all $x \in (a,b)$ when $p = 1$, and in L^p sense when $1 < p < \infty$.

We now introduce the following notations in order to define the generalized Stieltjes integrals. Assuming the limits exist and are finite, let

$f(a+) = \lim_{\epsilon\searrow 0} f(a+\epsilon)$,
$g(b-) = \lim_{\epsilon\searrow 0} g(b-\epsilon)$,
$f_{a+}(x) = [f(x) - f(a+)]1_{(a,b)}(x)$,
$g_{b-}(x) = [g(x) - g(b-)]1_{(a,b)}(x)$.

Definition 4 Suppose that f and g are functions such that $f(a+)$, $g(a+)$ and $g(b-)$ exist, $f_{a+} \in I^\alpha_{a+}(L^p)$ and $g_{b-} \in I^{1-\alpha}_{b-}(L^q)$ for some $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} \leq 1$ and $0 < \alpha < 1$. Then the generalized Stieltjes integral of f with respect to g is defined as follows, (using (14) and (15)),

$$\int_a^b f\,dg = (-1)^\alpha \int_a^b D^\alpha_{a+}f_{a+}(x)D^{1-\alpha}_{b-}g_{b-}(x)\,dx + f(a+)[g(b-) - g(a+)]. \tag{16}$$

Remark that if $\alpha p < 1$, under the assumptions of the above definition, we have $f \in I^\alpha_{a+}(L^p)$, and (16) can be written as

$$\int_a^b f\,dg = (-1)^\alpha \int_a^b D^\alpha_{a+}f_{a+}(x)D^{1-\alpha}_{b-}g_{b-}(x)\,dx. \tag{17}$$

For $a \leq c < d \leq b$, the restriction of $f \in I^\alpha_{a+}(L^p(a,b))$ to (c,d) belongs to $I^\alpha_{c+}(L^p(c,d))$ and the continuation of $f \in I^\alpha_{c+}(L^p(c,d))$ by zero beyond (c,d)

belongs to $I^{\alpha}_{a+}(L^p(a,b))$. Thus, if $f \in I^{\alpha}_{a+}(L^p)$ and $g_{b-} \in I^{1-\alpha}_{b-}(L^q)$, then the integral $\int_a^b 1_{(c,d)} f dg$ in the sense of (17) exists for any $a \leq c < d \leq b$, and whenever the left-hand side is defined in the sense of (17), we have

$$\int_c^d f\ dg = \int_a^b 1_{(c,d)} f\ dg. \tag{18}$$

3.2 A Priori Estimates

We have the following basic estimates. Let

$$\Lambda_\alpha(g) := \frac{1}{\Gamma(1-\alpha)} \sup_{0<\eta<\xi<1} |(D^{1-\alpha}_{\xi-} g_{\xi-})(\eta)|. \tag{19}$$

Then

$$\Lambda_\alpha(g) \leq \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \parallel g \parallel_{1-\alpha,\infty,0} < \infty. \tag{20}$$

If $f \in W^{\alpha,1}(0,1)$, and $g \in W_0^{1-\alpha,\infty}(0,1)$ then the integral $\int_0^\xi f\ dg$ exists for all $\xi \in [0,1]$.

Also, by (18) we have

$$\int_0^\xi f\ dg = \int_0^1 f 1_{(0,\xi)}\ dg.$$

Using (17) we get

$$\int_0^\xi f\ dg = (-1)^\alpha \int_0^\xi D^\alpha_{0+} f(\eta) D^{1-\alpha}_{\xi-} g_{\xi-}(\eta)\ d\eta.$$

Then

$$\left|\int_0^\xi f\ dg\right| \leq \sup_{0<\eta<\xi} |D^{1-\alpha}_{\xi-} g_{\xi-}(\eta)| \int_0^\xi |D^\alpha_{0+} f(\eta)|\ d\eta.$$

Hence

$$\left|\int_0^\xi f\ dg\right| \leq \Lambda_\alpha(g) \parallel f \parallel_{\alpha,1}. \tag{21}$$

4 Stochastic Integrals with Respect to a fBm

Let us recall the following result, for details and proofs see [37].

Lemma 1 *Let* $\{B_\eta^H : \eta \geq 0\}$ *be a real-valued fBm of Hurst parameter* $H \in (\frac{1}{2}, 1)$. *If* $1 - H < \alpha < \frac{1}{2}$, *then*

$$\mathbb{E} \sup_{0 \leq \gamma \leq \eta \leq 1} |D_{\eta-}^{1-\alpha} B_{\eta-}^H(\gamma)|^p < \infty, \tag{22}$$

for all $p \in [1, \infty)$.

Let $\{B_\eta^H : \eta \in [0, 1]\}$ be a real-valued fBm, with the Hurst parameter $\frac{1}{2} < H < 1$, defined on a complete probability space $(\Omega, \mathcal{F}, P)$. By (1), we have

$$\mathbb{E}(|B_\eta^H - B_\gamma^H|^2) = |\eta - \gamma|^{2H},$$

and for any $p \geq 1$ we have

$$\| B_\eta^H - B_\gamma^H \|_p = [\mathbb{E}(|B_\eta^H - B_\gamma^H|^p)]^{\frac{1}{p}} = c_p |\eta - \gamma|^H. \tag{23}$$

By Lemma (1) we know that the random variable

$$G = \frac{1}{\Gamma(1-\alpha)} \sup_{0 < \gamma < \eta < 1} |D_{\eta-}^{1-\alpha} B_{\eta-}^H(\gamma)| \tag{24}$$

has moments of all orders.

As a consequence for $1 - H < \alpha < \frac{1}{2}$, the pathwise integral $\int_0^\eta u_\gamma dB_\gamma^H$ exists when $u = \{u_\eta, \eta \in [0, 1]\}$ is a stochastic process whose trajectories belong to the space $W^{\alpha,1}$ and B_γ^H is a fBm with $H > \frac{1}{2}$. Moreover, we have the estimate

$$\left| \int_0^1 u_\gamma \, dB_\gamma^H \right| \leq G \|u\|_{\alpha,1}. \tag{25}$$

5 Main Results with Proofs

5.1 Main Result

We state the main result of the present work:

Theorem 1 *Let* $\alpha \in (1 - H, \frac{1}{2})$. *Assume that* $\phi \in W^{\alpha,\infty}[0, 1]$ *and the function* A *satisfies the assumptions A1, A2, A3, and A4. Then for every* $T > 0$, *there exists* *a unique stochastic process* $Y \in L^0((\Omega, \mathcal{F}, P), C([0, T], W^{\alpha,\infty}[0, 1]))$ *solution of the Eq.* (6).

As we already defined in Sect. 4, the stochastic integral with respect to the fBm is well defined pathwise. The above theorem will be proved using a contraction principle that will give us the existence and uniqueness of local solutions. In order to get the global solution, we will need to have an a priori estimate of the solution in some functional spaces. The proof will follow after several steps and will be stated in Sect. 5.3.

5.2 Fixed Point Argument

Consider the operator

$$F : C([0, T], W^{\alpha,\infty}[0, 1]) \longrightarrow C([0, T], W^{\alpha,\infty}[0, 1])$$

defined by

$$F(Y_\xi(t)) := \phi_\xi + \int_0^t \int_0^\xi A(Y_\gamma(s))\, dg_\gamma ds, \tag{26}$$

where $g \in C([0, T], W_0^{1-\alpha,\infty}[0, 1])$, $\phi \in W^{\alpha,\infty}[0, 1]$ and A satisfies the assumptions A1, A2, A3, and A4.

Remark 1 Let us remark that the function g in the Eq. (26) is a function of time t and γ and the fractional integration is with respect to the parameter $\gamma \in (0, 1)$.

For a given $R > 0$, let us define the ball $B_{R,T}$ in $C([0, T], W^{\alpha,\infty}[0, 1])$ as

$$B_{R,T} = \{Y \in C([0, T], W^{\alpha,\infty}[0, 1]) : ||Y||_{\alpha,\infty} \leq R\}.$$

We state the following lemmas and propositions without proof. Detailed proofs for this particular equation can be found in [7].

Lemma 2 *Given a positive constant* $R_1 > \|\phi\|_{\alpha,\infty}$, *there exists* $T_1 > 0$ *such that* $F(B_{R_1,T_1}) \subseteq B_{R_1,T_1}$. *The time* T_1 *depends on* R_1, α *and the initial condition* $\|\phi\|_{\alpha,\infty}$.

Now in order to prove that the operator F is a contraction, we will need the following two propositions.

Proposition 1 *Let* $f \in C([0, T], W_0^{\alpha,\infty}[0, 1])$ *and* $g \in C([0, T], W_0^{1-\alpha,\infty}[0, 1])$. *Then, for all* $\xi \in [0, 1]$ *and* $t \in [0, T]$

$$
\begin{aligned}
&\left|\int_0^t \int_0^\xi f_\gamma(s)\, dg_\gamma ds\right| \\
&+ \int_0^\xi (\xi-\eta)^{-\alpha-1} \left|\int_0^t \int_0^\xi f_\gamma(s)\, dg_\gamma ds - \int_0^t \int_0^\eta f_\gamma(s)\, dg_\gamma ds\right| d\eta \\
&\leq \sup_{0\leq t\leq T} \Lambda_\alpha(g) b_\alpha^{(3)} \int_0^t \int_0^\xi [(\xi-\gamma)^{-2\alpha} + \gamma^{-\alpha}] \Bigg(|f_\gamma(s)| \\
&+ \int_0^\gamma \frac{|f_\gamma(s) - f_\delta(s)|}{(\gamma-\delta)^{\alpha+1}}\, d\delta \Bigg) d\gamma ds. \qquad (27)
\end{aligned}
$$

where $b_\alpha^{(3)}$ is a constant which depends on α, its explicit expression is given below.

Proposition 2 *Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying the assumptions A3 and A4. Then for all $N > 0$ and $|X_1|, |X_2|, |X_3|, |X_4| \leq N$ for all $X_1, X_2, X_3, X_4 \in \mathbb{R}$,*

$$
\begin{aligned}
&|h(X_1) - h(X_2) - h(X_3) + h(X_4)| \\
&\leq M_1|X_1 - X_2 - X_3 + X_4| + M_N|X_1 - X_3|(|X_1 - X_2| + |X_3 - X_4|). \qquad (28)
\end{aligned}
$$

Lemma 3 *Given a positive constant $R_2 > \|\phi\|_{\alpha,\infty}$, there exists $T_2 > 0$ and a constant $0 < C < 1$ such that,*

$$||F(Y_1) - F(Y_2)||_{\alpha,\infty} \leq C||Y_1 - Y_2||_{\alpha,\infty}$$

for all $Y_1, Y_2 \in B_{R_2,T_2}$.

Now, we are able to state the following theorem:

Theorem 2 *Let $0 < \alpha < \frac{1}{2}$, $g \in C([0, T], W_0^{1-\alpha,\infty}[0, 1])$. Consider the integrodifferential equation*

$$Y_\xi(t) = \phi_\xi + \int_0^t \int_0^\xi A(Y_\eta(s))\, dg_\eta(s) ds \qquad (29)$$

where $t \in [0, T]$, $\xi \in [0, 1]$. Assume that A satisfies assumptions A1, A2, A3, and A4 and that $\phi \in W^{\alpha,\infty}([0, 1])$. Then, there exists $T_0 > 0$ such that the above equation has a unique solution

$$Y \in C([0, T], W^{\alpha,\infty}[0, 1]),$$

for all $T \leq T_0$.

Proof Choose $T_0 = \min\{T_1, T_2\}$, and $R = \min\{R_1, R_2\}$. Then, using Lemma 2 and Lemma 3 the operator F is a contraction on $B_{R,T}$ for all $T \leq T_0$ and this completes the proof. □

Now, we can show that the solution of (29) is global in time.

Theorem 3 *Let* $1 - H < \alpha < \frac{1}{2}$, $g \in C([0,T], W^{1-\alpha,\infty,0}[0,1])$. *Assume that* A *satisfies assumptions A1, A2, A3, and A4 and that* $\phi \in W^{\alpha,\infty}([0,1])$. *Then for all* $T > 0$, *there exists a unique* $Y \in C([0,T], W^{\alpha,\infty}[0,1])$ *solution of* (29).

Proof It is enough to get an estimate in $C([0,T], W^{\alpha,\infty}[0,1])$. We can write

$$||Y(t)||_{\alpha,\infty} = \sup_{\xi\in[0,1]} \left(Y_\xi(t) + \int_0^\xi \frac{|Y_\xi(t) - Y_\eta(t)|}{(\xi-\eta)^{\alpha+1}}\, d\eta \right). \tag{30}$$

Consider the first term on the right side of the above equality,

$$|Y_\xi(t)| \le |\phi(\xi)| + \int_0^t \left| \int_0^\xi A(Y_\eta(s))\, dg(s,\eta) \right| ds.$$

From Lemma (2) we can obtain

$$|Y_\xi(t)| \le |\phi_\xi| + \int_0^t \Lambda_\alpha(g) t \left(\frac{M_2}{1-\alpha} + M_1 ||Y(s)||_{\alpha,\infty} ds \right). \tag{31}$$

Consider the second term. We have

$$\begin{aligned}
&\left| \int_0^\xi \frac{|Y_\xi(t) - Y_\eta(t)|}{(\xi-\eta)^{\alpha+1}}\, d\eta \right| \\
&= \int_0^\xi \frac{1}{(\xi-\eta)^{\alpha+1}} \\
&\left| \phi(\xi) + \int_0^t \int_0^\xi A(Y_\gamma(s))\, dg_\gamma ds - \left(\phi(\eta) + \int_0^t \int_0^\eta A(Y_\gamma(s))\, dg_\gamma ds \right) \right| \\
&\le \int_0^\xi \frac{1}{(\xi-\eta)^{\alpha+1}} \left(|\phi_\xi - \phi_\eta| + \int_0^t \left| \int_\eta^\xi A(Y_\gamma(s))\, dg_\gamma \right| ds d\eta \right) \\
&\le \int_0^\xi \frac{|\phi_\xi - \phi_\eta|}{(\xi-\eta)^{\alpha+1}}\, d\eta + \int_0^t \int_0^\xi \frac{1}{(\xi-\eta)^{\alpha+1}} \left| \int_\eta^\xi A(Y_\gamma(s))\, dg_\gamma \right| d\eta ds.
\end{aligned}$$

From Lemma (2) we can obtain

$$\begin{aligned}
&\left| \int_0^\xi \frac{|Y_\xi(t) - Y_\eta(t)|}{(\xi-\eta)^{\alpha+1}}\, d\eta \right| \\
&\le \int_0^\xi \frac{|\phi_\xi - \phi_\eta|}{(\xi-\eta)^{\alpha+1}}\, d\eta \\
&+ \Lambda_\alpha(g) \int_0^t \left(\frac{M_2 b_\alpha^{(1)}}{1-2\alpha} \xi^{1-2\alpha} + \frac{M_1}{\alpha(1-\alpha)} ||Y(s)||_{\alpha,\infty} \xi^{1-\alpha} \right) ds.
\end{aligned} \tag{32}$$

By (31) and (32) we get,

$$
\begin{aligned}
&||Y(t)||_{\alpha,\infty} \\
&\leq ||\phi||_{\alpha,\infty} \\
&+ \Lambda_\alpha(g) \int_0^t \left(M_2 \left(\frac{1}{1-\alpha} + \frac{b_\alpha^{(1)}}{1-2\alpha} \right) + M_1 \left(1 + \frac{1}{\alpha(1-\alpha)} \right) ||Y(s)||_{\alpha,\infty} \right) ds.
\end{aligned}
$$

By Gronwall's inequality we obtain,

$$
||Y||_{\alpha,\infty} \leq ||\phi||_{\alpha,\infty} \exp(KT),
$$

where $K = \sup_{0 \leq t \leq T} \Lambda_\alpha(g) \left[M_2 \left(\frac{1}{1-\alpha} + \frac{b_\alpha^{(1)}}{1-2\alpha} \right) + M_1 \left(1 + \frac{1}{\alpha(1-\alpha)} \right) \right]$.

Hence, the local solution is global in time. □

5.3 *Proof of Theorem 1*

For every $t \in [0, T]$, the random variable

$$
G = \frac{1}{\Gamma(1-\alpha)} \sup_{0<\eta<\xi<1} |(D_{\xi-}^{1-\alpha} B_{\xi-})_\eta(t)|
$$

has moments of all orders by Proposition 1. Hence, the pathwise integral

$$
\int_0^1 A(Y_\eta(t)) dB_\eta(t)
$$

exists for $1 - H < \alpha < \frac{1}{2}$ and the existence and uniqueness of solutions follows from Theorem 3 which completes the proof.

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References

1. Alòs E, Mazet O, Nualart D (2001) Stochastic calculus with respect to Gaussian processes. Ann Probab 29(2):766–801
2. Alós E, Nualart D (2003) Stochastic integration with respect to the fractional Brownian motion. Stoch Stoch Rep 75(3):129–152

3. Berselli LC, Bessaih H (2002) Some results for the line vortex equation. Nonlinearity 15:1729–1746
4. Berselli LC, Gubinelli M (2007) On the global evolution of vortex filaments, blobs, and small loops in 3d ideal flows. Commun Math Phys 269(3):693–713
5. Bertozzi AL, Majda AJ (2002) Vorticity and incompressible flow. Cambridge University Press, Cambridge
6. Bessaih H, Gubinelli M, Russo F (2005) The evolution of a random vortex filament. Ann Probab 33(5):1825–1855
7. Bessaih H, Wijeratne C (2011) An integrodifferential equation driven by fractional Brownian motion. arXiv:1103.3489
8. Biagini F, Hu Y, Øksendal B, Zhang T (2008) Stochastic calculus for fractional Brownian motion and applications. Springer, Berlin
9. Carmona P, Coutin L, Montseny G (2003) Stochastic integration with respect to fractional Brownian motion. Ann Inst H Poincaré Probab Stat 39(1):27–68
10. Cheridito P (2003) Arbitrage in fractional Brownian motion models. Financ Stoch 7(4):533–553
11. Chorin AJ (1994) Vorticity and turbulence. Springer, New York
12. Comte F, Renault E (1996) Long memory continuous time models. J Econom 73(1):101–149
13. Coutin L (2007) An introduction to (stochastic) calculus with respect to fractional Brownian motion. Séminaire de Probabilités XL. Lectures Notes in Math 1899:3–65
14. Coutin L, Lejay A (2005) Semi-martingales and rough paths theory. Electron J Probab 10(23):761–785
15. Coutin L, Qian Z (2002) Stochastic analysis, rough path analysis and fractional Brownian motions. Probab Theory Relat Fields 122(1):108–140
16. Decreusefond L, Üstünel AS (1998) Stochastic analysis of the fractional Brownian motion. Potential Anal 10(2):177–214
17. Doukhan P, Oppenheim G, Taqqu MS (eds) (2003) Theory and applications of long-range dependence. Birkhäuser, Boston
18. Duncan TE, Hu Y, Pasik-Duncan B (2000) Stochastic calculus for fractional Brownian motion i. Theory. SIAM J Control Optim 38(2):582–612
19. Duncan TE, Yan Y, Yan P (2001) Exact asymptotics for a queue with fractional Brownian input and applications in ATM networks. J Appl Probab 36(4):932–945
20. Embrechts P, Maejima M (2002) Selfsimilar processes. Princeton University Press, Princeton
21. Flandoli F (2002) On a probabilistic description of small scale structures in 3D fluids. Ann Inst H Poincaré Probab Stat. 38(2):207–228
22. Gubinelli M, Lejay A, Tindel S (2006) Young integrals and SPDEs. Potential Anal 25(4):307–326
23. Hurst HE (1951) Long-term storage capacity of reservoirs. Trans Am Soc Civ Eng 116:770–799
24. Kilbas AA, Marichev OI, Samko SG (1993) Fractional integrals and derivatives: theory and applications. Princeton University Press, Princeton
25. Klingenhöfer F, Zähle M (1999) Ordinary differential equations with fractal noise. Proc Am Math Soc 127(4):1021–1028
26. Kolmogorov AN (1940) Wienersche Spiralen und einige andere interessante kurven im Hilbertschen Raum. Comptes Rendus (Doklady) Acad Sci USSR (NS) 30:299–303
27. Kruka I, Russoa F, Tudorb CA (2007) Wiener integrals, Malliavin calculus and covariance measure structure. J Funct Anal 249(1):92–142
28. Leland WE, Taqqu MS, Willinger W, Wilson DV (1993) On the self-similar nature of ethernet traffic. ACM SIGCOMM Comput Commun Rev 23(4):183–193
29. Lyons T (1994) Differential equations driven by rough signals. I. An extension of an inequality of LC young. Math Res Lett 1(4):451–464
30. Lyons TJ (1998) Differential equations driven by rough signals. Rev Mat Iberoamericana 14(2):215–310
31. Mandelbrot B, Van Ness J (1968) Fractional Brownian motions, fractional noises and applications. SIAM Rev 10:422–437

32. Marchioro C, Pulvirenti M (1994) Mathematical theory of incompressible nonviscous fluids. Springer, New York
33. Mishura YS (2008) Stochastic calculus for fractional Brownian motion and related processes. Springer, Berlin
34. Norros I (1995) On the use of fractional Brownian motion in the theory of connectionless networks. IEEE J Sel Areas Commun 13(6):953–962
35. Nualart D (2003) Stochastic integration with respect to fractional Brownian motion and applications. Stoch Stoch Rep 336:3–40
36. Nualart D (2006) The Malliavin calculus and related topics. Springer, Berlin
37. Nualart D, Răscanu A (2001) Differential equations driven by fractional Brownian motion. Collectanea Mathematica 53(1):55–81
38. Rogers LCG (1997) Arbitrage with fractional Brownian motion. Math Financ 7:95–105
39. Saffman PG (1992) Vortex dynamics. Cambridge University Press, New York
40. Young LC (1936) An inequality of the Hölder type connected with Stieltjes intergration. Acta Math 67(1):251–282

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